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# COMMUTANTS OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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**This paper describes the commutants of certain analytic Toeplitz operators. To underline the difference between the Bergman and Hardy spaces, we first prove that on the Bergman space $L_a^2$ the only isometric Toeplitz operators with harmonic symbols are scalar multiples of the identity. If T denotes the norm closed subalgebra of $\mathrm{L}(L_a^2)$ generated by Toeplitz operators, we show that for each positive integer $n$, $\{T_{z^n}\}' \cap \mathrm{T}$ is the set of all analytic Toeplitz operators. This result is also valid for the Hardy space. Here $\{T_{z^n}\}'$ denotes the commutant of $T_{z^n}$. Finally we prove the analogous result for $T_{u^n}$, where $u$ is an analytic, one-to-one map of the unit disk onto itself.**

**Introduction.** Let $D$ denote the open unit disk in the complex plane and let $dA$ denote the usual Lebesgue area measure on $D$. The complex space $L^2(D, dA)$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_D f \bar{g} \, dA.$$

The Bergman space $L_a^2$ is the set of those functions in $L^2(D, dA)$ that are analytic on $D$. The Bergman space is a closed subspace of $L^2(D, dA)$, and so there is an orthogonal projection $P$ from $L^2(D, dA)$ onto $L_a^2$. For $\varphi \in L^\infty(D, dA)$, the Toeplitz operator with symbol $\varphi$, denoted $T_\varphi$, is the operator from $L_a^2$ to $L_a^2$ defined by $T_\varphi f = P(\varphi f)$. For more information about the Bergman space and its operators see [**4**].

The algebra of bounded analytic functions on $D$ will be denoted by $H^\infty$. If $\varphi \in H^\infty$, then $T_\varphi$ is called an analytic Toeplitz operator.

For a Hilbert space $H$, $\mathrm{L}(H)$ denotes the algebra of all bounded linear operators on $H$. If $S \subset \mathrm{L}(H)$, then $S' = \{B \in \mathrm{L}(H) : AB = BA \text{ for all } A \in S\}$ is the commutant of $S$. In this paper we are interested in finding commutants of certain analytic Toeplitz operators acting on the Bergman space.

Much work has been done in studying commutants of analytic Toeplitz operators on the Hardy space. Some of those results can be extended to the Bergman space case. The complex space $L^2(\partial D)$

is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_{\partial D} f\bar{g}\frac{dt}{2\pi}.$$

For each integer $n$, let $e_n$ denote the function $e_n(z) = z^n$ for $|z| = 1$. Then $\{e_n\}$ is an orthonormal basis for $L^2(\partial D)$ and the Hardy space $H^2(\partial D)$ is, by definition, the subspace $\text{span}\{e_n \colon n \geq 0\}^-$. For $\varphi \in L^\infty(\partial D)$, the Toeplitz operator with symbol $\varphi$, denoted again by $T_\varphi$, on $H^2(\partial D)$ is defined in the analogous way. For the basic properties of the Hardy space Toeplitz operators see Douglas **[9]**. Shields and Wallen **[15]** studied commutants of certain multiplication operators in a Hilbert space of analytic functions and introduced interesting function theoretic methods. Deddens and Wong **[8]** studied the problem using operator theory and raised six questions. Abrahamse **[1]** answered some of Deddens-Wong questions negatively. Baker, Deddens and Ullman **[6]** found $\{T_f\}'$ if $f$ is an entire function. In a series of papers **[16]**–**[19]**, Thomson used function theoretic methods to find commutants or intersection of commutants of certain analytic Toeplitz operators. Finally, C. Cowen continued their work in **[7]** and found the commutant of Toeplitz operators whose symbol is a finite Blaschke product or a covering map.

It is well known that the Hardy space Toeplitz operator $T_\varphi$ is an isometry if and only if $\varphi$ is inner. If $\varphi$ is a nonconstant inner function, then $T_\varphi$ is a pure isometry and is unitarily equivalent to a unilateral shift, whose commutant can be characterized matricially. On the Bergman space, a Toeplitz operator whose symbol is a nonconstant inner function is not an isometry. We will prove even more (Theorem 1.1): The only Toeplitz operator with harmonic symbol that is an isometry is a scalar multiple of the identity.

Our first result about commutants concerns Toeplitz operator with symbol $z^n$. Let $\mathbf{T}$ be the norm closed subalgebra of $\mathbf{L}(L^2_a)$ generated by all Toeplitz operators. We show (Theorem 1.4) that for each positive integer $n$, $\{T_{z^n}\}' \cap \mathbf{T}$ is the set of all analytic Toeplitz operators. This result is also valid for the Hardy space. Then, we prove the analogous result for $T_{u^n}$, where $u$ is an analytic, one-to-one map of $D$ onto itself.

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**1. Commutants.** At first we will underline the difference between the Bergman and Hardy space Toeplitz operators. We will prove that on the Bergman space there are no nontrivial isometries with harmonic symbols. We need some facts about the maximal ideal space of $H^\infty$. Good references are Hoffman **[12]** and Garnett **[10]**.

The set of all multiplicative linear functionals on $H^\infty$ is called the maximal ideal space of $H^\infty$ and we denote it by $M$. The Gelfand transform $\hat{}: H^\infty \to C(M)$ is defined by $\hat{f}(\varphi) = \varphi(f)$, for $\varphi \in M$. The Gelfand transform is an isometry from $H^\infty \to C(M)$, so that we can identify $H^\infty$ with the uniformly closed subalgebra of $C(M)$. Hoffman ([**13**, Lemma 4.4]) has proved that $C(M)$ is identical to the sup norm closure of the algebra generated by the bounded harmonic functions. If $m_1$ and $m_2$ are in $M$, the pseudohyperbolic distance between $m_1$ and $m_2$ is defined as

$$\rho(m_1, m_2) = \sup\{|\hat{f}(m_2)| : f \in H^\infty, \ \|f\| \le 1, \ \hat{f}(m_1) = 0\}.$$

The relation $m_1 \sim m_2$ if and only if $\rho(m_1, m_2) < 1$ is an equivalence relation on $M$. The corresponding equivalence classes are called the Gleason parts of $M$. The set of one-point parts in $M$ will be denoted by $M_1$. Let

$$J = \{\varphi \in C(M) : \varphi = 0 \text{ on } M_1\}.$$

Let $\mathbf{T}(C(M))$ be the closed subalgebra of $\mathbf{L}(L_a^2)$ generated by $\{T_\varphi : \varphi \in C(M)\}$ and let $\mathbf{C}$ be the commutator ideal of $\mathbf{T}(C(M))$. McDonald and Sundberg **[14]** proved that the sequence

$$0 \to J \to C(M) \to \mathbf{T}(C(M))/\mathbf{C} \to 0$$

is exact. This implies that $C(M)/J$ is isomorphic to $\mathbf{T}(C(M))/\mathbf{C}$ with isomorphism

$$\Psi(\varphi + J) = T_\varphi + \mathbf{C}$$

for $\varphi \in C(M)$.

Another Banach algebra we need is $L^\infty(\partial D)$. The maximal ideal space of $L^\infty(\partial D)$, denoted by $M(L^\infty)$, plays an important role here. Since we may regard $H^\infty$ as a closed subalgebra of $L^\infty(\partial D)$, we may think of $M(L^\infty)$ as a subset of $M$. It turns out that $M(L^\infty)$ is a subset of $M_1$.

Now, we can prove our theorem.

THEOREM 1.1. *Suppose that $h \in L^\infty(D, dA)$ is harmonic and that $T_h$ is an isometry. Then $h$ is a constant function of modulus* 1.

*Proof*. Suppose that $T_h$ is an isometry, i.e., $T_h^* T_h = I$. Since $h$ is harmonic, Hoffman's result shows that $h$ and $\bar{h} \in C(M)$. Applying the isomorphism $\Psi^{-1}$, we obtain $(\bar{h} + J) \cdot (h + J) = 1 + J$, and by the definition of $J$ we have

$$|\varphi(h)| = 1 \tag{1.1}$$

for every $\varphi \in M(L^\infty)$. It is well known that the Gelfand transform of $L^\infty(\partial D)$ maps $L^\infty(\partial D)$ isometrically and isomorphically onto $C(M(L^\infty))$ (see Hoffman, [**12**, p. 170]), so that we have

$$\begin{aligned} \sup\{|h(z)| : z \in D\} &= \|h\|_{L^\infty(\partial D)} = \|\hat{h}\|_{C(M(L^\infty))} \\ &= \sup\{|\varphi(h)| : \varphi \in M(L^\infty)\}. \end{aligned}$$

Hence (1.1) implies $\sup\{|h(z)| : z \in D\} = 1$. If $|h(z)| = 1$ for some $z \in D$, then $h$ is constant by the Maximum Principle. If $|h(z)| < 1$ for all $z \in D$, then

$$\pi = \|1\|^2 = \|T_h 1\|^2 = \|Ph\|^2 \le \|h\|^2 = \int_D |h(z)|^2 \, dA < \int_D dA = \pi,$$

a contradiction. Here $\|\cdot\|$ denotes the $L^2(D, dA)$-norm. Therefore $h$ is a constant function, and since $T_h$ is an isometry, $|h| = 1$. □

We can slightly generalize this result and get the following:

COROLLARY 1.2. *Suppose that $T_h$ is an isometry, where $h = f \cdot g^n$, where $f$ is inner, $g \in L^\infty(D, dA)$ is harmonic, and $n \in \mathbf{N}$. Then $h$ is a constant function of modulus* 1.

J. Thukral asked for which harmonic $h$ is $T_h$ a partial isometry. If $T_h$ is a partial isometry, then by Halmos [**11**, Problem 98], $T_h = T_h T_h^* T_h$. Similarly as before, this means

$$\varphi(h)[1 - |\varphi(h)|^2] = 0$$

for every $\varphi \in M(L^\infty)$. If $\varphi(h) = 0$ for every $\varphi \in M(L^\infty)$, then $h = 0$. If $\varphi(h) \neq 0$ for some $\varphi \in M(L^\infty)$, then $\sup\{|h(z)| : z \in D\} = 1$ and $h$ must be a constant function. Thus we have proved the following theorem:

THEOREM 1.3. *Suppose that $h \in L^\infty(D, dA)$ is harmonic and that $T_h$ is a partial isometry. Then $h$ is either a constant function of modulus* 1 *or $h$ is identically* 0.

In [**5**] S. Axler and the author characterized commuting Toeplitz operators with harmonic symbols. Now, we are going to consider the related problem—the commutants of some analytic Toeplitz operators. At first we will be interested in finding the commutant of $T_{z^n}$, for arbitrary positive integer $n$. Before we state and prove our result, recall the following definitions.

For $\varphi \in L^\infty(D, dA)$, the Hankel operator with symbol $\varphi$, denoted $H_\varphi$, is the operator from $L_a^2$ to $(L_a^2)^\perp$ defined by $H_\varphi f = (I-P)(\varphi f)$.

For an analytic function $f$ on $D$ we set

$$\|f\|_B = \sup\{(1-|z|^2)|f'(z)| : z \in D\}.$$

The Bloch space $B$ is the set of all analytic functions $f$ on $D$ for which $\|f\|_B < \infty$. The quantity $|f(0)| + \|f\|_B$ defines a norm on $B$, and $B$ equipped with this norm is a Banach space. Contained in the Bloch space is the little Bloch space $B_0$, which is by definition the set of all analytic functions $f$ on $D$ for which

$$(1-|z|^2)f'(z) \to 0 \quad \text{as } |z| \to 1.$$

For the basic properties of the Bloch space see [**2**].

Let $n \in \mathbf{N}$ be fixed.

THEOREM 1.4. *Let $S \in \mathbf{T}$ commute with $T_{z^n}$. Then $S = T_\psi$ for some $\psi \in H^\infty$.*

*Proof*. The equation $ST_{z^n} = T_{z^n}S$ gives us the following:

Let $g_i = Sz^i$, for $i = 0, 1, \ldots, n-1$. Then for any such $i$

$$Sz^{n+i} = ST_{z^n}z^i = T_{z^n}Sz^i = z^n g_i$$
$$\ldots$$
$$Sz^{kn+i} = z^{kn} g_i, \quad \text{for } k = 0, 1, 2 \ldots.$$

Let

$$X_0 = \operatorname{span}\{e_{kn} : k = 0, 1, 2, \ldots\}^-,$$
$$X_1 = \operatorname{span}\{e_{kn+1} : k = 0, 1, 2, \ldots\}^-,$$
$$\ldots$$
$$X_{n-1} = \operatorname{span}\{e_{kn+(n-1)} : k = 0, 1, 2, \ldots\}^-.$$

Then $L_a^2 = X_0 \oplus X_1 \oplus \cdots \oplus X_{n-1}$, i.e., each $f \in L_a^2$ can be written as

$$f = f_0 + f_1 + \cdots + f_{n-1},$$

$f_i \in X_i$, $i = 0, 1, 2, \ldots, n-1$. Each $f_0 \in X_0$ has its Fourier series expansion

$$f_0 = \sum_{k=0}^{\infty} (f_0, e_{kn}) e_{kn}$$

i.e., $f_0 = \lim s_m$, where $s_m = \sum_{k=0}^{m}(f_0, e_{kn})e_{kn}$. Since the point evaluations are bounded on $L_a^2$, we have $f_0(z) = \lim s_m(z)$, for each $z \in D$, so that

$$(f_0 \cdot g_0)(z) = \lim(s_m \cdot g_0)(z) \tag{1.2}$$

for each $z \in D$. Since $Sz^{kn} = z^{kn}g_0$, $k = 0, 1, 2, \ldots$, it follows that $Ss_m = s_m \cdot g_0$, for every $m \in \mathbf{N}$. By continuity of $S$, we have

$$(Sf_0) = \lim Ss_m = \lim s_m \cdot g_0,$$

so that

$$(Sf_0)(z) = \lim(s_m \cdot g_0)(z) \tag{1.3}$$

for each $z \in D$. Comparing (1.2) and (1.3) we conclude that

$$Sf_0 = g_0 \cdot f_0,$$

for each $f_0 \in X_0$. Repeating the above reasoning, we get that $Sf_1 = g_1 f_1/z$, for $f_1 \in X_1$ and so on. Thus the operator $S$ can be described as

$$Sf = g_0 f_0 + \frac{g_1}{z} f_1 + \frac{g_2}{z^2} f_2 + \cdots + \frac{g_{n-1}}{z^{n-1}} f_{n-1}. \tag{1.4}$$

Let's observe another property of $S$, being an element of $\mathbf{T}$.

*Claim.* $ST_z - T_z S \in \mathbf{K}$, where $\mathbf{K}$ denotes the ideal of all compact operators.

At first, assume $S = T_\varphi$, $\varphi \in L^\infty(D, dA)$. Then

$$T_\varphi T_z - T_z T_\varphi = T_{z\varphi} - T_z T_\varphi = H_{\bar{z}}^* H_\varphi.$$

Because $z \in B_0$, the operator $H_{\bar{z}}^*$ is compact (see [3]), so that $T_\varphi T_z - T_z T_\varphi$ is compact. If $\Pi \colon \mathbf{L}(L_a^2) \to \mathbf{L}(L_a^2)/\mathbf{K}$ denotes the natural projection, then

$$\Pi(T_\varphi)\Pi(T_z) = \Pi(T_z)\Pi(T_\varphi) \tag{1.5}$$

for every $\varphi \in L^\infty(D, dA)$. If $S = T_{\varphi_1} \cdots T_{\varphi_l}$ then, because of (1.5), $\Pi(ST_z - T_z S) = 0$ so that $ST_z - T_z S \in \mathbf{K}$. An arbitrary operator $S$ in $\mathbf{T}$ is the limit of sums of operators of the form $T_{\varphi_1} \cdots T_{\varphi_l}$. In that case $S = \lim S_l$, each $S_l$ is of the form $\sum T_{\varphi_1} \cdots T_{\varphi_l}$, and so $ST_z - T_z S = \lim(S_l T_z - T_z S_l) \in \mathbf{K}$. Hence the claim is proved.

Now, let's express $ST_z - T_z S$ in terms of (1.4). It is easy to see that

$$ST_z f = g_1 f_0 + \frac{g_2}{z} f_1 + \frac{g_3}{z^2} f_2 + \cdots + z g_0 f_{n-1}.$$

From this and (1.4) we obtain

$$(ST_z - T_zS)f = f_0(g_1 - zg_0) + f_1\left(\frac{g_2}{z} - g_1\right) + f_2\left(\frac{g_3}{z^2} - \frac{g_2}{z}\right) + \cdots + f_{n-1}\left(zg_0 - \frac{g_{n-1}}{z^{n-2}}\right).$$

By the claim, $(ST_z - T_zS)|_{X_0} = M_{g_1 - zg_0}\colon X_0 \to L_a^2$ is compact ($M_{g_1-zg_0}$ is a multiplication operator). Let $\varphi = g_1 - zg_0$. Now, it immediately follows that $M_\varphi|_{X_j}$ is compact, for all $j = 1, \dots, n-1$, since $M_\varphi|_{X_j} = M_{z^j}(M_\varphi|_{X_0})(M_{z^{-j}}|_{X_j})$. Hence $M_\varphi$ is compact on $X_0 \oplus X_1 \oplus \cdots \oplus X_{n-1} = L_a^2$. Thus $\varphi = 0$ and therefore $g_0 = g_1/z$.

Similarly, $(ST_z - T_zS)|_{X_1} = M_{g_2/z - g_1}\colon X_1 \to L_a^2$ is a compact operator. Thus $M_{g_2/z-g_1}M_z|_{X_0} = M_{g_2 - zg_1}\colon X_0 \to L_a^2$ is compact. As before, this means that $g_2 - zg_1 = 0$ so that $g_1 = g_2/z$ and therefore $g_0 = g_2/z^2$. If we continue this way, (1.4) shows that

$$Sf = g_0 \cdot f = T_{g_0}f$$

for every $f \in L_a^2$. The function $g_0$ must be an $H^\infty$ function as a multiplier of $L_a^2$ (see **[15]**). If we let $\psi = g_0$, the theorem is proved. □

REMARK. From the proof of the theorem it is clear that we can replace the assumption $S \in \mathbf{T}$ by the weaker assumption $ST_z - T_zS \in \mathbf{K}$. Also, slightly modified arguments give a proof for the Hardy space case.

We can extend this result. Let $\mathrm{Aut}(D)$ denote the set of analytic, one-to-one maps of $D$ onto $D$. Let $u \in \mathrm{Aut}(D)$ and define an operator $V\colon L_a^2 \to L_a^2$ by $Vf = f \circ u^{-1}$. Clearly, $V$ is a bounded linear operator, with the inverse operator $V^{-1}f = f \circ u$. Observe that

$$T_zV = VT_u,$$

and therefore

$$T_{z^n}V = VT_{u^n} \tag{1.6}$$

for every $n \in \mathbf{N}$.

Suppose now that $S \in \mathbf{T}$ and $ST_{u^n} = T_{u^n}S$ for some $n$. Formula (1.6) implies

$$SV^{-1}T_{z^n}V = V^{-1}T_{z^n}VS$$

and we conclude that

$$VSV^{-1} \in \{T_{z^n}\}'.$$

The operator $B = VSV^{-1}$ has the property that $BT_z - T_zB$ is in $\mathbf{K}$ (because $u \in B_0$), so by the remark following Theorem 1.4, it follows that $B = T_\varphi$, for some $\varphi \in H^\infty$. Thus $S = V^{-1}T_\varphi V$. If we let $\psi = \varphi \circ u$, we have proved the following corollary:

COROLLARY 1.5. *Let* $u \in \mathrm{Aut}(D)$ *and let* $S \in \mathbf{T}$ *commute with* $T_{u^n}$, *for some* $n \in \mathbf{N}$. *Then* $S = T_\psi$ *for some* $\psi \in H^\infty$.

The results in this paper raise the following questions.

The McDonald-Sundberg functional calculus is the crucial tool in proving Theorem 1.1. Is this theorem true without the assumption on the symbol to be harmonic? If $h$ is in $C(M)$, then the McDonald-Sundberg calculus is still valid, but it is not true in general that $\sup\{|h(z)| : z \in D\} = \sup\{|\varphi(h)| : \varphi \in M(L^\infty)\}$. However, we guess that there are no nontrivial isometries among Toeplitz operators on the Bergman space.

Suppose $S \in \mathbf{L}(L^2_a)$ is such that $ST_z - T_zS \in \mathbf{K}$. If $ST_{z^n} - T_{z^n}S = 0$ for some $n$, then Theorem 1.4 shows that $S$ must be an analytic Toeplitz operator. What is the set of all functions $f$ such that $ST_f - T_fS = 0$ implies that $S$ is an analytic Toeplitz operator?

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