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POSITIVE 2-SPHERES IN 4-MANIFOLDS OF SIGNATURE $(1, n)$

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We sharpen Donaldson's theorem on the standardness of definite intersection forms of smooth 4-manifolds in the same sense as Kervaire and Milnor sharpened Rohlin's signature theorem. We then apply the result thus obtained to show that the homology classes of rational surfaces with $b_2^- \leq 9$ which can be represented by smoothly embedded 2-spheres S with $S \cdot S > 0$ are up to diffeomorphism represented by smooth rational curves. Furthermore, we not only extend part of the application to the case where $b_2^- > 9$, but also give an algorithm to see whether or not a given homology class of rational surfaces with $b_2^- \leq 9$ can be represented by a smoothly embedded 2-sphere.

1. Introduction. Let M be a closed oriented smooth 4-manifold. One of the most important facts in 4-dimensional differential topology is the following:

THEOREM R (*Rohlin's signature theorem* [**13**]). *If the second Stiefel-Whitney class $w_2(M)$ vanishes, then the signature $\sigma(M)$ is congruent to* 0 *modulo* 16.

Performing the topological blowing up/down operations and applying Theorem R, Kervaire and Milnor [**6**] extended Theorem R to deduce the following:

THEOREM KM. *If an integral homology class ξ of M, dual to $w_2(M)$, is represented by a smoothly embedded 2-sphere in M, then the self-intersection number $\xi \cdot \xi$ must be congruent to $\sigma(M)$ modulo* 16.

Note that, although used in their proof of Theorem KM, Theorem R can be regarded as a special case of Theorem KM with $\xi = 0$.

The primary purpose of this paper is to sharpen the following in the same sense as Kervaire and Milnor sharpened Theorem R:

THEOREM D (*Donaldson* [**2**]). *If the intersection form of M is negative-definite ($b_2^+ = 0$), then it is equivalent over the integers to $\bigoplus b_2^-(-1)$.*

We thus work through in the DIFF category. When the integral homology group $H_2(M)$ has torsion, we arbitrarily fix a splitting of $H_2(M)$, and accordingly of $\xi \in H_2(M)$, into free and torsion parts:

$$H_2(M) = F_2(M) \oplus T_2(M),$$
$$\xi = F_2\xi \oplus T_2\xi,$$

where $F_2\xi \in F_2(M)$, $T_2\xi \in T_2(M)$. We then regard $(F_2(M), \cdot)$ as the intersection form of M. We say that $\xi \in H_2(M)$ is represented by S^2 if it is represented by an embedded 2-sphere.

The primary result of this paper is then the following:

THEOREM 1. *Let M be a closed oriented smooth 4-manifold with $b_2^+ = 1$, $b_2^- = n \geq 1$, and ξ a class in $H_2(M)$ with $\xi \cdot \xi = s > 0$. If ξ is represented by S^2, then either of the following holds:*

(i) *there exist $\zeta_1, \dots, \zeta_n$ in $F_2(M)$ such that*

$$(F_2(M), \cdot) = (+1) \oplus n(-1)$$

with respect to the basis $\langle \eta; \zeta_1, \dots, \zeta_n \rangle$, where $F_2\xi = 2\eta$;

(ii) *there exist $\eta, \zeta_1, \dots, \zeta_{n-1}$ in $F_2(M)$ such that*

$$(F_2(M), \cdot) = \begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix} \oplus (n-1)(-1)$$

with respect to the basis $\langle F_2\xi, \eta; \zeta_1, \dots, \zeta_{n-1} \rangle$.

Note that Theorem D can be regarded as a special case of Theorem 1 with

$$M = \mathbf{C}P^2 \# N, \quad \xi = [\text{a quadric on } \mathbf{C}P^2],$$

where N is a closed oriented 4-manifold with $b_2^+(N) = 0$. We remark that Theorem 1 is an improvement over Lemma (2.1) of the author's previous paper [7], in which he, with relevance to the 11/8-conjecture, also proved another theorem (Theorem (1.3)) which implies Donaldson's theorem on even intersection forms of 4-manifolds.

The secondary purpose of this paper is to apply Theorem 1 to the problem of representing homology classes of complex rational surfaces by embedded 2-spheres.

Our results for this purpose are the following.

Theorem 2. *Let M be either $S^2 \times S^2$ or $\mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2$, $0 \leq n \leq 9$, and ξ a class in $H_2(M)$ with $\xi \cdot \xi = s > 0$. ξ is represented by S^2 if and only if either of the following diffeomorphisms f exists:*

(i) $f\colon \mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2 \to M$ *such that* $f_*([\mathbf{C}P^1]$ *or* $2[\mathbf{C}P^1]) = \xi$,

(ii) $f\colon \Sigma_s \# (n-1)\overline{\mathbf{C}P}^2 \to M$ *such that* $f_*([Z_s]) = \xi$,

where $\mathbf{C}P^1$ is a line on $\mathbf{C}P^2$, and Z_s is the "zero section" ($\cong \mathbf{C}P^1$) on the s-th Hirzebruch surface Σ_s with $Z_s \cdot Z_s = s$.

This reinterprets and improves all the known facts about that problem **[15, 9, 10, 12, 7]**. For Hirzebruch surfaces, see (3.1).

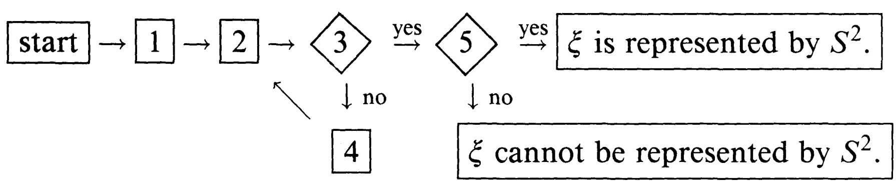
Theorem 3. *Let M be $\mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2$, $n \geq 2$, and ξ a class in $H_2(M)$ with $\xi \cdot \xi > 0$. Let $(x_0; x_1, \dots, x_n)$, $x_i \in \mathbf{Z}$, denote a class in $H_2(M)$ with respect to the natural basis of $H_2(M)$. If ξ is represented by S^2, then ξ is in the orbit of one of*

$$(2; 0, \dots, 0), \quad (k+1; k, 0, \dots, 0), \quad (k+1; k, 1, 0, \dots, 0)$$

under the action of the orthogonal group $O(M)$ of $(H_2(M), \cdot)$. Furthermore, the converse also holds if $n \leq 9$.

This improves Theorem (1.1) of **[7]**. When $n \leq 9$, there is an algorithm to ascertain whether a given ξ is in such an orbit or not:

Theorem 4. *Let M be $\mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2$, $2 \leq n \leq 9$, and ξ a class in $H_2(M)$ with $\xi \cdot \xi > 0$. Then one can see whether ξ is represented by S^2 or not by using the following algorithm:*



1. *Set* $\xi = (x_0; x_1, \dots, x_n)$, $x_i \in \mathbf{Z}$, *with respect to the natural basis of* $H_2(M)$.

2. *Set* $\eta = (y_0; y_1, \dots, y_n) = (|x_0|; |x'_1|, \dots, |x'_n|)$ *so that*

$$\{x'_1, \dots, x'_n\} = \{x_1, \dots, x_n\}, \quad y_1 \geq \cdots \geq y_n \geq 0.$$

3. *Does* η *satisfy* $y_0 \geq y_1 + y_2 + y_3$?

4. *Set*

$$\xi = \eta + \begin{cases} 2(y_0 - y_1 - y_2)(1; 1, 1), & n = 2, \\ (y_0 - y_1 - y_2 - y_3)(1; 1, 1, 1, 0, \dots, 0), & 3 \leq n \leq 9. \end{cases}$$

5. *Is* η *equal to* $(2; 0, \dots, 0)$, $(k+1; k, 0, \dots, 0)$ *or* $(k+1; k, 1, 0, \dots, 0)$?

Note that if one goes around once along the loop in the algorithm, one strictly reduces the absolute value $|x_0|$ of x_0, so that one must go down to step 5 after going around the loop finitely many times since $\xi \cdot \xi > 0$.

In §2 (resp. §3), we prove Theorem 1 (resp. Theorems 2–4); and in §4, we conclude by making some remarks about a deduction from Rohlin's genus theorem **[14]**, a modification to a theorem of B. H. Li **[11]**, and a conjecture on rationality of complex surfaces.

2. Proof of Theorem 1. We first recall some facts, indispensable for our proofs of Theorems 1–4, about Lorentzian spaces.

(2.1) *Facts.* Let $(\Lambda, \cdot)$ be Lorentzian $(1, n)$-space, i.e. the inner product space over $\mathbf{R}$ of signature $(1, n)$, $n \geq 1$.

(1) (Reverse Cauchy-Schwarz' inequality.) If $\xi \in \Lambda$ is timelike $(\xi \cdot \xi > 0)$, then $(\xi \cdot \eta)^2 \geq (\xi \cdot \xi)(\eta \cdot \eta)$ for any vector $\eta \in \Lambda$, where equality holds if and only if η is parallel to ξ.

(2) If $\xi, \eta \in \Lambda$ are future-pointing with respect to a certain timelike vector $\tau \in \Lambda$ $(\xi \cdot \xi \geq 0,\ \eta \cdot \eta \geq 0,\ \xi \cdot \tau > 0,\ \eta \cdot \tau > 0, \tau \cdot \tau > 0)$, then $\xi \cdot \eta \geq 0$, where equality holds if and only if ξ, η are lightlike $(\xi \cdot \xi = \eta \cdot \eta = 0)$ and proportional.

We next show a lemma, which we need in (2.7) and in (3.8).

LEMMA (2.2). *Let* $(\Xi, \cdot)$ *be an inner product space over* $\mathbf{Z}$ *of signature* $(1, n)$, $n \geq 1$, *and* ξ *a vector in* Ξ *with* $\xi \cdot \xi = s \geq 2$. *Let* Y *be the subset of* Ξ *of all vectors* η *with* $\xi \cdot \eta = 1$, $\eta \cdot \eta = 0$. *If* $\eta \in Y$, *then*

$$Y = \begin{cases} \{\eta, \xi - \eta\}, & s = 2, \\ \{\eta\}, & s \geq 3. \end{cases}$$

Proof. ξ and η generate a subspace of $(\Xi, \cdot)$ with orthogonal complement $(\Omega, \cdot)$ negative-definite. Let η' be another vector in Y. Then

$$\eta' = x\xi + y\eta + \zeta,$$

where $x, y \in \mathbf{Z}$ and $\zeta \in \Omega$. $\xi \cdot \eta' = 1$ and $\eta' \cdot \eta' = 0$ imply

$$sx + y = 1, \quad sx^2 + 2xy + \zeta \cdot \zeta = 0; \quad \therefore sx^2 - 2x - \zeta \cdot \zeta = 0.$$

Let d be the discriminant of the last equation. Then

$$d/4 = 1 + s(\zeta \cdot \zeta) \geq 0.$$

Since $s \geq 2$ and $(\Omega, \cdot)$ is negative-definite, we have $\zeta = 0$ and

$$(x, y) = \begin{cases} (0, 1) \text{ or } (1, -1), & s = 2, \\ (0, 1), & s \geq 3. \end{cases} \qquad \square$$

Now, we are ready to give the proof of Theorem 1, which is in fact obtained by improving that of Lemma (2.1) of [7]. We divide the proof into a series of steps: (2.3)–(2.7). Throughout the proof, for a finite set E, we denote by $\#E$ the number of elements in E.

LEMMA (2.3). *Let M, ξ be as in the hypothesis of Theorem 1. Let*

$$\Omega = \{(\zeta; z_1, \ldots, z_{s-1}) \in F_2(M) \oplus \mathbf{Z}^{s-1}; \xi \cdot \zeta - z_1 - \cdots - z_{s-1} = 0\},$$
$$Z = \{(\zeta; z_1, \ldots, z_{s-1}) \in \Omega; \zeta \cdot \zeta - z_1^2 - \cdots - z_{s-1}^2 = -1\}.$$

For $(\eta; y_1, \ldots, y_{s-1}) \in \Omega$ and $(\zeta; z_1, \ldots, z_{s-1}) \in \Omega$, define

$$(\eta; y_1, \ldots, y_{s-1}) \cdot (\zeta; z_1, \ldots, z_{s-1}) = \eta \cdot \zeta - y_1 z_1 - \cdots - y_{s-1} z_{s-1}.$$

Then, Theorem D *implies the following*:

(1) $(\Omega, \cdot) \cong \bigoplus (n + s - 1)(-1)$,
(2) $(1/2)\#Z = n + s - 1$.

Proof. Suppose that ξ is represented by an embedded 2-sphere S in M. "Blow up" $(s-1)$ distinct points of S, and then "blow down" the resulting "exceptional curve" of self-intersection $+1$, to construct a closed oriented 4-manifold N with $(b_2^+, b_2^-) = (0, n + s - 1)$:

$$(M, S)\#(s-1)(\overline{\mathbf{C}P}^2, \overline{\mathbf{C}P}^1) \cong (\mathbf{C}P^2, \mathbf{C}P^1)\#(N, \phi),$$

where $\mathbf{C}P^1$ (resp. $\overline{\mathbf{C}P}^1$) is a line on $\mathbf{C}P^2$ (resp. $\overline{\mathbf{C}P}^2$). Under the identifications

$$F_2(M\#(s-1)\overline{\mathbf{C}P}^2) = F_2(M) \oplus \mathbf{Z}^{s-1}, \quad (F_2(N), \cdot) = (\Omega, \cdot),$$

we see that Theorem D implies (1) and thus (2): for details, see [7]. $\square$

LEMMA (2.4). *Theorem* 1 *holds if $\xi \cdot \xi = s = 1$.*

Proof. By (2.3), there exist $\zeta_1, \ldots, \zeta_n \in F_2(M)$ such that

$$(F_2(M), \cdot) = (+1) \oplus n(-1)$$

with respect to the basis $\langle F_2\xi\,;\,\zeta_1\,,\,\ldots\,,\,\zeta_n\rangle$. Let $\eta = F_2\xi + \zeta_n$. Then

$$(F_2(M)\,,\,\cdot) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \oplus (n-1)(-1)$$

with respect to the basis $\langle F_2\xi\,,\,\eta\,;\,\zeta_1\,,\,\ldots\,,\,\zeta_{n-1}\rangle$. □

LEMMA (2.5). *Let ξ be as in the hypothesis of Theorem* 1, *and assume $\xi\cdot\xi = s \geq 2$. Let Z be as in* (2.3), *and let*

$$Z_0 = \{(\zeta\,;\,0\,,\,\ldots\,,\,0) \in Z\}\,, \quad Z_1 = Z - Z_0\,.$$

Choose and fix $(\zeta\,;\,z_1\,,\,\ldots\,,\,z_{s-1}) \in Z_1$ $(\#Z_1 \geq 2(s-1) \geq 2)$, *and let*

$$r = \#\{i\,;\,z_i \neq 0\}\,, \quad \Delta = (\xi\cdot\zeta)^2 - (\xi\cdot\xi)(\zeta\cdot\zeta)\,.$$

Then, the following equalities hold:
(1) $\xi\cdot\zeta = z_1 + \cdots + z_{s-1} = \pm r$,
(2) $\zeta\cdot\zeta = z_1^2 + \cdots + z_{s-1}^2 - 1 = r - 1$,
(3) $\Delta(\Delta - 1) = 0$.

Proof. We naturally embed $(F_2(M)\,,\,\cdot)$ into Lorentzian $(1\,,\,n)$-space $(\Lambda\,,\,\cdot)$. In light of (2.1)(1), we see $\Delta \geq 0$. Note $1 \leq r \leq s-1$. We then calculate as follows:

$$\begin{aligned} 0 \leq \Delta &= \left(\sum z_i\right)^2 - s\left(\sum z_i^2 - 1\right) \\ &\leq r\left(\sum z_i^2\right) - s\left(\sum z_i^2 - 1\right) = s - (s-r)\left(\sum z_i^2\right)\,, \end{aligned}$$

$$(s-r)r \leq (s-r)\left(\sum z_i^2\right) \leq s \leq (s-r)(r+1)\,,$$
$$\therefore 1 \leq r \leq \sum z_i^2 \leq r+1\,,$$

hence (2). Let $r_- = \#\{i\,;\,z_i = -1\}$. We further calculate:

$$\begin{aligned} 0 \leq \Delta &= (r - 2r_-)^2 - s(r-1) \\ &\leq (r-2r_-)^2 - (r+1)(r-1) = 1 - 4(r - r_-)r_- \leq 1\,, \end{aligned}$$

hence (1) and (3). □

LEMMA (2.6). *Let Δ be as in* (2.5). *Then Theorem* 1 *holds if* $\Delta = 0$ $(s \geq 2)$: *to be more precise, the case where $\Delta = 0$ corresponds to case* (i) *of Theorem* 1.

Proof. Note by (2.1)(1) that $F_2\xi\,,\,\zeta$ are proportional. We thus observe that $\Delta = r^2 - s(r-1) = 0$ implies

$$s = 4\,, \quad r = 2: \quad \xi\cdot\zeta = \pm 2\,, \quad \zeta\cdot\zeta = 1\,, \quad F_2\xi = \pm 2\zeta\,.$$

Let η be either of $\pm\zeta$ so that $F_2\xi = 2\eta$. We then see

$$Z_1 = \{\pm(\eta\,;\,0\,,\,1\,,\,1)\,,\,\pm(\eta\,;\,1\,,\,0\,,\,1)\,,\,\pm(\eta\,;\,1\,,\,1\,,\,0)\}:$$
$$(1/2)\#Z_1 = 3(= s-1)\,,\quad (1/2)\#Z_0 = n\,.$$

Note by (2.3) that, if $(\zeta_0\,;\,0\,,\,0\,,\,0)$ is an element in Z_0, then $\eta\cdot\zeta_0 = 0$, $\zeta_0\cdot\zeta_0 = -1$. The case where $\Delta = 0$ therefore corresponds to case (i). □

LEMMA (2.7). *Let Δ be as in* (2.5). *Then Theorem* 1 *holds if* $\Delta = 1$ $(s \geq 2)$: *to be more precise, the case where $\Delta = 1$ corresponds to case* (ii) *of Theorem* 1.

Proof. We first see that $\Delta = r^2 - s\,(r-1) = 1$ implies either of the following:

$$r = 1: \begin{cases} \xi\cdot\zeta = \pm 1\,, \\ \zeta\cdot\zeta = 0\,, \end{cases}$$
$$r = s-1: \begin{cases} \xi\cdot\zeta = \pm(s-1)\,, \\ \zeta\cdot\zeta = s-2\,. \end{cases}$$

We next observe the following equivalence:

$$\begin{cases} \xi\cdot\zeta = s-1\,, \\ \zeta\cdot\zeta = s-2\,, \end{cases} \Leftrightarrow \begin{cases} \xi\cdot(\xi-\zeta) = 1\,, \\ (\xi-\zeta)\cdot(\xi-\zeta) = 0\,. \end{cases}$$

In either case, we can choose $\eta \in F_2(M)$ such that

$$\begin{cases} \xi\cdot\eta = 1\,, \\ \eta\cdot\eta = 0\,. \end{cases}$$

Then the equivalence above and the uniqueness (2.2) of η show

$$Z_1 = \{\pm(\eta\,;\,1\,,\,0\,,\,\dots\,,\,0)\,,\,\pm(\eta\,;\,0\,,\,1\,,\,0\,,\,\dots\,,\,0)\,,\,\dots\,,$$
$$\pm\,(\eta\,;\,0\,,\,\dots\,,\,0\,,\,1)\,,\,\pm((F_2\xi)-\eta\,;\,1\,,\,1\,,\,\dots\,,\,1)\}:$$
$$(1/2)\#Z_1 = s\,,\quad (1/2)\#Z_0 = n-1\,.$$

Note by (2.3) that, if $(\zeta_0\,;\,0\,,\,\dots\,,\,0) \in Z_0$, then

$$\xi\cdot\zeta_0 = 0\,,\quad \eta\cdot\zeta_0 = 0\,,\quad \zeta_0\cdot\zeta_0 = -1\,.$$

The case where $\Delta = 1$ therefore corresponds to case (ii). □

We have completed the proof of Theorem 1.

3. Proofs of Theorems 2–4. To prove Theorem 2 and Theorem 3, we recall some facts about complex rational surfaces.

(3.1) *Facts.* Let Σ_k denote the k-th Hirzebruch surface, i.e., the total space of $\mathbf{C}P^1$-bundle over $\mathbf{C}P^1$ whose "zero section" Z_k ($\cong \mathbf{C}P^1$) and "fiber" F_k ($\cong \mathbf{C}P^1$) form a basis $\langle [Z_k], [F_k] \rangle$ of $(H_2(\Sigma_k), \cdot)$ such that

$$(H_2(\Sigma_k), \cdot) = \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix}.$$

(1) Σ_k is biholomorphic to Σ_l if and only if $|k| = |l|$, while Σ_k is diffeomorphic to Σ_l if and only if $k \equiv l \pmod 2$; in particular, Σ_{2k} (resp. Σ_{2k+1}) is diffeomorphic to $S^2 \times S^2$ (resp. $\mathbf{C}P^2 \# \overline{\mathbf{C}P}^2$): see [**1**, p. 141], [**17**, §1].

(2) If $n \geq 2$, then $\mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2$ is diffeomorphic to $\Sigma_k \# (n-1)\overline{\mathbf{C}P}^2$ for an arbitrary integer k: see [**17**, §3].

(3) Let M be either $\mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2$ or $\Sigma_k \# (n-1)\overline{\mathbf{C}P}^2$. If $n \leq 9$, then any automorph in the orthogonal group $O(M)$ of $(H_2(M), \cdot)$ can be represented by an orientation-preserving self-diffeomorphism of M: see [**17**, §3].

(3.2) *Proof of Theorem* 2. The "if" part is clear. Thus suppose that ξ is represented by S^2. Then it follows from Theorem 1 that there exists either of the following isomorphisms ϕ:

(i) $\phi: (H_2(\mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2), \cdot) \to (H_2(M), \cdot)$, $\phi([\mathbf{C}P^1]$ or $2[\mathbf{C}P^1]) = \xi$;

(ii) $\phi: (H_2(\Sigma_s \# (n-1)\overline{\mathbf{C}P}^2), \cdot) \to (H_2(M), \cdot)$, $\phi([Z_s]) = \xi$.

However, such an isomorphism ϕ is realized by an orientation-preserving diffeomorphism f because of (3.1)(2) and (3.1)(3). □

(3.3) *Proof of Theorem* 3. Let $X(\xi)$ be the subset of $H_2(M)$ which consists of those elements ξ' with $\xi' \cdot \xi' = \xi \cdot \xi$ such that $\xi'/2$ (resp. ξ') can be the first base of a basis of $(H_2(M), \cdot)$ of type (i) (resp. (ii)) in Theorem 1. Note that the orthogonal group $O(M)$ of $(H_2(M), \cdot)$ transitively acts on $X(\xi)$, and that

$$\xi_* = (2; 0, \dots, 0) \left(\text{resp. } \begin{cases} (k+1; k, 0, \dots, 0), \ \xi \cdot \xi = 2k+1 \\ (k+1; k, 1, 0, \dots, 0), \ \xi \cdot \xi = 2k \end{cases} \right)$$

can be a representative of $X(\xi)$: namely, $X(\xi)$ is the $O(M)$-orbit of ξ_*. The assertion follows from Theorem 1 and (3.1)(3), since ξ_* can be represented by a quadric on $\mathbf{C}P^2$ (resp. Z_s on Σ_s, $s = \xi \cdot \xi$ (cf. (3.1)(2))). □

To prove Theorem 4, we need the following.

LEMMA (3.4). *Let* $(\Xi, \cdot) = (+1) \oplus n(-1)$, $2 \leq n \leq 9$. *Let* ξ *be an element in* Ξ *denoted by* $(x_0; x_1, \dots, x_n)$, $x_i \in \mathbf{Z}$, *with*

$$\xi \cdot \xi > 0, \quad x_1 \geq \cdots \geq x_n \geq 0, \quad x_0 \geq x_1 + x_2 + x_3.$$

(1) *Suppose that* $(\Xi, \cdot)$ *is diagonalized as follows*:

$$(\Xi, \cdot) = (+1) \oplus n(-1)$$

with respect to $\langle \eta; \zeta_1, \dots, \zeta_n \rangle$, *where* $\eta = \xi$ (*resp.* $\xi/2$). *Then*

$$\xi = (1; 0, \dots, 0) \quad (\textit{resp. } (2; 0, \dots, 0)).$$

(2) *Suppose that* $\xi \cdot \xi = s \geq 2$, *and that*

$$(\Xi, \cdot) = \begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix} \oplus (n-1)(-1)$$

with respect to $\langle \xi, \eta; \zeta_1, \dots, \zeta_{n-1} \rangle$. *Then*

$$\xi = (k+1; k, 0, \dots, 0) \quad \textit{or} \quad (k+1; k, 1, 0, \dots, 0).$$

(3.5) *Proof of Theorem* 4 *assuming* (3.4). Note that operations 2, 4 in Theorem 4 are performed by automorphs in the orthogonal group $O(M)$ of $(H_2(M), \cdot)$: see [**16**, 1.5, 1.6], [7, (2.2)]. Thus the assertion immediately follows from Theorem 3 and (3.4). □

(3.6) *Proof of* (3.4)(1). Without loss of generality, we assume $n = 9$ and $\xi \cdot \xi = 1$. Since

$$0 \leq x_0^2 - (x_1 + x_2 + x_3)^2 \leq x_0^2 - x_1^2 - \cdots - x_9^2 = 1,$$

either $x_0 = 1$, $x_1 = \cdots = x_9 = 0$ (done); or $x_0 = x_1 + x_2 + x_3$. In the latter case, since

$$0 \leq (x_3^2 - x_4^2) + \cdots + (x_3^2 - x_9^2) \leq x_0^2 - x_1^2 - \cdots - x_9^2 = 1,$$

either (i) $x_3 = \cdots = x_8 = 1$, $x_9 = 0$; or (ii) $x_3 = \cdots = x_8 = x_9$. In case (i), $\xi \cdot \xi = 1$ implies

$$x_1 = x_2 = 1: \quad \xi = (3; 1, 1, 1, 1, 1, 1, 1, 1, 0).$$

However, this contradicts the diagonalizability of $(\Xi, \cdot)$, since the orthogonal complement of ξ turns out to be isomorphic to $(-E_8) \oplus (-1)$. In case (ii), $\xi \cdot \xi = 1$ yields

$$2(x_2 x_3 + x_3 x_1 + x_1 x_3 - 3x_3^2) = 1,$$

a contradiction. □

To prove (3.4)(2), we need the following, which holds even if $n > 9$.

SUBLEMMA (3.7). *Let* ξ, η *be as in the hypothesis of* (3.4)(2).
(1) ξ, η *are primitive and ordinary.*
(2) $(x_0 - 1)^2 \leq x_1^2 + \cdots + x_n^2$.
(3) $(s-1)(y_0^2 + 1) \leq x_0^2$, $y_0 > 0$ *if* $\eta = (y_0; y_1, \ldots, y_n)$.
(4) $(s-1)(y_i^2 - 1) \leq x_i^2$, $x_i y_i \geq 0$ $(i \geq 1)$ *if* $\eta = (y_0; y_1, \ldots, y_n)$.

Proof. (1) Clear since $n \geq 2$.
(2) Let $\eta = (y_0; y_1, \ldots, y_n)$. It follows:

$$\begin{aligned}(x_0 - 1)^2 y_0^2 &\leq (x_0 y_0 - 1)^2 \\ &= (x_1 y_1 + \cdots + x_n y_n)^2 \\ &\leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) \\ &= (x_1^2 + \cdots + x_n^2) y_0^2 .\end{aligned}$$

Since $\xi \cdot \eta = 1$ implies $y_0 \neq 0$, hence the inequality: cf. [7, (2.3)(2)].
(3) Embed $(\Xi, \cdot)$ into Lorentzian $(1, n)$-space. Since

$$\xi \cdot \xi > 0, \quad x_0 > 0, \quad \xi \cdot \eta = 1, \quad \eta \cdot \eta = 0,$$

it follows from (2.1)(2) that $y_0 > 0$. It also follows:

$$\begin{aligned}(x_0 y_0)^2 &= (x_1 y_1 + \cdots + x_n y_n + 1)^2 \\ &\leq (x_0^2 - s + 1)(y_0^2 + 1), \\ &\therefore (s-1)(y_0^2 + 1) \leq x_0^2 .\end{aligned}$$

(4) Embed $(\Xi, \cdot)$ into Lorentzian $(1, n)$-space. Assume $i \geq 1$. Let

$$\begin{aligned}\xi_i &= (x_0; x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n), \\ \eta_i &= (y_0; y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n).\end{aligned}$$

Note that $\xi_i \cdot \xi_i > 0$, and that $\eta_i \cdot \eta_i \geq 0$ if $y_i \neq 0$. Thus assume $y_i \neq 0$. Then, (2.1)(1) and (2.1)(2) imply

$$(s-1)(y_i^2 - 1) \leq x_i^2, \quad x_i y_i \geq 0$$

respectively, both of which are valid even if $y_i = 0$. □

(3.8) *Proof of* (3.4)(2). Assuming $n = 9$ as in (3.6), we divide the proof into a series of steps: (1)–(4).

Step (1). If $x_4 = 0$, then $x_0 = x_1 + 1$, $x_2 \leq 1$, $x_3 = 0$ (done).

Proof. Note that $\xi \cdot \xi \geq 2$ implies $x_1 + x_2 + x_3 \geq 1$. Thus by (3.7)(2),

$$\begin{aligned}(x_1 + x_2 + x_3 - 1)^2 \leq (x_0 - 1)^2 \leq x_1^2 + x_2^2 + x_3^2, \\ 2x_2(x_3 - 1) + 2x_3(x_1 - 1) + 2x_1(x_2 - 1) + 1 \leq 0,\end{aligned}$$

and hence $x_3 = 0$, $x_2 \leq 1$, $x_1 \geq 1$. Then by (3.7)(2) again,

$$0 \leq (x_0 - 1)^2 - x_1^2 \leq x_2^2 \leq 1,$$

hence $x_0 = x_1 + 1$. □

Step (2). If $x_4 > 0$, then $x_0 = x_1 + 2x_4$, $x_1 \leq x_4 + 1$, $x_2 = x_3 = x_4 \geq 2$.

Proof. First assume $x_0 \geq x_1 + x_2 + x_3 + 1$. By (3.7)(2),

$$(x_1 + x_2 + x_3)^2 \leq (x_0 - 1)^2 \leq x_1^2 + \cdots + x_9^2,$$
$$\therefore x_1 = \cdots = x_9 > 0: \quad \xi = (x_0 ; x_1, x_1, \ldots, x_1).$$

Since $\xi \cdot \xi \geq 6x_1 + 1 \geq 7$, η is unique by (2.2). Since ξ is then fixed by any permutation among $\{x_1, \ldots, x_9\}$, so is η: namely,

$$\eta = (y_0 ; y_1, y_1, \ldots, y_1).$$

However, $\eta \cdot \eta = 0$ implies $y_0 = \pm 3y_1$, which contradicts (3.7)(1): hence $x_0 = x_1 + x_2 + x_3$. Then, by (3.7)(2) again,

$$(x_1 + x_2 + x_3 - 1)^2 = (x_0 - 1)^2 \leq x_1^2 + \cdots + x_9^2,$$
$$L := 2x_2(x_3 - 1) + 2x_3(x_1 - 1) + 2x_1(x_2 - 1) + 1 \leq x_4^2 + \cdots + x_9^2 =: R.$$

Secondly, $x_1 \geq x_4 + 2$ implies

$$L \geq 2x_4(x_4 - 1) + 2x_4(x_4 + 1) + 2(x_4 + 2)(x_4 - 1) + 1 > R,$$

a contradiction, hence $x_1 \leq x_4 + 1$. Similarly, since $x_2 \geq x_4 + 1$ implies $L > R$, it follows $x_2 = x_3 = x_4$.

Lastly, to show $x_4 \geq 2$, assume $x_4 = 1$. The inequality $L = 2x_1 - 1 \leq R \leq 6$ implies $x_1 \leq 3$. If $x_1 = 1$, then

$$\xi = (3 ; 1, 1, 1, 1, x_5, \ldots, x_9).$$

Note by (3.7)(3) that, if $\eta = (y_0 ; y_1, \ldots, y_9)$, then $y_0 = 1$ or 2: this is impossible since $\xi \cdot \eta = 1$, $\eta \cdot \eta = 0$, and $1 \geq x_5 \geq \cdots \geq x_9 \geq 0$. Thus assume $x_1 = 2$ (resp. 3). Then

$$\xi = (4 ; 2, 1, 1, 1, x_5, \ldots, x_9), \quad \xi \cdot \xi \geq 4$$
$$(\text{resp. } (5 ; 3, 1, 1, 1, x_5, \ldots, x_9), \quad \xi \cdot \xi \geq 8).$$

From (3.7)(3), (3.7)(4) and the uniqueness (2.2) of η, it follows:

$$\eta = (y_0 ; y_1, y, y, y, y_5, \ldots, y_9),$$
$$y_0 = 1 \text{ or } 2 \text{ (resp. } 1), \quad y_1 = 0 \text{ or } 1, \quad y = 0 \text{ or } 1.$$

However, it is easily verified that each case contradicts either $\eta \cdot \eta = 0$ or $\xi \cdot \eta = 1$, which shows $x_4 \geq 2$. □

Step (3). ξ cannot be of form $(3x; x, x, x, x, x_5, \dots, x_9)$, $x \geq 2$.

Proof. Suppose so. Since $x > x_6$ contradicts (3.7)(2), it follows:

$$x_5 = x_6 = x: \quad \xi = (3x; x, x, x, x, x, x, x_7, x_8, x_9).$$

Note by (3.7)(1) that $x_9 \leq x-1$, $\xi \cdot \xi \geq 2x-1 \geq 3$. $\eta = (y_0; y_1, \dots, y_9)$ is hence unique by (2.2), and thus fixed both by reflection 4 in Theorem 4 (cf. [7, (2.2)]) and by any permutation among $\{y_1, \dots, y_6\}$. Thus

$$\eta = (3y; y, y, y, y, y, y, y_7, y_8, y_9).$$

However, $\eta \cdot \eta = 0$ implies:

$$3y^2 = y_7^2 + y_8^2 + y_9^2, \quad y \equiv y_7 \equiv y_8 \equiv y_9 \pmod 2,$$

which contradicts (3.7)(1). □

Step (4). ξ cannot be of form $(3x+1; x+1, x, x, x, x_5, \dots, x_9)$, $x \geq 2$.

Proof. Suppose so. As in (3), it follows:

$$\xi = (3x+1 : x+1, x, x, x, x, x, x, x, x_9),$$
$$\eta = (y_1 + 2y; y_1, y, y, y, y, y, y, y, y_9),$$
$$\eta \cdot \eta = 4y_1 y - 3y^2 - y_9^2 = 0,$$
$$\therefore (y_1, y, y_9) \equiv (0, 1, 1) \text{ or } (1, 0, 0) \pmod 2.$$

However, the former congruence and $\eta \cdot \eta = 0$ imply

$$0 \equiv 4y_1 y \equiv 3y^2 + y_9^2 \equiv 4 \pmod 8,$$

a contradiction, while the latter congruence and $\xi \cdot \eta = 1$ also give

$$0 \equiv x(2y_1 - y) + 2y - x_9 y_9 \equiv 1 \pmod 2,$$

a contradiction. □

We have completed the proof of Theorem 4.

4. Concluding remarks. We conclude by making some remarks about Theorem 1 and Theorem 2.

(4.1) Let M, ξ be as in the hypothesis of Theorem 1. Assume $H_1(M) = 0$ and ξ divisible. Then, it follows from Rohlin's genus theorem [**14**] that $\xi = 2\eta$ for some $\eta \in H_2(M)$ with $\eta \cdot \eta = 1$, which is only a part of Theorem 1. Note that in our proof of Theorem 1 we have applied only Theorem D (in (2.3)) without using Rohlin's genus theorem, and that the latter is theoretically level with the Atiyah-Singer index theorem on which the former partially depends about the calculation of the "virtual dimension" of the moduli space of instantons [**2**].

(4.2) Let M be as in the hypothesis of Theorem 1. Let η be a class in $H_2(M)$ with $\eta \cdot \eta = 0$, $F_2\eta$ being primitive. It is of great interest to compare with Theorem 1 the following slight generalization of a theorem of B. H. Li [**11**]: if η is represented by S^2, then there exist $\xi, \zeta_1, \dots, \zeta_{n-1} \in F_2(M)$ such that

$$(F_2(M), \cdot) = \begin{pmatrix} * & 1 \\ 1 & 0 \end{pmatrix} \oplus (n-1)(-1)$$

with respect to the basis $\langle \xi, F_2\eta; \zeta_1, \dots, \zeta_{n-1} \rangle$. In particular, consider the case where $M = S^2 \times S^2$ or $\mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2$, $1 \le n \le 9$. What corresponds to Theorem 2 is, then, the proposition that η is represented by S^2 if and only if, for some integer k, there exists a diffeomorphism f such that

$$f\colon \Sigma_k \# (n-1)\overline{\mathbf{C}P}^2 \to M, \quad f_*([F_k]) = \eta \quad \text{(cf. (3.1))}.$$

(4.3) Let M be a compact complex surface. One of the necessary and sufficient conditions for M to be rational is that M contains a smooth rational curve C with $C \cdot C > 0$ [**1**, p. 142]. We wish to conjecture that the phrase "smooth rational curve" might be substituted by "smoothly embedded 2-sphere". In fact, the following irrational surfaces have been proved not to contain any "positive 2-sphere" (2-sphere S with $[S] \cdot [S] > 0$):

(1) irrational ruled surfaces and their blown-ups [**3**],

(2) Dolgachev surfaces $S(p, q)$ and their blown-ups [**4**],

(3) simply connected projective surfaces with $p_g \ge 1$ [**8**].

We can now cite other instances: namely, generalized Dolgachev surfaces $S(p, q)$ with $(p, q) \equiv (p+q)/(p, q) \equiv 0 \pmod 2$ (e.g., Enriques surfaces) cannot contain any "positive 2-sphere" by Theorem 1, since $b_2^+ = 1$, $b_2^- = 9$ and their intersection forms are even, although they are not spin [**5**].

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