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ON THE NON-OCCURRENCE OF THE COXETER GRAPHS β_{2n+1}, D_{2n+1} AND E_7 AS THE PRINCIPAL GRAPH OF AN INCLUSION OF II_1 FACTORS

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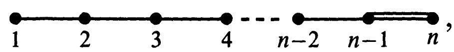
After discussing some preliminaries on the notion of an action of a hypergroup on a set, we present elementary proofs of the fact that the Coxeter graphs β_{2n+1}, D_{2n+1} and E_7 do not arise as Jones' principal graph invariant of an inclusion of II_1 factors. (Here, we use the symbol β_n to denote the graph that is normally denoted by B_n, the reason for this changed terminology being spelt out in the text.)

In this paper, we define and discuss some elementary consequences of the notion of an action of a hypergroup on a set and go on to use this notion to provide an elementary proof of the fact that the Coxeter graphs β_{2n+1}, D_{2n+1} and E_7 do not arise as Jones' principal graph invariant of an inclusion of II_1 factors. (The symbol β_n, rather than the symbol B_n, is used here to denote the graph



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for the reason, pointed out to us by the referee, that the double bond acquires different meanings depending upon whether the graph is viewed as a Coxeter-Dynkin diagram or as a Bratteli diagram describing the inclusion of a pair of finite-dimensional C^*-algebras.)

The assertion about the D and E graphs was announced, but without proof, in **[O1]**. After the preparation of the manuscript, it was brought to the attention of the authors that the recent preprint **[I]** also contains a proof of the above facts about the D and E graphs, and that the preprint **[Ka]** proves the occurrence of the D_{2n} diagrams as well as uses Ocneanu's concept of a flat connection to demonstrate the non-occurrence of the D_{2n+1} graphs.

One reason for presenting our proof is that it is elementary, it shows the use of hypergroups as convenient book-keeping devices, and it can be read easily by one who is not too familiar with index-theory of subfactors of type III factors or the work of Longo in this direction in the context of algebraic quantum field theory. It must be noted, however, that Izumi shows that the graphs E_7 and D_{2n+1} cannot arise as the graph-invariant of the finite-index inclusion of arbitrary factors, of type II as well as type III. Our proof, however, shows a subtle distinction between the cases D_{4n+1} and D_{4n+3}, besides containing more details.

For convenience of reference, we recall the definition of a hypergroup but refer to [S2] for basic facts concerning hypergroups.

DEFINITION. By a (discrete) hypergroup, we mean a set $\mathcal{G}$ equipped with a mapping $\mathcal{G} \times \mathcal{G} \times \mathcal{G} \to \mathbb{Z}^+$ $(= \{0, 1, 2, \dots\})$—denoted by $\langle \alpha, \beta, \gamma \rangle \to \langle \alpha \otimes \beta, \gamma \rangle$—that satisfies the following conditions:

(local finiteness): For all α, β in $\mathcal{G}$, $\langle \alpha \otimes \beta, \gamma \rangle > 0$ for at most finitely many γ;

(Associativity): for all α, β, γ, κ in $\mathcal{G}$, we have

$$\sum_{\lambda \in \mathcal{G}} \langle \alpha \otimes \beta, \lambda \rangle \langle \lambda \otimes \gamma, \kappa \rangle = \sum_{\lambda \in \mathcal{G}} \langle \alpha \otimes \lambda, \kappa \rangle \langle \beta \otimes \gamma, \lambda \rangle ;$$

(identity): there exists an element of $\mathcal{G}$, denoted by 1, such that

$\langle \alpha \otimes 1, \beta \rangle = \langle 1 \otimes \alpha, \beta \rangle = \delta_{\alpha\beta}$,

where the δ on the right side is the Kronecker symbol;

(contragredient): there is a self-map $\alpha \to \overline{\alpha}$ of $\mathcal{G}$ such that

$$\langle \alpha \otimes \beta, \gamma \rangle = \langle \overline{\alpha} \otimes \gamma, \beta \rangle, \qquad \text{for all } \alpha, \beta, \gamma \text{ in } \mathcal{G}.$$

We call a hypergroup $\mathcal{G}$ abelian if $\langle \alpha \otimes \beta, \gamma \rangle = \langle \beta \otimes \alpha, \gamma \rangle$ for all α, β, γ in $\mathcal{G}$. □

REMARK. It is a fact—analogous to the fact that a group in which every element has order two is necessarily abelian—that a hypergroup in which every element is self-conjugate is necessarily abelian. This fact, which is a consequence of the fact that the product of two real symmetric matrices is symmetric if and only if the two matrices commute, will be used in the sequel.

DEFINITION. An action of a hypergroup $\mathcal{G}_0$ on a set $\mathcal{G}_1$ is a mapping $\pi_1 : \mathbb{Z}^+\mathcal{G}_0 \to \text{End}(\mathbb{Z}^+\mathcal{G}_1)$ which is a homomorphism in the sense that it satisfies the following properties:

$$\pi_1(\alpha)\pi_1(\beta) = \sum_{\gamma \in \mathcal{G}_0} \langle \alpha \otimes \beta, \gamma \rangle \pi_1(\gamma) \qquad \forall \alpha, \beta \in \mathcal{G}_0$$

and

$$\pi_1(\overline{\alpha}) = \pi_1(\alpha)' \qquad \forall \alpha \in \mathcal{G}_0,$$

where $'$ denotes matrix transpose. (Here, we represent endomorphisms of $\mathbb{Z}^+\mathcal{G}_1$ by matrices with respect to the basis given by $\mathcal{G}_1$.)

EXAMPLE 1. Let $N \subset M$ be a pair of II_1 factors. The set $\mathcal{G}(N)$ of equivalence classes of irreducible N-N bimodules is a hypergroup, which acts on the collection $\mathcal{G}(N, M)$ of all equivalence classes of irreducible N-M bimodules, via tensor multiplication over N from the left.

EXAMPLE 2. Let $N = M_{-1}$, $M = M_0, M_1, M_2, \dots$ be the tower of the basic construction applied to the finite-index inclusion $N \subset M$ of II_1 factors. Let

$$\mathcal{G}_0 = \{\beta \in \mathcal{G}(N) \colon \beta \subseteq {}_N L^2(M_n)_N \text{ for some } n\}$$

and

$$\mathcal{G}_1 = \{\gamma \in \mathcal{G}(N, M) \colon \gamma \subseteq {}_N L^2(M_n)_M \text{ for some } n\}.$$

Then $\mathcal{G}_0$ acts on $\mathcal{G}_1$ and the inclusion $N \subset M$ has finite depth precisely when $\mathcal{G}_0$ and $\mathcal{G}_1$ are finite.

EXAMPLE 3. Suppose $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1$ is a $\mathbb{Z}_2$-graded hypergroup; i.e $\mathcal{G}_0$ is a subhypergroup of the hypergroup $\mathcal{G}$; the elements of $\mathcal{G}_0$ are thought of as having degree zero and the elements of $\mathcal{G}_1$ are thought of as having degree one and it is assumed that $\langle \alpha \otimes \beta, \gamma \rangle = 0$ unless $\deg \alpha + \deg \beta = \deg \gamma \pmod 2$. It is then easy to see that $\mathcal{G}_0$ acts on $\mathcal{G}_1$ in a natural fashion.

REMARK. Conversely suppose $\mathcal{G}_0$ acts on $\mathcal{G}_1$; then we could try to make $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1$ into a $\mathbb{Z}_2$-graded hypergroup by postulating that all the elements of $\mathcal{G}_1$ are self-contragredient as elements of $\mathcal{G}$, and defining $\langle \alpha \otimes \beta, \gamma \rangle$ to be the (α, β)th entry of the matrix $\pi_1(\gamma)$ whenever $\alpha, \beta \in \mathcal{G}_1$ and $\gamma \in \mathcal{G}_0$. It is a fairly easily verified fact that the above prescription makes $\mathcal{G}$ a $\mathbf{Z}_2$-graded hypergroup if and only if the following conditions are satisfied:

$$\pi_1(\mathcal{G}_0) \text{ is abelian}$$

and

$$\sum_{\beta \in \mathcal{G}_0} \pi_1(\beta)(\alpha_i, \alpha_j)\pi_1(\beta)(\alpha_l, \alpha_m) = \sum_{\beta \in \mathcal{G}_0} \pi_1(\beta)(\alpha_i, \alpha_l)\pi_1(\beta)(\alpha_j, \alpha_m) \quad \forall \alpha_i, \alpha_j, \alpha_l, \alpha_m \text{ in } \mathcal{G}_1. \quad \square$$

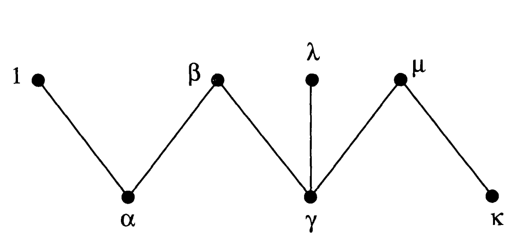
We recall now the description of one of the principal graphs corresponding to the finite-index inclusion $N \subset M$ of II_1 factors. We will be concerned with the Bratteli diagram of the tower $(N' \cap M_n : n = -1, 0, 1, 2, \ldots)$ of relative commutants of N in the members of the tower of the basic construction; this diagram admits the following alternative description—cf. **[GHJ]**, **[P]**, **[O1]**, **[O2]**—:

Let $\mathscr{G}_0^{(2\kappa)}$ denote the set of equivalence classes of irreducible N-N subbimodules of $L^2(M_{k-1})$, $k = 0, 1, 2, \ldots$, and let $\mathscr{G}_1^{(2\kappa+1)}$ denote the set of equivalence classes of irreducible N-M subbimodules of $L^2(M_k)$, $k = 0, 1, 2, \ldots$; connect a vertex β of $\mathscr{G}_0^{(2\kappa)}$ to a vertex γ of $\mathscr{G}_1^{(2\kappa\pm1)}$ by $\langle \beta \otimes_N L^2(M), \gamma\rangle$ bonds. Then, for $k \geq 0$, the Bratteli diagram of $(N' \cap M_{2k-1}) \subset (N' \cap M_{2k}) \subset (N' \cap M_{2k+1})$ is given by the nodes of $\mathscr{G}_0^{(2\kappa)}$, $\mathscr{G}_1^{(2\kappa+1)}$ and $\mathscr{G}_0^{(2\kappa+2)}$ with adjacency of nodes as described above. As discussed in Example 2 above, take $\mathscr{G}_0 = \bigcup_\kappa \mathscr{G}_0^{(2\kappa)}$ and $\mathscr{G}_1 = \bigcup_\kappa \mathscr{G}_1^{(2\kappa+1)}$.

The principal graph is the bipartite graph with even vertices indexed by $\mathscr{G}_0$ and odd vertices indexed by $\mathscr{G}_1$, and with $\langle \beta \otimes_N L^2(M), \gamma\rangle$ bonds between a vertex β in $\mathscr{G}_0$ and a vertex γ in $\mathscr{G}_1$. Thus the inclusion $N \subset M$ has finite depth if and only if the prinicpal graph is finite.

It was shown in **[J]** that the Jones subfactor of the hyperfinite II_1 factor R with index equal to $4\cos^2 \frac{\pi}{n+1}$ has principal graph given by the Coxeter diagram A_n (see also **[Ka]**). It has also been shown—see **[B-N]**—that the Coxeter diagram E_6 arises as the principal graph of a suitable subfactor of R. It was announced long ago by Ocneanu—see **[O1]**—that the graphs D_{2n} and E_8 also arise as principal graphs of subfactors of R, whilst the graphs D_{2n+1} and E_7 cannot arise as the principal graphs of any inclusion of II_1 factors. We present fairly elementary proofs of the negative statements contained in the above statement.

The case of E_7. Suppose there is an inclusion $N \subset M$ of II_1 factors with principal graph E_7. We label the vertices of the graph as indicated.



It is a fact that if a finite graph arises as the principal graph of a finite-index inclusion of II_1 factors, then the smallest co-ordinate of the Perron-Frobenius eigenvector of the adjacency matrix of the graph must occur at the distinguished vertex (which Ocneanu labels as $*$). (One proof of this fact goes as follows: exactly as one proves (cf. **[S2]**) the existence and uniqueness of a *dimension function* for a hypergroup—i.e., a function $\alpha \mapsto d_\alpha$ from the hypergroup to the positive real numbers satisfying $d_\alpha d_\beta = \sum_\gamma \langle \alpha \otimes \beta, \gamma \rangle d_\gamma$—one can prove the existence and uniqueness of a dimension function on an M_2-graded hypergroup (which is a pair of hypergroups $\mathscr{G}_0$ and $\mathscr{H}_0$ acting on the left and right respectively on a set $\mathscr{G}_1$ as described in Case (ii) $(n = 2k+1)$ of the discussion of the case of the D_{2n+1} graph); since $\langle \alpha \otimes \overline{\alpha}, 1 \rangle = 1$ and $d_\alpha = d_{\overline{\alpha}}$ for all α in $\mathscr{G}_1$ (one of the requirements of a dimension on an M_2-graded hypergroup), it follows that $d_{\alpha^2} \geq 1$ for all α in $\mathscr{G}_1$; a similar argument for hypergroups shows that $d_{\beta^2} \geq 1$ for every element β of a finite hypergroup. Finally, since the value of the dimension function agrees with the prescription given by the Perron-Frobenius eigenvalues, and since the vertex $*$ corresponds to the identity element of $\mathscr{G}_0$ which has dimension 1, the proof of the fact is complete.)

It follows from the above reasoning and an inspection of the Perron eigenvector of the E_7 graph that if the diagram E_7 occurs as a principal graph, then the distinguished vertex—which corresponds to the identity element of the hypergroup $\mathscr{G}_0$—must occur as labelled above.

Then, in the language of Example 2, we have $\mathscr{G}_0 = \{1, \beta, \lambda, \mu\}$ and $\mathscr{G}_1 = \{\alpha, \gamma, \kappa\}$. From the description given above of the principal graph, it is seen that α is nothing but the isomorphism class of the irreducible N-M bimodule $L^2(M)$; furthermore, the adjacency relations of the graph show that

$$(*) \qquad 1\alpha = \alpha, \quad \beta\alpha = \alpha + \gamma, \quad \lambda\alpha = \gamma, \quad \mu\alpha = \gamma + \kappa \quad \text{and}$$
$$\alpha\overline{\alpha} = 1 + \beta, \quad \gamma\overline{\alpha} = \beta + \lambda + \mu, \quad \kappa\overline{\alpha} = \mu$$

where we have used natural abbreviations: thus, the second equations in the two sets of equations are short-hand for $\beta \otimes_N \alpha \simeq \alpha \oplus \gamma$ and $\gamma \otimes_M \overline{\alpha} \simeq \beta \oplus \lambda \oplus \mu$ respectively.

If we write R_α for the map from $\mathbb{Z}^+\mathscr{G}_0$ to $\mathbb{Z}^+\mathscr{G}_1$ defined by (tensor-)multiplication on the right by α, and if we similarly write $R_{\overline{\alpha}}$ for the map from $\mathbb{Z}^+\mathscr{G}_1$ to $\mathbb{Z}^+\mathscr{G}_0$ defined by (tensor-)multiplication on the right by $\overline{\alpha}$, and if we represent these two linear maps with respect to the ordered bases given by $\mathscr{G}_0 = \{1, \beta, \lambda, \mu\}$ and $\mathscr{G}_1 = (\alpha, \gamma, \kappa)$, we then find from the equations $(*)$ that

$$R_\alpha = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad R_{\overline{\alpha}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} .$$

We shall also write R_ρ for the matrix representing the self-map of $\mathbb{Z}^+\mathscr{G}_0$ defined by (tensor-)multiplication on the right by ρ, for each ρ in $\mathscr{G}_0$. Deduce from $\alpha\overline{\alpha} = 1 + \beta$ that

$$R_1 + R_\beta = R_{\overline{\alpha}} R_\alpha = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} , \quad \text{so that } R_\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} .$$

The fact that R_β is a symmetric matrix means that β is self-contragredient. A look at the second column shows that $\beta^2 = 1 + \beta + \lambda + \mu$, which implies that

$$R_\lambda + R_\mu = (R_\beta)^2 - R_1 - R_\beta = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} .$$

Since R_1 and R_β are symmetric matrices, the matrices R_λ and R_μ must either both be symmetric or must be transposes of one another. These matrices have integral entries and their sum is seen to have an odd diagonal entry; hence they cannot be transposes of one another, and so, they must each be symmetric. Then it follows that the matrices $\{R_\rho : \rho \in \mathscr{G}_0\}$ commute pairwise. (Reason: the product of any two of these symmetric matrices is an integral linear combination of symmetric matrices and hence symmetric; and the product of two symmetric matrices is symmetric if and only if they commute.)

Since $\mathscr{G}_0$ is a commutative hypergroup, we see, from the last two columns of the matrix R_β, that $\beta\lambda = \lambda\beta = \beta + \mu$ and $\beta\mu = \mu\beta = \beta + \lambda + \mu$. These equations determine the first two columns of the matrices R_λ and R_μ. Also, since $\mathscr{G}_0$ is a commutative hypergroup, we find that the fourth column of R_λ must equal the third column of R_μ. Since R_λ and R_μ are both symmetric matrices, it follows that

$$R_\lambda = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & x & y \\ 0 & 1 & y & z \end{bmatrix} \quad \text{and} \quad R_\mu = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & y & z \\ 1 & 1 & z & w \end{bmatrix}$$

for some non-negative integers x, y, z, w which must satisfy—in view of the equation we have already obtained for $(R_\lambda + R_\mu)$—the equations $x+y = y+z = z+w = 1$; i.e., $x = z$, $y = w$ and $x+y=1$.

The third column of R_λ shows that $\lambda^2 = 1 + x\lambda + y\mu$, so $(R_\lambda)^2 = R_1 + xR_\lambda + yR_\mu$; comparing the (β, μ)th $(= (2, 4)$th) entry of the two sides of this matrix equation, it is seen that we must have $x = 0$ and $y = 1$.

We have thus determined the multiplication table for the hypergroup $\mathscr{G}_0$;

$$(**)\qquad \beta^2 = 1+\beta+\lambda+\mu, \quad \beta\lambda = \beta+\mu, \quad \beta\mu = \beta+\lambda+\mu,$$
$$\lambda^2 = 1+\mu, \quad \lambda\mu = \beta+\lambda, \quad \mu^2 = 1+\beta+\mu.$$

Note next that, in view of equations $(*)$, we have

$$\beta\alpha = \alpha+\gamma \Rightarrow \beta\gamma = \beta^2\alpha - \beta\alpha = (1+\lambda+\mu)\alpha = \alpha+2\gamma+\kappa$$

and

$$\mu\alpha = \gamma+\kappa \Rightarrow \beta\kappa = \beta\mu\alpha - \beta\gamma = (\beta+\lambda+\mu)\alpha - (\alpha+2\gamma+\kappa) = \gamma.$$

This shows that

$$\pi_1(\beta) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

which implies, since $\beta^2 = 1+\beta+\lambda+\mu$, that

$$\pi_1(\lambda) + \pi_1(\mu) = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Note next that

$$\lambda\gamma = \lambda^2\alpha = \alpha+\mu\alpha = \alpha+\gamma+\kappa; \quad \mu\gamma = \mu\lambda\alpha = (\beta+\lambda)\alpha = \alpha+2\gamma.$$

Since the fact that the elements λ and μ are self-contragredient elements of $\mathscr{G}_0$ implies that the matrices $\pi_1(\lambda)$ and $\pi_1(\mu)$ are symmetric, conclude that

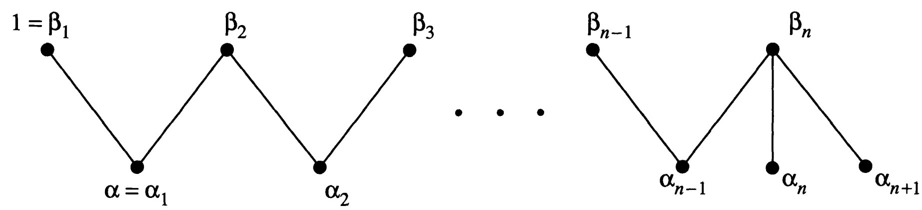
$$\pi_1(\lambda) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \pi_1(\mu) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Finally, the equation $\beta\lambda = \beta + \mu$ should imply that $\pi_1(\beta)\pi_1(\lambda) = \pi_1(\beta) + \pi_1(\mu)$, but it is seen that the matrices on the two sides of the alleged equality differ in the $(2, 3)$ as well as the $(3, 3)$ places. This contradiction finally completes the proof that E_7 cannot arise as the principal graph of any inclusion.

The case of D_{2n+1}. Suppose that there exists an inclusion $N \subset M$ of II_1 factors with principal graph D_{2n+1} and that the vertices of the principal graph are indexed as shown.

(For the same reasons as in the case of E_7, the identity of the hypergroup $\mathscr{G}_0$ must occur at the indicated vertex.) Thus the even vertices of D_{2n+1} are represented by $1 = \beta_1, \beta_2, \ldots, \beta_n$ and the odd vertices are represented by $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}$; i.e.,

$$\mathscr{G}_0 = \{1 = \beta_1, \ldots, \beta_n\} \quad \text{and} \quad \mathscr{G}_1 = \{\alpha_1, \alpha_2, \ldots \alpha_n, \alpha_{n+1}\}.$$



The graph implies the following relations

$$\beta_i \alpha = \begin{cases} \alpha_{i-1} + \alpha_i, & \text{for } 1 \leq i < n, \\ \alpha_{n-1} + \alpha_n + \alpha_{n+1}, & \text{for } i = n \end{cases} \tag{1}$$

(with the convention that $\alpha_0 = 0$),

$$\alpha_i \bar{\alpha} = \begin{cases} \beta_i + \beta_{i+1}, & \text{for } i < n, \\ \beta_n, & \text{for } i = n \text{ and } n+1. \end{cases} \tag{2}$$

Arguing as in the case of E_7, we see that the self-map R_{β_2} of $\mathbb{Z}\mathscr{G}_0$ of right multiplication by β_2 is given by the $n \times n$ matrix

$$R_{\beta_2} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}. \tag{3}$$

It is known—cf. **[BS]**—that there exists a unique hypergroup $\mathscr{G}_0 = \{1 = \beta_1, \beta_2, \ldots, \beta_n\}$ satisfying (3), that every element of this hypergroup is self-contragredient and (consequently) that this hypergroup is abelian.

We know that $\mathscr{G}_0$ acts on $\mathscr{G}_1$ (as described in Example 2); if this action is given by $\pi_1\colon \mathbb{Z}^+\mathscr{G}_0 \to \mathrm{End}(\mathbb{Z}^+\mathscr{G}_1)$, let us write A_i for the matrix representing $\pi_1(\beta_i)$. Let A_i have the block decomposition

$$A_i = \begin{bmatrix} P_i & Q_i \\ R_i & S_i \end{bmatrix}$$

corresponding to the partition: $n+1 = (n-1)+2$.

It can be proved by induction and straightforward (though somewhat laborious and painful) computation, using the fact that each β_i is an appropriate polynomial in β_2, that these matrices have the following descriptions:

$$P_i = \begin{bmatrix} 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & & & & & & & 0 \\ 1 & & & 1 & & & & 1 \\ 1 & & & & & & & 1 \\ 0 & & & & & & & 2 \\ \vdots & & & & & & & \vdots \\ 0 & \cdots & 0 & 1 & 1 & 2 & \cdots & 2 \end{bmatrix}$$

i.e.,

$$P_i(l, m) = \begin{cases} 0, & \text{if } l+m \le i-1 \text{ or } |l-m| \ge i, \\ 2, & \text{if } l+m \ge 2n-i+1, \\ 1, & \text{otherwise;} \end{cases}$$

$R_i = Q_i'$ has identical rows, both being equal to $[0\ldots011\ldots1]$, the first 1 occurring in the $(n-i+1)$st column, i.e.,

$$Q_i(l, m) = R_i(m, l) = \begin{cases} 1, & \text{if } l \ge n-i+1, \\ 0, & \text{otherwise;} \end{cases}$$

and

$$(S_{4k+1}, S_{4k+2}, S_{4k+3}, S_{4k+4}) = (I, S_2, S_3, J); \quad \{S_2, S_3\} = \{I, J\},$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Most of the computation is straight-forward; but we do wish to point out what causes the ambiguity as well as the periodicity in the S_j's, which is crucial to our argument. The point is that the relations (1) and (2) are seen fairly easily to imply that

$$\beta_2\alpha_j = \begin{cases} \alpha_{j-1} + \alpha_j + \alpha_{j+1}, & \text{for } 1 \le j < n-1, \\ \alpha_{n-2} + \alpha_{n-1} + \alpha_n + \alpha_{n+1}, & \text{for } j = n-1, \end{cases}$$

thereby establishing that the first $(n-1)$ columns of A_2 are indeed as asserted. Since P_2 and R_2 have been determined, so also is Q_2 $(= R_2')$, since the fact that β_2 is self-contragredient implies that A_2 is a symmetric matrix.

The further information—about $\beta_2\alpha_n$ and $\beta_2\alpha_{n+1}$—that can be gleaned from the relations (1) and (2) is only that

$$\beta_2(\alpha_n + \alpha_{n+1}) = 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}.$$

This means that S_2 is a symmetric 2×2 matrix (with non-negative integral entries) whose two columns have sum equal to $[1\ 1]'$. This means that necessarily $S_2 = I$ or J.

Also since the two rows (respectively, columns) of R_2 (resp., Q_2) are identical, it means that $R_2 I = R_2 J$ (resp., $IQ_2 = JQ_2$), and consequently, the ambiguity in the $(2, 2)$ entry in the block-decomposition of A_2 only results in the ambiguity of the $(2, 2)$ entry in the block decomposition of any polynomial of A_2.

The fact that the S_i's exhibit the periodic behaviour ascribed to them is an easy consequence of the forms of the Q_i's and R_i's, and is established by induction. Begin by noting that—since the hypergroup $\mathscr{G}_0$ is abelian—we have, in view of the form of the matrix for R_{β_2}, $A_{k+1} = A_2 A_k - A_k - A_{k-1}$ for $1 \le k < n$ (with the convention that $A_{-1} = 0$). Comparing $(2, 2)$-entries, we find that

$$(*) \qquad S_{k+1} = R_2 Q_k + S_2 S_k - S_k - S_{k-1}, \qquad \text{for } 1 \le k < n;$$

if the assertions about the matrices P_i, Q_i, R_i, S_i have been verified for $i \le k$, it is seen that $R_2 Q_k$ is the 2×2 matrix with 1's everywhere—i.e., $R_2 Q_k = I + J$. On the other hand, if $S_2 = I$, it follows from $(*)$ that $S_3 = (I + J) + I \cdot I - I - I = J$, that $S_4 = (I+J)+I\cdot J - J - I = J$, and that $S_5 = (I+J)+I\cdot J - J - J = I$; while, if $S_2 = J$, $(*)$ implies that $S_3 = (I+J)+J\cdot J - J - I = I$, that $S_4 = (I+J)+J\cdot I - I - J = J$, and that $S_5 = (I+J)+J\cdot J - J - I = I$. Hence, in either case, we see that $S_4 = J$ and $S_5 = I$. Then, $(*)$ implies that $S_6 = (I + J) + S_2 \cdot I - I - J = S_2$, and it is clear from the recursion relation $(*)$ that the S_i's behave in the periodic fashion indicated.

To proceed further, we need to discuss two cases depending upon the parity of n.

Case (i): $n = 2k$. Suppose k is odd; then $n \equiv 2 \pmod 4$, so that $S_n = S_2$ and $S_{n-1} = I$. The last column of R_{β_2} shows that $\beta_2\beta_n = \beta_{n-1} + 2\beta_n$. Since π_1 is an action, we should have $A_2 A_n =$

$A_{n-1} + 2A_n$, and in particular, $R_2Q_n + S_2S_n = S_{n-1} + 2S_n$; i.e., $(I+J) + S_2 \cdot S_2 = I + 2S_2$, or $I + J = 2S_2$ which is not true.

Suppose k is even; then $n \equiv 0 \pmod 4$, and we have $S_n = J$ and $S_{n-1} = S_3$. As before, we should have $R_2Q_1 + S_2S_n = S_{n-1} + 2S_n$; i.e., $(I+J) + S_2 \cdot J = S_3 + 2J$, or $I + J = 2J$ which is also not true.

Case (ii): $n = 2k+1$. It turns out that in this case, the possibility $S_2 = J$ again leads to a contradiction to the equation $A_2A_n = A_{n-1} + 2A_n$. However, setting $S_2 = I$ does lead to an action of $\mathscr{G}_0$ and the contradiction is not yet reached. What we have shown however is that there is a unique hypergroup $\mathscr{G}_0$ and a unique action of this hypergroup on $\mathscr{G}_1$ that is consistent with the equations (1) and (2).

Only this much can be proved by only considering "one-sided" actions, or equivalently only one of the principal graphs. To proceed further, we must note that, corresponding to the tower $\{M' \cap M_n : n \geq 0\}$ of relative commutants of M in the members of the tower of the basic construction, there exists another principal graph whose even vertices yield a hypergroup $\mathscr{H}_0$, and whose odd vertices are in bijection with the $\mathscr{G}_1$ of the original principal graph, in such a way that $\mathscr{H}_0$ admits a right action π_1^0 on $\mathscr{G}_1$ which commutes with the left-action of $\mathscr{G}_0$, and such that

$$\sum_{\beta \in \mathscr{G}_0} \pi_1(\beta)(\alpha_i, \alpha_j)\pi_1(\beta)(\alpha_l, \alpha_m) = \sum_{\gamma \in \mathscr{H}_0} \pi_1^0(\gamma)(\alpha_i, \alpha_l)\pi_1^0(\gamma)(\alpha_j, \alpha_m)$$

for all α_i, α_j, α_l, α_m in $\mathscr{G}_1$.

(This last condition stems from the associative law:

$$\langle (\alpha_i \otimes_M \overline{\alpha}_j) \otimes_N \alpha_l, \alpha_n \rangle = \langle \alpha_i \otimes_M (\overline{\alpha}_j \otimes_N \alpha_l), \alpha_m \rangle.)$$

First note that the principal graph corresponding to $\mathscr{H}_0$ and $\mathscr{G}_1$ must be a Coxeter diagram the norm of whose associated adjacency matrix is the same as that of D_{2n+1}. This can only be D_{2n+1} or a suitable A_m. Since the set of odd vertices of the graph must have the same cardinality as $\mathscr{G}_1$, we find that the other principal graph must also be D_{2n+1}. Then, we deduce from the earlier analysis that we must have $\mathscr{G}_0 = \mathscr{H}_0$ and that $\pi_1 = \pi_1^0$.

We may now deduce from the remark following Example 3 that $\mathscr{G}_0 \cup \mathscr{G}_1$ must have the structure of a $\mathbb{Z}_2$-graded hypergroup $\mathscr{G}$ with every element of $\mathscr{G}_1$ self-conjugate. In this hypergroup, we would have: $L_\beta = \pi_0(\beta) \oplus \pi_1(\beta)$ for all β in $\mathscr{G}_0$, where π_0 denotes the action of $\mathscr{G}_0$ on itself given by left-multiplication, π_1 denotes the action of

$\mathcal{G}_0$ on $\mathcal{G}_1$, and L_β denotes the matrix of left-multiplication by β on $\mathcal{G}$ with respect to the ordered basis $\{\beta_1, \ldots, \beta_n, \alpha_1, \ldots, \alpha_{n+1}\}$. Since very element of $\mathcal{G}_0$ as well as of $\mathcal{G}_1$, is self-contragredient, we deduce that the hypergroup $\mathcal{G}$ is abelian. Note now that the equation $\alpha_1\alpha_n = \alpha_n\alpha_1 = \beta_n = \alpha_{n+1}\alpha_1 = \alpha_1\alpha_{n+1}$ implies that the last two columns of the matrix L_{α_1} are equal; since this matrix is symmetric, the last two rows are also equal. Then, since $L_{\alpha_1}L_{\alpha_n} = L_{\beta_n}$, it must be the case that also the last two rows of L_{β_n} must be equal. However the bottom 2×2 principal submatrix is S_n which is equal to I or J and the desired contradiction has been reached, thus finally completing the proof of the fact that whether n is odd or even, the Coxeter diagram D_{2n+1} cannot arise as the principal graph of any inclusion $N \subset M$ of II_1 factors.

The case of β_{2n+1}. Assume there exists a finite-index inclusion $N \subset M$ of II_1 factors with β_{2n+1} as the principal graph corresponding to the tower $\{N' \cap M_n\}$ of relative commutants of N in the tower of the basic construction.

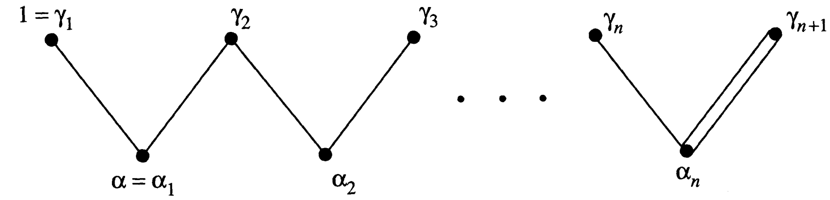
We first argue that the smallest entry of the Perron-Frobenius eigenvector of β_{n+2} occurs at the unique vertex of valency 1 in β_{n+1}, for all $n \geq 1$. For this, set $d = \|A\|$, where A denotes the adjacency matrix of the above graph. It is clear that $d > 2$. Let us assume, only for this paragraph, that the vertices are numbered linearly, in increasing order from the unique vertex with valency 1 (which is assigned label 1). It is fairly easy to see that, since d is the Perron eigenvalue of A, if v denotes the Perron-Frobenius eigenvector of A, then the kth co-ordinate of v is $P_{k-1}(d)$ if $k \leq n+1$, and $(2/d)P_n(d)$ if $k = n+2$, where $\{P_k: k = 0, 1, 2, \ldots\}$ are the (variant of) Chebyshev polynomials defined by $P_0(t) = 1$, $P_1(t) = t$ and $P_{k+1}(t) = tP_k(t) - P_{k-1}(t)$ for $k > 1$. Thus, in order to prove our assertion about the smallest co-ordinate of v occurring at vertex 1, we need to verify that $1 = P_0(d) < P_1(d) < \cdots < P_n(d) > d/2$. On the other hand, we know that $P_k(2\cos(z)) = (\sin(k+1)z)/(\sin z)$ for all complex numbers z. Since $d > 2$, we may pick a positive real number y such that $2\cosh(y) = d$. Now put $z = iy$, and note that, for any k, we have

$$\begin{aligned} P_k(d) - P_{k-1}(d) &= (\sin(k+1)z - \sin kz)/(\sin z) \\ &= (\cos(k+\tfrac{1}{2})z)/(\cos\tfrac{1}{2}z) = \frac{\cosh(k+\frac{1}{2})y}{\cosh\frac{y}{2}} > 0 \end{aligned}$$

while the inequalities $n \geq 1$ and $y > 0$ imply that

$$\begin{aligned} P_n(d) &= P_n(2\cos(iy)) \\ &= (\sinh(n+1)y)/(\sinh y) \geq (\sinh 2y)/(\sinh y) \\ &= 2\cosh y > \cosh y = d/2. \end{aligned}$$

Now, we may argue as we did in the cases of E_7 and D_{2n+1} that the identity of the hypergroup $\mathcal{G}_0$ must occur where indicated. Assume the other vertices are labelled as below.



As before, if we let R_α denote the matrix of the operator of right-multiplication by $\alpha = \alpha_1$ from $\mathbb{Z}^+\mathcal{G}_0$ to $\mathbb{Z}^+\mathcal{G}_1$, with respect to the ordered bases $\{\gamma_1, \dots, \gamma_{n+1}\}$ and $\{\alpha_1, \dots, \alpha_n\}$ respectively, we see that R_α is given, by the $n \times (n+1)$ matrix

$$R_\alpha = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix},$$

from which it may be deduced via the equation $\alpha\bar{\alpha} = 1 + \gamma_2$ that the matrix of the operator R_{γ_2} of $\mathbb{Z}\mathcal{G}_0$ of right multiplication by γ_2 is given by the $(n+1) \times (n+1)$ matrix

$$R_{\gamma_2} = (R_\alpha)' R_\alpha - I = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 1 & 2 \\ 0 & 0 & \cdots & 0 & 2 & 3 \end{bmatrix}.$$

The second column of the above equation, for instance, says that $\gamma_2^2 = \gamma_1 + \gamma_2 + \gamma_3$, and hence that $\gamma_3 = \gamma_2^2 - \gamma_1 - \gamma_2$, whence $R_{\gamma_3} = R_{\gamma_2}^2 - R_{\gamma_1} - R_{\gamma_2}$. A similar analysis, of columns 2 through n, yields the formulae

$$R_{\gamma_{i+1}} = \begin{cases} R_{\gamma_2}R_{\gamma_i} - R_{\gamma_{i-1}} - R_{\gamma_i}, & \text{for } 1 < i < n, \\ \frac{1}{2}(R_{\gamma_2}R_{\gamma_n} - R_{\gamma_{n-1}} - R_{\gamma_n}), & \text{for } i = n. \end{cases}$$

It is clear that the R_{γ_i}'s can now be recursively computed from the above equations. The desired contradiction stems from the fact that the matrix $R_{\gamma_{n+1}}$ turns out to have a non-integral entry. The computations are as follows:

For $k = 1, \dots, n+1$, let v_K denote the $n \times 1$ column-vector defined by

$$v_k' = \langle 0, 0, \dots, 0, 2, 6, 18, 54, \dots, (2 \cdot 3^m), \dots, (2 \cdot 3^{k-2}) \rangle,$$

where the first $(n-k+1)$ entries are equal to 0, and $'$ denotes transpose.

For $k = 1, \dots, n+1$, let P_k denote the $n \times n$ matrix defined by

$$P_k(i, j) = \begin{cases} 0, & \text{if } i+j \le k \text{ or } |i-j| \ge k, \\ 1, & \text{if } k < i+j < 2n-k+3 \text{ and } |i-j| < k, \\ 4 \cdot 3^m, & \text{if } i+j = 2n-k+3+m \\ & \qquad \text{with } m = 0, 1, 2, \dots. \end{cases}$$

(Thus, for instance, if $n = 7$ and $k = 5$, P_5 would be the following matrix:

$$P_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 1 & 1 & 4 & 12 \\ 0 & 0 & 1 & 1 & 4 & 12 & 36 \end{bmatrix} .)$$

Finally, define the $(n+1) \times (n+1)$ matrices $A_1, \dots, A_{n+1}$ by the block-decomposition given by

$$A_k = \begin{bmatrix} P_k & V_k \\ v_k' & 3^{k-1} \end{bmatrix}.$$

It can be verified by a straightforward, if somewhat laborious, induction argument that

$$R_{\gamma_k} = \begin{cases} A_k, & \text{for } 1 \le k \le n, \\ \frac{1}{2}A_{n+1}, & \text{for } k = n+1. \end{cases}$$

This would imply that $\langle \gamma_{n+1} \otimes \gamma_{n+1}, \gamma_{n+1} \rangle = 3^n/2$, which contradicts the requirement that these numbers should all be non-negative integers.

We conclude finally that the graph β_{2n+1} could not have arisen as the principal graph of a finite-index inclusion index of II_1 factors.

We remark that an almost identical argument also shows that the graph $\beta_{2n+1}^{(k)}$ cannot arise as the principal graph of a finite-index inclusion of a pair of II_1 factors, where the only distinction between $\beta_{2n+1}^{(k)}$ and β_{2n+1} $(= \beta_{2n+1}^{(2)})$ is that the unique double bond in the latter is substituted by k bonds in the former. We also remark that these arguments fail in the case of β_{2n}—or $\beta_{2n}^{(k)}$, for that matter—since it turns out in that case the adjacency matrix arises as the matrix R_{γ_2} of right-multiplication by the second element of a unique hypergroup $\{1 = \gamma_1, \gamma_2, \dots, \gamma_{2n}\}$ with $2n$ elements.

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