PACIFIC JOURNAL OF MATHEMATICS
Vol. 161, No. 1, 1993

CONTACT STRUCTURES ON $(n-1)$-CONNECTED $(2n+1)$-MANIFOLDS

HANSJÖRG GEIGES

A contact structure on a $(2n+1)$-dimensional manifold M is a completely non-integrable hyperplane distribution in the tangent bundle TM, i.e. a distribution which is (at least locally) defined by a 1-form α satisfying $\alpha \wedge (d\alpha)^n \neq 0$. An almost contact structure is a reduction of the structure group of TM to $\mathrm{U}(n) \times 1$. Every contact structure induces an almost contact structure.

Applying results of Eliashberg and Weinstein on contact surgery, we show that an $(n-1)$-connected $(2n+1)$-manifold is contact (to be precise: almost diffeomorphic to a contact manifold) if and only if it is almost contact.

1. Introduction. This paper is a sequel to [**4**], where we proved the following result.

THEOREM 1. *Let M be a simply-connected 5-manifold. Then M admits a contact structure in every homotopy class of almost contact structures.*

As in [**4**], all manifolds are assumed to be closed, oriented and smooth.

After the publication of [**4**], Eliashberg pointed out to me that he had obtained results similar to mine (but far more general) in [**1**]. We shall use his results to extend Theorem 1 to all $(n-1)$-connected $(2n+1)$-manifolds, which were classified by Wall [**10**] and Wilkens [**12**]. This classification is only up to almost diffeomorphism, that is, up to the connected sum with a homotopy sphere $\Sigma^{2n+1} \in \Theta_{2n+1}$, so the statement corresponding to Theorem 1 has to be weakened slightly in higher dimensions. Denote by bP_{2n+2} the subgroup of the group of homotopy spheres Θ_{2n+1} consisting of elements which bound a parallelizable manifold. Our extension of Theorem 1 can then be stated as

THEOREM 2. *Let M be an $(n-1)$-connected $(2n+1)$-manifold.*
If n is even (or $n = 1$), then M is almost diffeomorphic to a manifold M' which admits a contact structure in every homotopy class of almost contact structures.

If n is odd, then M is almost diffeomorphic to a manifold M' which admits a contact structure in every stable homotopy class of almost contact structures. Furthermore, in each stable homotopy class of almost contact structures, there are infinitely many contact structures (on M') with pairwise non-homotopic underlying almost contact structure.

The properties of M' are preserved under the connected sum with elements of bP_{2n+2}. In particular, "almost diffeomorphism" may be replaced by "diffeomorphism" for $n = 1, 2, 3, 5$, where $\Theta_{2n+1}/bP_{2n+2} = 0$.

See **[9]** for what was known on this problem a few years back.

The case $n = 1$ is the classical result of Lutz and Martinet **[6]** (and there are no obstructions to an almost contact structure, since all (orientable) 3-manifolds are parallelizable), so for the rest of this paper we shall assume $n \geq 2$.

2. Contact surgery. In **[4]** we showed that an n-sphere embedded in a $(2n+1)$-dimensional contact manifold can be C^0-approximated by an embedded Legendre sphere, that is, a sphere which is an integral submanifold of the contact distribution . A result of Weinstein **[11]** then allowed us to perform surgery along this Legendre sphere under preservation of the contact structure. However, the applications of this result were limited by the fact that Weinstein's construction does not allow any choice of framing.

Eliashberg's **[1]** essential improvement on our result is achieved by keeping track of the framing under this C^0-approximation. In other words, there is still no choice of framing in the basic surgery construction along a Legendre sphere, but different framings are realized by different C^0-approximations. Eliashberg states his theorem in terms of Stein manifolds and pseudoconvex boundaries; for our purposes we can rephrase it as

THEOREM 3 (*Eliashberg*). *Let M be a $(2n+1)$-manifold, $n \geq 2$, obtained from S^{2n+1} by surgery along spheres of dimension $\leq n$. Then M admits a contact structure in every stable homotopy class of almost contact structures.*

REMARK. The precise statement of Eliashberg's theorem and our proof of Theorem 2 imply that the contact structures we obtain on M' are symplectically (in fact, even holomorphically) fillable in the sense of **[2]**.

3. Almost contact structures. Let M be a $(2n+1)$-dimensional manifold. The obstructions to reducing the structure group of TM to $\mathrm{U}(n) \times 1$ lie in $H^q(M; \pi_{q-1}(\mathrm{SO}_{2n+1}/\mathrm{U}_n))$, $1 \leq q \leq 2n+1$. All the coefficient groups are stable, so M is almost contact, if, and only if, it is stably almost complex. This allows us to use $\widetilde{KU}$-theory to study almost contact structures.

Suppose M is $(n-1)$-connected. Denote by $M^{(n+1)}$ the $(n+1)$-skeleton of M and let x_0 be a point in $M^{(n+1)}$. The Atiyah-Hirzebruch spectral sequence in K-theory for the pair $(M^{(n+1)}, x_0)$ collapses at the E_2-term, and we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{n+1}(M; \pi_n(\mathrm{U})) & \longrightarrow & \widetilde{KU}(M^{(n+1)}) & \longrightarrow & H^n(M; \pi_{n-1}(\mathrm{U})) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^{n+1}(M; \pi_n(\mathrm{SO})) & \longrightarrow & \widetilde{KO}(M^{(n+1)}) & \longrightarrow & H^n(M; \pi_{n-1}(\mathrm{SO})) & \longrightarrow & 0, \end{array}$$

where the vertical maps are the obvious ones.

The coefficient groups are as follows.

n mod 8	$\pi_n(\mathrm{U})$	$\pi_n(\mathrm{SO})$	$\pi_n(\mathrm{U}) \to \pi_n(\mathrm{SO})$
0	0	$\mathbb{Z}_2$	
1	$\mathbb{Z}$	$\mathbb{Z}_2$	mod 2 reduction
2	0	0	
3	$\mathbb{Z}$	$\mathbb{Z}$	identity
4	0	0	
5	$\mathbb{Z}$	0	
6	0	0	
7	$\mathbb{Z}$	$\mathbb{Z}$	multiplication by 2.

The spectral sequence for (M, x_0) in KU-theory also collapses at the E_2-term, and since $H^{2n+1}(M; \pi_{2n}(\mathrm{U})) = 0$, there are no new entries of total degree 0 in the E_2-page, so we may replace $\widetilde{KU}(M^{(n+1)})$ by $\widetilde{KU}(M)$ in the diagram above.

For $n \not\equiv 0, 4 \bmod 8$, the long exact sequence in K-theory of the pair $(M, M^{(n+1)})$ reduces to the commutative diagram

$$\begin{array}{ccccc} & & \widetilde{KU}(M) & \xrightarrow{\cong} & \widetilde{KU}(M^{(n+1)}) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \widetilde{KO}(M) & \longrightarrow & \widetilde{KO}(M^{(n+1)}), \end{array}$$

so there are no obstructions to an almost contact structure in $H^{2n+1}(M; \pi_{2n}(\mathrm{SO/U}))$ (note that here $\pi_{2n}(\mathrm{SO/U}) \cong \mathbb{Z}$).

For $n \equiv 0, 4 \bmod 8$, when $\pi_{2n}(\mathrm{SO/U}) \cong \mathbb{Z}_2$, the vanishing of the top obstruction is not obvious, but it can be deduced from the results in the next section.

Following Wall [**10**], we denote by $\hat{\beta}$ and $\hat{\alpha}$ the element in $H^{n+1}(M; \pi_n(\mathrm{SO}))$ and $H^n(M; \pi_{n-1}(\mathrm{SO}))$ respectively corresponding to the stable tangent bundle of M in $\widetilde{KO}(M)$; in other words, $\hat{\alpha}$ and $\hat{\beta}$ are the obstructions to stable parallelizability.

For $n \equiv 1 \bmod 8$ the short exact sequence for $\widetilde{KO}(M^{(n+1)})$ does not split in general, so we cannot define $\hat{\beta}$. However, in this case $\hat{\alpha}$ is identified as an obstruction to an almost contact structure, and $\hat{\beta}$ is defined for $\hat{\alpha} = 0$.

A case by case study of the first commutative diagram above then yields

PROPOSITION 4. *An $(n-1)$-connected $(2n+1)$-manifold M is almost contact in the following cases*:

(i) $n \equiv 0 \bmod 8$, $\hat{\alpha}$ *even and* $\hat{\beta} = 0$,
(ii) $n \equiv 1 \bmod 8$ *and* $\hat{\alpha} = 0$,
(iii) $n \equiv 2 \bmod 8$ *and* $\delta\hat{\alpha} = 0$,
(iv) $n \equiv 3, 4, 5, 6 \bmod 8$,
(v) $n \equiv 7 \bmod 8$ *and* $\hat{\beta}$ *is even.*

In each case the mentioned conditions are necessary. The condition in (v) *is automatically satisfied for* $n = 7$.

Here δ denotes the Bockstein homomorphism associated to the coefficient sequence $0 \to \mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z} \to \mathbb{Z}_2 \to 0$.

Details of the proof of Proposition 4 can be found in [**3**].

4. $(n-1)$**-connected** $(2n+1)$**-manifolds.** We first recall Wall's classification of $(n-1)$-connected $(2n+1)$-manifolds [**10**, Theorem 7]. This classification contains certain exceptional cases, which have to be distinguished by additional invariants. However, it is easy to see that for $n \neq 4$ the exceptional cases violate the necessary conditions for an almost contact structure, so we can ignore them.

For $n = 4$ there is an exceptional invariant $\omega \in \mathbb{Z}_2$. Wall classifies *almost closed* manifolds, that is, manifolds bounded by a homotopy sphere Σ^{2n}. If $\Sigma^{2n} \notin bP_{2n+1}$, then the almost closed manifold cannot be smoothly closed by attaching a $(2n+1)$-disc. It can be shown that the invariant ω distinguishes between manifolds which can be closed and those which cannot. Since the existence problem for contact structures on open manifolds is covered by Gromov's h-principle (cf. the remarks in [**4**]), we only have to consider closed manifolds; hence we may treat the case $n = 4$ as non-exceptional.

THEOREM 5 (*Wall*). *Let* M *be an* $(n-1)$*-connected* $(2n+1)$*-manifold in the non-exceptional case,* $n \geq 2$, $n \neq 3, 7$. *Then* M *is, up to almost diffeomorphism, determined by*

A. *homology invariants (essentially* $H_n(M;\mathbb{Z})$ *with its quadratic structure),*

B. *tangential invariants*

(i) $\hat{\alpha} \in H^n(M; \pi_{n-1}(\mathrm{SO}))$,

(ii) $\hat{\beta} \in H^{n+1}(M; \pi_n(\mathrm{SO}))$,

(iii) *For* $n \neq 2, 6$ *even,* $\hat{\phi} \in H^{n+1}(M; \mathbb{Z}_2) \cong H_n(M;\mathbb{Z}) \otimes \mathbb{Z}_2$.

REMARKS. (i) Wall erroneously states that $\hat{\phi}$ has to be defined for $n \neq 2, 4, 8$ even, but see the correction in [**8**]. There one can also find a definition of $\hat{\phi}$, for which Wall referred to a paper that has never been published.

(ii) In [**10**] it was left undecided whether the case $n = 4$ is exceptional. The fact that it is indeed exceptional, and that the exceptional invariant ω has the described properties, can be deduced from the existence of an almost closed 4-connected 9-manifold M with $H_4(M) \cong \mathbb{Z}$ and $\hat{\alpha} = 1 \in \pi_3(\mathrm{SO}) \cong \mathbb{Z}$ which cannot be closed. This example is due to D. L. Frank, see [**8**].

The cases $n = 3$ and $n = 7$ are studied in [**12**]; for us it is enough to know that here $\hat{\alpha}$ is always zero.

We shall now give various geometric models of $(n-1)$-connected $(2n+1)$-manifolds M.

I. Manifolds M with $\hat{\alpha} = 0$. By [**10**, Theorem 8], M is (up to almost diffeomorphism) obtained from S^{2n+1} by surgery along a link of n-spheres. Eliashberg's theorem tells us that M will be contact if the tangential condition for an almost contact structure (as given in Proposition 4) is satisfied.

We note one specific example. If $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$, $\hat{\alpha} = \hat{\beta} = 0$, $\hat{\phi} \neq 0$, then M is the cotangent sphere bundle of S^{n+1}, which is well known to be contact.

II. If $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$, $\hat{\phi} = \hat{\beta} = 0$, then M is an S^{n+1}-bundle over S^n, classified by $\hat{\alpha} \in \pi_{n-1}(\mathrm{SO})$. So M is obtained from S^{2n+1} by surgery along a trivially embedded $(n-1)$-sphere, hence, by Eliashberg's theorem, it is again enough to satisfy the tangential condition.

III. Given an arbitrary M, write $M = M_1 \# M_2$ with $H_n(M_1)$ free, $H_n(M_2)$ pure torsion (such a splitting exists by Wall's classification). Choose generators for $H_n(M_1) \cong \mathbb{Z}^{\oplus k}$. With respect to these generators, $\hat{\alpha}(M_1)$ can be written as $(\alpha_1, \dots, \alpha_k)$ with $\alpha_i \in \pi_{n-1}(\mathrm{SO})$; $\hat{\phi}(M_1)$ can be written as $(\phi_1, \dots, \phi_k)$ with $\phi_i \in \mathbb{Z}_2$.

By changing our choice of generators, if necessary, we can always ensure $\phi_2 = \cdots = \phi_k = 0$. Assume $\hat{\beta}(M_1) = 0$. Then M_1 can be written as the connected sum of manifolds of type II with possibly one additional summand of the following type.

IV. Manifolds M with $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$, $\hat{\alpha} \neq 0$, $\hat{\phi} \neq 0$, $\hat{\beta} = 0$. Let M' be the cotangent sphere bundle of S^{n+1}. Then

$$H_n(M \# M'; \mathbb{Z}) \cong H_n(M;\mathbb{Z}) \oplus H_n(M';\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$$

with generators e_1, e_2, say. In terms of these generators, we have

$$\hat{\alpha}(M \# M') = (\hat{\alpha}(M), 0)$$

and

$$\hat{\phi}(M \# M') = (1, 1).$$

Now change the generators to $\tilde{e}_1 = e_1$ and $\tilde{e}_2 = e_1 + e_2$. In terms of these generators we have

$$\hat{\alpha}(M \# M') = (\hat{\alpha}(M), 0)$$

and

$$\hat{\phi}(M \# M') = (0, 1).$$

This proves that $M \# M'$ is almost diffeomorphic to $N \# N'$, where N is of type II and N' of type I (in fact, N' is the cotangent sphere bundle of S^{n+1} again).

Suppose M satisfies the tangential conditions as laid down in Proposition 4. Then the same holds for N, so $N \# N'$ is obtained from S^{2n+1} by surgery along a trivially embedded $(n-1)$-sphere and a trivially embedded n-sphere and admits a contact structure by Eliashberg's theorem. To obtain M from $M \# M'$, we have to perform surgery along an embedded n-sphere representing the fibre of the cotangent sphere bundle M' of S^{n+1}. Hence M is contact (up to almost diffeomorphism) by Eliashberg's theorem.

We can now prove Theorem 2. Write $M = M_1 \# M_2$ as in III. It is a classical result of Meckert that the connected sum of two contact manifolds is contact; this is also a special case of Weinstein's theorem, namely, surgery along S^0. We can therefore consider M_1 and M_2 separately. Furthermore, it is enough to deal with the case $H_n(M_1; \mathbb{Z}) \cong \mathbb{Z}$.

(i) $n \equiv 0 \bmod 8$. We have $\hat{\alpha} \in H^n(M; \mathbb{Z})$, so $\hat{\alpha}(M_2) = 0$. A necessary condition for the existence of an almost contact structure is $\hat{\beta} = 0$. This implies that M_1 is covered by II and IV, M_2 is covered by I.

(ii) $n \equiv 1 \bmod 8$. The obstruction $\hat{\alpha}$ has to vanish, so we are in case I.

(iii) $n \equiv 2 \bmod 8$. The condition $\delta\hat{\alpha} = 0$ implies $\hat{\alpha}(M_2) = 0$. So the same remarks as in (i) apply (since $\hat{\beta} = 0$).

(iv) $n \equiv 3, 5, 6, 7 \bmod 8$. Here trivially $\hat{\alpha} = 0$, and hence we are in case I again.

(v) $n \equiv 4 \bmod 8$. Here we have $\hat{\alpha} \in H^n(M; \mathbb{Z})$ and $\hat{\beta} = 0$, so we are in the same situation as in (i).

Eliashberg's theorem gives us a contact structure in every *stable* homotopy class of almost contact structures. The stronger statements in Theorem 2 follow from the work of Sato [7]. He showed that an element Σ^{2n+1} of bP_{2n+2} admits a contact structure in every homotopy class of almost contact structures if n is even (and there are only finitely many such classes, classified by $\pi_{2n+1}(\mathrm{SO}_{2n+1}/\mathrm{U}_n)$), and that Σ^{2n+1} admits a contact structure in infinitely many different homotopy classes of almost contact structures if $n \geq 3$ is odd.

By taking the connected sum with the (differentiably) standard sphere S^{2n+1} which is equipped with a contact structure whose underlying almost contact structure is not homotopically standard, we can thus change the homotopy class of the almost contact structure underlying a contact structure on M' (in the notation of Theorem 2).

The last statement of Theorem 2 is also clear from the remarks above.

It remains to fill the small gap in the proof of Proposition 4; that is, we have to show the vanishing of the top obstruction to an almost contact structure in $H^{2n+1}(M; \pi_{2n}(\mathrm{SO}/\mathrm{U}))$ for $n \equiv 0, 4 \bmod 8$. We need the following lemma.

LEMMA 6. (i) *The top obstruction to an almost contact structure is additive under the connected sum of manifolds (in particular: the connected sum of two almost contact manifolds is almost contact).*

(ii) *A manifold which is almost diffeomorphic to an almost contact manifold admits an almost contact structure.*

Proof. (i) This follows directly from **[5]**, where it is shown that the top-dimensional obstruction to extending an X-structure (this concept includes stable almost complex structures, non-singular vector fields etc.), that is, the obstruction to extending an X-structure from $M^m - D^m$ to a closed manifold M^m, is additive if S^m admits an X-structure. Now use the fact that S^{2n+1} admits an almost contact structure for all n.

(ii) All homotopy spheres are stably parallelizable, so the odd-dimensional ones admit an almost contact structure. Then apply (i).

The proof of Theorem 2 showed that, given a manifold M which satisfies the tangential conditions for an almost contact structure over the $(n+1)$-skeleton, we can find contact manifolds M', N and N' such that $M \# M'$ is almost diffeomorphic to $N \# N'$. Since the top obstruction vanishes for M', N and N', Lemma 6 implies that the same is true for M. This completes the proof of Proposition 4.

REFERENCES

[1] Ya. Eliashberg, *Topological characterization of Stein manifolds of dimension* > 2, Internat. J. Math., **1** (1990), 29–46.

[2] Ya. Eliashberg and M. Gromov, *Convex symplectic manifolds*, Proc. Sympos. Pure Math., **52** (1991), Part 2, 135–162.

[3] H. Geiges, *Contact structures on* $(n-1)$*-connected* $(2n+1)$*-manifolds*, J. T. Knight Prize essay, Cambridge (1991).

[4] ——, *Contact structures on* 1*-connected* 5*-manifolds*, Mathematika, **38** (1991), 303–311.

[5] P. J. Kahn, *Obstructions to extending almost* X*-structures*, Illinois J. Math., **13** (1969), 336–357.

[6] J. Martinet, *Formes de contact sur les variétés de dimension* 3, Proc. Liverpool Singularities Symp. II (C. T. C. Wall, ed.), LNM **209** (Springer-Verlag, 1971), 142–163.

[7] H. Sato, *Remarks concerning contact manifolds*, Tôhoku Math. J., **29** (1977), 577–584.

[8] S. Stolz, *Hochzusammenhängende Mannigfaltigkeiten und ihre Ränder*, LNM **1116** (Springer-Verlag, 1985).

[9] C. B. Thomas, *Contact structures on* $(n-1)$*-connected* $(2n+1)$*-manifolds*, Banach Center Publ., **18** (1986), 255–270.

[10] C. T. C. Wall, *Classification problems in differential topology*, VI–*Classification of* $(s-1)$*-connected* $(2s+1)$*-manifolds*, Topology, **6** (1967), 273–296.

[11] A. Weinstein, *Contact surgery and symplectic handlebodies*, Hokkaido Math. J., **20** (1991), 241–251.

[12] D. L. Wilkens, *Closed* $(s-1)$*-connected* $(2s+1)$*-manifolds,* $s = 3, 7$, Bull. London Math. Soc., **4** (1972), 27–31.

Received January 21, 1992.

UNIVERSITY OF CAMBRIDGE
16 MILL LANE
CAMBRIDGE CB2 1SB, ENGLAND

Current address: Stanford University
Stanford, CA 94305-2125