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ON THE UNIFORM APPROXIMATION PROBLEM FOR THE SQUARE OF THE CAUCHY-RIEMANN OPERATOR

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Let X be a compact subset of the plane and f a continuous function on X satisfying the equation $\overline{\partial}^2 f = 0$ in the interior of X. It is unknown whether f can be uniformly approximated on X by functions g satisfying the equation $\overline{\partial}^2 g = 0$ in some neighbourhood (depending on g) of X. We show that this is the case under the additional assumption that f satisfies a Dini-type continuity condition.

1. Introduction. Let L be a constant coefficients elliptic differential operator in $\mathbb{R}^d$. Given a compact $X \subset \mathbb{R}^d$ let $H(X, L)$ be the closure in $C(X)$ of the set

$$\{f|_X : Lf = 0 \text{ on some neighbourhood of } X\}.$$

It is clear that a function in $H(X, L)$ necessarily belongs to

$$h(X, L) = C(X) \cap \{Lf = 0 \text{ on the interior of } X\}.$$

The uniform approximation problem for the operator L consists in characterizing those X for which $H(X, L) = h(X, L)$. Since $h(X, L) = C(X)$ if and only if X is nowhere dense, our problem restricted to nowhere dense compact sets becomes that of describing those X for which $H(X, L) = C(X)$.

A complete solution for $L = \Delta$ (the Laplacian) was independently obtained in the forties by Deny [2] and Keldysh [6] using a duality approach relying on potential theoretic methods. Denoting by Cap the Wiener capacity of classical potential theory, their result can be stated as follows.

THEOREM (*Deny-Keldysh*). *The identity* $H(X, \Delta) = h(X, \Delta)$ *occurs if and only if one has* $\operatorname{Cap}(B \setminus \overset{\circ}{X}) = \operatorname{Cap}(B \setminus X)$ *for each open ball* B.

Vitushkin [11] solved in the sixties the problem for $L = \overline{\partial}$ (the Cauchy-Riemann operator in the plane) introducing his far reaching constructive scheme for the approximation, based on a localization procedure and the so called matching coefficients technique.

Let α stand for continuous analytic capacity (see **[3]** or **[11]**). We then have

THEOREM (*Vitushkin*). *The identity* $H(X, \overline{\partial}) = h(X, \overline{\partial})$ *occurs if and only if one has* $\alpha(\Delta \setminus \overset{\circ}{X}) = \alpha(\Delta \setminus X)$ *for each open disc* Δ.

In spite of the formal analogy between the above two statements, no unified proof of them has been found yet. This fact helps to explain why very little is known for other operators, in particular for Δ^n, $n > 1$, $d \geq 2$ or for $\overline{\partial}^n$, $n > 1$, in the plane.

In this paper the special case $d = 2$ and $L = \overline{\partial}^2$ is considered. Although we have not been able to solve the problem we do prove a result which seems to be quite close to the most plausible conjecture one is able to formulate.

THEOREM. *Let* $X \subset \mathbb{C}$ *be compact. Then each Dini-continuous function in* $h(X, \overline{\partial}^2)$ *is in* $H(X, \overline{\partial}^2)$.

Conjecture. For each compact $X \subset \mathbb{C}$ one has

$$H(X, \overline{\partial}^2) = h(X, \overline{\partial}^2). \tag{1}$$

To understand the conjecture one must realize that the capacitary conditions in the Deny-Keldysh and Vitushkin theorems arise because the fundamental solutions of the involved operators are unbounded functions. This is not the case for the fundamental solution $\frac{1}{\pi}\frac{\overline{z}}{z}$ of $\overline{\partial}^2$ and so we get a first reason to rule out the existence of capacitary conditions on X necessary for (1). But it has been pointed out [7] that one should also take into account the capacities associated to continuous functions and the kernels given by the first partial derivatives of the fundamental solution, for example, $1/z = \overline{\partial}(\overline{z}/z)$ and $\overline{z}/z^2 = -\partial(\overline{z}/z)$. One can easily check that these capacities vanish on lines and thus no examples like those constructed by Hedberg **[5]** in connection with the L^p approximation problem for $\overline{\partial}^2$, $2 \leq p < \infty$, can be used to disprove the conjecture (see §6).

We list now some particular instances in which (1) has been verified:

1. X is nowhere dense **[10]**.
2. The complement of X has finitely many connected components **[1]**.
3. The inner boundary of X is countable **[12]**.

The interested reader is urged to consult [7] in which a formal analog of the problem considered here is solved. In fact the scheme of the proof of our theorem is parallel to that of [7]: there is a first constructive part exploiting a covering lemma due to Mateu and then a duality argument in which the differentiability properties of certain potentials play an important role. The Dini type condition allows us to surmount the difficulties related to the bad behaviour of classical operators (more concretely Cauchy and Beurling transforms) under the supremum norm.

In §2 we state some known results and establish some notation to be used later. Section 3 is devoted to the proof of a technical lemma. In §4 we show how to approximate some Cauchy potentials of measures. The proof of the theorem is presented in §5. To support the conjecture we prove in §6 that (1) holds for the string of beads.

2. Background notation and results. In this section we establish some notation and we recall some known facts to be used throughout the rest of the paper.

A. *Newtonian capacity.* Newtonian capacity is the set function defined on Borel sets $E \subset \mathbb{C}$ by

$$C(E) = \sup \|\mu\| \tag{2}$$

where the supremum is taken over all positive measures μ with compact support contained in E and satisfying $\frac{1}{|z|} * \mu \leq 1$ on E (or, equivalently, on $\mathbb{C}$). In fact, one can besides require the continuity of $\frac{1}{|z|} * \mu$ without altering the supremum in (2).

There is a dual expression for newtonian capacity, namely

$$C(E) = \inf \|\lambda\|$$

the infimum being over all positive measures λ such that $\frac{1}{|z|} * \lambda \geq 1$ on E. From that it readily follows that $\frac{1}{|z|} * \lambda$ is finite C-almost everywhere for all complex Borel measures λ.

One can easily get a lower bound for $C(E)$ in terms of Hausdorff content. Let $h(t)$, $t > 0$, be a measure function (that is, nondecreasing and continuous) of the form $h(t) = w(t)t$ with

$$\int \frac{w(t)}{t}\, dt < \infty .$$

Set

$$\varepsilon(\delta) = w(\delta) + \int_0^\delta \frac{w(t)}{t}\, dt, \qquad \delta > 0 .$$

Then

$$M^h(E) \le \varepsilon(d)C(E)$$

where d is the diameter of E and $M^h(E) = \inf \sum_j h(d_j)$, the infimum being taken over all coverings of E by squares of diameter d_j.

We refer the reader to **[4]** for a convenient account on newtonian capacity.

B. *A covering lemma.* A family of discs (Δ_j) is said to be almost disjoint with constant N provided each point in $\mathbb{C}$ belongs to at most N of the discs Δ_j.

Given a set E and a disc Δ with radius δ we say that Δ has the three points property (with constant η, $0 < \eta < 1$) with respect to E if we can find points z_1, z_2 and z_3 in $\Delta \cap E$ such that $|z_1 - z_2| \ge \eta\delta$ and $d(z_3, l(z_1, z_2)) \ge \eta\delta$, where $l(z_1, z_2)$ is the straight line through z_1 and z_2.

The relevance of this notion in this paper is due to the fact that it allows us to produce a function annihilated by $\overline{\partial}^2$ off a disc and with a given expansion at ∞ up to order 2. More concretely we have

2.1. LEMMA. *Let $\delta > 0$ and $0 < \eta < 1$. Let z_1, z_2, z_3 be three points satisfying, for some positive constant C,*

$$|z_j - z_k| \le C\delta, \qquad j, k = 1, 2, 3,$$

and

$$\min\{|z_1 - z_2|, d(z_3, l(z_1, z_2))\} \ge \eta\delta.$$

Assume $a_0, a_1, b_1 \in \mathbb{C}$ and

$$|a_0| \le Cw, \quad |a_1|, |b_1| \le C\delta w,$$

for some $w > 0$. Then there exists a function g satisfying $\overline{\partial}^2 g = 0$ on $\mathbb{C} \backslash \{z_1, z_2, z_3\}$,

$$g(z) = a_0 \frac{\overline{z}}{z} + a_1 \frac{1}{z} b_1 \frac{\overline{z}}{z^2} + 0(|z|^{-2}), \qquad \text{as } z \to \infty,$$

and

$$\|g\|_\infty \le C\eta^{-2} w.$$

REMARK. In our application of the above statement, w will be $w(\delta)$, the modulus of continuity of the function to be approximated, and C and η constants depending only on $w(\delta)$.

Proof of 2.1. Set $g = \frac{\bar{z}}{z} * \mu$, where $\mu = \lambda_1\delta_{z_1} + \lambda_2\delta_{z_2} + \lambda_3\delta_{z_3}$ and the coefficients λ_j are chosen so that g has the required expansion at ∞. □

2.2. LEMMA (*Mateu* [7]). *Let $h(t) = tw(t)$ be a measure function with w nondecreasing and satisfying $w(2t) \leq Cw(t)$. Then for any compact $K \subset \mathbb{C}$ there exist a finite family of discs (Δ_j) which can be divided into two subfamilies (Δ_j^g) and (Δ_j^b) (the superscripts g and b stand for good and bad) in such a way that the following holds (with constants depending on w but not on K).*

(a) $K \subset \bigcup_j \Delta_j$.

(b) *For some $\lambda = \lambda(w) > 1$, $(\lambda\Delta_j)$ is almost disjoint (with constant depending only on w).*

(c) *Each Δ_j^g has the three points property (with constant depending only on w) with respect to K.*

(d) $\sum_j h(\delta_j^b) \leq CM^h(K)$, *where δ_j^b denotes the radius of the disc Δ_j^b.*

(e) *For each disc Δ of radius δ*

$$\sum_{\Delta_j^b \subset \Delta} h(\delta_j^b) \leq Ch(\delta).$$

REMARKS. 1. It is not difficult to realize that moreover one has

$$\text{(f)} \qquad \sum_j w(\delta_j)\delta_j^2 \leq C\delta M^h(K).$$

2. It has been recently shown in [8] that is not possible, for $h(t) = t$, to construct a family (Δ_j) satisfying (a), (b) and (d) (with Δ_j^b replaced by Δ_j).

3. A lemma. In this section we give a proof of the following technical lemma.

3.1. LEMMA. *Let $w(t)$, $t > 0$, be a non-decreasing continuous function satisfying $w(2t) \leq Cw(t)$ and $\int_0^1 \frac{w(t)}{t}\,dt < \infty$. Suppose that (Δ_j) is a finite family of discs with centers z_j and radii δ_j such that $(\lambda\Delta_j)$ is almost disjoint for some $\lambda > 1$. Then the following holds.*

(i) *For some constant $C > 0$*

$$(3) \qquad \sum_j \min\left(w(\delta_j), \frac{w(\delta_j)\delta_j^2}{|z - z_j|^2}\right) \leq C\varepsilon(d), \qquad z \in \mathbb{C},$$

where d *is the diameter of* $\bigcup_j \Delta_j$ *and*

$$\varepsilon(\delta) = w(\delta) + \int_0^\delta \frac{w(t)}{t}\, dt, \qquad \delta > 0. \tag{4}$$

(ii) *If moreover the family* (Δ_j) *satisfies the packing condition*

$$\sum_{\Delta_j \subset \Delta} w(\delta_j)\delta_j \le C w(\delta)\delta,$$

for each disc Δ *of radius* δ, *then*

$$\sum_j \min\left(w(\delta_j), \frac{w(\delta_j)\delta_j}{|z - z_j|}\right) \le C\varepsilon(\delta), \qquad z \in \mathbb{C}. \tag{5}$$

Proof. To prove (3) (and (5)) it is enough to assume $z \in 2D$, D being a disc of radius d containing $\bigcup_j \Delta_j$. In fact if $z \notin 2D$ then

$$\sum_j \min\left(w(\delta_j), \frac{w(\delta_j)\delta_j^2}{|z - z_j|^2}\right) \le \sum_j \frac{w(\delta_j)\delta_j^2}{|z - z_j|^2} \le \frac{w(d)}{d^2} \sum_j \delta_j^2 \le C w(d)$$

where in the last inequality almost disjointness was used.

We argue similarly for (5) applying the packing condition to the test disc D.

Fix then $z \in 2D$. Set $J = \{j : z \notin \lambda\Delta_j]\}$ and $J^* = \{j : z \in \lambda\Delta_j\}$. Thus

$$\sum_{j \in J^*} \min\left(w(\delta_j), \frac{w(\delta_j)\delta_j^2}{|z - z_j|^2}\right) \le \# J^* w(d).$$

To estimate the above sum over the indices $j \in J$ we proceed as follows. Set $\mu_j = w(\delta_j)\chi_j(\zeta)\, dx\, dy$, $j \in J$, where χ_j stands for the characteristic function of Δ_j. Hence

$$\frac{w(\delta_j)\delta_j^2}{|z - z_j|^2} \le C \int \frac{d\mu_j(\zeta)}{|z - \zeta|^2}$$

and so

$$\sum_{j \in J} \min\left(w(\delta_j), \frac{w(\delta_j)\delta_j^2}{|z - z_j|^2}\right) \le C \int \frac{d\mu(\zeta)}{|z - \zeta|^2},$$

where $\mu = \sum_{j \in J} \mu_j$. Write $\mu(r) = \mu(\Delta(z, r))$. Then $\mu(r) = 0$ provided r is small enough, because z does not belong to the support of μ. Consequently one has

$$\int \frac{d\mu(\zeta)}{|z - \zeta|^2} = \int_0^{3d} \frac{d\mu(r)}{r^2} = \frac{\mu(3d)}{(3d)^2} + 2\int_0^{3d} \frac{\mu(r)}{r^3}\, dr.$$

Now it is clearly sufficient to show that

(6) $$\mu(r) \leq Cw(r)r^2, \qquad r > 0.$$

To get (6) one writes

$$\begin{aligned}\mu(r) &= \sum_{j\in J} \mu_j(\Delta_j \cap \Delta(z, r)) \\ &\leq \sum_{\delta_j \leq r} w(\delta_j)\pi\delta_j^2 + \sum_{\delta_j > r} w(\delta_j)\pi r^2 \equiv \mathrm{I} + \mathrm{II},\end{aligned}$$

where only indices $j \in J$ with $\Delta_j \cap \Delta(z, r) \neq \varnothing$ are considered in the above sums.

If $\delta_j \leq r$ then $\Delta_j \subset \Delta(z, 3r)$ and so

$$\mathrm{I} \leq \sum_{\Delta_j \subset \Delta(z, 3r)} w(\delta_j)\delta_j^2 \leq Cw(r)r^2.$$

If $\delta_j > r$ then $r \geq (\lambda - 1)\delta_j$ and so $\mathrm{II} \leq Cw(r)r^2 N$, where N is the number of indices $j \in J$ with $\delta_j > r$ and $\Delta_j \cap \Delta(z, r) \neq \varnothing$. It turns out that N can be estimated by some constant, because

$$\begin{aligned}4\pi r = \text{ length } \partial\Delta(z, 2r) &\geq \text{ length } \bigcup_j (\Delta_j \cap \partial\Delta(z, 2r)) \\ &\geq C \sum_j \text{ length } \Delta_j \cap \partial\Delta(z, 2r) \geq CrN.\end{aligned}$$

This completes the proof of (i).

The proof of (ii) is essentially the same. One uses the packing assumption in estimating the analog of the term I above. □

4. Approximation of some Cauchy transforms. This section is devoted to the proof of the following

4.1. THEOREM. *Let $X \subset \mathbb{C}$ be compact and let μ be a complex Borel measure whose support does not intersect the interior of X and such that $(1/|z|) * |\mu|$ is continuous on $\mathbb{C}$. Then if $P = (1/z) * \mu$ and $Q = (\overline{z}/z^2) * \mu$ we have that*

$$P, Q, P^2 \text{ and } PQ \in H(X, \overline{\partial}^2).$$

REMARK. The function P is continuous on X and analytic on $\overset{\circ}{X}$. Although its modulus of continuity may vanish at zero as slowly as we wish, some Dini type condition is implicit in the fact that $(1/|z|) * |\mu|$

is continuous. It is not known whether a function f, continuous on X and analytic on $\overset{\circ}{X}$, belongs to $H(X, \bar{\partial}^2)$, even under the additional assumption that f is the Cauchy transform of some measure.

The proof of 4.1 goes by duality. Let λ be a complex Borel measure on X annihilating $H(X, \bar{\partial}^2)$. Then the potential $F = (\bar{z}/z) * \lambda$ vanishes on the complement of X and, on the other hand, we have

$$\int P(z)\, d\lambda(z) = -\int \left(\frac{1}{z} * \lambda\right) d\mu = -\int \bar{\partial} F(z)\, d\mu(z).$$

It can be shown that F is continuous except at those points where λ has a positive mass **[10]**. This implies that F vanishes on $(\overset{\circ}{X})^c$ except, eventually, on a countable set.

Therefore, we are left with the task of proving that $\bar{\partial} F$ vanishes on $(\overset{\circ}{X})^c$ μ-almost everywhere and this will be achieved by studying ordinary differentiability properties of F.

4.2. LEMMA. *Let λ be a complex Borel measure and set $F = (\bar{z}/z) * \lambda$. Then F is differentiable in the ordinary sense except on a set of newtonian capacity zero.*

Proof. We have, in the distributions sense, $\bar{\partial} F = (1/z) * \lambda$ and $\partial F = -(\bar{z}/z^2) * \lambda$. Assume, without loss of generality, that λ is positive and let a be a point with $|z|^{-1} * \lambda(a) < \infty$. We are going to show that

(7) $\quad |F(z) - F(a) - \alpha(z - a) - \beta(\bar{z} - \bar{a})| = o(|z|), \quad$ as $z \to a$,

where $\alpha = (-(\bar{z}/z^2) * \lambda)(a)$ and $\beta = ((1/z) * \lambda)(a)$.

To prove (7) we assume $a = 0$ and according to **[9**, p. 244**]** we write

$$|F(z) - F(0) - (\alpha z + \beta \bar{z}| \le \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\mathrm{I} = \left| \int_{|\zeta| > 2|z|} \left(\frac{\bar{z} - \bar{\zeta}}{z - \zeta} - \frac{\bar{\zeta}}{\zeta} \right) d\lambda(\zeta) - (\alpha z + \beta z) \right|,$$

$$\mathrm{II} = \left| \int_{|\zeta| \le 2|z|} \frac{\bar{z} - \bar{\zeta}}{z - \zeta} d\lambda(\zeta) \right|$$

and

$$\mathrm{III} = \left| \int_{|\zeta| \le 2|z|} \frac{\bar{\zeta}}{\zeta} d\lambda(\zeta) \right|.$$

Set $\lambda(r) = \lambda(\Delta(0, r))$. Since the terms II and III are not greater than $\lambda(2|z|)$, to take care of them it is enough to realize that $\lambda(r) = o(r)$, which follows from

$$\lambda(r) \leq r \int_{|\zeta|<r} |\zeta|^{-1}\, d\lambda(\zeta).$$

We must now estimate I. Fix $\zeta \neq 0$ and apply Taylor's formula to the function $\frac{\overline{z}-\overline{\zeta}}{z-\zeta}$ around $z = 0$ to get

$$\frac{\overline{z}-\overline{\zeta}}{z-\zeta} =' \frac{\overline{\zeta}}{\zeta} + \frac{\overline{\zeta}}{\zeta^2} z - \frac{1}{\zeta}\overline{z} + O\left(\frac{|z|^2}{|\zeta|^2}\right).$$

Using the definition of α and β we obtain

$$\begin{aligned} \mathrm{I} \leq |z| \left| \int_{|\zeta|\leq 2|z|} \overline{\zeta}\zeta^{-2}\, d\lambda(\zeta) \right| + |z| \left| \int_{|\zeta|\leq 2|z|} \zeta^{-1}\, d\lambda(\zeta) \right| \\ + C|z|^2 \int_{|\zeta|>2|z|} |\zeta|^{-2}\, d\lambda(\zeta). \end{aligned}$$

The first two terms can be estimated by $|z| \int_{|\zeta|\leq 2|z|} |\zeta|^{-1}\, d\lambda(\zeta)$ which is $o(|z|)$. To take care of the third it is enough to show that $r\int_r^\infty d\lambda(t)/t^2 \to 0$ as $r \to 0$. This follows from

$$\int_r^\infty \frac{d\lambda(t)}{t^2} = -\frac{\lambda(r)}{r^2} + \int_r^\infty \frac{\lambda(t)}{t^3}\, dt,$$

and l'Hôpital's rule, which gives

$$\lim_{r\to 0} r \int_r^\infty \frac{\lambda(t)}{t^3}\, dt = \lim_{r\to 0} \frac{\lambda(r)}{r} = 0. \qquad \square$$

Proof of 4.1. Keep the notation in the argument following the statement of 4.1. Let E stand for the set of points at which F vanishes and has a non-zero ordinary differential.

Lemma 2.3 in [7] tells us that E has σ-finite length and consequently zero newtonian capacity. On the other hand, the hypothesis on μ guarantees that μ vanishes on sets of zero newtonian capacity. Therefore $\nabla F = 0$ μ-almost everywhere and so

$$\int P(z)\, d\lambda(z) = -\int \overline{\partial} F(z)\, d\mu(z) = 0,$$

$$\int Q(z)\, d\lambda(z) = \int \partial F(z)\, d\mu(z) = 0.$$

Since $\overline{\partial}(P^2) = 2\pi P\mu$,

$$\int P^2(z)\,d\lambda(z) = -2\int \overline{\partial}F(z)P(z)\,d\mu(z) = 0.$$

To complete the proof of 4.1 we are going to show that

$$\int PQ\,d\lambda = -\int \overline{\partial}F(z)Q(z)\,d\mu(z) + \int \partial F(z)P(z)\,d\mu(z). \tag{8}$$

Set $G = \frac{\overline{z}}{z} * \mu$ and consider regularizations λ_ε and F_ε of λ and F. Since F_ε is a smooth compactly supported function, one has

$$\begin{aligned}\int PQ\,d\lambda_\varepsilon &= -\frac{1}{\pi}\int \overline{\partial}G\,\partial G\,\overline{\partial}^2 F_\varepsilon\,dx\,dy \\ &= -\int \overline{\partial}F_\varepsilon\,Q\,d\mu + \frac{1}{\pi}\int \overline{\partial}F_\varepsilon(z)\overline{\partial}G(z)\overline{\partial}\partial G(z).\end{aligned}$$

The last integral above is equal to

$$\begin{aligned}\frac{1}{\pi}\int \overline{\partial}F_\varepsilon\,\overline{\partial}G\,\partial\overline{\partial}G &= \frac{1}{\pi}\int \overline{\partial}F_\varepsilon\,\partial\left(\frac{P^2}{2}\right) \\ &= \frac{1}{\pi}\int \partial F_\varepsilon\,\overline{\partial}\left(\frac{P^2}{2}\right) = \int \partial F_\varepsilon\,P\,d\mu,\end{aligned}$$

and thus

$$\int PQ\,d\lambda_\varepsilon = -\int \overline{\partial}F_\varepsilon\,Q\,d\mu + \int \partial F_\varepsilon\,P\,d\mu. \tag{9}$$

To get (8) from (9) by letting $\varepsilon \to 0$ one only has to apply the Lebesgue dominated convergence theorem to the right-hand side of (9), which is justified by the inequality

$$\frac{1}{|z|} * d|\lambda_\varepsilon| \le C\frac{1}{|z|} * d|\lambda|$$

and the fact that μ vanishes on sets of zero newtonian capacity. □

5. Proof of the theorem. Let $f \in h(X, \overline{\partial}^2)$ be Dini-continuous on X. We can extend f to a compactly supported Dini-continuous function on the whole plane $\mathbb{C}$ [**9**, Chapter VI] with modulus of continuity $w(\delta)$. Fix $\delta > 0$ and consider a δ-Vitushkin scheme $(\Delta_j, \varphi_j, f_j)$ for the approximation of f. This means that (Δ_j) is an almost disjoint covering of $\mathbb{C}$ by discs of radius δ, (φ_j) is a partition of the unity subordinated to (Δ_j) satisfying $|\nabla^l \varphi_j| \le C\delta^{-l}$, $0 \le l \le 2$, for all j, and $f_j = \frac{1}{\pi}\frac{\overline{z}}{z} * \varphi_j\overline{\partial}^2 f$. It turns out that f_j is continuous on $\mathbb{C}$,

$f_j \in h(X, \overline{\partial}^2)$, $\|f_j\|_\infty \le Cw(\delta)$ and $f = \sum_j f_j$. Each f_j has an expansion outside Δ_j of the form (we assume Δ_j is centered at 0)

$$f_j(z) = c_0\frac{\overline{z}}{z} + \frac{b_1}{z} + c_1\frac{\overline{z}}{z^2} + \cdots + \frac{b_n}{z^n} + c_n\frac{\overline{z}}{z^{n+1}} + \cdots,$$

where $c_n = \pi^{-1}\langle\overline{\partial}^2 f_j, z^n\rangle$ and $b_n = -\pi^{-1}\langle\overline{\partial}^2 f_j, z^{n-1}\overline{z}\rangle$, the brackets meaning the duality between compactly supported distributions and C^∞ functions. From these formulas one easily shows that $|c_n|$ and $|b_n|$ are estimated by some constant times $\delta^n w(\delta)$.

The goal of the next lemma is to find functions g_j in $H(X, \overline{\partial}^2)$ whose expansion outside Δ_j coincides with that of f_j up to order 2.

5.1. Lemma. *Let Δ be a disc of center a and radius δ and $\varphi \in C_0^2(\Delta)$ with $|\nabla^l\varphi| \le C\delta^{-l}$, $0 \le l \le 2$. Set $f_\Delta = \frac{1}{\pi}\frac{\overline{z}}{z} * \varphi\overline{\partial}^2 f$ and let E be the* (*compact*) *support of $\overline{\partial}^2 f_\Delta$. Then there exists $g_\Delta \in H(X, \overline{\partial}^2)$ such that $\|g_\Delta\|_\infty \le C\varepsilon(\delta)$ and*

(10) $$|f_\Delta(z) - g_\Delta(z)| \le C\varepsilon(\delta)\delta C(E)|z-a|^{-2}, \quad |z-a| \ge 3\delta,$$

where $\varepsilon(\delta)$ is defined by (4).

Proof. We apply Lemma 2.2 to the set E and the measure function $h(t) = w(t)t$, w being the modulus of continuity of f. We then get a family of discs $\{\Delta_j\}$ satisfying properties from (a) to (e) in 2.2 and also (f) in Remark 1.

Take now $\varphi_j \in C_0^2(\lambda\Delta_j)$, $|\nabla^l\varphi_j| \le C\delta^{-l}$, $0 \le l \le 2$, such that $\sum_j \varphi_j = 1$ on $\bigcup_j \Delta_j$. Set

$$F_j = \frac{1}{\pi}\frac{\overline{z}}{z} * \varphi_j\overline{\partial}^2 f_\Delta = \frac{1}{\pi}\frac{\overline{z}}{z} * \varphi_j\varphi\overline{\partial}^2 f,$$

so that $f_\Delta = \sum_j F_j$ and $\|F_j\|_\infty \le Cw(\delta_j)$. Let us call an index j good or bad according to whether the disc Δ_j is good or bad.

If j is a good index, then Δ_j has the three points property with respect to $E \subset \mathbb{C}\backslash\overset{\circ}{X}$, and consequently with respect to $\mathbb{C}\backslash X$. Therefore, by Lemma 2.1 we can find $G_j \in H(X, \overline{\partial}^2)$ satisfying $\|G_j\|_\infty \le Cw(\delta_j)$ and $F_j = G_j + O(|z|^{-2})$ as $z \to \infty$.

If j is a bad index, then $G_j \in H(X, \overline{\partial}^2)$ can be found [7, p. 310] with $\|G_j\|_\infty \le Cw(\delta_j)$ and $F_j = G_j + O(|z|^{-1})$ as $z \to \infty$.

Set $G_\Delta = \sum_j G_j$. Then

$$\|G_\Delta\|_\infty \le \|f_\Delta - G_\Delta\|_\infty + Cw(\delta)$$

and

$$\|f_\Delta - G_\Delta\|_\infty \le \left\| \sum_{j \text{ bad}} F_j - G_j \right\|_\infty + \left\| \sum_{j \text{ good}} F_j - G_j \right\|_\infty \le C\varepsilon(\delta),$$

because of Lemma 3.1 and the properties of $\{\Delta_j\}$.

We need now to modify G_Δ to get a g_Δ satisfying (10). Assume that the center of Δ is the origin and expand $H \equiv f_\Delta - G_\Delta$ at ∞:

$$H(z) = \frac{b_1}{z} + c_1 \frac{\overline{z}}{z^2} + \cdots + \frac{b_n}{z^n} + c_n \frac{\overline{z}}{z^{n+1}} + \cdots.$$

To estimate b_n and c_n we must also consider the expansions of $H_j \equiv F_j - G_j$ (z_j is the center of Δ_j):

$$H_j(z) = \frac{b_1^j}{z - z_j} + c_1^j \frac{\overline{z} - \overline{z}_j}{(z - z_j)^2} + \cdots + \frac{b_n^j}{(z - z_j)^n} + c_n^j \frac{\overline{z} - \overline{z}_j}{(z - z_j)^{n+1}} + \cdots.$$

Recall that if j is good then $b_1^j = c_1^j = 0$, so that

$$|b_1| = \left| \sum_{j \text{ bad}} b_1^j \right| \le \sum_{j \text{ bad}} C w(\delta_j)\delta_j \le C M^h(E) \le C\varepsilon(\delta) C(E),$$

where the last estimate comes from §2(A) and in the next to the last inequality (d) of 2.2 was used. Similarly one gets $|c_1| \le C\varepsilon(\delta)C(E)$.

For $n \ge 2$, $\pi|c_n|$ is equal to

$$|\langle \overline{\partial}^2 H, z^n \rangle| \le \sum_j |\langle \overline{\partial}^2 H_j, z^n \rangle|.$$

Writing $z^n = \sum_{m=0}^n \binom{n}{m} z_j^{n-m}(z - z_j)^m$, we can estimate $|\langle \overline{\partial}^2 H_j, z^n \rangle|$ by

$$\sum_{m=1}^n \binom{n}{m} |z_j|^{n-m} |\langle \overline{\partial}^2 H_j, (z - z_j)^m \rangle| = \pi \sum_{m=1}^n \binom{n}{m} |z_j|^{n-m} |c_m^j|.$$

Since $c_1^j = 0$ for good indexes j we have

$$|c_n| \le C\delta^{n-1} \sum_{j \text{ bad}} w(\delta_j)\delta_j + \sum_{m=2}^n \binom{n}{m} \delta^{n-m} \sum_j w(\delta_j)\delta_j^m.$$

Now properties (e) and (f) of the family $\{\Delta_j\}$ give

$$|c_n| \le C 2^{n-1} \varepsilon(\delta) \delta^{n-1} C(E).$$

One proves the same estimate for $|b_n|$ arguing similarly.

Let μ be a positive measure, with support in E, such that $\frac{1}{|z|} * \mu$ is continuous on $\mathbb{C}$, $\frac{1}{|z|} * \mu \le 1$ and $2\|\mu\| \ge C(E)$. Set

$$P = \frac{1}{z} * b_1(\mu/\|\mu\|) + \frac{\overline{z}}{z^2} * c_1(\mu/\|\mu\|).$$

Then $\|P\|_\infty \le C\varepsilon(\delta)$ and $P \in H(X, \overline{\partial}^2)$ according to Lemma 4.1.

Define $g_\Delta = G_\Delta + P$. Then the expansion of $f_\Delta - g_\Delta = f_\Delta - G_\Delta - P$ outside Δ is of the form

$$f_\Delta(z) - g_\Delta(z) = \frac{b_2^*}{z^2} + c_2^* \frac{\overline{z}}{z^3} + \cdots + \frac{b_n^*}{z^n} + c_n^* \frac{\overline{z}}{z^{n+1}} + \cdots$$

where $b_n^* = b_n - b_n(P)$ and $c_n^* = c_n - c_n(P)$, $b_n(P)$ and $c_n(P)$ being the coefficients in the expansion of P. We wish now to estimate $|b_n^*|$ and $|c_n^*|$ by $C2^{n-1}\varepsilon(\delta)\delta^{n-1}C(E)$, which is easy, because we have even better estimates for $|b_n(P)|$ and $|c_n(P)|$. For example

$$|c_n(P)| = \left| \int z^{n-1} b_1 \|\mu\|^{-1} d\mu(z) \right| \le \delta^{n-1}|b_1| \le C\delta^{n-1}\varepsilon(\delta)C(E).$$

Therefore for $|z| > 3\delta$ we get

$$|f_\Delta(z) - g_\Delta(z)| \le \sum_{n=2}^{\infty} C2^{n-1}\varepsilon(\delta)\delta^{n-1}C(E)|z|^{-n} \le C\varepsilon(\delta)\delta C(E)|z|^{-2},$$

and this completes the proof of the lemma. □

We go back now to the proof of the theorem. We used a δ-Vitushkin scheme to express f as $\sum_j f_j$. Fix j. Let E be the support of $\overline{\partial}^2 f_j$ and let $\gamma = C(E)$. We sublocalize f_j according to a γ-Vitushkin scheme (D_k, ψ_k, F_k), where $F_k = \frac{1}{\pi}\frac{\overline{z}}{z} * \psi_k \overline{\partial}^2 f_j = \frac{1}{\pi}\frac{\overline{z}}{z} * \psi_k \varphi_j \overline{\partial}^2 f$. Thus $f_j = \sum_k F_k$ and $\|F_k\|_\infty \le cw(\gamma)$. Apply now Lemma 5.1 to D_k and F_k to get $G_k \in H(X, \overline{\partial}^2)$ such that $\|G_k\|_\infty \le C\varepsilon(\gamma)$ and

$$|F_k(z) - G_k(z)| \le C\varepsilon(\gamma)\gamma C(E_k)|z - z_k|^{-2}, \qquad |z - z_k| > 3\gamma,$$

where z_k is the center of D_k and E_k the support of $\overline{\partial}^2 F_k$.

We would like at this point to apply a variant of a well known lemma of Vitushkin ([**3**, 2.7, p. 202]). We present a proof of the result we need for the reader's convenience and also because the argument gives a quick new proof of Vitushkin's lemma.

5.2. Lemma. *Let E be a Borel subset of the plane with newtonian capacity $C(E) = \gamma$. Assume that $\{\Delta_k\}$ is an almost disjoint family of discs of radius γ and set $E_k = E \cap \Delta_k$. Then for some constant C,*

$$\sum_k C(E_k) \le CC(E)$$

and

$$\sum_{|z-z_k|>2\gamma} \frac{C(E_K)}{|z - z_k|} \le C, \qquad z \in \mathbb{C}.$$

Proof. By Hölder's inequality with exponents 3 and $3/2$

$$\begin{aligned}\sum_{|z-z_k|>2\gamma} \frac{C(E_k)}{|z - z_k|} &\le \left(\sum_k \left(\frac{C(E_k)}{\gamma}\right)^{3/2}\right)^{2/3} \left(\sum_{|z-z_k|>2\gamma} \frac{\gamma^3}{|z - z_k|^3}\right)^{1/3} \\ &\le \left(\sum_k \frac{C(E_k)}{C(E)}\right)^{2/3} \left(\gamma \int_{|\zeta - z|>\gamma} \frac{dx\,dy}{|\zeta - z|^3}\right)^{1/3} \\ &= C\left(\sum_k \frac{C(E_k)}{C(E)}\right)^{2/3}.\end{aligned}$$

Let μ_k be a positive measure with support in E_k, $2\|\mu_k\| \ge C(E_k)$ and $\frac{1}{|z|} * \mu_k \le 1$ on $\mathbb{C}$. For $|z - z_k| > 2\gamma$ we have

$$\int \frac{d\mu_k(\zeta)}{|z - \zeta|} \le C \frac{C(E_k)}{|z - z_k|}$$

and hence, setting $\mu = \sum_k \mu_k$,

$$\int \frac{d\mu(\zeta)}{|z - \zeta|} \le C + C\left(\sum_k \frac{C(E_k)}{C(E)}\right)^{2/3}.$$

The definition of $C(E)$ now gives

$$\sum_k \frac{C(E_k)}{C(E)} \le C + C\left(\sum_k \frac{C(E_k)}{C(E)}\right)^{2/3},$$

and so

$$\sum_k C(E_k) \le CC(E),$$

which completes the proof of the lemma. □

We proceed with the last step in the proof of the theorem. We claim that

$$\left\|\sum_k G_k\right\|_\infty \le C\varepsilon(\delta).$$

Clearly

$$\left\|\sum_k G_k\right\|_\infty \le \left\|\sum_k F_k - G_k\right\|_\infty + Cw(\delta).$$

Using the decay estimate (10) and 5.2 we get

$$\sum_k |F_k(z) - G_k(z)| \le C\varepsilon(\gamma) + C\varepsilon(\gamma) \sum_{|z-z_k|>3\gamma} \frac{C(E_k)}{|z-z_k|} \le C\varepsilon(\gamma),$$

which proves the claim.

Set $g_j = \sum_k G_k$ and consider the expansions

$$f_j(z) - g_j(z) = \frac{b_2}{z_2} + c_2\frac{\overline{z}}{z^3} + O(|z|^{-3}) \quad \text{as } z \to \infty,$$
$$F_k(z) - G_k(z) = \frac{b_2^k}{(z-z_k)^2} + c_2^k\frac{\overline{z}-\overline{z}_k}{(z-z_k)^3} + O(|z|^{-3}) \quad \text{as } z \to \infty.$$

We have by 5.2

$$|b_2| = \left|\sum_k b_2^k\right| \le \sum_k C\varepsilon(\gamma)\gamma C(E_k) \le C\varepsilon(\gamma)\gamma C(E) \le C\varepsilon(\delta)C(E)^2,$$

and similarly $|c_2| \le C\varepsilon(\delta)C(E)^2$.

Let μ be a positive measure, with support in E, such that $\frac{1}{|z|} * \mu$ is continuous, $\frac{1}{|z|} * \mu \le 1$ and $2\|\mu\| \ge C(E)$.

Set $P = (1/z) * (\mu/\|\mu\|)$ and $Q = (\overline{z}/z^2) * (\mu/\|\mu\|)$. Then

$$b_2P^2(z) = \frac{b_2}{z^2} + O(|z|^{-3}) \quad \text{as } z \to \infty,$$
$$c_2P(z)Q(z) = c_2\frac{\overline{z}}{z^3} + O(|z|^{-3}) \quad \text{as } z \to \infty,$$

$\|b_2P^2\|_\infty \le C\varepsilon(\delta)$ and $\|c_2PQ\|_\infty \le C\varepsilon(\delta)$.

The function $h_j = g_j + b_2P^2 + c_2PQ$ belongs to $H(X, \overline{\partial}^2)$ because of 4.1. Since

$$f_j - h_j = O(|z|^{-3}) \quad \text{as } z \to \infty,$$

$$\left\| f - \sum_j h_j \right\|_\infty = \left\| \sum_j f_j - h_j \right\|_\infty \le C \max_j \|f_j - h_j\|_\infty \le C\varepsilon(\delta),$$

and thus $\sum_j h_j$ is the desired approximant in $H(X, \overline{\partial}^2)$.

6. The string of beads. It is a set of the form $X = \overline{D}\backslash(\bigcup_k D_k)$ where D is the open unit disc, the D_k are open discs centered on the interval $I = [-1/2, 1/2]$ such that $\overline{D}_k \subset D$, $\overline{D}_k \cap \overline{D}_l = \varnothing$ if $k \neq l$ and $\bigcup D_k$ is dense in I.

We wish to prove that (1) holds for X. Take $f \in h(X, \overline{\partial}^2)$ and write $f = \sum_j f_j$ using a δ-Vitushkin scheme $(\Delta_j, \varphi_j, f_j)$. If Δ_j does not intersect I then $f_j \in H(X, \overline{\partial}^2)$ (by [2, Theorem 2]). If $\Delta_j \cap I \neq \varnothing$, replacing Δ_j by a disc of comparable size, we can assume that the center c of Δ_j lies in I. Using two points in $\Delta_j \cap (\bigcup_k D_k)$ at distance not less than $\delta/2$, we find, by a variant of 2.1, a function $g_j \in H(X, \overline{\partial}^2)$ such that $\|f_j\|_\infty \le Cw(\delta)$ and $f_j(z) - g_j(z) = b/(z-c) + O(|z|^{-2})$ as $z \to \infty$.

We claim now that

$$|b| \le Cw(\delta)\alpha(\Delta_j\backslash X). \tag{11}$$

Once this is proved we consider h_j continuous on $\mathbb{C}$, analytic off a compact subset of $\Delta_j\backslash X$, $\|h_j\|_\infty \le Cw(\delta)$ and $h_j(z) = b/(z-c) + O(|z|^{-2})$ as $z \to \infty$. Then $K_j = g_j + h_j \in H(X, \overline{\partial}^2)$, $\|K_j\|_\infty \le Cw(\delta)$ and $f_j - K_j = O(|z|^{-2})$ as $z \to \infty$.

Consequently

$$\left\| \sum_{\Delta_j \cap I \neq \varnothing} f_j - K_j \right\|_\infty \le Cw(\delta),$$

and so $\sum_{\Delta_j \cap I = \varnothing} f_j + \sum_{\Delta_j \cap I \neq \varnothing} K_j$ is the desired approximant in $H(X, \overline{\partial}^2)$.

To prove (11), given $\varepsilon > 0$, we cover $(\Delta_j \cap I)\backslash(\bigcup_k D_k)$ by discs B_l of radius $r_l = \varepsilon$ such that $\sum_l r_l \le 2\delta$. It is not difficult to realize that $\bigcup D_k$ can be covered by a family of discs D_m^* of radii $r_m^* \le \delta$ such that $\sum r_m^* \le C$ length$((\Delta_j\backslash X) \cap I)$. Write now $\{B_l\} \cup \{D_m^*\} = \{D_n^{**}\}$ and consider a partition of the unity (φ_n) subordinated to the above covering and satisfying $|\nabla^i \varphi_n| \le C\delta^{-i}$, $0 \le i \le 2$. Set $d_j = f_j - g_j$.

Denoting by z_n the center of D_n^{**} we then have

$$\begin{aligned}
\pi|b| &= |\langle\overline{\partial}^2 d_j\,,\,\overline{z}\rangle| = |\langle\overline{\partial}^2 d_j\,,\,\overline{z}-z\rangle| \\
&\le 2\sum_n |\langle\overline{\partial}^2 d_j\,,\,\varphi_n(z)\operatorname{Im} z\rangle| \\
&\le 2\sum_n \int |d_j(z)-d_j(z_n)|\,|\operatorname{Im} z\,\overline{\partial}^2\varphi_n(z)+\overline{\partial}\varphi_n(z)|\,dx\,dy \\
&\le C\sum_l w(\varepsilon)r_l + C\sum_m w(\delta)r_m^* \\
&\le Cw(\varepsilon)\delta + Cw(\delta)\ \text{length}\ ((\Delta_j\backslash X)\cap I)\,.
\end{aligned}$$

Letting $\varepsilon \to 0$ we get

$$|b| \le Cw(\delta)\gamma((\Delta_j\backslash X)\cap I) \le Cw(\delta)\gamma(\Delta_j\backslash X) = Cw(\delta)\alpha(\Delta_j\backslash X)\,,$$

where we used that four times the analytic capacity γ of a subset of the line is equal to its length and that γ and α coincide on open subsets of the plane.

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