VALUE DISTRIBUTION OF THE GAUSS MAP AND THE TOTAL CURVATURE OF COMPLETE MINIMAL SURFACE IN R^m

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The aim of this paper is to prove the following

THEOREM. *Let S be a complete non-degenerate minimal surface in R^m such that its generalized Gauss map f intersects only a finite number of times the hyperplanes $A_1, \ldots, A_q$ in CP^{m-1} in general position. If $q > m(m+1)/2$, then S must have finite total curvature.*

1. Introduction. The study of the value distribution property of Gauss map of minimal surface began with a series of papers by Osserman [9], [11] and the results can be summarized in the following

THEOREM (*R. Osserman*). *Let S be a complete minimal surface in R^3. Then*

S has infinite total curvature $\Leftrightarrow$ the Gauss map of S takes on all directions infinitely often with the exception of at most a set of logarithmic capacity zero;

S has finite non-zero total curvature $\Leftrightarrow$ the Gauss map of S takes on all directions a finite number of times, omitting at most three directions;

S has zero total curvature $\Leftrightarrow$ S is a plane.

For a long time, the above theorem had been the best result on this direction. But all the known examples indicated that the exceptional set of logarithmic capacity should be a finite set. In 1981, Xavier made a surprising breakthrough by proving the following result, using a result of Yau about a differential equation on complete Riemannian manifold.

THEOREM (*F. Xavier* [13]). *Let S be a complete minimal surface in R^3. Then its Gauss map can omit at most six directions unless it is a plane.*

In 1988, Fujimoto finally found a way to arrive at the best possible number 4.

THEOREM (*H. Fujimoto* [4]). *Let* S *be a complete minimal surface in* R^3. *Then its Gauss map can omit at most* 4 *directions unless it is a plane.*

A combination of Osserman's early study and Fujimoto's above work gives the following

THEOREM (*X. Mo and R. Osserman* [8]). *Let* S *be a complete minimal surface in* R^3 *with infinite total curvature. Then its Gauss map must take every direction infinitely often except at most* 4 *directions.*

For a surface in R^m there is the following

THEOREM (*H. Fujimoto* [5]). *Let* S *be a complete minimal surface in* R^m *with nondegenerate Gauss map. Then the image of* S *under the Gauss map cannot fail to intersect more than* $m(m+1)/2$ *hyperplanes in general position in* CP^{m-1}.

And the result of this paper mentioned at the beginning of this section is the infinite covering property corresponding to the above theorem.

An oriented minimal surface S in R^m may be described by a conformal immersion

$$X\colon M \to R^m, \quad X = (x_1, \ldots, x_m),$$

where M is a Riemann surface and each x_k is a harmonic function on M.

By definition, the generalized Gauss map of S is the map that assigns to each point of S the tangent plane of S at that point. Because the tangent space of R^m at every point is naturally identified with R^m itself, the range of the Gauss map is the Grassmannian manifold consisting of all the oriented 2-subspaces of R^m. We can further identify the 2-plane spanned by the orthonormal basis X, Y with the line in C^m generated by $(X - iY)/2$. So the range of the Gauss map can be thought of as $P^{m-1}(C)$.

Let $z = u + iv$ be a holomorphic local coordinate of M. Denote

$$\frac{\partial x}{\partial z} = \frac{1}{2}\left(\frac{\partial x_1}{\partial u} - i\frac{\partial x_1}{\partial v}, \ldots, \frac{\partial x_m}{\partial u} - i\frac{\partial x_m}{\partial v}\right),$$

by $F = (f_0, \ldots, f_n)$, where $n = m - 1$; $f = (f_0 : f_1 : \cdots : f_n)$ is the point in CP^n represented by $(f_0, \ldots, f_n)$ in C^m. Then the

holomorphic map f represents the Gauss map, and the metric on M as a minimal surface is

$$ds^2 = 2|F(z)|^2|dz|^2,$$

where $|F|^2 = |f_0|^2 + \cdots + |f_n|^2$.

In this way, we turn the problem of the Gauss map partly into a problem on holomorphic curves. The value distribution property of the holomorphic curve may lead to corresponding results about the Gauss map.

In §2 we will summarize some of the basic ideas and notation of holomorphic curves. We will also introduce an important construction of Cowen and Griffiths [2] on holomorphic curves in CP^n which was the basis of their remarkable proof of Ahlfors' defect relation. In §3 we will present the proof of Fujimoto's inequality in such a way that will clarify the relation between Cowen and Griffiths' construction and Fujimoto's. Fujimoto's inequality is the key to both the proof of his theorem mentioned above and the proof of our result. In §4, we will give the proof of our result.

2. Some properties of holomorphic curves. Value distribution properties of holomorphic curves have been studied since the end of the 19th century. The central problem was to generalize the Picard theorem and the Nevanlinna defect relation for entire functions to the case of holomorphic curves. This was finally achieved in 1941 by L. Ahlfors, overcoming great technical difficulties.

In 1976, M. Cowen and P. Griffiths [2] gave a much simpler proof of Ahlfors' result using what they called a "negatively curved collection of metrics". Using their result, H. Fujimoto [5] was able to construct a single metric of negative curvature under certain conditions. Then by the Schwarz-Pick lemma, he derived an inequality which is the key to the study of the value distribution property of the Gauss map of minimal surface. In this section, we will give an outline of Cowen and Griffiths' result.

Let $\Delta_R = \{z \mid |z| < R\}$ be a disk in the complex plane, $f\colon \Delta_R \to P^n(C)$ be a holomorphic curve derived from a holomorphic map $F\colon \Delta_R \to C^{n+1}$ through homogeneous coordinates. $F(z) = (f_0(z), \ldots, f_n(z))$, $f_0, \ldots, f_n$ are holomorphic functions on Δ_R. We write $f = (f_0 : \cdots : f_n)$ and define $|F| = (\sum_{i=1}^n |f_i|^2)^{1/2}$; for our purposes, we assume that $|F| \neq 0$.

Take the l-th derivative:

$$F^{(l)}(z) = (f_0^{(l)}(z), \ldots, f_n^{(l)}(z)).$$

Define $F_k = F^{(0)} \wedge F^{(l)} \wedge \cdots \wedge F^{(k)} : \Delta_R \to \wedge^{k+1} C^{k+1} \subset G(n, k)$, where $G(n, k)$ is the Grassmannian manifold. By the Plücker embedding $G(n, k) \subset P^N(C)$, $N = \binom{n+1}{k+1} - 1$, F_k induces a map $f_k : \Delta_R \to P^N(C)$, called the kth derived curve of f.

We can define $|F_k|$ in a natural way. Let $e_0, \ldots, e_n$ be the standard basis of C^{n+1},

$$F(z) = F^{(0)} \wedge \cdots \wedge F^{(k)}(z) = \sum_{i_0 < \cdots i_k} F_{i_0 < \cdots < i_k} e_{i_0} \wedge \cdots \wedge e_{i_k},$$

and we define

$$|F_k(z)| = \left(\sum_{i_0 < \cdots < i_k} |F_{i_0 \cdots i_k}(z)|^2 \right)^{1/2}.$$

Now the Fubini-Study metrics on P^n and P^N naturally induce metrics on Δ_R by pulling back:

$$\Omega_0 = dd^c \log |F_0|^{1/2} = dd^c \log |F|^2,$$
$$\Omega_k = dd^c \log |F_k|^2, \qquad k = 1, \ldots, n,$$

where $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$. Because F_n is just a holomorphic function, $\Omega_n = 0$. We also set $|F_{-1}| = 1$ for convenience, so $\Omega_{-1} = 0$.

The metrics Ω_k will be used later to construct the negatively curved collection of metrics.

Let $a = (a_0, \ldots, a_n) \in C^{n+1}$, $|a| = (\sum_{i=1}^{n} |a_i|^2)^{1/2} = 1$. Then

$$a_0 w_0 + \cdots + a_n w_n = 0$$

defines a hyperplane A in both C^{n+1} and P_n, and

$$F(A) = a_0 f_0 + \cdots + a_n f_n$$

measures the distance from $F(z)$ to A; in a similar way

$$|F_k(A)|^2 = \sum_{i_1 < \cdots < i_k} \left| \sum_{j \neq i_1, \ldots, i_k} a_j F_{j_{i_1 \cdots i_k}} \right|^2$$

measures how far F_k is from A. Here $F_{j_{i_1 \cdots i_k}} = \text{sign}(\sigma) F_{j_0 \cdots j_k}$, σ is the permutation

$$\sigma = \begin{pmatrix} j & i_1 \cdots i_k \\ j_0 & j_1 \cdots j_k \end{pmatrix}.$$

In fact $|F_k(A)(z_0)| = 0$ means $F(z_0)$, $F^{(1)}(z_0)$, ..., $F^{(k)}(z_0)$ all lie in the hyperplane

$$a_0 w_0 + \cdots + a_n w_n = 0.$$

The corresponding quantities for the holomorphic curve f in P^n are

$$\phi_0(A) = \frac{|F(A)|^2}{|F_k|^2}, \qquad \phi_k(A) = \frac{|F_k(A)|^2}{|F_k|^2},$$

and if $\phi_k(A)(z_0) = 0$, the curve f is said to have contact of order $k+1$ with A at z_0.

Now if the holomorphic curve $f: \Delta_R \to P^n$ omits a certain number of hyperplanes $A_1, A_2, \ldots, A_q$, we want to construct a metric or a collection of metrics that is negatively curved.

If $n = 1$, $A_1, \ldots, A_q$ are points on P^1, we can just pull back the Poincaré metric of $P^1 - \{A_1, \ldots, A_q\}$. To be more explicit, let us take a local coordinate ζ of P^1 around a neighborhood of A_1 (or any other A_i, $i = 1, \ldots, q$), with $\zeta = 0$ at A^1. Then the Poincaré metric is asymptotically

$$\frac{d\zeta \wedge d\bar{\zeta}}{|\zeta|^2 \log^2(1/|\zeta|^2)}$$

around the point A_1. Cowen and Griffiths [2] found a way to generalize this construction to the case when $n \geq 2$. In that case, it becomes necessary to consider not only f but all of its derived curves f_k. The quantity $|\zeta|^2$ for A_1 will be replaced by $\phi_k(A_1)$ as defined above.

Let $\omega = (\sqrt{-1}/2\pi) h(z) \, dz \wedge d\bar{z}$ be a metric. Then the Ricci form is defined by $\mathrm{Ric}\, \omega = dd^c \log h(z)$, and $\mathrm{Ric}\, \omega \geq \omega$ is equivalent to the fact that the curvature of ω is less than -1.

Let $A_1, \ldots, A_q$ be hyperplanes in general position in P^n and $q \geq n + 2$. For $i = 0, \ldots, n-1$, following the indication of the Poincaré metric, define

$$\omega_i = c_i \prod_{\nu=1}^{q} \left(\frac{\phi_{i+1}(A_\nu)}{\phi_i(A_\nu) \log^2(\mu/\phi_i(A_\nu))} \right)^{1/(n-1)} \Omega_i.$$

Cowen and Griffiths [2] proved the following

PROPOSITION. *Given $\varepsilon > 0$, for a suitable choice of constants c_i, and μ, we have*

$$\sum_{i=0}^{n-1} (n-i) \mathrm{Ric}\, \omega_i \geq (q - (n+1)) \Omega_0 + \sum_{i=0}^{n-1} \omega_i - \varepsilon \left(\sum_{i=0}^{n-1} \Omega_i \right).$$

Aside from the term with the ε, this inequality illustrates what is meant by saying that the collection of metrics $\{\omega_i\}$ is negatively curved. Based on this, Fujimoto constructed a single metric with negative curvature under some additional assumptions. The next section will give a detailed presentation of Fujimoto's construction.

3. Fujimoto's inequality. This section will be centered around curvature computations. For this purpose, a few lemmas from [2] are collected here for convenience.

We have defined $\Omega_k = dd^c \log |F_k|^2$,

LEMMA 1.
$$\Omega_k = \frac{\sqrt{-1}}{2\pi} \frac{|F_{k-1}|^2 |F_{k+1}|^2}{|F_k|^4} \, dz \wedge d\bar{z}.$$

LEMMA 2. *Define*
$$h_k = \frac{|F_{k-1}^2| \, |F_{k+1}|^2}{|F_k|^4} \, ;$$

then
$$\operatorname{Ric} \Omega_k = dd^c \log h_k = \Omega_{k+1} + \Omega_{k-1} - 2\Omega_k.$$

In the process of computation, we will use these two lemmas whenever necessary without referring to them explicitly.

To help understanding, we give here an outline of the idea of the proof of this section. The motivation is to construct a single metric of negative curvature out of a collection of negatively curved metrics. Let $\omega_i = (\sqrt{-1}/2\pi)h_i(z) \, dz \wedge d\bar{z}$, and suppose
$$\sum \operatorname{Ric} \omega_i \geq \sum \omega_i.$$

Then
$$\sum_i dd^c \log h_i \geq \sum h_i dz \wedge d\bar{z},$$

$$dd^c \log \left(\prod h_i \right) \geq \left(\sum h_i \right) dz \wedge d\bar{z} \geq n \left(\prod h_i \right)^{1/n} dz \wedge d\bar{z},$$

$$dd^c \log \left(\prod h_i \right)^{1/n} \geq \left(\prod h_i \right)^{1/n} dz \wedge d\bar{z},$$

so $\omega = (\prod h_i)^{1/n} dz \wedge d\bar{z}$ satisfies $\operatorname{Ric} \omega \geq \omega$ and ω is the desired metric. In our situation, there are two other factors that complicate the proof. One is that in the proposition of the last section, the collection of metrics is not strictly negatively curved; the term with ε will cause some complications. The other factor is that there are many

computations and cancellations due to the special form of metrics that we have. Let us start with the inequality

$$\sum_{i=0}^{n-1}(n-i)\mathrm{Ric}\,\omega_i \geq (q-(n+1))\Omega_0 + \sum_{i=0}^{n-1}\omega_i - \varepsilon\left(\sum_{i=0}^{n-1}\Omega_i\right),$$

where

$$\omega_i = c_i \prod_{\nu=1}^{q}\left(\frac{\phi_{i+1}(A_\nu)}{\phi_i(A_\nu)\log^2(\mu/\phi_i(A_\nu))}\right)^{1(n-1)}\Omega_i.$$

We want to compute each term of the inequality explicitly.

Step 1.

$$\sum_{i=0}^{n-1}(n-i)\mathrm{Ric}\,\omega_i$$

$$= \sum_{i=0}^{n-1}(n-i)dd^c\log\prod_{\nu=1}^{q}\left(\frac{\phi_{i+1}(A_\nu)}{\phi_i(A_\nu)\log^2(\mu/\phi_i(A_\nu))}\right)^{1/(n-1)}$$

$$+ \sum_{i=0}^{n-1}(n-i)\mathrm{Ric}\,\Omega_i$$

$$= \sum_{i=0}^{n-1}dd^c\log\prod_{\nu=1}^{q}\frac{\phi_{i+1}(A_\nu)}{\phi_i(A_\nu)\log^2(\mu/\phi_i(A_\nu))}$$

$$+ \sum_{i=0}^{n-1}(\Omega_{i+1} + \Omega_{i-1} - 2\Omega_i)$$

$$= dd^c\prod_{\nu=1}^{q}\frac{\phi_n(A_\nu)}{\phi_0(A_\nu)\prod_i\log^2(\mu/\phi_i(A_\nu))} - (n+1)\Omega_0;$$

but $\phi_0(A_\nu) = 1$, $\phi_0(A_\nu) = |F(A_\nu)|^2/|F|^2$, so

$$\sum_{i=0}^{n-1}(n-i)\mathrm{Ric}\,\omega_i = dd^c\log\prod_{\nu=1}^{q}\left(\frac{|F|^2}{|F(A_\nu)|^2\prod_i\log^2(\mu/\phi_i(A_\nu))}\right)$$

$$- (n+1)dd^c\log|F|^2$$

$$= dd^c\log\left(\frac{|F|^{2q}}{\prod_{\nu=1}^{q}(|F(A_\nu)|^2\prod_i\log^2(\mu/\phi_i(A_\nu)))}\right)$$

$$- dd^c\log|F|^{2(n+1)}$$

$$= dd^c\log\left(\frac{|F|^{2(q-(n+1))}}{\prod_{\nu=1}^{q}(|F(A_\nu)|^2\prod_i\log^2(\mu/\phi_i(A_\nu)))}\right).$$

Step 2.

$$\sum_{i=0}^{n-1} \omega_i = \sum_{i=0}^{n-1} c_i \prod_{\nu=1}^{q} \left(\frac{\phi_{i+1}(A_\nu)}{\phi_i(A_\nu) \log^2(\mu/\phi_i(A_\nu))} \right)^{1(n-1)} \Omega_i$$

$$= \sum_{i=0}^{n-1} c_i \prod_{\nu=1}^{q} \left(\frac{\phi_{i+1}(A_\nu)}{\phi_i(A_\nu) \log^2(\mu/\phi_i(A_\nu))} h_i^{n-i} \right)^{1/(n-1)} dz \wedge d^c z,$$

where $\Omega_i = h_i dz \wedge d^c z$. Using the inequality

$$a_1 x_1 + \cdots + a_n x_n \geq (a_1 + \cdots + a_n)(x_1^{a_1} \cdots x_n^{a_n})^{1/(a_1 + \cdots + a_n)}$$

with $a_i = n - i$, $\sum_{i=0}^{n-1} a_i = n(n+1)/2$, we have

$$\sum_{i=0}^{n-1} \omega_i \geq C \left(\prod_{i=1}^{n-1} \left(\prod_{\nu=1}^{q} \left(\frac{\phi_{i+1}(A_\nu)}{\phi_i(A_\nu) \log^2(\mu/\phi_i(A_\nu))} \right) h_i^{n-i} \right)^{2/n(n+1)} \right) dz \wedge d^c z$$

$$= C \left(\prod_{\nu=1}^{q} \left(\frac{\phi_n(A_\nu)}{\phi_0(A_\nu) \prod_{i=1}^{n-1} \log^2(\mu/\phi_i(A_\nu))} \right) \right.$$

$$\left. \cdot \prod_{i=0}^{n-1} \left(\frac{|F_{i-1}|^2 |F_{i+1}|^2}{|F_i|^4} \right)^{n-1} \right)^{2/n(n+1)}$$

$$= C \left(\prod_{\nu=1}^{q} \left(\frac{|F|^2}{|F(A_\nu)|^2 \prod_{i=1}^{n-1} \log^2(\mu/\phi_i(A_\nu))} \right) \right.$$

$$\left. \cdot \prod_{i=0}^{n-1} \left(\frac{|F_n|^2}{|F_0|^{2(n+1)}} \right) \right)^{2/n(n+1)},$$

but $|F_0| = |F|$, so

$$\sum_{i=0}^{n-1} \omega_i \geq C \left(\frac{|F|^{2(q-(n+1))} |F_n|^2}{\prod_{\nu=1}^{q} (|F(A_\nu)|^2 \prod_{i=1}^{n-1} \log^2(\mu/\phi_i(A_\nu)))} \right)^{2/n(n+1)}.$$

Step 3.

$$\varepsilon \left(\sum_{i=1}^{n-1} \Omega_i \right) = \varepsilon \sum_{i=1}^{n-1} dd^c \log |F_k|^2 = dd^c \log |F_o|^{2\varepsilon} \cdots |F_{n-1}|^{2\varepsilon},$$

$$(q - (n+1))\Omega_0 = (q - (n+1)) dd^c \log |F_0|^2 = dd^c \log |F|^{2(q-(n+1))}.$$

Step 4. Combining the results of Steps 1, 2, 3, we have

$$dd^c \log \frac{|F_0|^{2\varepsilon} \cdots |F_{n-1}|^{2\varepsilon}}{\prod_{\nu=1}^q (|F(A_\nu)|^2 \prod_{i=1}^{n-1} \log^2(\mu/\phi_i(A_\nu)))}$$

$$\geq C \left(\frac{|F|^{2(q-(n+1))}|F_n|^2}{\prod_{\nu=1}^q (|F(A_\nu)|^2 \prod_{i=1}^{n-1} \log^2(\mu/\phi_i(A_\nu)))} \right)^{2/n(n+1)} dz \wedge d^c z.$$

Setting $G = \prod_{\nu=1}^q (|F(A_\nu)|^2 \prod_{i=1}^{n-1} \log^2(\mu/\phi_i(A_\nu)))$, we have

$$dd^c \log |F_0|^{2\varepsilon} \cdots |F_{n-1}|^{2\varepsilon} + dd^c \log(1/G^2)$$

$$\geq C \left(\frac{|F|^{2(q-(n+1))}|F_n|^2}{G^2} \right)^{2/n(n+1)} dz \wedge d^c z.$$

Step 5. Notice that F_n is a holomorphic function, so $dd^c \log |F| = 0$; also $\log |F|^2$ is subharmonic, so $dd^c \log |F|^2$, the -4ε in the exponent is necessary and we will see the reason in the arguments later. With $\eta = (|F|^{2(q-(n+1))}|F_n|^2)/G^2$, we have

$$\varepsilon dd^c \log |F_0|^2 \cdots |F_{n-1}|^2 + dd^c \log \frac{\eta}{|F|^{4\varepsilon P_{n+1}}} \geq C\eta^{2/n(n+1)} dz \wedge d^c z.$$

Step 6. Let $P_n = n(n+1)/2$, $Q_n = \sum_{k=1}^n P_k$. Then

$$P_n dd^c \log |F_0|^2 \cdots |F_{n-1}|^2$$

$$\geq \left(P_n \frac{|F_1|^2}{|F_0|^4} + P_{n-1} \frac{|F_0|^2|F_2|^2}{|F_1|^4} + \cdots + P_1 \frac{|F_{n-2}|^2|F_n|^2}{|F_{n-1}|^4} \right) dz \wedge d^c z$$

$$\geq Q_n \left(\left(\frac{|F_1|^2}{|F_0|^4} \right)^{P_n} \left(\frac{|F_0|^2|F_2|^2}{|F_1|^4} \right)^{P_{n-1}} \right.$$

$$\left. \cdots \left(\frac{|F_{n-2}|^2|F_n|^2}{|F_{n-1}|^4} \right)^{P_1} \right)^{1/Q_n} dz \wedge d^c z$$

$$= Q_n \left(\frac{|F_1|^2 \cdots |F_{n-1}|^2|F_n|^2}{|F_0|^{n^2+3n}} \right)^{1/Q_n} dz \wedge d^c z,$$

so

$$\varepsilon dd^c \log |F_0|^2 \cdots |F_{n-1}|^2 \geq \varepsilon \frac{Q_n}{P_n} \left(\frac{|F_0|^2 \cdots |F_n|^2}{|F_0|^{2P_{n+1}}} \right)^{1/Q_n} dz \wedge d^c z.$$

Step 7. Add up the results of Steps 5 and 6, replace the ε (which is arbitrary) with $\varepsilon/2$ and notice that $dd^c \log|F_n|^2 = 0$, we have

$$\varepsilon dd^c \log|F_0|^2 \cdots |F_n|^2 + dd^c \log \frac{\eta}{|F|^{2\varepsilon P_{n+1}}}$$

$$\geq \left(C\eta^{1/P_n} + \frac{\varepsilon}{2}\frac{Q_n}{P_n} \left(\frac{|F_0|^2 \cdots |F_n|^2}{|F_0|^{2P_{n+1}}} \right)^{1/Q_n} \right) dz \wedge d^c z,$$

using $a_1 x_1 + a_2 x_2 \geq (a_1 + a_2)(x_1^{a_1} x_2^{a_2})^{1/(a_1 + a_2)}$ with $a_1 = P_n$, $a_2 = \varepsilon Q_n$, we have

$$dd^c \log \frac{|F_0|^{2\varepsilon} \cdots |F_n|^{2\varepsilon}}{|F|^{2\varepsilon P_{n+1}}} \eta \geq C_1 \left(\frac{|F_0|^{2\varepsilon} \cdots |F_n|^{2\varepsilon} \eta}{|F|^{2\varepsilon P_{n+1}}} \right)^{1/(P_n + \varepsilon Q_n)} dz \wedge d^c z.$$

Set

$$h = \left(\frac{|F_0|^{2\varepsilon} \cdots |F_n|^{2\varepsilon} \eta}{|F|^{2\varepsilon M_{n+1}}} \right)^{1/(P_n + \varepsilon Q_n)};$$

then

$$dd^c h \geq C_2 h \, dz \wedge d^c z,$$

so $h \, dz \wedge d^c <$ is the desired metric.

Step 8. By the Schwarz-Pick lemma, we have a constant C_3 such that

$$h(z) \leq C_3 \frac{2R}{R^2 - |z|^2},$$

where $\frac{2R}{R^2 - |z|^2} dz \wedge d^c z$ is the Poincaré metric of the disk $\{z | |z| < R\}$. Writing out everything explicitly, we have

$$\frac{|F|^{q - (n+1) - \varepsilon P_{n+1}} |F_0|^\varepsilon \cdots |F_{n-1}|^\varepsilon |F_n|^\varepsilon}{\prod_{\nu=1}^q \prod_{i=1}^n \log(\mu/\phi_i(A_\nu))} \leq C_4 \left(\frac{2R}{R^2 - |z|^2} \right)^{P_n + Q_n \varepsilon}.$$

Step 9. We would like to get rid of the log terms. Knowing that

$$K = \sup_{0 \leq x \leq 1} x^{\varepsilon/2q} \log \frac{\mu}{x} < +\infty \quad \text{for } \mu > 1,$$

we have

$$\frac{1}{\log(\mu/\phi_k(A_\nu))} \geq \frac{1}{K} \phi_k(A_\nu)^{\varepsilon/2q} = \frac{1}{K} \frac{|F_k(A_\nu)|^{\varepsilon/q}}{|F_k|^{\varepsilon/q}},$$

substituting this into the result of Step 8, we have

PROPOSITION (*Fujimoto's inequality* [5]). *Let* $\Delta_R = \{z | |z| < R\}$ *be a disk in the complex plane,* $f : \Delta_R \to CP^n$ *be a holomorphic curve*

derived from a holomorphic map $F: \Delta_R \to C^{n+1}$, *using the notations introduced in the previous section, we have the following statement. For any* $\varepsilon > 0$, *there is a* $C > 0$, *such that*

$$\frac{|F|^{q-(n+1)-\varepsilon P_{n+1}} \prod_{i=1}^{n} (\prod_{\nu=1}^{q} |F_k(A_\nu)|)^{\varepsilon/q} |F_n|^{1+\varepsilon}}{\prod_{\nu=1}^{q} |F(A_\nu)|} \le C \left(\frac{2R}{R^2 - |z|^2}\right)^{P_n + Q_n \varepsilon}.$$

4. Minimal surfaces in R^m. We assume that all surfaces are orientable, since analogous theorems for non-orientable surfaces are easily formulated by taking the two sheeted orientable covering surface and applying the theorem to it. Following the notation of the previous section, we will prove the following

THEOREM. *Let S be a complete non-degenerate minimal surface in R^m such that the Gauss map $f = (f_0: \cdots : f_n)$ (here $n = m = 1$) intersects only a finite number of times the hyperplanes $A_1, \ldots, A_q$ (in CP^n) in general position. If $q > m(m + 1)/2 = (n + 1) + n(n + 1)/2$, then S must have finite total curvature.*

REMARK. If S is a generalized minimal surface with a finite number of branch points, all the arguments of our proof will not be affected. So the theorem is also true for the somewhat more general class of surfaces. This also applies to the similar theorem for surfaces in R^3 by Mo and Osserman [8].

It was already observed by Osserman (see R. Osserman, *A survey of minimal surfaces*, second edition, 1986, p. 73) that his classic results on the value distribution of Gauss map is true for simply connected surfaces with a finite number of branch points. An observation of Ahlfors implies that they are still true if a certain condition on the distribution of the branching points is satisfied. But there exist complete generalized minimal surfaces in R^3, not lying in a plane, whose Gauss map lies in an arbitrarily small neighborhood on the sphere. So the results are not true for arbitrary generalized minimal surfaces. The method of our proof is similar to the method of [8].

Proof. Step 1. Since f is non-degenerate, none of the $F_k(A_\nu)$ vanishes identically, where $\nu = 1, \ldots, q$, $k = 0, \ldots, n$. Let A be given by the equations

$$a_{\nu_0} z_0 + \cdots + a_{\nu_n} z_n = 0,$$

$$F_k = \sum_{i_0 < \cdots < i_k} F_{i_0 \cdots i_k} e_{i_0} \wedge \cdots \wedge e_{i_k},$$

$$F_{j_0 \cdots j_k} = \text{sign} \begin{pmatrix} i_0 \cdots i_k \\ j_0 \cdots j_k \end{pmatrix} F_{i_0 \cdots i_k},$$

then for each pair (ν, k), there is $i_1, \ldots, i_k$ such that

$$\psi_{\nu k} = \sum_{l \neq i_1, \ldots, i_k} a_{\nu l} F_{l_{i_1 \cdots i_k}}$$

does not vanish identically. Apparently $\psi_{\nu_0} = F(A_\nu)$, $\psi_{\nu n} = F_n$. Every $\psi_{\nu k}$ is holomorphic, so they have only isolated zeros.

Step 2. The hypothesis of the theorem implies that outside of a compact set D in S, f does not intersect any of the $A_1, \ldots, A_q$; therefore $F(A_\nu) \neq 0$. Let

$$S' = \{p \in S \backslash D : \psi_{\nu k} \neq 0 \text{ for any } (\nu, k)\}.$$

On S' we define a new metric

$$d\tilde{s}^2 = \left| \frac{\prod_{\nu=1}^q F(A_\nu)}{|F_n|^{1+\varepsilon} \prod_{\nu, k} |\psi_{\nu k}|^{\varepsilon/q}} \right|^{2p^*} |dz|^2$$

where

$$p^* = \frac{1}{(q - (n+1) - P_{n+1}\varepsilon) - (P_n + Q_n\varepsilon)},$$

$$\frac{q - (n+1) - P_n}{P_{n+1} + Q_n} > \varepsilon > \frac{q - (n+1) - Q_n}{P_{n+1} + Q_n + 1/q};$$

the last inequality is equivalent to $\varepsilon p^*/q > 1$.

Here the definition of $d\tilde{s}^2$ would be valid if S' has a global coordinate z. Take a hyperplane A (out of $A_1, \ldots, A_q$). Then on S', f does not intersect A, namely

$$a_0 \frac{\partial x_1}{\partial z} + \cdots + a_n \frac{\partial x_n}{\partial z} \neq 0;$$

this means that if $\xi = a_0 x_1(z) + \cdots + a_m x_m(z)$ is a global coordinate on S', call it z, then $d\tilde{s}^2$ is well defined.

Step 3. Since $F(A_\nu)$, F_n and $\psi_{\nu k}$ are all holomorphic, the metric $d\tilde{s}^2$ is flat, and it can be smoothly extended over D. We thus obtain a metric, still call it $d\tilde{s}^2$, on

$$S'' = S' \cup D$$

that is flat outside the compact set D. The key to our proof is showing that S'' is complete in that metric.

Step 4. We proceed by contradiction. If S'' is not complete, then there is a divergent curve $\gamma(t)$ on S'' with finite length. By removing an initial segment, if necessary, we may assume that there is a positive distance d between the curve γ and the compact set D. Thus $\gamma\colon [0, 1) \to S'$, and since γ is divergent on S'', with finite length, it follows that from the point of view of S, either $\gamma(t)$ tends to a point z_0 where

$$|F_n|^{1+\varepsilon} \prod_{\nu,k} |\psi_{\nu k}|^{\varepsilon/q} = 0,$$

or else $\gamma(t)$ tends to the boundary of S as $t \to 1$. But the former case cannot occur, because if

$$|F_n(z_0)|^{1+\varepsilon} \prod_{\nu,k} |\psi_{\nu k}(z_0)|^{\varepsilon/q} = 0,$$

then by the fact that $\varepsilon p^*/q > 1$ (here q is the number of hyperplanes) we have

$$|d\tilde{s}| \sim \frac{c}{|z - z_0|^{\delta_0}} dz$$

around z_0 where $c > 0$, $\delta_0 > 1$. Thus

$$\int_0^1 d\tilde{s} = \infty,$$

contradicting the finite length of γ.

Step 5. We conclude that $\gamma(t)$ must tend to the boundary of S when $t \to 1$. Choose t_0 such that

$$\int_{t_0}^1 d\tilde{s} < \frac{d}{3};$$

that is, the length of $\gamma([t_0, 1))$ is less than $d/3$. Consider a small disk Δ with center $\gamma(t_0)$. Since $d\tilde{s}^2$ is flat, Δ is isometric to an ordinary disk in the plane. Let G be an isometry of $|w| < \eta$ onto Δ with $G(0) = \gamma(t_0)$. Extend G, as a local isometry into S', to the largest disk possible, say $|w| < R$. (Note that G may be viewed simply as the exponential map to S'' at $\gamma(t_0)$.) In view of $\int_{t_0}^1 d\tilde{s} < \frac{d}{3}$, and the fact that γ is a divergent curve on S, we have $R \leq d/3$. Hence the image under G must be bounded away from D by a distance of at least $2d/3$. Thus, the reason that the map G cannot be extended to a larger disk must be that the image goes to the boundary of S''. Since the zeros of $|F_n|^{1+\varepsilon} \prod_{\nu,k} |\psi_{\nu k}|^{\varepsilon/q}$ have been shown to be infinitely far away in the metric, the image must actually go to the boundary

of S. More specifically, there must be a point w_0 with $|w_0| = r$, such that the image under G of the line segment from 0 to w_0 is a divergent curve Γ on S. Our goal is to show that Γ has finite length in the *original* metric ds^2 on S, contradicting the completeness of the original surface.

Step 6. We know that

$$|dw| = |d\tilde{s}| = \left| \frac{\prod_{\nu=1}^{q} F(A_\nu)}{|F_n|^{1+\varepsilon} \prod_{\nu,k} |\psi_{\nu k}|^{\varepsilon/q}} \right|^{p^*} |dz|.$$

Instead of z, we change to the coordinate w for the right-hand side of the above expression. Precisely speaking, we let

$$\overline{F}(w) = (\overline{f}_0(w), \dots, \overline{f}_n(w)) = (f_0(z(w)), \dots, f_n(z(w))) = F(z(w)),$$

and let $\overline{\psi}_{\nu k}(w)$ be defined from $\overline{F}(z)$ in the same way the $\psi_{\nu k}$ was defined from $F(z)$. Then a little computation shows that

$$\left| \frac{dw}{dz} \right| = \left| \frac{\prod_{\nu=1}^{q} \overline{F}(A_\nu)}{|\overline{f}_n (\frac{dw}{dz})^{P_N}|^{1+\varepsilon} \prod_{\nu,k} |\overline{\psi}_{\nu k}(\frac{dw}{dz})^{P_k}|^{\varepsilon/q}} \right|^{P^*}$$

$$= \left| \frac{\prod_{\nu=1}^{q} \overline{F}(A_\nu)}{|\overline{F}_n|^{1+\varepsilon} \prod_{\nu,k} |\overline{\psi}_{\nu k}|^{\varepsilon/q}} \right|^{p^*} \frac{1}{|\frac{dw}{dz}|^{p^*(P_n+\varepsilon Q_n)}},$$

$$\left| \frac{dw}{dz} \right|^{1+p^*(P_n+\varepsilon Q_n)} = \left| \frac{\prod_{\nu=1}^{q} \overline{F}(A_\nu)}{|\overline{F}_n|^{1+\varepsilon} \prod_{\nu,k} |\overline{\psi}_{\nu k}|^{\varepsilon/q}} \right|^{p^*}$$

by

$$p^* = \frac{1}{(q-(n+1)-P_{n+1}\varepsilon) - (P_n+Q_n\varepsilon)};$$

we have

$$\left| \frac{dw}{dz} \right| = \left| \frac{\prod_{\nu=1}^{q} \overline{F}(A_\nu)}{|\overline{F}_{n+1}|^{1+\varepsilon} \prod_{\nu,k} |\overline{\psi}_{\nu k}|^{\varepsilon/q}} \right|^{1/(q-(n+1)-P_{n+1}\varepsilon)}.$$

Step 7. We now denote by C the line segment from 0 to w_0, and by Γ, the image of C on S. Then for the length L of Γ, we have

$$L = 2 \int_C |F(z(w))| \, |dz(w)|$$

$$= 2 \int_C |\overline{F}(w)| \left| \frac{dz}{dw} \right| |dw|$$

$$= 2 \int \left| \frac{|\overline{F}|^{q-(n+1)-P_{n+1}\varepsilon} |\overline{F}_{n+1}|^{1+\varepsilon} \prod_{\nu,k} |\overline{\psi}_{\nu k}|^{\varepsilon/q}}{\prod_{\nu=1}^{q} |\overline{F}(A_\nu)|} \right|^{1/(q-(n+1)-\varepsilon P_{n+1})}.$$

By the definition of $\overline{\psi}_{\nu k}$, $|\overline{\psi}_{\nu k}| \leq |\overline{F}_k(A_\nu)|$, and using the proposition of the previous section, the Fujimoto inequality, we have

$$L \leq 2C \int \left(\frac{2R}{R^2 - |w|^2} \right)^{(P_n + Q_n \varepsilon)/(q - (n+1) - P_{n+1}\varepsilon)} |dw|.$$

Because $0 < (P_n + Q_n\varepsilon)/(q - (n+1) - Q_{n+1}\varepsilon) < 1$, L is finite.

Step 8. To sum up, we have shown that if the surface S'' were not complete, then we could find a divergent curve on S with finite length in the original metric, so that S would not be complete. We therefore conclude that S'' is complete. Since the metric on S'' is flat outside of a compact set, we are in a familiar situation (see [11] p. 3564, or Osserman, *A survey...*, p. 81). By a theorem of Huber [7], the fact that S'' has finite total curvature implies that S'' is finitely connected. We conclude first that $|F_n|^{1+\varepsilon} \prod_{\nu,k} |\psi_{\nu k}|^{\varepsilon/q}$ can have only a finite number of zeros, and second, that the original surface S is finitely connected. Further, by [10, Theorem 2.1] (or the argument in [11, pp. 354]) each annular end of S'', hence of S, is conformally equivalent to a punctured disk. Thus, the Riemann surface M on which S is based must be conformally equivalent to a compact Riemann surface $\overline{M}$ with a finite number of points removed. In a neighborhood of each of those points the Gauss map f does not intersect $q \geq n(n-1)/2 + 1 \geq n + 2$ hyperplanes. By a generalized Picard theorem (see [2, p. 136]), the Gauss map f can be extended to a holomorphic map from $\overline{M}$ to $P^n(C)$. If the homology class represented by the image of $f: \overline{M} \to P^n(C)$ is m times the fundamental homology class of $P^n(C)$, then we have

$$\iint K \, dA = -2\pi m$$

as the total curvature of S. This proves the theorem.

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