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# A NONEXISTENCE RESULT FOR THE $n$-LAPLACIAN

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**Let $P$ be a point in $\mathbb{R}^n$, $n \geq 2$; then the problem** $\operatorname{div}(|\nabla u|^{n-2}\nabla u) = e^u$ **with** $u \in W^{1,n}_{\text{loc}} \cap L^\infty_{\text{loc}}$ **has no subsolutions in** $\mathbb{R}^n \backslash \{P\}$.

**Introduction.** Let $P = P(x_1, x_2, \ldots, x_n)$ be a point in $\mathbb{R}^n$, $n \geq 2$, and $\Omega = \mathbb{R}^n \backslash \{P\}$. Without any loss of generality we will take $P$ to be the origin. Consider the problem

$$\text{(1.1)} \qquad \begin{cases} L_p u = e^u & \text{in } \Omega, \\ u \in W^{1,p}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega); & p > 1. \end{cases}$$

Here $L_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian with $1 < p < \infty$. By a subsolution $u$ of (1.1) we will mean that $u \in W^{1,p}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$, and

$$\int_\Omega |\nabla u|^{p-2}\nabla u, \ \nabla \psi + \int_\Omega e^u \psi \leq 0, \quad \forall \psi \in C_0^\infty(\Omega) \text{ and } \psi \geq 0.$$

It is known that for $1 < p < n$, (1.1) has no subsolutions in the exterior of a compact set [AW]. However, for $p = n$ there exist radial subsolutions for large values of $|x|$. We show that (1.1) has no subsolutions in $\Omega$, thus extending the results of [AW], namely

THEOREM 1. *The following problem*

$$L_n u = e^u \quad in\ \Omega, \ \ n \geq 2,$$

*has no subsolutions in* $W^{1,n}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$.

The proof of Theorem 1 will be a consequence of a comparison principle and nonexistence of global radial solutions. The proof is presented in §4.

## 2. Preliminary results.

LEMMA 2.1. *Consider*

$$C(x) = \frac{(1+x)^{1/n}}{1+x^{1/n}} \quad in\ 0 \leq x \leq 1.$$

*Then* $C(x)$ *is decreasing on* $[0, 1]$.

*Proof*. Elementary computations show that

$$\frac{dC}{dx} = \frac{(1+x)^{1/n}(1-x^{(1-n)/n})}{n(1+x^{1/n})^2(1+x)} \leq 0$$

in $0 \leq x \leq 1$. Furthermore, $C(0) = 1$ and $C(1) = 2^{1-n/n}$, and $C(x) \to 1$ as $x \to 0$. □

We now state an elementary inequality that is easy to prove

$$x^n - b^n \geq (x-b)^n, \quad \text{for } x \geq b \geq 0. \tag{2.1}$$

LEMMA 2.2. *Suppose $u(r) \in C^1$ satisfies the following differential inequality in $(a, R)$,*

$$\dot{u} \geq A\left(e^{u/n} + \frac{B-b}{R-r}\right),$$

*where $\dot{u}$ represents differentiation with respect to $r$, $0 < A < 1$, $0 < b < 1$, $0 < a < R$ and $B \geq \frac{n}{A} + b$. Then there is an $\bar{r}$ in $(a, R)$ such that $u(r) \to \infty$ as $r \to \bar{r}$.*

*Proof*. Setting $v = e^{-u/n}$, we obtain that

$$\dot{v} + \frac{c}{R-r}v \leq -\frac{A}{n}, \quad a < r < R,$$

where $c = \frac{A(B-b)}{n}$. Using the integrating factor $\phi(r) = (\frac{1}{R-r})^c$ and setting $Z = v(r)\phi(r) - v(a)\phi(a)$, we obtain

$$Z \leq \begin{cases} \left(-\frac{A}{n}\right) \ln\frac{R-a}{R-r}; & c = 1, \\ \left(-\frac{A}{n}\right)\left(\frac{1}{c-1}\right)\left\{\left(\frac{1}{R-r}\right)^{c-1} - \left(\frac{1}{R-a}\right)^{c-1}\right\}; & c > 1. \end{cases}$$

It is clear that for each $c \geq 1$, there is an $\bar{r} \in (a, R)$ such that $v(r) \to 0$ as $r \to \bar{r}$, and hence $u(r) \to \infty$ as $r \to \bar{r}$. □

We present a comparison lemma; please refer to [**AW**] for its proof.

LEMMA 2.3. *In a region $(\Omega) \subseteq R^n$, $n \geq 2$, suppose $u, v \in W^{1,p}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$, and $(u-v)^+ \in W^{1,p}_0(\Omega)$. If $g$ is a nondecreasing function and*

$$L_p u \geq g(u) \quad \text{in } D'(\Omega),$$
$$L_p v \leq g(v) \quad \text{in } D'(\Omega),$$

*then $u \leq v$ a.e. in $(\Omega)$.*

**3. Nonexistence of radial subsolutions.** Consider the following problem

$$(3.1)\qquad (n-1)|\dot{u}|^{n-2}\left(\ddot{u}+\frac{\dot{u}}{r}\right)=e^{u}, \qquad 0<r<\infty,$$
$$u(R)=a, \quad \text{and} \quad \dot{u}(R)=b; \qquad a, b\in R.$$

LEMMA 3.1. *For the problem in* (3.1), *there exists a* $C^1$ *radial solution* $u(r)$ *such that at least one of the following occurs.*

(i) *There is an* $\bar{r}$ *in* $(0, R)$ *such that* $u(r)\to\infty$ *as* $r\to\bar{r}$.

(ii) *There is an* $\bar{r}$ *in* $(R, \infty)$ *such that* $u(r)\to\infty$ *as* $r\to\bar{r}$.

*Furthermore, there are values of* $b$ *for which both* (i) *and* (ii) *occur.*

*Proof.* We divide the proof into three parts.

*Case* 1. Take $b=0$. Let $u(r)$ be the solution defined by

$$(3.2)\qquad u(r)=a+\int_R^r\frac{1}{t}\left\{\int_R^t s^{n-1}e^{u(s)}\,ds\right\}^{1/(n-1)}dt,$$

in $r>R$. The existence and uniqueness in a small interval follows from Picard's iteration. It can be shown by differentiating that $u$ solves (3.1). From (3.2) it is clear that $r\dot{u}$ is increasing and thus $\dot{u}\geq 0$ in $(R, r)$, and hence $u$ is increasing. Continue $u$ by (3.2). By differentiating (3.2) once,

$$\dot{u}(r)=\frac{1}{r}\left\{\int_R^r s^{n-1}e^{u(s)}\,ds\right\}^{1/(n-1)}.$$

Thus,

$$\frac{d}{dr}\left\{\frac{(\dot{u})^{n-1}}{r}\right\}=\frac{r^n e^{u(r)}-n\int_R^r s^{n-1}e^{u(s)}\,ds}{r^{n+1}}$$
$$\geq\frac{r^n e^{u(r)}-e^{u(r)}(r^n-R^n)}{r^{n+1}}\geq 0.$$

By simplifying the left side of the foregoing inequality,

$$(n-1)\ddot{u}\geq\frac{\dot{u}}{r}.$$

Note that $u$ is $C^2$ except possibly where $\dot{u}=0$. Noting that $\dot{u}\geq 0$, (3.1) yields

$$n(n-1)(\dot{u})^{n-1}\ddot{u}\geq e^{u}, \qquad R<r<\infty.$$

Multiplying both sides by $\dot{u}$ and integrating once from $R$ to $r$,

$$(\dot{u})^n \geq \frac{e^u - e^a}{n-1}. \tag{3.3}$$

For $\varepsilon > 0$, small enough, it follows from (3.2) and the fact that $u$ is increasing that

$$u(r) > a + \int_{R+\varepsilon}^{r} \frac{1}{t}\left\{\int_R^{R+\varepsilon} s^{n-1}e^{u(s)}\,ds\right\}^{1/(n-1)} dt.$$

Hence for some appropriate constant $A > 0$,

$$u(r) > a + A\ln\frac{r}{R+\varepsilon}$$

implying that $u(r) \to \infty$ as $r$ gets large. Thus in (3.3) we may take $r > R_1$, where $R_1$ is large enough so that $e^u/2 \leq e^u - e^a$ for $r > R_1$. If $u(r) \to \infty$ as $r \to R_1$, then we are done. Otherwise, continue $u$ using (3.2) past $r = R_1$. Hence

$$\dot{u} \geq Ce^{u/n}, \quad \text{in } r > R_1,$$

for some $C > 0$. Integrating,

$$\int_{u(R_1)}^{u(r)} e^{-u/n}\,du \geq C(r - R_1).$$

It is clear that there exists an $\bar{r} > R$, such that $u(r) \to \infty$ as $r \to \bar{r}$. The case $b > 0$ follows similarly.

*Case* 2. Without any loss of generality, take $a = 0$. Take $b < 0$. Now $\dot{u}(r) < 0$ near $r = R$, so we obtain that $\dot{u}(r)$ satisfies

$$\dot{u}(r) = -\frac{1}{r}\left\{|bR|^{n-1} - \int_R^r t^{n-1}e^{u(t)}\,dt\right\}^{1/(n-1)}, \tag{3.4}$$

in $r > R$. We show that there is $\bar{b} < 0$ such that if $\bar{b} < b < 0$, there is an $\hat{r} > R$ such that $\dot{u}(r) \to 0$ as $r \to \hat{r}$. It follows from (3.4) that $r\dot{u}$ is increasing and thus

$$\frac{bR}{r} \leq \dot{u} \leq 0, \quad \text{for } r > R.$$

Set $c = bR$. Integrating, we find

$$e^u \geq r^c,$$

and so (3.4) yields

$$\dot{u}(r) \geq -\frac{1}{r}\left\{|c|^{n-1} - \int_R^r t^{n-1+c}\,dt\right\}^{1/(n-1)}.$$

Therefore,

$$\dot{u}(r) \geq \begin{cases} -\frac{1}{r}\left\{|c|^{n-1} - \frac{r^{n+c} - R^{n+c}}{n+c}\right\}^{1/(n-1)}; & -n < c < 0, \\ -\frac{1}{r}\left\{|c|^{n-1} - \ln\frac{r}{R}\right\}^{1/(n-1)}; & c = -n. \end{cases}$$

It is clear that there is an $\hat{r} > R$ for which $\dot{u}(r) \to 0$ as $r \to \hat{r}$. Now, take $c < -n$, satisfying

$$|c|^{n-1} - \frac{1}{|c| - n}\left(\frac{1}{R}\right)^{|c|-n} < n^{n-1}. \tag{3.5}$$

Now, (3.4) yields

$$\dot{u}(r) \geq -\frac{1}{r}\left[|c|^{n-1} - \frac{1}{|c|-n}\left\{\left(\frac{1}{R}\right)^{|c|-n} - \left(\frac{1}{r}\right)^{|c|-n}\right\}\right]^{1/(n-1)}.$$

Using (3.5), there is an $\tilde{r}$ such that $\dot{u}(r) \geq -\frac{n}{r}$ for $r > \tilde{r}$. If $\dot{u}(r) \to 0$ as $r \to \tilde{r}$, then we are done. Otherwise, continue $u$ past $r = \tilde{r}$. Repeating the arguments preceding (3.5), we see that $\dot{u}(r) \to 0$ as $r \to \hat{r}$ for some $\hat{r} > R$. Continuing $u$ past $r = \hat{r}$ using

$$u(r) = u(\hat{r}) + \int_{\hat{r}}^{r} \frac{1}{t}\left\{\int_{\hat{r}}^{t} s^{n-1} e^{u(s)}\, ds\right\}^{1/(n-1)} dt,$$

we may show, as in Case 1, that there is an $\bar{r} > R$ where $u$ blows up.

*Case* 3. We may again take $a = 0$. Let $c < -n$, $t = R - r$, and $v(t) = u(r)$, where $0 < r \leq R$. Then $\dot{v}(t) = -\dot{u}(r)$, where $\dot{v}$ represents differentiation with respect to $t$. Then

$$(n-1)|\dot{v}|^{n-2}\left(\ddot{v} - \frac{\dot{v}}{R-t}\right) = e^{v}, \qquad 0 \leq t < R, \tag{3.6}$$
$$v(0) = 0 \quad \text{and} \quad \dot{v}(0) = -b.$$

A solution of (3.6) is given by

$$v(t) = \int_0^t \frac{1}{R-s}\left\{|c|^{n-1} + \int_0^s (R-w)^{n-1} e^{v(w)}\, dw\right\}^{1/(n-1)} ds.$$

Equation (3.6) yields that $\frac{d}{dt}\{(R-t)\dot{v}\} \geq 0$, thus $\dot{v} \geq 0$ in $t > 0$. Integrating this inequality from 0 to $t$, we obtain

$$\dot{v}(t) \geq \frac{|c|}{(R-t)}.$$

Hence,

$$e^{v(t)} \geq \left(\frac{1}{R-t}\right)^{|c|}. \tag{3.7}$$

Let $0 < \varepsilon_0 < 1$ be such that

$$|c| \geq n\left\{\frac{1+\varepsilon^{1/n}}{(1+\varepsilon)^{1/n}}\right\} + \varepsilon$$

for every $\varepsilon$ in $(0, \varepsilon_0)$. It follows from (3.7) that there is a $t_1 < R$ such that

$$\left(\frac{|c|}{R-t}\right)^n e^{-v(t)} < \varepsilon_0,$$

for $t > t_1$. If $v(t) \to \infty$ as $t \to t_1$, then we are done; otherwise continue $v(t)$ past $t = t_1$. Furthermore, we may take $t_1$ such that $R - t_1 < \varepsilon_0$. Rearranging the terms in (3.6), and multiplying by $\dot{v}(t)$ yields

$$(n-1)(\dot{v})^{n-1}\ddot{v} = e^v\dot{v} + \frac{n-1}{R-t}(\dot{v})^n, \qquad 0 \leq t < R.$$

Integrating both sides from 0 to $t$, and noting that $\dot{v} \geq \frac{|c|}{R-t}$, we find

$$(\dot{v})^n \geq e^v - 1 + \left(\frac{|c|}{R-t}\right)^n, \qquad 0 \leq t < R.$$

By the definition of $t_1$, it follows that

$$(\dot{v})^n \geq e^v + \left(\frac{|c| - \varepsilon_0}{R-t}\right)^n, \qquad t_1 < t < R.$$

Setting

$$x = \left(\frac{|c| - \varepsilon_0}{R-t}\right)^n e^{-v},$$

the above may be rewritten as

$$(\dot{v})^n \geq e^v\{1 + x\}.$$

Hence,

$$\dot{v} \geq e^{v/n}\{1 + x\}^{1/n}.$$

Using Lemma 2.1 and the definition of $t_1$,

$$\dot{v} \geq C(\varepsilon_0)e^{v/n}\{1 + x^{1/n}\}.$$

Thus we obtain

$$\dot{v} \geq C(\varepsilon_0)\left\{e^{v/n} + \frac{|c| - \varepsilon_0}{R-t}\right\}, \qquad t_1 < t < R.$$

By Lemma 2.2, there is a $t_2 > t_1$ such that $v(t) \to \infty$ as $t \to t_2$. Hence there is an $\bar{r} \in (0, R)$ for which $u(r) \to \infty$ as $r \to \bar{r}$. Thus for every $c < -n$, we have a vertical asymptote in $(0, R)$. It is clear from (3.5) that there are values of $b$ for which both (i) and (ii) happen. Call one such value to be $b_R$.

For the case $a \neq 0$, we introduce the following change of variables. Let $v(r) = u(r) - a$; then

$$(n-1)|\dot{v}|^{n-2}\left(\ddot{v} + \frac{n-1}{r}\dot{v}\right) = e^a e^v.$$

Setting $t = re^{a/n}$, and $w(t) = v(r)$, and differentiating with respect to $t$, we have

$$(n-1)|\dot{w}|^{n-2}\left(\ddot{w} + \frac{n-1}{t}\dot{w}\right) = e^w,$$
$$w(\overline{R}) = 0 \quad \text{and} \quad \dot{w}(\overline{R}) = e^{-a/n} b,$$

where $\overline{R} = e^{a/n}R$. There is a $b_{\overline{R}}$ so that the corresponding solution which we continue to call $w(t)$, blows up near zero and at a point past $\overline{R}$. Then $u(t) = a + w(e^{-a/n}t)$ is such a solution for the original problem. □

**4. Proof of Theorem 1.** This follows easily from Lemma 2.3 and Lemma 3.1.

*Proof of Theorem* 1. Assume to the contrary. Let $U(x)$ be such a subsolution in (1.2). Let

$$a = \inf_{1/2 \le |x| \le 3/2} U(x).$$

By Lemma 3.1, there is a radial solution $u(r)$ such that $u(1) = a - 1$, and $u(r)$ blows up at some $\underline{r} \in (0, 1)$ and $\bar{r} \in (1, \infty)$. Let

$$M = \sup_{\underline{r} \le |x| \le \bar{r}} U(x),$$

$\underline{\underline{r}} \in (\underline{r}, 1)$ and $\bar{\bar{r}} \in (1, \bar{r})$ be such that $u(\underline{\underline{r}}),\ u(\bar{\bar{r}}) \ge M + 1$. Using Lemma 2.3, $u(x) \ge U(x)$ in $\underline{\underline{r}} \le |x| \le \bar{\bar{r}}$, a contradiction. □

REMARK. In Theorem 1, $1 < p \le n$ is the best possible. For $p > n$, take $u = \ln(\frac{A}{r^p})$, where $0 < A \le (p-n)p^{p-1}$. Then

$$L_p u = \frac{(p-n)p^{p-1}}{r^p} \ge \frac{A}{r^p}.$$

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