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BROWNIAN MOTION AND THE HEAT SEMIGROUP ON THE PATH SPACE OF A COMPACT LIE GROUP

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Let G be a compact connected Lie group with identity element e, and let P_eG denote the space of continuous maps $y: [0, 1] \to G$ such that $y(0) = e$. When equipped with the natural group structure and sup metric, P_eG becomes an interesting example of an infinite dimensional nonlinear topological group. The purpose of this paper is to consider certain aspects of analysis on P_eG. Stimulated by a theorem of M. Malliavin and P. Malliavin, we prove the existence of a natural Brownian motion on P_eG which depends only on a choice of bi-invariant metric for G. Our main results, however, concern the heat semigroup associated to the Brownian motion on P_eG. We identify the action of the generator of this semigroup when applied to certain highly regular functions, with a result similar to that obtained earlier by L. Gross in the (linear) abstract Wiener space context.

1. Introduction. Let G be a compact connected Lie group whose identity element we denote e. The purpose of this paper is to construct a natural Brownian motion and associated heat semigroup on the infinite dimensional nonlinear space of continuous maps $y: [0, 1] \to G$ such that $y(0) = e$. We refer to this space as P_eG. The Brownian motion on P_eG depends only on a choice of bi-invariant metric for G.

Note that P_eG inherits a group structure from G: for $y_1, y_2 \in P_eG$ define y_1y_2 by $(y_1y_2)(t) = y_1(t)y_2(t)$. The constant path at the identity is the identity element in P_eG. Given a Riemannian metric g on G, let $d_g(\cdot, \cdot)$ denote the associated distance function on $G \times G$, which induces a metric on P_eG given by $\sup_{t\in[0,1]} d_g(y_1(t), y_2(t))$. In any such metric P_eG becomes a Polish topological group. This structure leads to a convolution law for probability measures on the Borel field of P_eG. In the special case where the metric g is bi-invariant on G, it happens that the associated bi-invariant Wiener measures on P_eG form a convolution semigroup. This fact, discovered by M. Malliavin and P. Malliavin [**16**], is the origin of Brownian motion on P_eG. Lemma 2.2 of this paper supplies the additional required estimate for continuity of sample paths. We also provide an elementary analytic proof of the Malliavins' theorem in Lemma 2.1.

Many of the previous constructions of diffusion processes on infinite dimensional nonlinear spaces have relied on the use of abstract Wiener spaces. The notion of an abstract Wiener space was formulated by L. Gross [7, **8**], and used in his study of Brownian motion and potential theory in Hilbert space [**9**]. H. H. Kuo [**13**, **14**] subsequently developed a theory of Brownian motion on infinite dimensional manifolds modeled on an abstract Wiener space. Other authors have considered diffusion processes on submanifolds of an abstract Wiener space. For example, following Kusuoka's construction of the Ornstein-Uhlenbeck process on Wiener space [**15**], Getzler [**6**] considered the use of Malliavin calculus to define Dirichlet forms (and the associated Ornstein-Uhlenbeck process) on the based loop space of a Riemannian manifold. The Dirichlet form approach was also applied by Airault and Van Biesen [**1**] to finite codimension submanifolds of Wiener space. Another approach was taken by Epperson and Lohrenz [**5**] in constructing diffusion processes (with explosions) on the based finite-energy loop space of a compact submanifold of $\mathbf{R}^n$ with trivial normal bundle. This was accomplished by considering the loop space to be a submanifold of the based loop space of $\mathbf{R}^n$, for which a theory of stochastic differential equations in abstract Wiener space applies. Finally, we mention P. Malliavin's construction [**17**] of Brownian motions on the loop space of a compact Lie group, via stochastic differential equations.

We begin this paper with a quick review of bi-invariant Brownian motion on a compact Lie group. The earliest reference on this subject is probably Itô [**11**]. Our account is elementary and analytic, in the sense that it avoids stochastic differential equations. This exacts a price; namely, that we use an explicit Gaussian upper bound for heat kernels on G when it comes to establishing continuity of sample paths. Next we prove the Malliavins' product law for independent bi-invariant Brownian motions on G. Together with an estimate on moments of these Brownian motions, the product law results in a heat semigroup for P_eG, which is the content of Theorem 2.1. Section 3 begins our study of the generator of this heat semigroup. Proposition 3.1 considers the action of this generator on exceptionally regular functions. One of our main goals for the future is to demonstrate the extension of previous results concerning regularity of potentials (e.g. on classical Wiener space) to the present nonlinear setting.

2. Brownian motions on G and P_eG. Let G be a compact connected Lie group with bi-invariant metric g. The associated bi-invariant distance function on $G\times G$ will be denoted $d(x, y)$, and the measure associated with the natural volume form on G will be denoted dx. In local coordinates $dx = g^{1/2}\,dx_1\cdots dx_n$, where $g = \det g_{ij}$ and $n = \dim G$. Note that the bi-invariance of g implies that dx is actually a Haar measure on G. The volume of a measurable set $B \subset G$ will be denoted $|B|$.

Consider the bi-invariant Laplacian Δ associated with g. It is given in local coordinates by $g^{-1/2}\partial_i g^{1/2} g^{ij}\partial_j$, with $C^\infty(G)$ as a dense domain of definition in $L^2(G)$. According to Davies [**4**], the nonnegative operator $-\Delta$ is essentially self-adjoint on $C^\infty(G)$. We let $-\Delta$ also denote its closure. The semigroup $e^{t\Delta}$ on $L^2(G)$ is known to have a strictly positive C^∞ kernel $K_t(x, y)$ on $(0, \infty)\times G\times G$. This kernel is bi-invariant, since it is the unique fundamental solution of the heat operator $\partial_t - \Delta$. We will review the associated diffusion process, following the conventions of references [**10, 20**] regarding continuous stochastic processes.

DEFINITION. A *Brownian motion* on G of variance parameter $s > 0$ starting at the identity element e is a continuous G-valued stochastic process $y(t, \omega)$, $t \in [0, 1]$, $\omega \in \Omega$, defined over a probability space $(P, \mathscr{B}, \Omega)$ such that

1. $y(0, \omega) = e$ almost surely,
2. for every $0 \le t_1 < t_2$, the G-valued random variable $y(t_1, \omega)^{-1}y(t_2, \omega)$ is distributed according to the law $K_{s(t_2-t_1)}(e, y)\,dy$,
3. for every $0 \le t_1 < t_2 < t_3 < t_4$, the random variables $y(t_1, \omega)^{-1}y(t_2, \omega)$ and $y(t_3, \omega)^{-1}y(t_4, \omega)$ are independent.

NOTATION. (1) We will frequently suppress the dependence of a stochastic process on its argument ω. (2) If S is a metric space with metric δ, then for $x \in S$, $r > 0$, we let $B_S(x, r) = \{y \in S : \delta(x, y) < r\}$.

PROPOSITION 2.1. *Brownian motion on G exists.*

Proof. This is a standard result in probability theory. One constructs, via Kolmogorov's extension theorem, a family of G-valued random variables $\{x(t), t \in [0, 1]\}$ such that $x(0) = e$, and such that for every $0 < t_1 < t_2 < \cdots < t_k$ and sequence of Borel sets

$B_i \in \mathscr{B}(G)$, $i = 1, \dots, k$,

$$\begin{aligned}(1) \qquad \Pr\{x(t_1) \in B_1, x(t_1) \in B_2, \dots, x(t_k) \in B_k\} \\ = \int_{B_1} K_{st_1}(e, x_1)\, dx_1 \int_{B_2} K_{s(t_2-t_1)}(x_1, x_2)\, dx_2 \\ \cdots \int_{B_k} K_{s(t_k-t_{k-1})}(x_{k-1}, x_k)\, dx_k.\end{aligned}$$

Using the bi-invariance of the kernels $K_t(x, y)$, it is straightforward to verify that any stochastic process distributed according to (1) satisfies conditions 1–3 in the definition of Brownian motion. The converse is also true.

The only difficulty is to show that the process x has a continuous version. To do this, we use an explicit upper bound on the kernel $K_t(x, y)$. According to Davies (reference [3], Theorem 16), for every $\delta > 0$ there exists a constant c_δ such that

$$K_t(x, y) \le c_\delta |B_G(x, t^{1/2})|^{-1/2} |B_G(y, t^{1/2})|^{-1/2} e^{-d(x,y)^2/(4+\delta)t}$$

for all $x, y \in G$ and $0 < t < 1$. Let r_0 denote a positive number less than the injectivity radius of G with respect to the bi-invariant metric g. By calculating in normal coordinates (and using the compactness of G) it can be shown that if $r_0 > 0$ is sufficiently small, then there exist constants $0 < c_1, c_2 < \infty$ such that $|B_G(x, r)| \ge c_1 r^n$, and $\rho_x(r) := \partial_r |B_G(x, r)| \le c_2 r^{n-1}$ for all $r < r_0$. Combining the kernel and volume estimates, we have for $t^{1/2} < r_0$,

$$\begin{aligned}\int_G d(x, y)^4 K_t(x, y)\, dy &\le c_\delta c_1 t^{-n/2} \int_G d(x, y)^4 e^{-d(x,y)^2/(4+\delta)t}\, dy \\ &= c_\delta c_1 t^{-n/2} \left(\int_0^{r_0} r^4 e^{-r^2/(4+\delta)t} \rho_x(r)\, dr \right. \\ &\qquad \left. + \int_{G \setminus B_G(x, r_0)} d(x, y)^4 e^{-d(x,y)^2/(4+\delta)t}\, dy \right) \\ &\le c_\delta c_1 t^{-n/2} \left(c_2 \int_0^\infty r^{n+3} e^{-r^2/(4+\delta)t}\, dr + (\operatorname{diam} G)^4 |G| e^{-r_0^2/(4+\delta)t} \right) \\ &= k_1 t^2 \int_0^\infty r^{n+3} e^{-r^2}\, dr + k_2 t^{-n/2} e^{-r_0^2/(4+\delta)t}.\end{aligned}$$

The integral in this last expression converges, and the second term has rapid decay in t as $t \searrow 0$. Since $\int_G d(x, y)^4 K_t(x, y)\, dy \le (\operatorname{diam} G)^4$

for all $t > 0$, it follows that there exists a constant $c < \infty$ such that

$$\int_G d(x, y)^4 K_t(x, y)\, dy \le ct^2 \tag{2}$$

for all $0 < t \le 1$. It is well known that this sort of estimate leads to a continuous version y of the process x. □

Let P_eG be the space of continuous maps $y: [0, 1] \to G$ such that $y(0) = e$. We define a metric ρ on P_eG by

$$\rho(y_1, y_2) = \sup_{t\in[0,1]} d(y_1(t), y_2(t)).$$

This makes P_eG a complete separable metric space. It is known that in this setting, the σ-algebra $\sigma(\mathscr{C})$ generated by the family of Borel cylinder sets $\mathscr{C}$ coincides with the topological σ-algebra $\mathscr{B}(P_eG)$. Moreover, every probability measure on $(P_eG, \mathscr{B}(P_eG))$ is uniquely determined by its values on $\mathscr{C}$. In this way, the distribution law of a Brownian motion on G of variance parameter s determines a unique measure $P_s(dy)$ on $(P_eG, \mathscr{B}(P_eG))$, which we refer to as the *Wiener measure* of variance parameter s.

Let $\mathscr{A}$ denote the Banach space of real valued, bounded, uniformly continuous functions on the Polish space (P_eG, ρ), together with the sup norm $\|\cdot\|$. For $y_1, y_2 \in P_eG$, let $y_1y_2 \in P_eG$ denote the pointwise product path given by $y_1y_2(t) = y_1(t)y_2(t)$. With this product, we define for every $s > 0$ the convolution operator

$$p_sf(x) = \int_{P_eG} f(yx)P_s(dy), \qquad f \in \mathscr{A}.$$

The following theorem is the impetus for this paper

THEOREM 2.1. *The operators p_s, $s > 0$, form a strongly continuous contraction semigroup on $\mathscr{A}$.*

Before proving this theorem, we need two lemmas. In reference **[16]**, Malliavin and Malliavin deduced the first lemma by manipulation of stochastic differentials. We provide an independent proof, since our definition of Brownian motion avoids stochastic differential equations.

LEMMA 2.1. *If y_1, y_2 are two independent Brownian motions on G of variance parameters s_1, s_2, respectively, then the product process $y = y_1y_2$ is a Brownian motion of variance parameter $s_1 + s_2$.*

Proof. The product process y is continuous, and verifies condition 1, that $y(0) = e$ almost surely. As for condition 2, we calculate that for $0 \leq t_1 < t_2$ and $B \in \mathscr{B}(G)$

$$\begin{aligned}
&\Pr\{y(t_1)^{-1}y(t_2) \in B\} \\
&\quad = \int_G K_{s_1t_1}(e, a_1)\,da_1 \int_G K_{s_1(t_2-t_1)}(a_1, a_2)\,da_2 \int_G K_{s_2t_1}(e, b_1)\,db_1 \\
&\qquad \cdot \int_G \chi(b_1^{-1}a_1^{-1}a_2b_2 \in B) K_{s_2(t_2-t_1)}(b_1, b_2)\,db_2 \\
&\quad = \int_G K_{s_1t_1}(e, a_1)\,da_1 \int_G K_{s_1(t_2-t_1)}(e, a_2)\,da_2 \int_G K_{s_2t_1}(e, b_1)\,db_1 \\
&\qquad \cdot \int_G \chi(b_1^{-1}a_2b_1b_1^{-1}b_2 \in B) K_{s_2(t_2-t_1)}(b_1, b_2)\,db_2
\end{aligned}$$

by the change of variables $a_1^{-1}a_2 \to a_2$,

$$\begin{aligned}
&\quad = \int_G K_{s_2t_1}(e, b_1)\,db_1 \int_G K_{s_1(t_2-t_1)}(e, a_2)\,da_2 \\
&\qquad \cdot \int_G \chi(a_2b_2 \in B) K_{s_2(t_2-t_1)}(e, b_2)\,db_2
\end{aligned}$$

by the changes of variables $b_1^{-1}a_2b_1 \to a_2$ and $b_1^{-1}b_2 \to b_2$,

$$\begin{aligned}
&\quad = \int_G K_{s_1(t_2-t_1)}(e, a_2)\,da_2 \int_G \chi(a_2b_2 \in B) K_{s_2(t_2-t_1)}(a_2, a_2b_2)\,db_2 \\
&\quad = \int_G \chi(b_2 \in B) K_{(s_1+s_2)(t_2-t_1)}(e, b_2)\,db_2
\end{aligned}$$

by the change of variables $a_2b_2 \to b_2$. This proves condition 2. To prove condition 3, let $0 \leq t_1 < t_2 < t_3 < t_4$ and $A, B \in \mathscr{B}(G)$. Then we have

$$\begin{aligned}
(3)\ &\Pr\{y(t_1)^{-1}y(t_2) \in A,\ y(t_3)^{-1}y(t_4) \in B\} \\
&\quad = \int_G K_{s_1t_1}(e, a_1)\,da_1 \int_G K_{s_1(t_2-t_1)}(a_1, a_2)\,da_2 \int_G K_{s_2t_1}(e, b_1)\,db_1 \\
&\qquad \cdot \int_G \chi(b_1^{-1}a_1^{-1}a_2b_2 \in A) F(a_2, b_2) K_{s_2(t_2-t_1)}(b_1, b_2)\,db_2,
\end{aligned}$$

where

$$\begin{aligned}
F(a_2, b_2) = &\int_G K_{s_1(t_3-t_2)}(a_2, a_3)\,da_3 \int_G K_{s_1(t_4-t_3)}(a_3, a_4)\,da_4 \\
&\cdot \int_G K_{s_2(t_3-t_2)}(b_2, b_3)\,db_3 \\
&\cdot \int_G \chi(b_3^{-1}a_3^{-1}a_4b_4 \in B) K_{s_2(t_4-t_3)}(b_3, b_4)\,db_4.
\end{aligned}$$

Making the changes of variables $a_2^{-1}a_3 \to a_3$ and $b_2^{-1}b_3 \to b_3$,

$$\begin{aligned} F(a_2, b_2) = &\int_G K_{s_1(t_3-t_2)}(e, a_3)\,da_3 \int_G K_{s_1(t_4-t_3)}(a_2a_3, a_4)\,da_4 \\ &\cdot \int_G K_{s_2(t_3-t_2)}(e, b_3)\,db_3 \\ &\cdot \int_G \chi(b_3^{-1}b_2^{-1}a_3^{-1}a_2^{-1}a_4b_2b_2^{-1}b_4 \in B) K_{s_2(t_4-t_3)}(b_2b_3, b_4)\,db_4. \end{aligned}$$

Making the changes of variables $a_2^{-1}a_4 \to a_4$ and $b_2^{-1}b_4 \to b_4$,

$$\begin{aligned} F(a_2, b_2) = &\int_G K_{s_1(t_3-t_2)}(e, a_3)\,da_3 \int_G K_{s_1(t_4-t_3)}(a_3, a_4)\,da_4 \\ &\cdot \int_G K_{s_2(t_3-t_2)}(e, b_3)\,db_3 \\ &\cdot \int_G \chi(b_3^{-1}b_2^{-1}a_3^{-1}a_4b_2b_4 \in B) K_{s_2(t_4-t_3)}(b_3, b_4)\,db_4. \end{aligned}$$

Finally, making the change of variables $a_3b_2^{-1}a_3^{-1}a_4b_2 \to a_4$,

$$\begin{aligned} F(a_2, b_2) = &\int_G K_{s_1(t_3-t_2)}(e, a_3)\,da_3 \int_G K_{s_1(t_4-t_3)}(a_3, a_4)\,da_4 \\ &\cdot \int_G K_{s_2(t_3-t_2)}(e, b_3)\,db_3 \\ &\cdot \int_G \chi(b_3^{-1}a_3^{-1}a_4b_4 \in B) K_{s_2(t_4-t_3)}(b_3, b_4)\,db_4 \\ = &\Pr\{y(t_3 - t_2)^{-1}y(t_4 - t_2) \in B\} \\ = &\Pr\{y(t_3)^{-1}y(t_4) \in B\}. \end{aligned}$$

Substituting this in (3) proves condition 3. □

For the next lemma we appeal to some basic martingale theory. The reader is advised to consult reference **[20]** for missing details. Fix the variance parameter s, and let

$$\mathcal{F}_t = \begin{cases} \bigcap_{\varepsilon>0} \sigma\{y(u),\, 0 \le u \le t+\varepsilon\} & \text{for } 0 \le t < 1, \\ \sigma\{y(u),\, 0 \le u \le 1\} & \text{for } t = 1, \end{cases}$$

denote the natural right continuous filtration of the Brownian motion. It is known that for every $f \in C^2(G)$, the stochastic process

$$f(y(t)) - \int_0^t s\Delta f(y(u))\,du \tag{4}$$

is a martingale with respect to $\{\mathscr{F}_t\}$. We recall the ingredients of the proof. First, Brownian motion has the Markov property

$$\Pr(y(t_2) \in B \mid \mathscr{F}_{t_1}) = \int_B K_{s(t_2-t_1)}(y(t_1), z)\,dz, \quad \text{a.s.}$$

for all $0 \le t_1 < t_2 \le 1$ and $B \in \mathscr{B}(G)$. Next, for each $0 \le t_1 < t_2 \le 1$ and $f \in C^{1,2}([0, t_2] \times G)$, there holds the backward equation

$$f(t_1, x) = \int_G f(t_2, y) K_{s(t_2-t_1)}(x, y)\,dy - \int_{t_1}^{t_2} dt \cdot \int_G ((\partial_t + s\Delta) f(t, y)) K_{s(t-t_1)}(x, y)\,dy.$$

Now we calculate, for $0 \le t_1 < t_2 \le 1$ and $f \in C^2(G)$,

$$\begin{aligned} &\mathrm{E}(f(y(t_2)) - f(y(t_1)) - \int_{t_1}^{t_2} s\Delta f(y(t))\,dt \mid \mathscr{F}_{t_1}) \\ &= \int_G f(z) K_{s(t_2-t_1)}(y(t_1), z)\,dz - f(y(t_1)) - \int_{t_1}^{t_2} dt \cdot \int_G s\Delta f(z) K_{s(t-t_1)}(y(t_1), z)\,dz \end{aligned}$$

which vanishes according to the backward equation.

NOTATION. If x is a real valued stochastic process defined on the time interval $[0, 1]$, then define the random variable $x_* = \sup_{t\in[0,1]} |x(t)|$. If y is a Brownian motion on G and τ is a stopping time, then let $y_\tau(t) := y(t \wedge \tau)$.

LEMMA 2.2. *Let y be a Brownian motion on G of variance parameter $0 < s < 1$ starting at the identity element e. Then for every $q > 1$ there exists a constant c_q independent of s such that $E(d(e, y)_*^q) \le c_q s^{q/2}$.*

Proof. In reference [12] it is shown that if $r > 0$ is sufficiently small, then there exists a bounded coordinate mapping $\phi: B_G(e, r) \to \mathbf{R}^n$ whose components ϕ^i satisfy $\Delta\phi^i = 0$ in $B_G(e, r)$. For each $i = 1, \dots, n$ let $\tilde{\phi}^i: G \to \mathbf{R}$ denote a smooth extension of the harmonic function ϕ^i to all of G. Also let y be a Brownian motion on G of variance parameter s. Then according to the discussion of (4), the stochastic process

$$\tilde{\phi}^i(y(t)) - \int_0^t s\Delta\tilde{\phi}^i(y(u))\,du$$

is a martingale with respect to $\{\mathcal{F}_t\}$. Let

$$\tau = \begin{cases} \inf\{u : d(e, y(u)) > r\} & \text{if } y \text{ exits } \overline{B_G(e, r)}, \\ 1 & \text{otherwise.} \end{cases}$$

Then by Doob's stopping time theorem,

$$\phi^i(y(t \wedge \tau)) = \tilde{\phi}^i(y(t \wedge \tau)) - \int_0^{t\wedge\tau} s\Delta\tilde{\phi}^i(y(u))\, du$$

is also a martingale with respect to $\{\mathcal{F}_t\}$.

We may assume that $\phi(e) = 0$. Also, by decreasing r if necessary, we may assume that there exist constants c_1, c_2 such that for all $x \in B_G(e, r)$,

$$c_1 \|\phi(x)\|^2_{\mathbf{R}^n} \le d(e, x)^2 \le c_2 \|\phi(x)\|^2_{\mathbf{R}^n}.$$

Now, using Doob's inequality for martingales,

$$\begin{aligned}
(5)\quad & \mathrm{E}(d(e, y_\tau)^q_*) \\
&\le c_2^{q/2} n^{q/2} \mathrm{E}(\|\phi^1(y_\tau)\|^q_* + \cdots + \|\phi^n(y_\tau)\|^q_*) \\
&\le c_2^{q/2} n^{q/2} \left(\frac{q}{q-1}\right)^q \mathrm{E}(\|\phi^1(y(\tau))\|^q + \cdots + \|\phi^n(y(\tau))\|^q) \\
&\le \left(\frac{c_2}{c_1}\right)^{q/2} n^{(q+2)/2} \left(\frac{q}{q-1}\right)^q \mathrm{E}(d(e, y(\tau))^q).
\end{aligned}$$

We use this to bound moments of the unstopped process,

$$\begin{aligned}
(6)\quad \mathrm{E}(d(e, y)^q_*) &= \mathrm{E}(\chi(\tau = 1) d(e, y)^q_* + \chi(\tau < 1) d(e, y)^q_*) \\
&\le \mathrm{E}(d(e, y_\tau)^q_*) + \Pr(\tau < 1)(\operatorname{diam} G)^q \\
&\le c(\mathrm{E}(d(e, y(\tau))^q) + \Pr(\tau < 1)), \\
&\qquad\qquad \text{the last inequality following from (5),} \\
&= c(\mathrm{E}(d(e, y(1))^q + \chi(\tau < 1) d(e, y(\tau))^q) + \Pr(\tau < 1)) \\
&\le c(\mathrm{E}(d(e, y(1))^q) + (1 + r^q)\Pr(\tau < 1)).
\end{aligned}$$

The second term in the last line of (6) is bounded using

$$\begin{aligned}
\Pr(\tau < 1) &\le \Pr(d(e, y(1)) > r/2) + \Pr(\tau < 1, d(e, y(1)) \le r/2) \\
&\le \Pr(d(e, y(1)) > r/2) + \Pr(\tau < 1, d(y(\tau), y(1)) \ge r/2).
\end{aligned}$$

Using Gaussian upper bounds on the kernels $K_t(x, y)$ and the strong Markov property for Brownian motions on G, the last two terms are seen to have rapid decay in the variance parameter s, for fixed r, as $s \searrow 0$. The principal term $E(d(e, y(1))^q)$ in (6) is bounded by

$cs^{q/2}$, again using Gaussian upper bounds on the kernels $K_t(x, y)$, as in the derivation of (2). □

NOTATION. We let e also denote the constant path in P_eG mapping $[0, 1]$ to the identity element e. Thus, if $y \in P_eG$, we write $\rho(e, y)$ instead of $\sup_{t\in[0,1]} d(e, y(t))$. The conclusion of Lemma 2.2 may be written $\int_{P_eG} \rho(e, y)^q P_s(dy) \leq c_q s^{q/2}$.

Proof of Theorem 2.1. Since $P_s(dy)$ is a probability measure, $\|p_s f\| \leq \|f\|$ for every $f \in \mathscr{A}$. Now for $f \in \mathscr{A}$, $\varepsilon > 0$, let $\delta > 0$ be such that $|f(x) - f(y)| < \varepsilon$ whenever $\rho(x, y) < \delta$. Then we have, for $\rho(x, y) < \delta$,

$$\begin{aligned} |p_s f(x) - p_s f(y)| &\leq \int_{P_eG} |f(zx) - f(zy)| P_s(dz) \\ &\leq \sup_{z\in P_eG} |f(zx) - f(zy)| < \varepsilon, \end{aligned}$$

since $\rho(zx, zy) = \rho(x, y)$. Thus p_s is a contraction mapping $\mathscr{A}$ to itself. The semigroup law $p_s p_t = p_{s+t}$ is an immediate consequence of Lemma 2.1. As for strong continuity, again fix $f \in \mathscr{A}$, $\varepsilon > 0$, and let $\delta > 0$ be such that $|f(x) - f(y)| < \varepsilon$ whenever $\rho(x, y) < \delta$. Then we have

$$\begin{aligned} |p_s f(x) - f(x)| &= \left| \int_{P_eG} (f(yx) - f(x)) P_s(dy) \right| \\ &\leq \int_{P_eG} |f(yx) - f(x)| \chi(\rho(e, y) < \delta) P_s(dy) \\ &\quad + \int_{P_eG} 2\|f\| \chi(\rho(e, y) \geq \delta) P_s(dy) \\ &\leq \varepsilon + 2\|f\| \delta^{-2} \int_{P_eG} \rho(e, y)^2 P_s(dy) \\ &\leq \varepsilon + c\|f\| \delta^{-2} s, \end{aligned}$$

in which the last two lines use Chebyshev's inequality and Lemma 2.2. This shows that $\lim_{s\to 0} \|p_s f - f\| \leq \varepsilon$ for every $\varepsilon > 0$. □

There is a stochastic process associated with the semigroup p_s, $s \in [0, \infty)$.

DEFINITION. A *Brownian motion* on P_eG starting at $x \in P_eG$ is a ρ-continuous stochastic process $Y_x(t, \omega)$, $t \in [0, \infty)$, $\omega \in \Omega$, defined over a probability space $(P, \mathscr{B}, \Omega)$ such that

1. $Y_x(0, \omega) = x$ almost surely,
2. for every $0 \leq t_1 < t_2$, the P_eG-valued random variable $Y_x(t_1, \omega)^{-1}Y_x(t_2, \omega)$ is distributed according to the law $P_{t_2-t_1}(dy)$,
3. for every $0 \leq t_1 < t_2 < t_3 < t_4$, the random variables $Y_x(t_1, \omega)^{-1}Y_x(t_2, \omega)$ and $Y_x(t_3, \omega)^{-1}Y_x(t_4, \omega)$ are independent.

The existence of this process is an immediate consequence of Lemmas 2.1, and 2.2. Continuity of sample paths is obtained from Lemma 2.2, e.g., by using $q = 4$. The finite dimensional distributions are easy to describe. If $0 < t_1 < t_2 < \cdots < t_k$ and $B_i \in \mathscr{B}(P_eG)$, $i = 1, \ldots, k$, then the Brownian motion satisfies

$$\begin{aligned}&\Pr\{Y_x(t_1) \in B_1, \ldots, Y_x(t_k) \in B_k\}\\ &\quad = \int_{x^{-1}B_1} P_{t_1}(dy_1) \int_{y_1^{-1}B_2} P_{t_2-t_1}(dy_2) \cdots \int_{y_{k-1}^{-1}B_k} P_{t_k-t_{k-1}}(dy_k).\end{aligned}$$

3. The generator of Brownian motion on P_eG. The purpose of this section is to begin describing the generator of the semigroup p_s as an operator on the real Banach space $\mathscr{A}$. Let $\mathscr{G} = T_eG$ be the Lie algebra of G and $\exp: \mathscr{G} \to G$ the exponential map. We regard $\mathscr{G}$ as a real inner product space using $g_e(\cdot, \cdot)$, with $\|\cdot\|_{\mathscr{G}}$ the resulting norm. Let r_0 denote the injectivity radius of G, and recall that exp is a diffeomorphism from $B_{\mathscr{G}}(0, r_0)$ onto $B_G(e, r_0)$ such that $\|x\|_{\mathscr{G}} = d(e, \exp x)$ for every $x \in B_{\mathscr{G}}(0, r_0)$. Note that if we fix $r_1 < r_0$ sufficiently small, then for every $r \leq r_1$ there exists a constant $c_r < \infty$ such that $\|\exp^{-1} y_1 - \exp^{-1} y_2\|_{\mathscr{G}} \leq c_r d(y_1, y_2)$ for all $y_1, y_2 \in B_G(e, r)$. The space of continuous maps $z: [0, 1] \to \mathscr{G}$ with $z(0) = 0$ will be denoted $P_0\mathscr{G}$; it is a real separable Banach space under the natural vector space operations and sup norm $\|z\| = \sup_{t\in[0,1]} \|z(t)\|_{\mathscr{G}}$. The exponential map induces a homeomorphism Exp from $B_{P_0\mathscr{G}}(0, r_0)$ onto $B_{P_eG}(e, r_0)$ given by $(\operatorname{Exp} z)(t) = \exp(z(t))$. Clearly $\|z\| = \rho(e, \operatorname{Exp} z)$ for every $z \in B_{P_0\mathscr{G}}(0, r_0)$.

For each $s > 0$, $r \leq r_1$ we will define a measure $Q_{r,s}$ on $(P_0\mathscr{G}, \mathscr{B}(P_0\mathscr{G}))$ derived from the Wiener measure P_s on $(P_eG, \mathscr{B}(P_eG))$. These measures will be used in Proposition 3.1, which describes the action of the generator of p_s on certain very regular functions. In reference **[9]**, Gross proved an abstract Wiener space version of this proposition. In his linear context, where the Exp mapping did not appear, no analogue of the measure $Q_{r,s}$ was needed. Lemma 3.2 involving these measures essentially takes care of the nonlinear difficulties that Gross did not face in his early paper.

Let $T_{r,s}$ denote the homeomorphism from $B_{P_0\mathscr{G}}(0, s^{-1/2}r)$ onto

$B_{P_eG}(e, r)$ given by $T_{r,s}(z) = \operatorname{Exp} s^{1/2} z$. Then for $B \in \mathscr{B}(P_0\mathscr{G})$ let

$$Q_{r,s}(B) = P_s \circ T_{r,s}(B \cap B_{P_0\mathscr{G}}(0, s^{-1/2}r)).$$

Note that according to Lemma 2.2, along with Chebyshev's inequality,

$$1 - \frac{c_q s^{q/2}}{r^q} \le P_s(B_{P_eG}(0, r)) = Q_{r,s}(P_0\mathscr{G}) \le 1,$$

so that $\lim_{s\to 0} Q_{r,s}(P_0\mathscr{G}) = 1$.

LEMMA 3.1. *For every $q > 1$ there exists a constant $c_q < \infty$ such that*

$$\int_{P_0\mathscr{G}} \|z\|^q Q_{r,s}(dz) \le c_q$$

for every $r \le r_1$, $0 < s \le 1$.

Proof. We have

$$\begin{aligned}\int_{P_0\mathscr{G}} \|z\|^q Q_{r,s}(dz) &= s^{-q/2} \int_{B_{P_0\mathscr{G}}(0, s^{-1/2}r)} \rho(e, T_{r,s}z)^q Q_{r,s}(dz) \\ &= s^{-q/2} \int_{B_{P_eG}(0,r)} \rho(e, y)^q P_s(dy) \\ &\le s^{-q/2} c_q s^{q/2}\end{aligned}$$

in which the inequality uses Lemma 2.2. □

Now let W denote the ordinary Wiener measure on $(P_0\mathscr{G}, \mathscr{B}(P_0\mathscr{G}))$, which is uniquely determined by our choice $g_e(\cdot, \cdot)$ of inner product for $\mathscr{G}$ and the condition

$$\int_{P_0\mathscr{G}} \|z(t)\|^2 W(dz) = 2t, \qquad 0 < t \le 1.$$

LEMMA 3.2. *For every $r \le r_1$, the measures $Q_{r,s}$ converge weakly to W as $s \to 0$.*

Proof. We begin by showing that for every $0 < t_1 < t_2 < \cdots < t_k \le 1$ and collection of *bounded* Borel sets $B_i \in \mathscr{B}(\mathscr{G})$, $i = 1, \ldots, k$, it holds that

$$\lim_{s\to 0} \int_{P_0\mathscr{G}} \chi(z(t_1) \in B_1, \ldots, z(t_k) \in B_k) Q_{r,s}(dz) = \int_{P_0\mathscr{G}} \chi(z(t_1) \in B_1, \ldots, z(t_k) \in B_k) W(dz). \tag{7}$$

First note that

$$(8)\quad \int_{P_0\mathscr{G}} \chi(z(t_1)\in B_1, \dots, z(t_k)\in B_k) Q_{r,s}(dz)$$
$$= \int_{P_eG} \chi(y(t_1)\in \exp(s^{1/2}B_1), \dots, y(t_k)\in \exp(s^{1/2}B_k))$$
$$\cdot \chi(\rho(e, y)\le r) P_s(dy)$$
$$= \int_{\exp(s^{1/2}B_1)} K_{st_1}(e, y_1)\, dy_1 \int_{\exp(s^{1/2}B_2)} K_{s(t_2-t_1)}(y_1, y_2)\, dy_2$$
$$\cdots \int_{\exp(s^{1/2}B_k)} K_{s(t_k-t_{k-1})}(y_{k-1}, y_k)\, dy_k + O(s^m)$$

for $m > 1/2$, as $s \to 0$. Let $G_t(x, y) = (4\pi t)^{-n/2}e^{-d(x,y)^2/4t}$. According to well-known asymptotics for heat kernels (see references [**18**, **19**]), for every $N \ge 1$ there exists a constant $C_N < \infty$ such that if r_2 is sufficiently small, then

$$|K_t(x, y) - G_t(x, y)| \le c_N((d(x, y) + t)G_t(x, y) + t^{N-n/2+1})$$

for every $0 < t \le 1$ and $x, y \in B_G(e, r_2)$. We use this in bounding the difference between the leading expression in (8) and

$$(9)\quad \int_{\exp(s^{1/2}B_1)} G_{st_1}(e, y_1)\, dy_1 \int_{\exp(s^{1/2}B_2)} G_{s(t_2-t_1)}(y_1, y_2)\, dy_2$$
$$\cdots \int_{\exp(s^{1/2}B_k)} G_{s(t_k-t_{k-1})}(y_{k-1}, y_k)\, dy_k$$

by the sum of k quantities of the form

$$(10)\quad \int_{\exp(s^{1/2}B_1)} G_{st_1}(e, y_1)\, dy_1$$
$$\cdots \int_{\exp(s^{1/2}B_{i-1})} G_{s(t_{i-1}-t_{i-2})}(y_{i-2}, y_{i-1})\, dy_{i-1}$$
$$\cdot \int_{\exp(s^{1/2}B_i)} |G_{s(t_i-t_{i-1})}(y_{i-1}, y_i) - K_{s(t_i-t_{i-1})}(y_{i-1}, y_i)|\, dy_i$$
$$\cdot \int_{\exp(s^{1/2}B_{i+1})} K_{s(t_{i+1}-t_i)}(y_i, y_{i+1})\, dy_{i+1}$$
$$\cdots \int_{\exp(s^{1/2}B_k)} K_{s(t_k-t_{k-1})}(y_{k-1}, y_k)\, dy_k.$$

Thus, if we let $N \ge n/2$, then the middle integral in this last expression is $O(s^{1/2})$ as $s \to 0$. The important point is that $d(y_{i-1}, y_i)$ is $O(s^{1/2})$ when $y_{i-1} \in \exp s^{1/2}B_{i-1}$ and $y_i \in \exp s^{1/2}B_i$. The other integrals in (10) are bounded uniformly in s, with the result that (8) and (9) differ by $O(s^{1/2})$ as $s \to 0$.

We focus now on expression (9), which must be transformed from an iterated integral on G to an iterated integral on $\mathscr{G}$. According to the Baker-Campbell-Hausdorff formula (see reference **[21]**), if s is sufficiently small, then there exists a constant $k < \infty$ such that for $y_i \in \exp s^{1/2} B_i$, $i = 1, \dots, k$,

$$\| \exp^{-1} y_{i-1}^{-1} y_i - (\exp^{-1} y_{i-1} - \exp^{-1} y_i) \|_{\mathscr{G}} \leq ks.$$

Consequently, for s sufficiently small and y_i as above,

$$\begin{aligned} d(y_{i-1}, y_i)^2 &= \| \exp^{-1} y_{i-1}^{-1} y_i \|_{\mathscr{G}}^2 \\ &= \| \exp^{-1} y_i - \exp^{-1} y_{i-1} \|_{\mathscr{G}}^2 + O(s^{3/2}). \end{aligned} \tag{11}$$

Using (11), the Gaussian kernels appearing in (9) may be rewritten as

$$\begin{aligned} &G_{s(t_i - t_{i-1})}(y_{i-1}, y_i) \\ &\quad = (4\pi s(t_i - t_{i-1}))^{-n/2} \\ &\quad\quad \cdot \exp\left(- \frac{\| \exp^{-1} y_i - \exp^{-1} y_{i-1} \|_{\mathscr{G}}^2}{4s(t_i - t_{i-1})} \right) (1 + O(s^{1/2})). \end{aligned} \tag{12}$$

Let $d\tilde{y}_i$ denote the pullback by $\exp^{-1}$ of the Haar measure dy_i on $B_G(e, r)$ to $B_{\mathscr{G}}(0, r)$. By calculating in normal coordinates it is easy to see that the Radon-Nikodym derivative of $d\tilde{y}_i$ with respect to the Lebesgue measure dz_i on $B_{\mathscr{G}}(0, r)$ satisfies

$$\frac{d\tilde{y}_i}{dz_i}(z_i) = 1 + O(\|z_i\|_{\mathscr{G}}^2). \tag{13}$$

Combining (12, 13) we see that as $s \to 0$, the iterated G integral (9) differs only by $O(s^{1/2})$ from the iterated $\mathscr{G}$ integral

$$\begin{aligned} &\int_{s^{1/2}B_1} (4\pi s t_1)^{-n/2} e^{-\|z_1\|_{\mathscr{G}}^2 / 4st_1} \, dz_1 \\ &\quad \cdot \int_{s^{1/2}B_2} (4\pi s(t_2 - t_1))^{-n/2} e^{-\|z_2 - z_1\|_{\mathscr{G}}^2 / 4s(t_2 - t_1)} \, dz_2 \\ &\quad \cdots \int_{s^{1/2}B_k} (4\pi s(t_k - t_{k-1}))^{-n/2} e^{-\|z_k - z_{k-1}\|_{\mathscr{G}}^2 / 4s(t_k - t_{k-1})} \, dz_k. \end{aligned}$$

The claim (7) now follows from the changes of variables $s^{-1/2} z_i \to z_i$.

Fix $\varepsilon > 0$ and let $B_1, \dots, B_k$ be Borel sets (possibly unbounded) in $\mathscr{G}$. As a consequence of Lemma 3.1 we may pick R large enough that $\int \chi(\|z\|_{\mathscr{G}} > R) Q_{r,s}(dz) < \varepsilon/3$ uniformly in $s \in (0, 1]$. Of course we may also pick R large enough that $\int \chi(\|z\|_{\mathscr{G}} > R)W(dz) < \varepsilon/3$. With this value of R fixed, we have for s sufficiently small,

$$\left| \int_{P_0\mathscr{G}} \chi(z(t_1) \in B_1 \cap B_{\mathscr{G}}(0, R), \dots, z(t_k) \in B_k \cap B_{\mathscr{G}}(0, R))Q_{r,s}(dz) \right.$$
$$\left. - \int_{P_0\mathscr{G}} \chi(z(t_1) \in B_1 \cap B_{\mathscr{G}}(0, R), \dots, z(t_k) \in B_k \cap B_{\mathscr{G}}(0, R))W(dz) \right| < \varepsilon/3.$$

Hence, by an $\varepsilon/3$ argument we obtain the limit (7) for arbitrary Borel sets $B_1, \dots, B_k$.

The proof is finished with a compactifying estimate. Let $0 < t_1 < t_2 \le 1$ and consider, for $q > 1$,

$$\begin{aligned}
&\int_{P_0\mathscr{G}} \|z(t_2) - z(t_1)\|_{\mathscr{G}}^q Q_{r,s}(dz) \\
&= s^{-q/2} \int_{P_eG} \chi(\rho(e, y) \le r) \| \exp^{-1} y(t_2) - \exp^{-1} y(t_1)\|_{\mathscr{G}}^q P_s(dy) \\
&\le s^{-q/2} \int_{P_eG} \chi(d(e, y(t_1)), d(e, y(t_2)) \le r) \\
&\qquad\qquad \cdot \| \exp^{-1} y(t_2) - \exp^{-1} y(t_1)\|_{\mathscr{G}}^q P_s(dy) \\
&= s^{-q/2} \int_{P_eG} \chi(d(e, y_1) \le r) K_{st_1}(e, y_1)\, dy_1 \\
&\quad \cdot \int_{P_eG} \chi(d(e, y_2) \le r) \| \exp^{-1} y_2 - \exp^{-1} y_1\|_{\mathscr{G}}^q K_{s(t_2-t_1)}(y_1, y_2)\, dy_2 \\
&\le c_r^q s^{-q/2} \int_{P_eG} K_{st_1}(e, y_1)\, dy_1 \int_{P_eG} d(y_1, y_2)^q K_{s(t_2-t_1)}(y_1, y_2)\, dy_2 \\
&\le \mathrm{const}(t_2 - t_1)^{q/2}.
\end{aligned}$$

The reader may consult reference **[10]** to be convinced that this estimate finishes the proof. □

REMARK. Lemmas 3.1 and 3.2 imply that if f is a real valued continuous function on $P_0\mathscr{G}$ with a polynomial bound $|f(z)| \le c\|z\|^k$, then

$$\lim_{s\to 0} \int_{P_0\mathscr{G}} f(z) Q_{r,s}(dz) = \int_{P_0\mathscr{G}} f(z) W(dz). \tag{14}$$

To see this, fix $\varepsilon > 0$, and for every $R > 0$ let m_R be a nonnegative continuous function on $P_0\mathscr{G}$ such that $|m_R(z)| \le 1$ and

$$m_R(z) = \begin{cases} 1 & \text{for } \|z\| \le R, \\ 0 & \text{for } \|z\| \ge R + 1. \end{cases}$$

By taking R large enough we have

$$\int_{P_0\mathscr{G}} |f(z)|(1 - m_R(z))Q_{r,s}(dz) < \varepsilon/3$$

uniformly in $s \in (0, 1]$, and also

$$\int_{P_0\mathscr{G}} |f(z)|(1 - m_R(z))W(dz) < \varepsilon/3.$$

With this value of R fixed we have

$$\left|\int_{P_0\mathscr{G}} f(z)m_R(z)Q_{r,s}(dz) - \int_{P_0\mathscr{G}} f(z)m_R(z)W(dz)\right| < \varepsilon/3$$

for s sufficiently small. Since ε was arbitrary, the limit (14) results.

LEMMA 3.3. *For $B \subseteq P_eG$ let $B^{-1} = \{y^{-1} : y \in B\}$. Each of the Wiener measures on $(P_eG, \mathscr{B}(P_eG))$ has the property $P_s(B) = P_s(B^{-1})$ for every $B \in \mathscr{B}(P_eG)$.*

Proof. It suffices to prove the equality for cylinder sets. Two ingredients are required for this simple calculation which we leave to the reader. First, Haar measure is invariant under group inversion in G. Second, the heat kernels have the property $K_t(x, y) = K_t(x^{-1}, y^{-1})$. This follows from bi-invariance of the kernels, and the fact that the kernels are symmetric in their G-arguments. (Symmetry in the G-arguments can be seen in the eigenfunction expansion for the kernel.) □

REMARK. Different proofs of this lemma have appeared in **[2]**, §4, and **[16]**, Lemma 2.2.5.

Lemmas 3.1–3 put us in a position to discuss the generator of p_s. Let $\mathscr{D}(L)$ consist of those functions $f \in \mathscr{A}$ for which the limit

$$\lim_{s\to 0} \frac{p_s f - f}{s}$$

exists in $\mathscr{A}$. On these functions define Lf to be the above limit. Theorem 2.1 implies that $\mathscr{D}(L)$ is dense in $\mathscr{A}$ and $L: \mathscr{D}(L) \to \mathscr{A}$ is a closed operator. Before stating the next proposition, we need some notation and a concept of differentiability for functions on P_eG.

NOTATION. If A and B are Banach spaces, let $\mathscr{B}(A, B)$ denote the space of bounded linear maps from A to B. Recall that we regard $P_0\mathscr{G}$ as a Banach space; let $P_0\mathscr{G}^*$ denote its topological dual space, and let (z, y) denote the canonical pairing between $z \in P_0\mathscr{G}^*$, $y \in P_0\mathscr{G}$.

Now let f be a function defined in a neighborhood U of a path $x \in P_eG$, taking values in a Banach space B. We say that f is *differentiable* at x if there exists an element $w \in \mathscr{B}(P_0\mathscr{G}, B)$ such that

$$\|f(yx) - f(x) - w(\mathrm{Exp}^{-1} y)\|_B = o(\|\mathrm{Exp}^{-1} y\|)$$

for all y with $yx \in U$ and $\rho(e, y)$ sufficiently small. Clearly this element w is unique when it exists, so we write $f'(x) = w$.

PROPOSITION 3.1. *Let $f \in \mathscr{A}$ be such that f' and f'' exist everywhere, f'' is bounded, and the map $f'': P_eG \to \mathscr{B}(P_0\mathscr{G}, P_0\mathscr{G}^*)$ is uniformly P_eG-continuous in the weak operator topology. Then*

$$\lim_{s\to 0} \frac{(p_s f)(x) - f(x)}{s} = \frac{1}{2}\int_{P_0\mathscr{G}} (f''(x)z, z)W(dz). \tag{15}$$

Moreover, if for some fixed r, the functions

$$\frac{1}{2}\int_{P_0\mathscr{G}} (f''(x)z, z)Q_{r,s}(dz)$$

converge uniformly in $x \in P_eG$ as $s \to 0$ to the function on the right side of (15), *then $f \in \mathscr{D}(L)$ and the limit* (15) *holds uniformly in x.*

REMARKS. We follow the proof of Proposition 8 in Gross, reference [9]. As in his proof, we could identify the right side of (15) as a trace with respect to a suitable abstract Wiener space structure on $P_0\mathscr{G}$. However, this would only introduce additional notation that is unimportant in the present context.

Proof. Fix a positive number $r \le r_1$ (recall that the measures $Q_{r,s}$ are defined for $0 < r \le r_1$). We have

$$\begin{aligned}\frac{(p_s f)(x) - f(x)}{s} &= \frac{1}{s}\int_{P_eG} (f(yx) - f(x))P_s(dy) \\ &= \frac{1}{s}\int_{P_eG} \chi(\rho(e, y) \le r)(f(yx) - f(x))P_s(dy) \\ &\quad + \frac{1}{s}\int_{P_eG} \chi(\rho(e, y) > r)(f(yx) - f(x))P_s(dy).\end{aligned} \tag{16}$$

Since f is bounded, the last term decays rapidly to zero as $s \to 0$, uniformly in x. (Use Lemma 2.2.) For every $y \in P_eG$ define $y_t =$

$\operatorname{Exp}(t\operatorname{Exp}^{-1}y)$. Then the function $f(y_t x)$ is a twice differentiable function of t on the interval $[0, 1]$, with

$$\frac{df(y_t x)}{dt} = (f'(y_t x), \operatorname{Exp}^{-1} y) \quad \text{and}$$
$$\frac{d^2 f(y_t x)}{dt^2} = (f''(y_t x)\operatorname{Exp}^{-1} y, \operatorname{Exp}^{-1} y).$$

Since the second derivative is bounded, the first derivative is absolutely continuous, so two integrations by parts yield

$$\begin{aligned} f(yx) = f(x) &+ (f'(x), \operatorname{Exp}^{-1} y) \\ &+ \int_0^1 (1-t)(f''(y_t x)\operatorname{Exp}^{-1} y, \operatorname{Exp}^{-1} y)\,dt. \end{aligned}$$

We insert this in line (16). According to Lemma 3.3

$$\begin{aligned} &\int_{P_e G} \chi(\rho(e, y) \le r)(f'(x), \operatorname{Exp}^{-1} y) P_s(dy) \\ &\quad = \int_{P_e G} \chi(\rho(e, y^{-1}) \le r)(f'(x), \operatorname{Exp}^{-1} y^{-1}) P_s(dy) \\ &\quad = -\int_{P_e G} \chi(\rho(e, y) \le r)(f'(x), \operatorname{Exp}^{-1} y) P_s(dy). \end{aligned}$$

Hence, line (16) reduces to

(17)
$$\begin{aligned} &\int_0^1 (1-t) \int_{P_e G} \chi(\rho(e, y) \le r)(f''(y_t x)\operatorname{Exp}^{-1} y, \operatorname{Exp}^{-1} y) s^{-1} P_s(dy)\,dt \\ &\quad = \frac{1}{2}\int_{P_e G} \chi(\rho(e, y) \le r)(f''(x)\operatorname{Exp}^{-1} y, \operatorname{Exp}^{-1} y) s^{-1} P_s(dy) \\ &\qquad + \int_0^1 (1-t) \int_{P_e G} \chi(\rho(e, y) \le r) \\ &\qquad\qquad \cdot ((f''(y_t x) - f''(x))\operatorname{Exp}^{-1} y, \operatorname{Exp}^{-1} y) s^{-1} P_s(dy)\,dt \\ &\quad = \frac{1}{2}\int_{P_0 \mathscr{G}} (f''(x)z, z) Q_{r,s}(dz) \\ &\qquad + \int_0^1 (1-t) \int_{P_0 \mathscr{G}} ((f''(\operatorname{Exp}(s^{1/2}tz)x) - f''(x))z, z) Q_{r,s}(dz)\,dt. \end{aligned}$$

For each fixed z, the integrand on the last line is continuous in the parameters s, t and x. Let $\{x_i\}$ denote a countable dense set in

$P_0\mathscr{G}$. Then

$$\sup_{t\in[0,1]}\sup_{u\in[0,s]}\sup_{x\in P_0\mathscr{G}}|((f''(\operatorname{Exp}(u^{1/2}tz)x)-f''(x))z\,,\,z)|$$
$$=\sup_{\substack{u\in[0,s]\\ u\text{ rational}}}\sup_i|((f''(\operatorname{Exp}(u^{1/2}z)x_i)-f''(x_i))z\,,\,z)|$$

is a measurable function of z. This allows us to write

$$(18)\quad \limsup_{s\to 0}\sup_x\left|\int_0^1(1-t)\int_{P_0\mathscr{G}}((f''(\operatorname{Exp}(s^{1/2}tz)x)-f''(x))z\,,\,z)Q_{r,s}(dz)\,dt\right|$$
$$\le\limsup_{s\to 0}\int_{P_0\mathscr{G}}\sup_{u\in[0,s]}\sup_x|((f''(\operatorname{Exp}(u^{1/2}z)x)-f''(x))z\,,\,z)|Q_{r,s}(dz)$$
$$\le\lim_{\tau\to 0}\limsup_{s\to 0}\int_{P_0\mathscr{G}}\sup_{u\in[0,\tau]}\sup_x|((f''(\operatorname{Exp}(u^{1/2}z)x)-f''(x))z\,,\,z)|Q_{r,s}(dz).$$

Note that the last line requires the natural extension of the map Exp from our original domain of definition $B_{P_0\mathscr{G}}(0\,,\,r_0)$ to all of $P_0\mathscr{G}$. (This causes no problem; it is the map Exp^{-1} that generally cannot be defined everywhere on P_eG.) We claim now that for each $\tau>0$,

$$(19)\quad \lim_{s\to 0}\int_{P_0\mathscr{G}}\sup_{u\in[0,\tau]}\sup_x|((f''(\operatorname{Exp}(u^{1/2}z)x)-f''(x))z\,,\,z)|Q_{r,s}(dz)$$
$$=\int_{P_0\mathscr{G}}\sup_{u\in[0,\tau]}\sup_x|((f''(\operatorname{Exp}(u^{1/2}z)x)-f''(x))z\,,\,z)|W(dz).$$

The integrand in this expression will be shown to be a $P_0\mathscr{G}$-continuous function of z. First we show that for each u and x,

$$F(u\,,\,x\,,\,z):=((f''(\operatorname{Exp}(u^{1/2}z)x)-f''(x))z\,,\,z)$$

is continuous in z. Fix z_0 and use the inequality

$$\begin{aligned}&|F(u\,,\,x\,,\,z)-F(u\,,\,x\,,\,z_0)|\\ &\quad\le|((f''(\operatorname{Exp}(u^{1/2}z)x)-f''(x))z\,,\,(z-z_0))|\\ &\qquad+|((f''(\operatorname{Exp}(u^{1/2}z)x)-f''(x))(z-z_0)\,,\,z_0)|\\ &\qquad+|((f''(\operatorname{Exp}(u^{1/2}z)x)-f''(\operatorname{Exp}(u^{1/2}z_0)x))z_0\,,\,z_0)|.\end{aligned}$$

Since f'' is bounded, the first two terms in this inequality tend to zero as $\|z-z_0\|\to 0$, uniformly in u, x. For the last term note that

$$\rho(\operatorname{Exp}(u^{1/2}z)x\,,\,\operatorname{Exp}(u^{1/2}z_0)x)$$
$$=\sup_{t\in[0,1]}d(\exp(u^{1/2}z(t))\,,\,\exp(u^{1/2}z_0(t)))\,,$$

which tends to zero as $\|z-z_0\|\to 0$, uniformly in $u\in[0\,,\,\tau]$ and x. Therefore, by the uniform weak continuity of f'', the last term in the inequality tends to zero as $\|z - z_0\| \to 0$, uniformly in $u \in [0, \tau]$ and x. In summary, we have shown that for each z_0, $F(u, x, z)$ tends to $F(u, x, z_0)$ as $\|z - z_0\| \to 0$, uniformly in $u \in [0, \tau]$ and x. This implies in particular that

$$\lim_{z \to z_0} \sup_{u \in [0, \tau]} \sup_x |F(u, x, z)| = \sup_{u \in [0, \tau]} \sup_x |F(u, x, z_0)|.$$

Since f'' is bounded, the integrand in (19) is bounded by $c\|z\|^2$ for some $c < \infty$. Therefore, according to the remark after Lemma 3.2, the limit (19) holds. Substituting this in the last line of (18), we have

$$\lim_{\tau \to 0} \int_{P_0 \mathscr{G}} \sup_{u \in [0, \tau]} \sup_x |((f''(\operatorname{Exp}(u^{1/2} z)x) - f''(x))z, z)| W(dz)$$

which vanishes by the dominated convergence theorem. This means that

$$\begin{aligned} \lim_{s \to 0} \frac{(p_s f)(x) - f(x)}{s} &= \frac{1}{2} \lim_{s \to 0} \int_{P_0 \mathscr{G}} (f''(x)z, z) Q_{r,s}(dz) \\ &= \frac{1}{2} \int_{P_0 \mathscr{G}} (f''(x)z, z) W(dz). \end{aligned}$$

(See line (17).) In view of the uniform estimates on the last lines of (16, 17), the last statement of the proposition is now clear. □

It is a consequence of Proposition 3.1 that every smooth cylinder function belongs to $\mathscr{D}(L)$.

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