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LACUNARY STATISTICAL CONVERGENCE

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The sequence x is statistically convergent to L provided that for each $\varepsilon > 0$,

$$\lim_n n^{-1}\{\text{the number of } k \le n\colon |x_k - L| \ge \varepsilon\} = 0.$$

In this paper we study a related concept of convergence in which the set $\{k\colon k \le n\}$ is replaced by $\{k\colon k_{r-1} < k \le k_r\}$, for some lacunary sequence $\{k_r\}$. The resulting summability method is compared to statistical convergence and other summability methods, and questions of uniqueness of the limit value are considered.

1. Introduction. A complex number sequence x is said to be *statistically convergent* to the number L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n}|\{k \le n\colon |x_k - LK| \ge \varepsilon\}| = 0, \tag{1}$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write $S\text{-}\lim x = L$ or $x_k \to L(S)$. We shall also use S to denote the set of all statistically convergent sequences. The idea of statistical convergence was introduced by Fast [4] and studied by several authors [2], [3], [5], [6], [11]. There is a natural relationship [2] between statistical convergence and strong Cesàro summability:

$$|\sigma_1| := \left\{x\colon \text{ for some } L,\ \lim_n \left(\frac{1}{n}\sum_{k=1}^{n}|x_k - L|\right) = 0\right\}.$$

By a *lacunary sequence* we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r := (k_{r-1}, k_r]$, and the ratio k_r/k_{r-1} will be abbreviated by q_r. There is a strong connection [7] between $|\sigma_1|$ and the sequence space N_θ, which is defined by

$$N_\theta := \left\{x\colon \text{for some } L,\ \lim_r \left(\frac{1}{h_r}\sum_{k\in I_r}|x_k - L|\right) = 0\right\}.$$

The purpose of this paper is to introduce and study a concept of convergence that is related to statistical convergence (1) in the same way that N_θ is related to $|\sigma_1|$.

DEFINITION. Let θ be a lacunary sequence; the number sequence x is *S_θ-convergent* to L provided that for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r}|\{k \in I_r \colon |x_k - L| \geq \varepsilon\}| = 0. \tag{2}$$

In this case we write $S_\theta\text{-}\lim x = L$ or $x_k \to L(S_\theta)$, and we define

$$S_\theta := \{x \colon \text{ for some } L,\ S_\theta\text{-}\lim x = L\}.$$

The limits in (1) and (2) can be expressed using matrix transformations of the characteristic function χ_K of the set

$$K = K(x, L, \varepsilon) := \{k \in \mathbb{N} \colon |x_k - L| \geq \varepsilon\}.$$

The limit in (1) is $\lim_n (C_1\chi_K)_n = 0$, where C_1 is the Cesàro mean; the limit in (2) is $\lim_n (C_\theta \chi_K)_n = 0$, where C_θ is the matrix given by

$$C_\theta[n, k] := \begin{cases} \frac{1}{h_r}, & \text{if } k \in I_r, \\ 0, & \text{if } k \notin I_r. \end{cases}$$

In this form S_θ-convergence is seen to be a part of "A-density convergence" as defined in **[8]** and **[3]**.

In the next section we establish inclusion relations between S_θ and N_θ and also between S_θ and S. In §3 we show that the S_θ-limit of a given sequence x is not necessarily unique for different θ's, but different S_θ-limits cannot occur if $x \in S$. In the final section we get a relationship between S_θ-convergence and strong almost convergence, a concept introduced by Maddox **[10]** and (independently) by Freedman et al. **[7]**.

2. Inclusion theorems. In this section we first give some inclusion relations between N_θ- and S_θ-convergence and show that they are equivalent for bounded sequences. We also study the inclusions $S \subseteq S_\theta$ and $S_\theta \subseteq S$ under certain restrictions on $\theta = \{k_r\}$.

THEOREM 1. *Let $\theta = \{k_r\}$ be a lacunary sequence; then*

(i) (a) $x_k \to L(N_\theta)$ *implies* $x_k \to L(S_\theta)$, *and*
(b) N_θ *is a proper subset of* S_θ;
(ii) $x \in l_\infty$ *and* $x_k \to L(S_\theta)$ *imply* $x_k \to L(N_\theta)$;
(iii) $S_\theta \cap l_\infty = N_\theta \cap l_\infty$,

where l_∞ denotes the set of bounded sequences.

Before proving this theorem we remark that this result is included by Theorem 8 in **[3]**, where Connor bases the proof on the concept of ideals in l_∞; we give a direct proof.

Proof. (a) If $\varepsilon > 0$ and $x_k \to L(N_\theta)$ we can write

$$\sum_{k\in I_r} |x_k - L| \geq \sum_{\substack{k\in I_r \\ |x_k - L|\geq\varepsilon}} |x_k - L| \geq \varepsilon|\{k \in I_r : |x_k - L| \geq \varepsilon\}|,$$

which yields the result.

(b) In order to establish that the inclusion $N_\theta \subseteq S_\theta$ in (i) is proper, let θ be given and define x_k to be $1, 2, \ldots, [\sqrt{h_r}]$ at the first $[\sqrt{h_r}]$ integers in I_r, and $x_k = 0$ otherwise. Note that x is not bounded. We have, for every $\varepsilon > 0$,

$$\frac{1}{h_r}|\{k \in I_r : |x_k - 0| \geq \varepsilon\}| = \frac{[\sqrt{h_r}]}{h_r} \to 0 \qquad \text{as } r \to \infty,$$

i.e., $x_k \to 0(S_\theta)$. On the other hand,

$$\frac{1}{h_r}\sum_{k\in I_r} |x_k - 0| = \frac{1}{h_r}\frac{[\sqrt{h_r}]([\sqrt{h_r}] + 1)}{2} \to \frac{1}{2} \neq 0;$$

hence $x_k \nrightarrow 0(N_\theta)$.

(ii) Suppose that $x_k \to L(S_\theta)$ and $x \in l_\infty$, say $|x_k - L| \leq M$ for all k. Given $\varepsilon > 0$, we get

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} |x_k - L| &= \frac{1}{h_r}\sum_{\substack{k\in I_r \\ |x_k - L|\geq\varepsilon}} |x_k - L| + \frac{1}{h_r}\sum_{\substack{k\in I_r \\ |x_k - L|<\varepsilon}} |x_k - L| \\ &\leq \frac{M}{h_r}|\{k \in I_r : |x_k - L| \geq \varepsilon\}| + \varepsilon,\end{aligned}$$

from which the result follows.

We remark that the example given in (i) shows that the boundedness condition cannot be omitted from the hypothesis of Theorem 1 (ii).

(iii) This is an immediate consequence of (i) and (ii).

Since any N_θ-summable sequence is C_θ-summable, we conclude from Theorem 1 (ii) that any bounded S_θ-summable sequence is also C_θ-summable.

LEMMA 2. *For any lacunary sequence* θ, $S\text{-}\lim x = L$ *implies* $S_\theta\text{-}\lim x = L$ *if and only if* $\liminf_r q_r > 1$. *If* $\liminf_r q_r = 1$, *then there exists a bounded* S_θ*-summable sequence that is not* S*-summable* (*to any limit*).

Proof. Suppose first that $\liminf_r q_r > 1$; then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}.$$

If $x_k \to L(S)$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r}|\{k \leq k_r\colon |x_k - L| \geq \varepsilon\}| &\geq \frac{1}{k_r}|\{k \in I_r\colon |x_k - L| \geq \varepsilon\}| \\ &\geq \frac{\delta}{1+\delta} \cdot \frac{1}{h_r}|\{k \in I_r\colon |x_k - L| \geq \varepsilon\}|;\end{aligned}$$

this proves the sufficiency.

Conversely, suppose that $\liminf_r q_r = 1$. Proceeding as in [7; p. 510] we can select a subsequence $\{k_{r(j)}\}$ of the lacunary sequence θ such that

$$\frac{k_{r(j)}}{k_{r(j)-1}} < 1 + \frac{1}{j} \quad \text{and} \quad \frac{k_{r(j)-1}}{k_{r(j-1)}} > j, \quad \text{where } r(j) \geq r(j-1) + 2.$$

Now define a bounded sequence x by $x_i = 1$ if $i \in I_{r(j)}$ for some $j = 1, 2, \ldots$ and $x_i = 0$ otherwise. It is shown in [7; p. 510] that $x \notin N_\theta$ but $x \in |\sigma_1|$. The above Theorem 1 (ii) implies that $x \notin S_\theta$, but it follows from Theorem 2.1 of [2] that $x \in S$. Hence $S \not\subseteq S_\theta$, and the proof is complete.

LEMMA 3. *For any lacunary sequence* θ, S-$\lim x = L$ *implies* S_θ-$\lim x = L$ *if and only if* $\limsup_r q_r < \infty$. *If* $\limsup_r q_r = \infty$, *then there exists a bounded* S-*summable sequence that is not* S_θ-*summable* (*to any limit*).

Proof. If $\limsup_r q_r < \infty$, then there is an $H > 0$ such that $q_r < H$ for all r. Suppose that $x_k \to L(S_\theta)$, and let $N_r := |\{k \in I_r\colon |x_k - L| \geq \varepsilon\}|$. By (2), given $\varepsilon > 0$, there is an $r_0 \in \mathbb{N}$ such that

$$\frac{N_r}{h_r} < \varepsilon \quad \text{for all } r > r_0. \tag{3}$$

Now let $M := \max\{N_r\colon 1 \leq r \leq r_0\}$ and let n be any integer satisfying

$k_{r-1} < n \leq k_r$; then we can write

$$\begin{aligned}
\frac{1}{n}|\{k \leq n\colon |x_k - L| \geq \varepsilon\}| &\leq \frac{1}{k_{r-1}}|\{k \leq k_r\colon |x_k - L| \geq \varepsilon\}| \\
&= \frac{1}{k_{r-1}}\{N_1 + N_2 + \cdots + N_{r_0} + N_{r_0+1} + \cdots + N_r\} \\
&\leq \frac{M}{k_{r-1}} \cdot r_0 + \frac{1}{k_{r-1}}\left\{h_{r_0+1}\frac{N_{r_0+1}}{h_{r_0+1}} + \cdots + h_r\frac{N_r}{h_r}\right\} \\
&\leq \frac{r_0 \cdot M}{k_{r-1}} + \frac{1}{k_{r-1}}\left(\sup_{r>r_0}\frac{N_r}{h_r}\right)\{h_{r_0+1} + \cdots + h_r\} \\
&\leq \frac{r_0 \cdot M}{k_{r-1}} + \varepsilon \cdot \frac{k_r - k_{r_0}}{k_{r-1}}, \quad \text{by (3)}, \\
&\leq \frac{r_0 \cdot M}{k_{r-1}} + \varepsilon \cdot q_r \leq \frac{r_0 \cdot M}{k_{r-1}} + \varepsilon H,
\end{aligned}$$

and the sufficiency follows immediately.

Conversely, suppose that $\limsup_r q_r = \infty$. Following the idea in [7; p. 511] we can select a subsequence $\{k_{r(j)}\}$ of the lacunary sequence $\theta = \{k_r\}$ such that $q_{r(j)} > j$, and define a bounded sequence by $x_i = 1$ if $k_{r(j)-1} < i \leq 2k_{r(j)-1}$ for some $j = 1, 2, \ldots$, and $x_i = 0$ otherwise. It is shown in [7; p. 5.11] that $x \in N_\theta$ but $x \notin |\sigma_1|$. By Theorem 1 (i) we conclude that $x \in S_\theta$, but Theorem 2.1 of [2] implies that $x \notin S$. Hence, $S_\theta \not\subseteq S$.

Combining Lemma 2 and Lemma 3 we get

THEOREM 4. *Let θ be a lacunary sequence; then $S = S_\theta$ if and only if*

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty;$$

then S-$\lim x = L$ implies S_θ-$\lim x = L$.

For an example of a lacunary sequence satisfying the conditions of Theorem 4, we can take $k_r = 2^r$ for $r > 0$, whence $S_{\{2^r\}} = S$. We remark that the examples given in Lemmas 2 and 3 illustrate the difference between S-convergence and S_θ-convergence.

We conclude this section with the following observation. Buck [1, Theorem 3.2] proved that if a real sequence is C_1-summable to its finite limit inferior, then the sequence "converges to that point for almost all n" (i.e., it is statistically convergent to its limit inferior [2]). Note that this result remains true if we replace limit inferior by limit superior. For each subset K of $\mathbb{N}$, define

$$D(K) := \lim_r (C_\theta \chi_K)_r = \lim_r \frac{|K \cap I_r|}{h_r};$$

then D is a density [**8**; p. 296], and it is not hard to get a result for S_θ-convergence that is analogous to Buck's. To be precise, the following result is such an analogue.

PROPOSITION 5. *If the real number sequence x is C_θ-summable to either its finite limit inferior or finite limit superior, then x is S_θ-convergent to that value.*

3. Uniqueness of S_θ-limit and lacunary refinements. It is easy to see that, for any fixed θ, the S_θ-limit is unique. It is possible, however, for a sequence—even a bounded one—to have different S_θ-limits for different θ's. This can be seen by applying Theorem 1 (i) to the sequence x given in [7, proof of Theorem 2.1] for which $N_{\theta_1}\text{-}\lim x = 0$ and $N_{\theta_2}\text{-}\lim x = 1$. The next theorem shows that this situation cannot occur if $x \in S$; in other words, every S_θ method is consistent with the S-method.

THEOREM 6. *If $x \in S \cap S_\theta$, then $S_\theta\text{-}\lim x = S\text{-}\lim x$.*

Proof. Suppose $S\text{-}\lim x = L$ and $S_\theta\text{-}\lim x = L'$, and $L \neq L'$. For $\varepsilon < \frac{1}{2}|L - L'|$ we get

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L'| \geq \varepsilon\}| = 1.$$

Consider the k_mth term of the statistical limit expression $n^{-1}|\{k \leq n : |x_k - L'| \geq \varepsilon\}|$:

$$(4) \quad \frac{1}{k_m}\left|\left\{k \in \bigcup_{r=1}^{m} I_r : |x_k - L'| \geq \varepsilon\right\}\right|$$
$$= \frac{1}{k_m}\sum_{r=1}^{m} |\{k \in I_r : |x_k - L'| \geq \varepsilon\}| = \frac{1}{\sum_{r=1}^{m} h_r}\sum_{r=1}^{m} h_r t_r,$$

where $t_r = h_r^{-1}|\{k \in I_r : |x_k - L'| \geq \varepsilon\}| \to 0$ because $x_k \to L'(S_\theta)$. Since θ is a lacunary sequence, (4) is a regular weighted mean transform of t, and therefore it, too, tends to zero as $m \to \infty$. Also, since this is a subsequence of $\{n^{-1}|\{k \leq n : |x_k - L'| \geq \varepsilon\}|\}_{n=1}^{\infty}$, we infer that

$$\frac{1}{n}|\{k \leq n : |x_k - L'| \geq \varepsilon\}| \nrightarrow 1,$$

and this contradiction shows that we cannot have $L \neq L'$.

We now consider the inclusion of $S_{\theta'}$ by S_θ, where θ' is a lacunary refinement of θ. Recall [7] that the lacunary sequence $\theta' = \{k'_r\}$ is called a *lacunary refinement* of the lacunary sequence $\theta = \{k_r\}$ if $\{k_r\} \subseteq \{k'_r\}$.

THEOREM 7. *If θ' is a lacunary refinement of θ and $x_k \to L(S_{\theta'})$, then $x_k \to L(S_\theta)$.*

Proof. Suppose each I_r of θ contains the points $\{k'_{r,i}\}_{i=1}^{\nu(r)}$ of θ' so that

$$k_{r-1} < k'_{r,1} < k'_{r,2} < \cdots < k'_{r,\nu(r)} = k_r, \quad \text{where } I'_{r,i} = (k'_{r,i-1}, k'_{r,i}].$$

Note that for all r, $\nu(r) \geq 1$ because $\{k_r\} \subseteq \{k'_r\}$. Let $\{I^*_j\}_{j=1}^{\infty}$ be the sequence of abutting intervals $\{I'_{r,i}\}$ ordered by increasing right end points. Since $x_k \to L(S_{\theta'})$, we get, for each $\varepsilon > 0$,

$$\lim_j \sum_{I^*_j \subset I_r} \frac{1}{h^*_r} |\{k \in I^*_j : |x_k - L| \geq \varepsilon\}| = 0. \tag{5}$$

As before we write, $h_r = k_r - k_{r-1}$, $h'_{r,i} = k'_{r,i} - k'_{r,i-1}$, and $h'_{r,1} = k'_{r,1} - k_{r-1}$. For each $\varepsilon > 0$ we have

$$\begin{aligned} &\frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| \\ &\quad = \frac{1}{h_r} \sum_{I^*_j \subseteq I_r} h^*_j \frac{1}{h^*_j} |\{k \in I^*_j : |x_k - L| \geq \varepsilon\}| \\ &\quad = \frac{1}{h_r} \sum_{I^*_j \subseteq I_r} h^*_j (C_{\theta'} \chi_K)_j, \end{aligned} \tag{6}$$

where χ_K is the characteristic function of the set $K := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$. By (5), $C_{\theta'}\chi_K$ is a null sequence, and (6) is a regular weighted mean transform of $C_{\theta'}\chi_K$. Hence, the transform (6) also tends to zero as $r \to \infty$.

We conclude this section by observing that Theorem 7 establishes inclusion between two lacunary methods *only* when one sequence is a lacunary refinement of the other. The example cited at the beginning of this section shows that S_θ can be inconsistent with $S_{\theta'}$. A general description of inclusion between two arbitrary lacunary methods is left as an open problem.

4. Strong almost convergence and S_θ-convergence. The idea of almost convergence was introduced by Lorentz [9]: the sequence x is said to be *almost convergent* to L if

$$\lim_n \frac{1}{n} \sum_{i=m+1}^{m+n} (x_i - L) = 0, \quad \text{uniformly in } m.$$

Maddox [10] and (independently) Freedman et al. [7] introduced the notion of strong almost convergence: the sequence x is said to be *strongly almost convergent* to L if

$$\lim_n \frac{1}{n} \sum_{i=m+1}^{m+n} |x_i - L| = 0, \quad \text{uniformly in } m.$$

Let c, AC and $[AC]$, respectively, denote the sets of all convergent, almost convergent, and strongly almost convergent sequences. It is known [10] that

$$c \subsetneqq [AC] \subsetneqq AC \subsetneqq l_\infty. \tag{7}$$

THEOREM 8. *If $\mathscr{L}$ denotes the set of all lacunary sequences, then*

$$[AC] = l_\infty \cap \left(\bigcap_{\theta \in \mathscr{L}} S_\theta \right).$$

Proof. By [7, Theorem 3.1], the relations (7) and Theorem 1 (iii), we have

$$\begin{aligned} l_\infty \supset [AC] &= \bigcap_{\theta\in\mathscr{L}} N_\theta = l_\infty \cap \left(\bigcap_{\theta\in\mathscr{L}} N_\theta \right) \bigcap_{\theta\in\mathscr{L}} (l_\infty \cap N_\theta) \\ &= \bigcap_{\theta\in\mathscr{L}} (l_\infty \cap S_\theta) = l_\infty \cap \left(\bigcap_{\theta\in\mathscr{L}} S_\theta \right). \end{aligned}$$

Finally we remark that in contrast to [7, Theorem 3.1] where it was proved that $[AC] = \bigcap N_\theta$, the factor l_∞ cannot be omitted from Theorem 8. For, $\bigcap S_\theta \not\subseteq l_\infty$ and $\bigcap N_\theta = [AC]$ is a proper subset of $\bigcap S_\theta$. To see this consider the sequence x defined by $x_k = m$, if $k = m^2$ for $m = 1, 2, \ldots$, and $x_k = 0$ otherwise. Observe that x is not bounded, so it is not strongly almost convergent. On the other hand, for any lacunary sequence θ, we have

$$\frac{1}{h_r} |\{k \in I_r : x_k \neq 0\}| \leq \frac{\sqrt{h_r}}{h_r} \to 0, \quad \text{as } r \to \infty;$$

hence, $x_k \to O(S_\theta)$.

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