PACIFIC JOURNAL OF MATHEMATICS
Vol. 161, No. 2, 1993

COMMUTATIVITY OF SELFADJOINT OPERATORS

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Nonnegative bounded operators A and B on a Hilbert space $\mathscr{H}$ commute if $AB^n + B^nA \geq 0$ for $n = 1, 3, \ldots,$, or if $e^{tA} \leq e^{tA+sB} \leq e^{tA+s\|B\|}$ for every $s, t > 0$.

In this paper A and B represent (not necessarily bounded) selfadjoint operators with spectral families $\{E_\lambda\}$ and $\{F_\lambda\}$, respectively, on a Hilbert space $\mathscr{H}$. We study some conditions which imply that A and B commute.

1. In general, $AB + BA$ is not necessarily nonnegative for some nonnegative operators A and B (cf. [3]).

THEOREM 1. *Let A and B be nonnegative and bounded operators. Then $AB = BA$ if and only if*

$$0 \leq AB^n + B^nA \quad \textit{for } n = 1, 2, \ldots.$$

To prove this theorem, we need the following:

LEMMA. *If a projection P satisfies $0 \leq AP + PA$, then $AP = PA$.*

Proof. For arbitrary vectors $x \in P\mathscr{H}$, $y \in (1-P)\mathscr{H}$, and arbitrary complex numbers s and t, we have

$$\begin{aligned} 0 &\leq ((AP + PA)(tx + sy), (tx + sy)) \\ &= 2|t|^2(Ax, x) + 2\operatorname{Re} t\bar{s}(Ax, y), \end{aligned}$$

from which it follows that $0 = (Ax, y)$. Thus we get $AP = PA$.

Proof of Theorem 1. The "only if" part is clear, so we show the "if" part. We may assume that $\|B\| \leq 1$, which means $0 \leq B \leq 1$. Since $0 \leq AB^n + B^nA$, we get

(1) $$0 \leq A\exp(tB) + \exp(tB)A \quad \text{for every } t > 0,$$

from which it follows that

$$0 \leq \exp(-tB)A + A\exp(-tB).$$

Thus (1) is valid for $-\infty < t < \infty$. Since $0 \leq A\exp(tB)\exp(sB) + \exp(sB)\exp(tB)A$ for $-\infty < s$, $t < \infty$, we have

$$0 \leq \exp(-sB)A\exp(tB) + \exp(tB)A\exp(-sB).$$

By the Laplace transform relation

$$(2) \quad \int_0^\infty s^{n-1}\exp(-\lambda s)\exp(-sB)\,ds = (n-1)!(B+\lambda)^{-n} \quad \text{for } \lambda > 0,$$

we obtain

$$0 \leq (B+\lambda)^{-n}A\exp(tB) + \exp(tB)A(B+\lambda)^{-n} \quad \text{for } \lambda > 0,$$

which implies that

$$0 \leq A\exp(tB)(B+\lambda)^n + (B+\lambda)^n\exp(tB)A.$$

Since A and B are continuous, by letting $\lambda \to 0$, we get

$$\begin{aligned} 0 &\leq A\exp(tB)B^n + B^n\exp(tB)A \\ &= AB^n\exp(tB) + \exp(tB)B^nA \quad \text{for } -\infty < t < \infty. \end{aligned}$$

It is easy to show that

$$0 \leq \exp(-t(I-B))AB^n + B^nA\exp(-t(I-b)) \quad \text{for } t > 0,$$

from which, using (2) again, we obtain

$$0 \leq AB^n(1-B)^m + (1-B)^mB^nA \quad \text{for } m, n = 0, 1, 2, \ldots.$$

By Bernstein's theorem, each polynomial $p(x)$ which is positive on the interval $[0, 1]$ is a linear combination of polynomials of the form $x^n(1-x)^m$ with real nonnegative coefficients. Thus we have

$$0 \leq Ap(B) + p(B)A.$$

For each continuous function $f(x)$ which is > 0 on $[0, 1]$ we can select a sequence of polynomials as above which uniformly converges to $f(x)$. Therefore we have

$$0 \leq Af(B) + f(B)A.$$

It is easy to show that the latter inequality holds for any continuous function $f(x)$ which is ≥ 0 on $[0, 1]$, and hence that $0 \leq AF_\lambda + F_\lambda A$, where $\{F_\lambda\}$ is the spectral family corresponding to b. From the lemma we obtain $AF_\lambda = F_\lambda A$ and hence $AB = BA$. This concludes the proof.

COROLLARY 2. *Let A and B be nonnegative bounded operators. Then $AB = BA$ if $A^2 \leq (A+tB)^2$ for every $t > 0$ and $n = 1, 2, \ldots$.*

Proof. From the assumption, it follows that

$$0 \leq (AB^n + B^n A) + tB^{2n} \quad \text{for } t > 0.$$

Letting $t \to 0$, we get $0 \leq AB^n + B^n A$.

COROLLARY 3. *Let $0 \leq A$ and $0 \leq B$. Suppose B is bounded. Then $BA \subset AB$ if for $n = 1, 2, \ldots$,*

(3) $\quad B\mathscr{D}(A) \subset \mathscr{D}(A)$ *and* $0 \leq ((AB^n + B^n A)x, x)$ *for every $x \in \mathscr{D}(A)$.*

Proof. For $t > 0$, $(t + A)^{-1}$ is bounded and nonnegative. From (3) it follows that $0 \leq (t + A)^{-1}B^n + B^n(t + A^{-1})$, which implies $(t+A)^{-1}B = B(t+A)^{-1}$ and hence $BA \subset AB$.

COROLLARY 4. *Let A be unbounded selfadjoint, and let B be selfadjoint and bounded from below. Then $E_\lambda F_\mu = F_\mu E_\lambda$ for every λ, μ if $0 \leq \exp(A)\exp(-nB) + \exp(-nB)\exp(A)$ for $n = 1, 2, \ldots$, where the inequality should be interpreted like* (3).

Proof. Clearly $\exp(-B)$ is bounded and nonnegative. Since $\exp(-nB) = \{\exp(-B)\}^n$ (cf. §128 of **[9]**), we have

$$\exp(-B)\exp(A) \subset \exp(A)\exp(-B).$$

Since the spectral family corresponding to $\exp(A)$ is $\{E_{\log t}\}_{0<t<\infty}$, $\exp(-B)$ and E_λ commute. Thus we get $E_\lambda F_\mu = F_\mu E_\lambda$.

For a C^*-algebra $\mathscr{A}$, Ogasawara [7] showed that $\mathscr{A}$ is abelian if the condition $0 \leq a \leq b$, $a, b \in \mathscr{A}$ implies $a^2 \leq b^2$. In other words, $\mathscr{A}$ is abelian if $0 \leq ab + ba$ for every $0 \leq a$, $b \in \mathscr{A}$. Clearly Theorem 1 and Corollary 2 are true for nonnegative a, b in $\mathscr{A}$. Consequently we can consider them to be extensions of Ogasawara's theorem.

2. Let us recall that if A and B are unbounded, then $A \leq B$ means that $\mathscr{D}(B^{1/2}) \subset \mathscr{D}(A^{1/2})$ and $\|A^{1/2}x\| \leq \|B^{1/2}x\|$ for $x \in \mathscr{D}(B^{1/2})$. We have

$$0 \leq A \leq B \Rightarrow 0 \leq B^{-1} \leq A^{-1}. \tag{4}$$

PROPOSITION 5. *Let A and B be bounded from below, and suppose $A \geq -\zeta$, $B \geq -\zeta$. Then the following are equivalent:*

(a) $(A+\zeta)^n \leq (B+\zeta)^n$ *for every* $n = 1, 2, \ldots$.
(b) $F_\lambda \leq E_\lambda$ *for every* λ.
(c) $\exp(tA) \leq \exp(tB)$ *for every* $t > 0$.
(d) $\exp(-tB) \leq \exp(-tA)$ *for every* $t > 0$.

Proof. Olson **[8]** (cf. **[12]**) showed that (a) and (b) are equivalent if A and B are bounded and $\zeta = 0$. We can easily apply his proof to this case. To show (a) $\Rightarrow$ (d), we need the following (cf. Chap. 9 of **[5]**):

$$\exp(-tA) = \lim_{m\to\infty} (I + t/mA)^{-m}. \tag{5}$$

If $m > t\zeta$, then each term in the right side is positive and bounded. From (a) we get

$$(1 + t/mA)^{-m} \geq (1 + t/mB)^{-m} \quad \text{for } m > t\zeta.$$

By using (5) we have (d). We show (d) $\Rightarrow$ (a). Since (d) is equivalent to

$$\exp(-t(B+\zeta)) \leq \exp(-t(A+\zeta)),$$

from (2) it follows that

$$(B+\zeta+\lambda)^{-n} \leq (A+\zeta+\lambda)^{-n} \quad \text{for } \lambda > 0,\ n = 1, 2, \ldots.$$

Thus for $x \in \mathscr{D}((A+\zeta)^{-n/2})$ we have

$$\|(B+\zeta+\lambda)^{-n/2}x\| \leq \|(A+\zeta+\lambda)^{-n/2}x\| \leq \|(A+\zeta)^{-n/2}x\|.$$

By using Fatou's lemma we obtain

$$\|(B+\zeta)^{-n/2}x\| \leq \varliminf_{\lambda\to 0} \|(B+\zeta+\lambda)^{-n/2}x\| \leq \|(A+\zeta)^{-n/2}x\|,$$

that is, $(B+\zeta)^{-n} \leq (A+\zeta)^{-n}$. Taking their inverses, we obtain (a). Now we have only to show (c) $\Leftrightarrow$ (d). But since

$$I = \exp(tA)\exp(-tA) \supset \exp(-tA)\exp(tA)$$

(cf. §128 of **[9]**), $\exp(tA)$ is the inverse of $\exp(-tA)$; by (4) we obtain it. This concludes the proof.

THEOREM 6. *Let A and B be unbounded selfadjoint operators with spectral families $\{E_\lambda\}$ and $\{F_\lambda\}$, respectively. Then the following are equivalent:*

(b) $F_\lambda \leq E_\lambda$ *for every* λ.
(c) $\exp(tA) \leq \exp(tB)$ *for every* $t > 0$.
(d) $\exp(-tB) \leq \exp(-tA)$ *for every* $t > 0$.

Proof. (b) implies that for every $\mu > 0$, $F_{\log\mu} \leq E_{\log\mu}$. Since these operators are the spectral families corresponding to $\exp(B)$ and $\exp(A)$, respectively, by Proposition 5 we obtain

$$0 \leq (\exp(A))^n \leq (\exp(B))^n \quad \text{for } n = 1, 2, \ldots. \tag{6}$$

To see that the above inequalities hold for all $t > 0$, we use Heinz's inequality [**6**]. Since $\exp(tA) = (\exp(A))^t$, we have (c). Conversely, (c) implies (6). By using Proposition 5 again, we arrive at (b). (c) $\Leftrightarrow$ (d) is obvious. This concludes the proof.

THEOREM 7. *Let A be a (not necessarily bounded) selfadjoint operator. Let X be a bounded operator which is nonnegative. If there is a real number $\alpha \geq \|X\|$ such that*

$$\exp(tA) \leq \exp(t(A+\varepsilon X)) \leq \exp(t(A+\varepsilon\alpha I)) \quad \textit{for every } t, \varepsilon > 0, \tag{7}$$

then $XA \subset AX$.

Proof. Set $B = A+\varepsilon X$. Then B is selfadjoint and $\mathscr{D}(B) = \mathscr{D}(A)$. Now let us denote the spectral families corresponding A and B by $E(\lambda)$ and $F(\lambda)$, respectively. From Theorem 6, it follows that

$$E(\lambda - \varepsilon\alpha) \leq F(\lambda) \leq E(\lambda) \quad \text{for } -\infty < \lambda < \infty.$$

The above inequalities are equivalent to

$$E(\lambda)\mathscr{H} \subset F(\lambda + \varepsilon\alpha)\mathscr{H} \subset E(\lambda + \varepsilon\alpha)\mathscr{H} \quad \text{for } -\infty < \lambda < \infty.$$

Since $BE(\lambda)\mathscr{H} \subset BF(\lambda + \varepsilon\alpha)\mathscr{H} \subset F(\lambda + \varepsilon\alpha)\mathscr{H} \subset E(\lambda + \varepsilon\alpha)\mathscr{H}$, we have $XE(\lambda)\mathscr{H} \subset E(\lambda + \varepsilon\alpha)\mathscr{H}$. Since $E(\lambda)$ is continuous from the right, we obtain $XE(\lambda)\mathscr{H} \subset E(\lambda)\mathscr{H}$ and hence $XE(\lambda) = E(\lambda)X$, which implies $XA \subset AX$. Thus the proof is complete.

COROLLARY 8. *Let A and X be nonnegative operators. Suppose X is bounded. If there is a real number $\alpha \geq \|X\|$ such that*

$$A^n \leq (A+\varepsilon X)^n \leq (A+\varepsilon\alpha I)^n \quad \textit{for every } \varepsilon > 0,\ n = 1, 2, \ldots, \tag{8}$$

then $XA \subset AX$.

Proof. It is clear.

For finite matrices or compact operators, we can get better conditions than (7) or (8). From now on, A and B are nonnegative finite matrices or compact operators which are represented as $A = \sum \mu_i(A) e_i \otimes e_i$ and $B = \sum \mu_i(B) d_i \otimes d_i$, where $\{\mu_i(\cdot)\}$ is a decreasing sequence of eigenvalues. It is easy to see that, in this case, the condition (b) in Proposition 5 is equivalent to

(b′) $\mu_i(A) \leq \mu_i(B)$, and if $\mu_i(A) > \mu_j(B)$, then $e_i \perp d_j$.

PROPOSITION 9. *Let A be a nonnegative finite matrix. Set $\delta(A) := \min\{|\lambda - \mu| : \lambda \neq \mu,\ \lambda, \mu \in \sigma_p(A)\}$.*

(i) *If $0 \leq X < \delta(A)$, and $(A+X)^n \geq A^n$ for $n = 1, 2, \ldots$, then $AX = XA$.*

(ii) *If $0 \leq X < \delta(A)$, and $A^n \geq (A-X)^n \geq 0$ for $n = 1, 2, \ldots$, then $AX = XA$.*

Proof. (i) Set $B = A + X$ and suppose $\mu_1(A) = \cdots = \mu_i(A) > \mu_{i+1}(A)$. Then, by Ky Fan **[4]** (cf. **[10]**), we obtain

$$\mu_{i+1}(B) \leq \mu_{i+1}(A) + \mu_1(X) \leq \mu_{i+1}(A) + \delta(A) < \mu_i(A).$$

(b′) implies $\{e_1, \ldots, e_i\} \perp \{d_{i+1}, d_{i+2}, \ldots\}$ and hence the subspace $\{e_1, \ldots, e_i\} = \{d_1, \ldots, d_i\}$ reduces A and B. Since the reduced operator of A is constant, A and B commute there. Repeating this procedure in the same way to the other restrictions of A and B, we can derive $AB = BA$, which means $AX = XA$.

(ii) To prove this in the same way as (i), we need only to start with the smallest eigenvalue of A. Thus the proof is complete.

COROLLARY 10. *Let A be a selfadjoint finite matrix which is not necessarily nonnegative.*

(i) *If $0 \leq X < \delta(A)$, and $\exp(tA) \leq \exp(t(A+X))$ for every $t > 0$, then $AX = XA$.*

(ii) *If $0 \leq X < \delta(A)$, and $\exp(t(A-X)) \leq \exp(tA)$ for every $t > 0$, then $AX = XA$.*

Proof. (i) Take a real number $\zeta > 0$ so that $A + \zeta I \geq 0$. From $\exp(t(A+\zeta I)) \leq \exp(t(A+\zeta I+X))$, using Proposition 5.9. $AX = XA$ follows.

(ii) Take $\zeta > 0$ such that $A + \zeta I - X \geq 0$. Then we can derive $AX = XA$.

PROPOSITION 11. *Let A and X be nonnegative compact operators. If $A^n \leq (A+sX)^n$ for every $s > 0$ and $n = 1, 2, \ldots$, then $AX = XA$.*

Proof. Suppose $\mu_1(A) = \cdots = \mu_j(A) > \mu_{i+1}(A)$ as in the proof of Proposition 7. Let us take s which satisfies $s\|X\| < \mu_i(A) - \mu_{i+1}(A)$. Then the subspace $\{e_1, \ldots, e_i\}$ reduces A and $A + sX$, where they commute. We have only to repeat this procedure to get $AXe_m = XAe_m$ for every m.

Let us end this paper by giving an example. Let A and B be nonnegative matrices. Set $V = \{rA + sB + tI; r, s, t > 0\}$. Then $AB = BA$ if

$$(9) \quad \exp(\tfrac{1}{2}(X + Y)) \leq \tfrac{1}{2}(\exp(X) + \exp(Y)) \quad \text{for every } X, Y \in V,$$

In fact, take $r > 0$ such that $A \leq rI \leq B + rI$. Then we have $\exp(tA) \leq \exp(t(B + rI))$ for every $t > 0$. From this and (9) it follows that

$$\exp\left(t(B + rI)(\tfrac{1}{2} + (\tfrac{1}{2})^2 + \cdots + (\tfrac{1}{2})^n) + t(\tfrac{1}{2})^n A\right) \leq \exp(t(B + rI)).$$

By Corollary 10(ii), we get $AB = BA$. This example shows that we cannot regard $\exp(\frac{1}{2}(X + Y))$ as the geometric mean of $\exp X$ and $\exp Y$ if they do not commute (cf. **[1]**).

Acknowledgment. This paper was written while the author was at the Department of Mathematics of the University of California, San Diego as a visiting scholar. He is grateful to its faculty members for their support. Especially he would like to express his gratitude to Professor J. W. Helton for his hospitality and useful discussions. He also thanks the referee for pointing out many grammatical errors.

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Received November 20, 1991 and in revised form October 20, 1992.

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