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A TRANSVERSE STRUCTURE FOR THE LIE-POISSON BRACKET ON THE DUAL OF THE VIRASORO ALGEBRA

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KdV equations can be described as Hamiltonian systems on the dual of the Virasoro algebra with the canonical Lie-Poisson (also called Berezin-Kirillov-Kostant) bracket. In this paper we give an explicit transverse structure for this Poisson manifold along a finite dimensional submanifold. The structure is linearizable and equivalent to the Lie-Poisson structure on $\mathrm{sl}(2, \mathbf{R})^*$. This problem is closely related to the classification of Hill's equations.

1. Introduction and main definitions. It was known since Lie's time that if a manifold has a Poisson structure and the rank of the Poisson tensor is constant around a point (that is, the point is *regular*), then the manifold can be locally described at such a point as foliated into leaves of maximum rank or *symplectic* leaves. If the Poisson manifold is the dual of a Lie algebra with its *Lie-Poisson bracket*, then the symplectic leaves coincide with the orbits under the coadjoint action of the group. If the point is singular the local description can be achieved by finding a section which is transversal to the orbit of the point and which is endowed with a Poisson structure induced by the global Poisson bracket. This induced bracket, or *transverse structure*, was initially introduced by A. Weinstein for finite dimensional Poisson manifolds (see [**20**]) and it describes the relation between the symplectic structures on the different leaves as we cross them transversally to the orbit of a singular point. Weinstein also proved that transverse structures were unique in the following sense: if we have two sections transversal to the orbit of a singular point with Poisson brackets induced on them and with dimensions equal to the codimension of the orbit, then there exists a Poisson isomorphism of the manifold, defined between two neighbourhoods of the intersections with the orbit, which will clearly preserve the two transverse structures.

The aim of this paper is to show the geometrical description of the coadjoint orbits on the dual of the Virasoro algebra as we move transversally through them and to use this description to find an explicit transverse structure for its Lie-Poisson bracket. Descriptions and classifications of the coadjoint orbits have been given by different authors (see **[8]**, **[9]**, **[17]**, or **[21]**). The problem is closely related to finding normal forms for Hill's equations as we will see later.

In §2 we try to find a suitable transversal section in which we will define our structure. This direction might have a role in this work. In the case of the Virasoro algebra (a Frechèt manifold) there are no results on uniqueness of transverse structures available to us, so that, in principle, the transverse structure we get might not have been canonically chosen. We will show that it is enough to describe a transverse structure for constant potentials of the form $p = \frac{n^2}{2}$, for all integers n (these are analogous to *singular* points in finite dimensions). An orbit that does not contain such a potential will automatically possess a trivial transverse structure (potentials on these orbits are analogous to finite dimensional *regular* points). A direction transversal to an orbit which goes through a potential of the form $p = \frac{n^2}{2}$ is given by a 3-dimensional submanifold which is isomorphic to $\mathrm{sl}(2, \mathbf{R})$. In §3 we find a transverse structure along that section and we show how, although it is nonlinear, it can be linearized along the submanifold and therefore it is equivalent to the standard Lie-Poisson structure on $\mathrm{sl}(2, \mathbf{R})^*$. We also discuss how this fact does not imply a uniqueness result. The definition of transverse structure is also revised, to make it easier to adapt to the infinite dimensional case.

In the last section we provide an expression for the Taylor expansion of the transverse structure in terms of the even moments corresponding to a certain moment functional. This linear functional is defined as follows: the symplectic structure on the intersection of the coadjoint orbits with the transverse section can be, in some sense, represented by a Jacobi matrix. There exists a Jacobi fraction (continued fraction) associated to such a matrix and its corresponding partial denominators can be described as orthogonal polynomials with respect to certain discrete measure. The linear functional we are looking for is given by integrating against that measure.

Finally we show how the transverse structure can also be expressed in terms of the Fourier coefficients of a periodic solution of a nonhomogeneous equation whose homogeneous part is given by the coadjoint action of the algebra on its dual.

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Central extensions: The Virasoro algebra and the Lie-Poisson structure on its dual.

Let G be a Lie group, g its associated Lie algebra and g^* the dual space of g. We define the *Chevalley-Eilenberg complex* associated to a representation (V, ρ) for g, as the chain complex given by

$$\cdots \longrightarrow V \otimes \Lambda^1 g^* \xrightarrow{\delta_1} V \otimes \Lambda^2 g^* \xrightarrow{\delta_2} V \otimes \Lambda^3 g^* \longrightarrow \cdots$$

with the coboundaries defined as

$$\delta_1(\alpha)(\xi_1 \wedge \xi_2) = \rho(\xi_1)\alpha(\xi_2) - \rho(\xi_2)\alpha(\xi_1) - \alpha([\xi_1, \xi_2]),$$
$$\delta_2(\beta)(\xi_1 \wedge \xi_2 \wedge \xi_3) = \sum_{\sigma \in A_3} \rho(\xi_{\sigma_1})\beta(\xi_{\sigma_2} \wedge \xi_{\sigma_3}) + \sum_{\sigma \in A_3} \beta(\xi_{\sigma_1} \wedge [\xi_{\sigma_2}, \xi_{\sigma_3}]),$$

and where $\xi_1, \xi_2, \xi_3 \in g$, $[\ ,\ ]$ is the Lie bracket in the algebra, and A_3 is the space of cyclic permutations of $\{1, 2, 3\}$.

In particular, if $(V, \rho) = (\mathbf{R}, 0)$, the conditions above become

$$\delta_1(\alpha)(\xi_1 \wedge \xi_2) = -\alpha([\xi_1, \xi_2]),$$
$$\delta_2(\beta)(\xi_1 \wedge \xi_2 \wedge \xi_3) = \sum_{\sigma \in A_3} \beta(\xi_{\sigma_1} \wedge [\tau_{\sigma_2}, \tau_{\sigma_3}]).$$

We will denote by $H^2(g, (V, \rho)) \equiv H^2(g, V)$ the second cohomology group associated to the Chevalley-Eilenberg complex.

Given a nontrivial 2-cocycle $c \in H^2(g, \mathbf{R})$, define the Lie algebra $g_0 = g \oplus \mathbf{R}$ with Lie bracket

$$[(\xi, t), (\mu, s)]_0 = ([\xi, \mu], c(\xi, \mu)).$$

g_0 is called a *central extension* for g.

Let S^1 be the unit circle and G be the group of diffeomorphisms of S^1, $\mathrm{diff}(S^1)$, with the composition $\circ$ as the operation of the group. We can naturally identify g with the space of vector fields of the circle, $\mathrm{vect}(S^1)$ (for more information about infinite dimensional Lie algebras see **[14]** and **[16]**). The Lie bracket on g is given by the usual commutator

$$\left[\xi(\theta)\frac{\partial}{\partial\theta}, \eta(\theta)\frac{\partial}{\partial\theta}\right] = (\xi\eta' - \xi'\eta)\frac{\partial}{\partial\theta},$$

and the adjoint action of the group is carried out through a simple change of variables in the vector field, $\mathrm{Ad}(\phi)(\xi(\theta)\frac{\partial}{\partial\theta}) = (\phi'\xi)\circ\phi^{-1}\frac{\partial}{\partial\theta}$.

On the other hand, g^* can be identified with the space of 2-tensors on S^1 acting as

$$\left\langle p(\theta)\, d\theta \otimes d\theta\,,\, \xi(\theta)\frac{\partial}{\partial\theta}\right\rangle = \int_0^{2\pi} p(\theta)\xi(\theta)\, d\theta\,,$$

and the coadjoint action of the group is then given by $\mathrm{Ad}(\phi)(p(\theta)\, d\theta^2) = \frac{p}{\phi'^2}\circ\phi^{-1}\, d\theta^2$, which is the usual change of variable for 2-tensors.

It is known that $H^2(\mathrm{vect}(S^1)\,,\,\mathbf{R})\cong\mathbf{R}$ and a generator is given by

$$c\left(\xi\frac{\partial}{\partial\theta}\,,\,\mu\frac{\partial}{\partial\theta}\right) = \int_0^{2\pi}\xi'\mu''\, d\theta = \int_0^{2\pi}\xi'''\mu\, d\theta\,.$$

c is the so-called *Gelfand-Fuks cocycle.*

In the case when $g=\mathrm{vect}(S^1)$ and c is the Gelfand-Fuks cocycle, the central extension g_0 is called the *Virasoro algebra.*

c can be integrated to a cocycle in the group

$$B(\varphi\,,\,\phi) = \int_0^{2\pi}[\ln(\varphi\circ\psi)]'\, d(\ln\psi')\,,$$

called *Bott's cocycle.* The group $G_0=\mathrm{diff}(S^1)\times\mathbf{R}$ with operation

$$(\varphi\,,\,s)*(\psi\,,\,t) = (\varphi\circ\psi\,,\,t+s+B(\varphi\,,\,\psi))$$

is the Lie group that has g_0 as its corresponding Lie algebra. It is called the *Virasoro group.* Finally, g_0^* can be viewed as

$$g_0^* = \{(p(\theta)\, d\theta^2\,,\,s)\,,\,p(\theta)\ 2\pi\text{-periodic function, } s\in\mathbf{R}\} = g^*\oplus\mathbf{R}\,,$$

acting on g_0 as

$$\langle(p\,,\,s)\,,\,(\xi\,,\,t)\rangle = \int_0^{2\pi} p(\theta)\xi(\theta)\, d\theta + ts\,,$$

where, for convenience, we have denoted $p(\theta)\, d\theta\otimes d\theta$ and $\xi(\theta)\frac{\partial}{\partial\theta}$ by p and ξ, as we will often do from now on.

Let $\mathscr{H}$ be an element of $C^\infty(g^*)$. Define the *gradient* of $\mathscr{H}$ to be the element of g given by $\delta_p\mathscr{H}(\theta)\frac{\partial}{\partial\theta}\in g$, where $\delta_p\mathscr{H}(\theta)$ is a 2π-periodic function such that

$$\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0}\mathscr{H}(p+\varepsilon h) = \int_0^{2\pi} h(\theta)\delta_p\mathscr{H}(\theta)\, d\theta\,,$$

for any 2π-periodic function h.

This definition establishes a correspondence between elements of $C^\infty(g^*)$ and elements of the Lie algebra. We can define the classical

Lie-Poisson structure on g^* as the one induced on g^* by the Lie bracket on g through the correspondence above, i.e.

$$\{\mathscr{H}, \mathscr{P}\}(\alpha) = \left\langle \alpha, \left[\delta_p\mathscr{H}(\theta)\frac{\partial}{\partial\theta}, \delta_p\mathscr{P}(\theta)\frac{\partial}{\partial\theta}\right]\right\rangle,$$

for any $\mathscr{H}, \mathscr{P} \in C^\infty(g^*)$, and any $\alpha \in g^*$.

If we denote by $\mathscr{H}$ and $\mathscr{P}$ two elements of $C^\infty(g_0^*)$, their gradients will have two partial components $(\delta_p\mathscr{H}, \delta_t\mathscr{H})$. By definition, the Lie-Poisson bracket on g_0^* is given by

$$\begin{aligned}\{\mathscr{H}, \mathscr{P}\}_0(p, s) &= \langle (p, s), [(\delta_p\mathscr{H}, \delta_t\mathscr{H}), (\delta_p\mathscr{P}, \delta_t\mathscr{P})]_0\rangle \\ &= \int_0^{2\pi} [\delta_p\mathscr{H}, \delta_p\mathscr{P}]\, p(\theta)\, d\theta + sc(\delta_p\mathscr{H}, \delta_p\mathscr{P}),\end{aligned}$$

for all $\mathscr{H}, \mathscr{P} \in C^\infty(g_0^*)$. Since the expression above does not depend on the value of $\mathscr{H}$ and $\mathscr{P}$ in the central direction, we can rewrite it in the usual way

$$\{\mathscr{H}, \mathscr{P}\}_0(p, s) = \{\mathscr{H}, \mathscr{P}\}(p) + sc(\delta\mathscr{H}, \delta\mathscr{P}). \tag{1.1}$$

The KdV equation $u_t = 3uu_x - u_{xxx}$ can be interpreted as a Hamiltonian system with respect to $\{\ ,\ \}_0$ in the following sense:

Consider the *evaluation operator* $\mathscr{D}$ defined as $\mathscr{D}(p) = p(\theta)$. That is, $\mathscr{D}$ has Dirac's delta function as gradient (Dirac's delta function does not give rise to a differentiable operator but it can be expressed as a series of differential kernels, so we view it in such an approximate way). If we consider the Hamiltonian operator $\mathscr{H}$ defined as

$$\mathscr{H}(p) = \frac{1}{2}\int_0^{2\pi} p^2(\theta)\, d\theta,$$

it is straightforward to check that the KdV equation is equal to the Hamiltonian system $u_t = \{\mathscr{H}, \mathscr{D}\}_0(u)$, with central charge $s = -1$ (for more information see **[1]**, [2], **[6]** or [7]).

2. A transverse section to the orbits: Classification of Hill's equations. An explicit expression for the coadjoint action of the Virasoro group on the dual of the Virasoro algebra can be found in Kirillov's paper **[8]** and it is given by

$$K^*(\varphi)(p, s) = \left(\frac{p + sS(\varphi)}{\varphi'^2} \circ \varphi^{-1}, s\right), \tag{2.1}$$

where $S(\varphi)$ denotes the *Schwartz derivative* of φ, $S(\varphi) = (\varphi'''\varphi' - \frac{3}{2}\varphi''^2)/\varphi'^2$. One can obtain the coadjoint action of the Virasoro algebra on its dual by differentiating the expression (2.1)

$$k^*(\xi)(p, s) = (s\xi''' - 2p\xi' - p'\xi, 0). \tag{2.2}$$

The central charge s remains invariant under the action, that is, g_0^* stratifies into a family of Poisson submanifolds with constant central parameter. Each one of them is isomorphic to g^* with Poisson structure given as in (1.1) and they are all geometrically equivalent, except for the case $s = 0$. This is the usual change in the Poisson Geometry of the dual of a Lie algebra, produced by a central extension. Let's fix once and for all an adequate hyperplane inside g_0^*, namely $s = -1$.

Define *the stabilizer of a point* p to be the set of diffeomorphisms of the circle that fix the point p under the coadjoint action, i.e.,

$$\mathrm{Stab}(p) = \{\varphi \in \mathrm{diff}(S^1) \text{ such that } K^*(\varphi)(p, -1) = (p, -1)\}.$$

From (2.2) we deduce that the tangent of the stabilizer of p at the identity element is given by the vector space

$$\begin{aligned} (2.3) \quad & T_{\mathrm{id}}(\mathrm{Stab}(p)) \\ & = \left\{\xi \frac{\partial}{\partial\theta} \in \mathrm{vect}(S^1) \text{ such that } \xi''' + 2p\xi' + p'\xi = 0\right\}. \end{aligned}$$

A classification of the stabilizers of potentials was given in **[8]**. It was shown there that the set of solutions of (2.3) has a structure of Lie algebra which is isomorphic to $\mathrm{sl}(2, \mathbf{R})$, and that, furthermore, the number of periodic solutions of (2.3) is either 1 or 3, i.e., $\mathrm{Stab}(p)$ is either 1 or 3 dimensional. This dimension coincides with the codimension of the coadjoint orbit.

Let g be a Lie algebra with Lie bracket $[\ ,\ ]$. If $\mathrm{ad}(\xi)(\mu) = [\xi, \mu]$ is the usual adjoint action of the algebra, we define the *Killing form of* g to be the bilinear form $B(\xi, \mu) = \mathrm{tr}(\mathrm{ad}\,\xi(\mathrm{ad}\,\mu))$, for any $\xi, \mu \in g$.

PROPOSITION 2.1. *In the case of codimension* 3, *the coadjoint orbit contains a point of the form* $\frac{n^2}{2}\,d\theta^2$ *for some integer* n.

Proof. Assume that the codimension is three and let us consider $T_{\mathrm{id}}(\mathrm{Stab}(p))$ with its $\mathrm{sl}(2, \mathbf{R})$-structure. An expression for its Killing form was given in **[8]** and it is equal to

$$B(\xi) = -\frac{1}{4}\left(\xi\xi'' + p\xi^2 - \frac{1}{2}\xi'^2\right).$$

We know that the Killing form of $\mathrm{sl}(2, \mathbf{R})$ takes positive, negative and zero values; so does the Killing form of $T_{\mathrm{id}}(\mathrm{Stab}(p))$. If ξ is a periodic function with simple zeros we obtain

$$I(\xi) = \xi\xi'' + p\xi^2 - \frac{1}{2}\xi'^2 = -\frac{1}{2}\xi'^2 < 0.$$

If ξ has double zeros then $I(\xi) \equiv 0$, so that $I(\xi) > 0$ implies that ξ never vanishes.

For such a nonvanishing vector field $\xi \in T_{\mathrm{id}}(\mathrm{Stab}(p))$ choose φ to be equal to

$$\varphi(\theta) = \int_0^\theta \frac{A}{\xi(\theta)}\, d\theta,$$

$A =$ constant chosen so that $\varphi(2\pi) = 2\pi$.

It is immediate that $\mathrm{Ad}(\varphi)(\xi) = (\varphi'\xi)\circ\varphi^{-1} = A$ (remember that we denote by $\mathrm{Ad}(\varphi)$ the coadjoint action of $\mathrm{diff}(S^1)$). It is straightforward to check that the constant A should be a solution of the equation $\mu''' + 2K^*(\varphi)(p)\mu' + [K^*(\varphi)(p)]'\mu = 0$, and therefore $K^*(\varphi)(p) = p_1$ is also constant. Since the number of periodic solutions of (2.3) is preserved along the orbit, the equation $\mu''' + 2p_1\mu' = 0$ must have three independent periodic solutions. The only choice is $p_1 = \frac{n^2}{2}$ for some integer n and we are done. □

This last result entitles us to restrict the problem of finding a transverse structure to the case $p = \frac{n^2}{2}$: if the orbit does not go through $\frac{n^2}{2}$ for some n, the transverse section would be 1-dimensional and the transverse structure trivial. In fact, the codimension of the orbit is constant around a point different from $p = \frac{n^2}{2}$, and therefore we can refer to them as *regular* potentials. Furthermore, if $\mathrm{Or}(p)$ goes through $\frac{n^2}{2}$ for some n we can immediately obtain a transverse structure at p translating from $\frac{n^2}{2}$ to p using the coadjoint action.

When $p = \frac{n^2}{2}$, three independent solutions for equation (2.3) are $\xi_1 = \cos(n\theta)$, $\xi_2 = \sin(n\theta)$, $\xi_3 =$ *constant.* Consider the linear section

$$Q_n = \left\{ \left(\frac{n^2}{2} + a\cos(n\theta) + b\sin(n\theta) + c \right) d\theta^2, \quad a, b, c \in \mathbf{R},\ |c|, |a|, |b| < \delta \right\} \tag{2.4}$$

for some fixed integer n and some small δ that we will fix later on.

PROPOSITION 2.2. *Q_n is transversal to the orbit of $\frac{n^2}{2}$ at $\frac{n^2}{2}$.*

Proof. Denote the orbit through p by $\mathrm{Or}(p)$ and define the *annihilator of $T_{\mathrm{id}}(\mathrm{Or}(p))$* as the subset of g_0 given by

$$T_{\mathrm{id}}(\mathrm{Or}(p))^\perp = \{\xi \in g_0 \text{ such that } \langle \xi, k^*(\nu)(p)\rangle = 0 \text{ for all } \nu \in g_0\}.$$

It is easy to see that $T_{\mathrm{id}}(\mathrm{Or}(p))^{\perp} =$ kernel of $k^*(p)$, since

$$\begin{aligned}\langle \xi , k^*(\nu)p\rangle &= \int_{S^1} -\xi(\nu''' + 2p\nu' + p'\nu)\, d\theta \\ &= \int_{S^1} \nu(\xi''' + 2p\xi' + p'\xi)\, d\theta = -\langle k^*(\xi)(p) , \nu\rangle .\end{aligned}$$

Besides, the pairing

$$T_{\mathrm{id}}\left(\mathrm{Or}\left(\frac{n^2}{2}\right)\right)^{\perp} \times Q_n \to \mathbf{R}, \qquad (\xi , p) \to \langle p , \xi\rangle$$

is nondegenerate, so that Q_n has to necessarily be transversal to $\mathrm{Or}(\frac{n^2}{2})$. □

Since the Virasoro algebra is a Frechèt manifold, there are no general inverse function theorems we could apply at this point to deduce straightforwardly that Q_n intersects all nearby orbits. This is an important condition on Q_n if we wish to describe the Poisson structure around $\frac{n^2}{2}$. To avoid this problem we need a description of the invariants of the coadjoint orbits to later check that they are all locally reached along Q_n. The classification of the orbits has been studied by several authors. Kirillov gave a classification of the stabilizers in his paper **[8]**. Lazutkin and Pankratova provided a partial description in **[9]**. Later on, Segal **[17]** pointed out a discrete invariant that was missing in **[9]** and gave the complete set of invariants which we are going to describe next.

First of all, we can identify g_0^* with the space of Hill's operators associating to a tensor $p\, d\theta^2$ the equation

$$\xi'' + \frac{p}{2}\xi = 0. \tag{2.5}$$

If ξ is a solution of (2.5), it is straightforward to prove that its *Liouville-Green transform*, $\mu = [(\varphi')^{1/2}\xi] \circ \varphi^{-1}$, is a solution of

$$\xi'' + \frac{K^*(\varphi)(p)}{2}\xi = 0. \tag{2.6}$$

Moreover, this is the only transform which preserves Hill's equations. In that sense we will view our manifold as the manifold of Hill's operators and the coadjoint action as a change of variable in the corresponding equation. Using this interpretation it is immediate to check that, if F_p is the *Floquet matrix* or *monodromy* associated to (2.5), then RF_pR^{-1} is the monodromy associated to (2.6) where

$$R = \begin{pmatrix} \varphi'^{1/2} & 0 \\ \frac{1}{2}\varphi'^{-3/2} & \varphi'^{-1/2} \end{pmatrix} (2\pi) \in \mathrm{SL}(2, \mathbf{R}).$$

That is, the $\mathrm{SL}(2, \mathbf{R})$-conjugation class of the monodromy matrix is preserved along the coadjoint orbit. This is one of the invariants of the orbit, in fact the only one that changes continuously. There exists a second invariant which we can describe in the following way:

Consider $u: \mathbf{R} \to \mathbf{R}^2\backslash\{0\}$ to be an immersion given by two independent solutions of (2.5); we can assume that $u(0) = (1, 0)$ and $u'(0) = (0, 1)$. Let $\hat{u}: \mathbf{R} \to S^1$ be its radial projection and let n_p be the number of complete turns that $\hat{u}$ makes in a period. n_p is another invariant of the orbit, called a *discrete invariant* since it does not change continuously (again we can easily check that n_p is invariant using a Liouville-Green's transformation).

THEOREM 2.1. *Let u_1 and u_2 be two orientation-preserving immersions given as above by the solutions of two equations $\xi'' + \frac{p_1}{2}\xi = 0$ and $\xi'' + \frac{p_2}{2}\xi = 0$, respectively. If $n_{p_1} = n_{p_2} = m$ and $F_{p_1} = RF_{p_2}R^{-1}$ for some $R \in \mathrm{SL}(2, \mathbf{R})$, then p_1 is in the same orbit as p_2. That is, up to a Liouville-Green transformation, each Hill's equation corresponds to a different conjugacy class of the universal covering space of* $\mathrm{SL}(2, \mathbf{R})$ *under the* $\mathrm{SL}(2, \mathbf{R})$*-action.*

Proof. 1*st case*. Assume that $F_{p_1} = F_{p_2} = F$.

Then $\hat{u}_1$ and $\hat{u}_2$ make the same number of turns in a period and $\hat{u}_1(2\pi) = \hat{u}_2(2\pi)$. Divide the interval $[0, 2\pi]$ into several subintervals $[0, \theta_1], \ldots, [\theta_i, \theta_{i+1}], \ldots, [\theta_{m-1}, \theta_m], [\theta_m, 2\pi]$, such that $\hat{u}_1$ covers a complete turn on S^1 at each subinterval, except for $[\theta_m, 2\pi]$. Repeat the subdivision for $\hat{u}_2$. Then $\varphi = \hat{u}_2^{-1} \circ \hat{u}_1$ is smooth and well defined if we map each one of the $\hat{u}_1$-subintervals diffeomorphically into the corresponding $\hat{u}_2$-subinterval, that is, following in the mapping a natural order.

Because of the condition $F_{p_1} = F_{p_2} = F$, we get that $\varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi$, and therefore φ is a diffeomorphism of the circle with $\hat{u}_2 \circ \varphi = \hat{u}_1$.

Finally, since $\hat{u}_s(\theta)$ is the radial projection of $u_s(\theta)$, $s = 1, 2$, we obtain that $u_1(\theta) = f(\theta)(u_2 \circ \varphi)$, for some differentiable and real-valued function f. Both u_1 and u_2 were given by solutions of Hill's equations. Through a straight substitution in the equations one can check that this condition imposes a unique possible choice, $f = \varphi'^{-1/2}$. Therefore, $p_1 = K^*(\varphi)p_2$ and this case is proved.

2nd case.

$$F_{p_2} = RF_{p_1}R^{-1}.$$

We know that

$$\hat{u}_i(\theta + 2\pi) = (F_{p_i}u_i)\hat{}(\theta), \qquad i = 1, 2. \tag{2.7}$$

Therefore

$$(R^{-1}u_2)\hat{}(\theta + 2\pi) = (F_{p_1}R^{-1}u_2)\hat{}(\theta).$$

Denote the image of u as subset of $\mathbf{R}^2$ by $\mathrm{Im}(u)$. It is not hard to check that R can be chosen so that $\mathrm{Im}(\hat{u}_1)$ and $\mathrm{Im}(R^{-1}u_2)\hat{}$ intersect at some point. In fact, we could use above $-R$ instead of R if they do not intersect (the sets $\mathrm{Im}(\hat{u}_1) \cap \mathrm{Im}(R^{-1}u_2)\hat{}$ and $\mathrm{Im}(\hat{u}_1) \cap \mathrm{Im}(-R^{-1}u_2)\hat{}$ cannot be simultaneously void). By translation in the argument, we can make the initial values coincide. $R^{-1}u_2$ and any translation of it is given by solutions of the same Hill's equation as u_2. We can now obtain this case as a corollary of the previous one. □

PROPOSITION 2.3. *The space of Hill's equations, up to Green-Liouville's transformations, is in one-to-one correspondence with the space of* $\mathrm{SL}(2, \mathbf{R})$ *conjugation classes of the Universal covering space of* $\mathrm{SL}(2, \mathbf{R})$, *with the point (Identity,* $n = 0$*) removed.*

Proof. First of all notice that if a matrix $M \in \mathrm{SL}(2, \mathbf{R})$ has two different eigenvalues (that is, $|\mathrm{trace}(M)| > 2$), then its $\mathrm{GL}(2, \mathbf{R})$ and $\mathrm{SL}(2, \mathbf{R})$ conjugation classes coincide. If $\mathrm{trace}(M) = \pm 2$, then the two different $\mathrm{SL}(2, \mathbf{R})$-Jordan forms are $\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\pm\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, and if $|\mathrm{trace}(M)| < 2$, then both eigenvalues are imaginary and there are also two different $\mathrm{SL}(2, \mathbf{R})$-Jordan forms, namely $\pm\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $\pm\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a, b > 0$.

Next, consider the potential

$$p_{\alpha,\beta}(\theta) = \begin{cases} \alpha^2 & \text{if } 0 \le \theta < \theta_0, \\ -\beta^2 & \text{if } \theta_0 \le \theta \le 2\pi \end{cases}$$

and consider its associated Hill's equation $\xi'' + p_{\alpha,\beta}\xi = 0$. A funda-

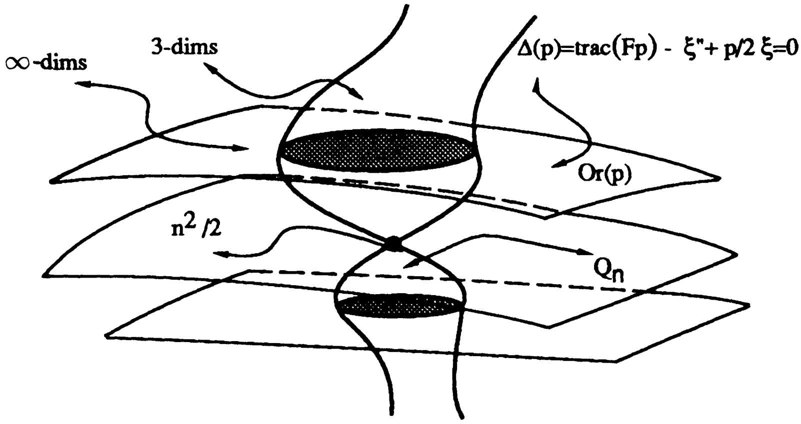

FIGURE 1

mental matrix of solutions for it is given by $X = X_2X_1$ where

$$X_1 = \begin{pmatrix} \cos(\alpha\theta_0) & \frac{1}{\alpha}\sin(\alpha\theta_0) \\ -\alpha\sin(\alpha\theta_0) & \cos(\alpha\theta_0) \end{pmatrix},$$

$$X_2 = \begin{pmatrix} \cosh(\beta(2\pi-\theta_0)) & \frac{1}{\beta}\sinh(\beta(2\pi-\theta_0)) \\ \beta\sinh(\beta(2\pi-\theta_0)) & \cosh(\beta(2\pi-\theta_0)) \end{pmatrix} \quad \text{if } \beta \neq 0, \text{ or}$$

$$X_2 = \begin{pmatrix} 1 & 2\pi-\theta_0 \\ 0 & 1 \end{pmatrix} \quad \text{if } \beta = 0.$$

The rotation number (in the above sense) of this equation is $\alpha\theta_0/2\pi$ plus an angle ω_0 with $\tan(\omega_0) < \frac{1}{\beta}$ and which is, in any case, less than $\frac{\pi}{2}$. Next, we will show that, for different values of α, β and θ_0 we obtain all possible $SL(2, \mathbf{R})$-Jordan forms, with all possible rotation numbers, except for the case of no complete turns (rotation number 0) and monodromy equals the identity.

If $\theta_0 = 2\pi$, then

$$X = \begin{pmatrix} \cos(2\pi\alpha) & \frac{1}{\alpha}\sin(2\pi\alpha) \\ -\alpha\sin(2\pi\alpha) & \cos(2\pi\alpha) \end{pmatrix}.$$

We can therefore cover the four possible Jordan forms corresponding to complex conjugated eigenvalues by choosing different values of α from the intervals $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$ and $[\frac{3}{4}, 2]$, respectively. Considering values $m\alpha$ with m being an integer $m \geq 1$, we would obtain the same Jordan forms but the rotation number would be $m-1$. The identity matrix is reached here whenever α is a nonzero whole number. It is never reached for rotation number equals zero, since the solution curve is in this case periodic and it should, at least, give a complete turn around S^1.

If $\theta_0 = \pi$ and $\alpha = 2, 4, \ldots$, then $\operatorname{trace}(X) = e^{2\pi\beta} + e^{-2\pi\beta} > 2$. Changing the values of β and α we will reach all possible values of the trace and all rotation numbers. If $\theta_0 = \pi$ and $\alpha = 1, 3, 5, \ldots$, then $\operatorname{trace}(X) = -(e^{2\pi\beta} + e^{-2\pi\beta}) < -2$ and the same result holds.

Therefore, we are left with the classes which have Jordan forms $\pm\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\pm\left(\begin{smallmatrix}1&-1\\0&1\end{smallmatrix}\right)$. If $\theta_0 = \alpha$, $\alpha = 0, 2, 4, \ldots$ and $\beta = 0$, we obtain the classes with Jordan form $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ (and $-\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ for the choices $\alpha = 1, 3, 5, \ldots$) and all different rotation numbers.

Finally, consider $\theta_0 = \frac{\pi}{2}$ and $\alpha = 3, 7, \ldots$. Then,

$$X = \begin{pmatrix} \cosh(\frac{3\pi\beta}{2}) & \frac{1}{\beta}\sinh(\frac{3\pi\beta}{2}) \\ \beta\sinh(\frac{3\pi\beta}{2}) & \cosh(\frac{3\pi\beta}{2}) \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{\alpha} \\ \alpha & 0 \end{pmatrix}$$

has a double eigenvalue whenever $(\frac{\alpha}{\beta} - \frac{\beta}{\alpha})^2 \sinh^2(\frac{3\pi\beta}{2}) = 4$. Its eigenvalues would be $\pm 1 = \pm\frac{1}{2}(\frac{\alpha}{\beta} - \frac{\beta}{\alpha})\sinh(\frac{3\pi\beta}{2})$ depending on the sign of $(\frac{\alpha}{\beta} - \frac{\beta}{\alpha})\sinh(\frac{3\pi\beta}{2})$. In this case, X will have Jordan form $\pm\left(\begin{smallmatrix}1&-1\\0&1\end{smallmatrix}\right)$. On the other hand,

$$\lim_{\beta\to 0+} \left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)\sinh\left(\frac{3\pi\beta}{2}\right) = \frac{3\pi\alpha}{2} > 2$$

and

$$\lim_{\beta\to +\infty} \left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)\sinh\left(\frac{3\pi\beta}{2}\right) = -\infty .$$

From here it is obvious that this last case is also covered.

If we approximate $p_{\alpha,\beta}$ by C^∞ periodic functions we will immediately obtain the claim of the proposition. □

Using this geometrical description it is easy to prove the following theorem.

THEOREM 2.2. *The transverse section* Q_n (*see* (2.4)) *intersects all orbits nearby the one going through the potential* $\frac{n^2}{2}\,d\theta^2$.

Proof. It suffices to prove that the map $Q_n \to \mathrm{SL}(2, \mathbf{R})$, which associates to each potential the monodromy of equation (2.5), is locally surjective. If we expand the monodromy as a function of (a, b, c) we obtain

$$F_p = \text{Identity } + \pi\left\{\begin{pmatrix} 0 & -1 \\ \frac{4}{n^2} & 0 \end{pmatrix} c + \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{2}{n^2} & 0 \end{pmatrix} a + \begin{pmatrix} -\frac{1}{n} & 0 \\ 0 & \frac{1}{n} \end{pmatrix} b\right\}$$
$$+ \text{ higher order terms},$$

which has maximal rank. □

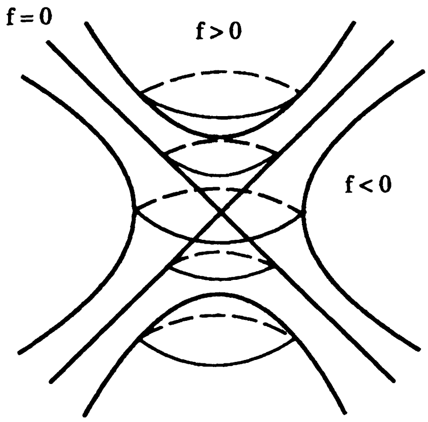

FIGURE 2

Assume that $p = \frac{n^2}{2} + c + a\cos(n\theta) + b\sin(n\theta) \in Q_n$. We can also calculate the Taylor expansion of $\Delta(p)$ in a, b and c up to second order (see **[10]**). We are required to solve two ordinary second order differential equations for each Taylor coefficient we want to find.

After some long calculations one gets

$$\Delta(p) = (-1)^n \left\{ 2 + \frac{2\pi^2}{n^2} \left[\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 - c^2 \right] \right\}$$
$$+ \text{ higher order terms in } (a, b, c).$$

Let us have a closer look to the real function $f(x, y, z) = x^2+y^2-z^2$. Its level sets are given as in Figure 2. Recall that f is a nontrivial Casimir element for the Lie-Poisson bracket of $\mathrm{sl}(2, \mathbf{R})^*$.

Recall also that $\Delta(p)$ is constant along each orbit, in particular along each intersection of the orbit with Q_n. If there exists a transverse structure for $(g_0, \{\ ,\ \}_0)$ on Q_n, say $\{\ ,\ \}_Q$, one expects the Kirillov leaves of $\{\ ,\ \}_Q$ to be such an intersection. Since a function that is constant along the symplectic leaves is a Casimir function, $\Delta(p)$ would be a Casimir for $\{\ ,\ \}_Q$. Therefore, we can make a guess and claim that $(Q_n, \{\ ,\ \}_Q)$ is locally isomorphic to $\mathrm{sl}(2, \mathbf{R})^*$ with its canonical Lie-Poisson structure. This is actually one of the main results in the next section.

Theorem 2.2 partially proves a claim by Lazutkin and Pankratova about normal forms of Hill's equations. In their paper (**[9]**) they claim that any Hill's equation has normal form $\xi''+(d+e\cos(n\theta))\xi = 0$, for some real numbers d and e. This normal form can be achieved under a Liouville-Green transformation. From Theorem 2.2 any potential p can be taken to the intersection of the leaf with Q_n using the coadjoint action, as far as it belongs to an orbit close enough to $\mathrm{Or}(\frac{n^2}{2})$, for some integer n. Besides $c + a\cos(n\theta) + b\sin(n\theta) = c + \beta\cos(n\theta + \alpha)$ for some $\alpha, \beta \in \mathbf{R}$. The result follows. The methods used on ([9]) are different from the ones in this paper.

3. A transverse structure for g_0^*, $\{\ ,\ \}_0$.

3.1. *Induced Poisson structures: Transverse structures and Dirac formalism. Transverse structures in infinite dimensions.* In the finite dimensional case, transverse structures were introduced by A. Weinstein [20] and they were proved to be unique. Some results have already been proved in the infinite dimensional case, whenever the manifold is modelled by a Hilbert or Banach space (see [11]). That is not our case either since $\mathrm{vect}(S^1)$ is a Frechèt manifold (it is not only that the Fourier series of an element has to converge, but all the series of its derivatives). Therefore, we now encounter one of the obstacles in this work: it is not clear how to induce a Poisson structure in this kind of space.

The idea we will follow is to imitate the finite dimensional procedure, covering any gap we find in some appropriate way. In particular, we will find the analogue of Dirac's formula for transverse structures in finite dimensions and we will check that it actually defines a Poisson structure on Q_n which is induced by the Lie-Poisson structure of the Virasoro algebra (for more information about induced Poisson structures see for example [12], [18], or [13], or [20]).

DEFINITION. Let $L_p = Q_n \cap \mathrm{Or}(p)$. Assume (1) $\{\ ,\ \}_0$ induces a nondegenerate (symplectic) structure on L_p, for all $p \in Q_n$.

(2) There exists a smooth (resp. analytic) Poisson structure on Q_n, $\{\ ,\ \}_Q$, that induces the same structure as $\{\ ,\ \}_0$ on L_p.

$\{\ ,\ \}_Q$ is called a *smooth* (*resp. analytic*) *transverse structure for* $(g_0^*, \{\ ,\ \}_0)$ in the direction of Q_n.

THEOREM 3.1 [20] (*Induced Poisson structures in finite dimensions*). *Let M be a finite dimensional Poisson manifold with Poisson tensor P. Let Q be an immersed submanifold of M. Assume that, for all $x \in Q$,*

$$\text{(a)} \qquad P(x)(T_x(Q)^{\perp}) \cap T_x(Q) = \{0\},$$

$$\text{(b)} \qquad \mathrm{Ker}(P(x)) \cap T_x(Q)^{\perp} = 0.$$

Then Q canonically inherits a Poisson structure from M, which we will denote by P_Q.

We will make some comments on how the induced Poisson structure is found.

Using (a) and (b) it is not hard to show that

$$T_x(M) = T_x(Q) \oplus P(x)(T_x(Q)^{\perp})$$

which provides itself a smooth projection $\pi\colon T_x(M) \to T_x(Q)$ whenever $x \in Q$. The induced Poisson structure is then defined as

$$P_Q = \pi \circ P \circ \pi^*\colon T_x(Q)^* \to T_x(Q).$$

In other words, given a Hamiltonian function $\alpha \in T_x(Q)^*$ we can find an extension of it, $\pi^*(\alpha) = \hat{\alpha} \in T_x(M)^*$. The vector field $P(x)(\hat{\alpha}) \in T_x(M)$ has a component on $T_x(Q)$. Such a component is the value of $P_Q(x)(\alpha)$, and it is found taking away from $P(x)(\hat{\alpha})$ a linear combination of elements in $P(x)(T_x(Q)^{\perp})$.

In local coordinates the idea is as follows:

Let $\{z_1, \dots, z_{2s}\}$ be independent defining functions for Q near x. That is, $Q = \{x \in M\colon z_1(x) = z_2(x) = \cdots = z_{2s}(x) = 0\}$. Denote by $C(y) = (C_{ij}(y))$ the matrix $C_{ij}(y) = \{z_i, z_j\}(y)$, with $i, j = 1, \dots, 2s$ and $y \in Q$. This matrix has smooth (resp. analytic) entries and it is nonsingular. Let $C^{-1}(y) = (C^{ij}(y))$ be its inverse matrix, which also has smooth (resp. analytic) entries. Let f be a smooth function on Q and $\hat{f}$ be any extension of f to M. Due to the invertibility of C one can easily show that there exist unique smooth functions $\{g_i(y)\}_{i=1}^{2s}$ defined on a neighbourhood of x such that, if

$$P_Q(f)(y) = P(\hat{f})(y) + \sum_{i=1}^{2s} g_i(y)P(z_i)(y)$$

then $P_Q(f)(y) \in T_y(Q)$ for all $y \in Q$ in a neighbourhood of x. Imposing the tangency condition on $P_Q(f)$ we can uniquely solve for g_i in terms of the entries of C^{-1}.

The final expression for P_Q is

$$\text{(3.1)} \quad \{f, g\}_Q(y) = \{\hat{f}, \hat{g}\}(y) + \sum_{i=1}^{2s}\{\hat{f}, z_i\}(y)C^{ij}(y)\{z_j, \hat{g}\}(y),$$

for all $y \in Q$ around x. This formula is referred as *Dirac's formula for induced structures.*

P_Q immediately induces a structure on Q whose symplectic leaves coincide with the intersection of Q and the symplectic leaves of P. Notice that, in order to find an expression for P_Q, we need not only nondegeneracy of the bracket along the coadjoint orbits but also to invert it locally.

PROPOSITION 3.1. *For δ small enough, $\{\ ,\ \}_0(p)$ is nondegenerate (symplectic) on $T_p(Q_n)^\perp$, for all $p \in Q_n$, and locally invertible when considered as a linear operator from l^2 to l^2 (δ as in (2.4)). Coordinates can be chosen such that $\{\ ,\ \}_0$ and its inverse are represented by infinite matrices with analytic entries.*

Proof. Consider Fourier coefficients as coordinates for the dual of the Virasoro algebra, $\varepsilon_m\colon g_0^* \to \mathbf{R}$ defined as

$$\varepsilon_m(p) = \frac{1}{2\pi}\int_0^{2\pi} e^{im\theta} p(\theta)\, d\theta, \quad \text{for any integer } m,$$

Q_n is locally defined as the zero set of $\{\varepsilon_m\}_{m\neq\pm n,0}$.

In order to prove the proposition, we need to check that the linear operator $C\colon l^2 \to l^2$, represented by the infinite matrix $C(p) = (\{\varepsilon_m, \varepsilon_k\}_0(p))_{m,k\neq\pm n,0}$, is invertible for any $p \in Q_n$.

Assume that $p = \frac{n^2}{2} + c + a\cos(n\theta) + b\sin(n\theta) \in Q_n$, so that $\varepsilon_0(p) = \frac{n^2}{2} + c$, $\varepsilon_n(p) = \frac{1}{2}(a+bi)$, $\varepsilon_{-n}(p) = \frac{1}{2}(a-bi)$ $(i^2 = -1)$. Straightforwardly, one can show that

$$\begin{aligned} \{\varepsilon_{-k}, \varepsilon_k\}_0(p) &= \frac{k}{2\pi i}(2\varepsilon_0(p) - k^2), \\ \{\varepsilon_{-(n+k)}, \varepsilon_k\}_0(p) &= \frac{(2k+n)}{2\pi i}\varepsilon_{-n}(p), \\ \{\varepsilon_{n-k}, \varepsilon_k\}_0(p) &= \frac{(2k-n)}{2\pi i}\varepsilon_n(p), \\ \{\varepsilon_m, \varepsilon_k\}_0(p) &= 0, \quad \text{otherwise}. \end{aligned} \tag{3.2}$$

To invert the matrix above is equivalent to solve for $\{\gamma_m\}_{m=-\infty}^{+\infty}$ in the system

$$\sum_m \gamma_m \{\varepsilon_m, \varepsilon_k\}_0(p) = b_k, \tag{3.3}$$

for all k, and for some $B = \{b_k\}$ given. Let us assume that $\{\gamma_k\}$ and $\{b_k\}$ are elements of l^2. We can rescale so that system (3.3) becomes

$$\begin{aligned} \gamma_{-k}k(2\varepsilon_0(p) - k^2) + \gamma_{-(n+k)}(2k+n)\varepsilon_{-n}(p) \\ + \gamma_{n-k}(2k-n)\varepsilon_n(p) = b_k, \end{aligned} \tag{3.4}$$

for any integer k.

Observe that the system (3.4) can be divided into a finite number of autonomous subsystems. Each one of them involves only the γ's whose subindices belong either to the set $\{k = sn + r\,,\ s = 1\,, 2\,, \ldots\}$ for a fixed integer $r \neq 0\,,\ -n < r < n$ or to the set $\{k = 2n\,, s = 2\,, 3\,, \ldots\}$. There are a finite number of subsystems so that it is enough to prove that each one can be solved in l^2. The reasoning is the same for any one of them. We will prove it here only for the system given by the subindices $\{k = sn\,,\ s = 2\,, 3\,, \ldots\}$. For simplicity call $\gamma_{-sn} = \gamma_s$ since no confusion is possible from now on.

We can rewrite the system to solve as $A(p)\gamma = b$, where $A(p)$ is given by the infinite tridiagonal matrix (Jacobi matrix)

$$\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & (s-1)(2\varepsilon_0(p) - [[s-1]n]^2) & [2s-1]\varepsilon_{-n}(p) & 0 & \cdots \\ \cdots & [2s-1]\varepsilon_n(p) & s(2\varepsilon_0(p) - [sn]^2) & [2s+1]\varepsilon_{-n}(p) & \cdots \\ \cdots & 0 & [2s+1]\varepsilon_n(p) & (s+1)(2e_0(p) - [[s+1]n]^2) & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Observe that $A(\frac{n^2}{2})$ is a diagonal matrix with nonvanishing diagonal entries. Therefore, $A(\frac{n^2}{2})$ is an invertible matrix and its inverse is a diagonal matrix with diagonal entries $\frac{1}{s(n^2-(sn)^2)}$, $s = 2\,, 3\,, \ldots$. Although $A(\frac{n^2}{2})$ does not take l^2 into l^2, its inverse does.

Observe also that, if we define the matrix $D(p)$ through the relation $A(p) = A(\frac{n^2}{2}) + D(p)$, to solve the system $A(p)\gamma = b$ is equivalent to solve for γ in

$$A\left(\frac{n^2}{2}\right)^{-1} b = \left[I + A\left(\frac{n^2}{2}\right)^{-1} D(p)\right]\gamma,$$

where $A(\frac{n^2}{2})^{-1}$ is the inverse matrix of $A(\frac{n^2}{2})$. The matrix $A(\frac{n^2}{2})^{-1}D(p)$ is given by the tridiagonal matrix

$$\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \frac{(s-1)(2\varepsilon_0(p)-[[s-1]n]^2)}{(1-(s-1)^2)(s-1)n^2} & \frac{[2s-1]\varepsilon_{-n}(p)}{(1-(s-1)^2)(s-1)n^2} & 0 & \cdots \\ \cdots & \frac{[2s-1]\varepsilon_n(p)}{s(1-s^2)n^2} & \frac{s(2\varepsilon_0(p)-[sn]^2)}{s(1-s^2)n^2} & \frac{[2s+1]\varepsilon_{-n}(p)}{s(1-s^2)n^2} & \cdots \\ \cdots & 0 & \frac{[2s+1]\varepsilon_n(p)}{[1-(s+1)^2](s+1)n^2} & \frac{[s+1](2\varepsilon_0(p)-[[s+1]n]^2)}{[1-(s+1)^2](s+1)n^2} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}.$$

The infinite dimensional operator $A(\frac{n^2}{2})^{-1}D(p)\colon l^2 \to l^2$ represented by this matrix is clearly bounded with norm bounded by $|\delta|$

(δ as in (2.4)). We can apply standard theorems on invertibility of linear operators on Hilbert spaces that are perturbations of the identity to obtain that, for δ small enough, the matrix $I + A(\frac{n^2}{2})^{-1}D(p)$ is invertible, for any $p \in Q_n$.

The matrix $I + A(\frac{n^2}{2})D(p)$ has analytic entries in ε_0, ε_n and ε_{-n}. Therefore, $A(p)^{-1} = [I + A(\frac{n^2}{2})^{-1}D(p)]^{-1}A(\frac{n^2}{2})^{-1}$ also has analytic entries and the result of the proposition is now proved. □

Denote by $(C^{ij}(p)) = C^{-1}$ the inverse matrix of C as in Proposition 3.1.

LEMMA 3.1. *Let* $p = (a, b, c) = (\varepsilon_0(p), \varepsilon_n(p), \varepsilon_{-n}(p)) \in Q_n$, *and let* $C^{-2n2n}(p)$ *be the entry in place* $(-2n, 2n)$ *of* C^{-1}.

Then, $\frac{1}{2\pi i}C^{-2n2n}$ *is a real analytic function of* $(\varepsilon_0(p), (\varepsilon_n(p)\varepsilon_{-n}(p)))$. *That is, it depends only on* c *and the ratio* $a^2 + b^2$.

Proof. This lemma is a corollary of §3.2, Theorem 3.7, in which we give an explicit Taylor expansion for it. A shorter proof can be given but we will avoid it. □

THEOREM 3.2. *A transverse structure for the dual of the Virasoro algebra at the point* $\frac{n^2}{2}\,d\theta^2$ *is given locally by an antisymmetric tensor,* $\{\ ,\ \}_Q$, *defined as*

$$\{\varepsilon_0, \varepsilon_n\}_Q(p) = \frac{n}{2\pi i}\varepsilon_n(p),$$
$$\{\varepsilon_0, \varepsilon_{-n}\}_Q(p) = \frac{-n}{2\pi i}\varepsilon_{-n}(p),$$
$$\{\varepsilon_n, \varepsilon_{-n}\}_Q(p) = \frac{1}{2\pi i}\left[-2n\varepsilon_0(p) + n^3 - 9n^2\varepsilon_n(p)\varepsilon_{-n}(p)\frac{C^{-2n2n}(p)}{2\pi i}\right].$$

The structure is analytic in $\{\varepsilon_0, \varepsilon_n, \varepsilon_{-n}\}$, *linearizable and equivalent to the Lie-Poisson structure on* $\mathrm{sl}(2, \mathbf{R})^*$.

Proof. Notice that we are actually copying the formula in coordinates given by Theorem 3.1. Define $\{\ ,\ \}_Q$ as

$$\{\varepsilon_i, \varepsilon_j\}_Q(p) = \{\varepsilon_i, \varepsilon_j\}_0(p) + \sum_{k,l\neq\pm n,0}\{\varepsilon_i, \varepsilon_k\}_0(p)C^{kl}(p)\{\varepsilon_l, \varepsilon_j\}_0(p).$$

Applying commutation relations (3.2), the formula above gives the expression in the statement of the theorem. This expression is found following formally the finite dimensional reasoning in Theorem 3.1.

On the other hand, since this is only a formal approach we will have to check straightforwardly that P_Q defines a Poisson structure on Q_n and that the intersections of Q_n with the symplectic leaves are symplectic with respect to both structures. Notice also that (a) and (b) on Theorem 3.1 are also true here due to the nondegeneracy of $\{\ ,\ \}_0$ along the leaves. Nevertheless, one cannot get the splitting of the tangent space into a direct sum as it happened in the finite dimensional case.

Leibniz's rule is obvious from the definition. To check Jacobi's identity for $\{\ ,\ \}_Q$ reduces to prove that

$$(3.5)\qquad \{\varepsilon_0, \{\varepsilon_n, \varepsilon_{-n}\}_Q\} + \{\varepsilon_n, \{\varepsilon_{-n}, \varepsilon_0\}_Q\} + \{\varepsilon_{-n}, \{\varepsilon_0, \varepsilon_n\}_Q\}_Q = 0,$$

on Q_n. Substituting we reduce (3.5) to

$$(3.6)\qquad \{\varepsilon_0, (\varepsilon_n\varepsilon_{-n})C^{-2n2n}\}_Q(p) = 0.$$

As a result of Lemma 3.1, $(\varepsilon_n\varepsilon_{-n})C^{-2n2n}$ restricted to Q_n is actually a function of the ratio $(\varepsilon_n(p)\varepsilon_{-n}(p))$. Applying Leibniz's rule and the definition of $\{\ ,\ \}_Q$ one gets that ε_0 commutes with the ratio along Q, and therefore (3.6) holds.

The last part is to check that P is symplectic on the intersections of Q_n with the symplectic leaves, L_p. As it happened in the finite dimensional case, that is a consequence of property (a) in Theorem 3.1, since the intersection $P(p)(T_p(Q_n)^\perp) \cap T_p(Q_n)$ is equal to the kernel of P along L_p, and in this case it vanishes.

Finally, we apply the following result by J. Conn **[4]** (see **[5]** for the smooth case): *if a Lie algebra g is semisimple (as* $\mathrm{sl}(2, \mathbf{R})$ *is), then any analytic Poisson structure on g^*, which is a perturbation of the Lie-Poisson structure by a tensor of order at least 2 that vanishes at the origin, is linearizable.*

It is now obvious that P_Q is linearizable and equivalent to the Lie-Poisson structure on $\mathrm{sl}(2, \mathbf{R})^*$. □

One comment on the linearization. Notice that by being linearizable we mean linearizable as structure on Q_n, not as a structure induced by g_0^*. That is, this result does not imply that a canonical transverse structure for the Lie-Poisson structure on the dual of the Virasoro algebra is the Lie-Poisson structure on $\mathrm{sl}(2, \mathbf{R})^*$, since no uniqueness result has been proved yet. What the result really means is that we can find coordinates (only) on Q_n such that $\{\ ,\ \}_Q$ on those coordinates is linear. In order to prove uniqueness we would need to extend that change of coordinates to g_0^* obtaining in this way an automorphism of the Lie-Poisson structure on g_0^*. We will comment more about uniqueness at the end of the section.

3.2. *The explicit expression for* $\{\ ,\ \}_Q$. In this section we will give an explicit expression for the Taylor expansion of the function $C^{-2n2n}(p)$.

Recall that our function is the entry in place $(-2n, 2n)$ of the matrix $(C^{km}(p))$, inverse of $(C_{km}(p)) = (\{\varepsilon_k, \varepsilon_m\}_0(p))_{k,m\neq\pm n,0}$. If again we set the system of equations $C\gamma = e_{2n}$, where $\gamma = \{\gamma_k\}$ and e_{2n} is a vector that has all its components equal to zero except for 1 in place $2n$, then $\gamma_{-2n} = C^{-2n2n}(p)$. Again, if for simplicity we write γ_k instead of γ_{-nk}, we get the recurrence relation

$$k(2\varepsilon_0 - (kn)^2)\gamma_k + (2k-1)\varepsilon_n\gamma_{k-1} + (2k+1)\varepsilon_{-n}\gamma_{k+1} = 0, \tag{3.7}$$

for any $k > 2$, and

$$2(2\varepsilon_0 - (2n)^2)\gamma_2 + 5\varepsilon_{-n}\gamma_3 = \beta,$$

where $\beta = \frac{2\pi i}{n}$.

Proposition 3.2. *For any* $k \geq 2$,

$$F_k(\varepsilon_n\varepsilon_{-n}, \varepsilon_0)\gamma_2 = H_k(\varepsilon_n\varepsilon_{-n}, \varepsilon_0)\beta - \delta_k\gamma_k$$

where F_k, H_k *satisfy the recurrence relation*

$$G_{k+1} = XG_k - \frac{(2k-1)^2}{k(k-1)((kn)^2 - Y)(((k-1)n)^2 - Y)}G_{k-1}, \tag{1}$$

with

$$X = \frac{1}{(\varepsilon_n\varepsilon_{-n})^{\frac{1}{2}}}, \quad Y = 2\varepsilon_0, \quad \delta_k = -\frac{(2k-1)\varepsilon_{-n}X}{(k-1)(Y - ((k-1)n)^2)}\delta_{k-1}$$

and initial conditions

$$F_1 = 0, \quad F_2 = 1, \quad H_2 = 0, \quad H_3 = \frac{1}{2(Y-(2n)^2)},$$
$$\delta_3 = -\frac{5\varepsilon_{-n}X}{2(Y-(2n)^2)}.$$

Proof. The proof of this proposition is by induction on k. □

The solutions of the recurrence relation (1) can be interpreted as orthogonal polynomials in X as we will show next.

A. *Jacobi fractions and orthogonal polynomials: Definitions and some results.*

DEFINITIONS. (a) Let $\{a_n\}$ and $\{b_n\}$ be arbitrary sequences of complex numbers and write

$$C_0 = b_0\,, \quad C_1 = b_0 + \frac{a_1}{b_1}\,, \quad C_2 = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2}}\,, \ldots$$

$$C_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots\, (a_n/b_n)}}\,.$$

C_n is called the nth *approximant of the continued fraction associated to the sequences* $\{a_n\}$, $\{b_n\}$. We will denote C_n as

$$C_n = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n}\,.$$

(b) A continued fraction of the form

$$\frac{\lambda_1|}{|x - c_1} - \frac{\lambda_2|}{|x - c_2} - \frac{\lambda_3|}{|x - c_3} - \cdots$$

is called a *Jacobi type continued fraction* (J*-fraction*).

(c) If C is a continued fraction and $C_n = A_n/B_n$, then A_n and B_n are called nth *partial numerator and* nth *partial denominator*, respectively.

Note. If A_n and B_n are the partial numerators and denominators for a J-fraction

$$\frac{\lambda_1|}{|x - c_1} - \cdots - \frac{\lambda_n|}{|x - c_n} - \cdots$$

it is very simple to prove that they satisfy the recurrence relations

$$\begin{gathered} B_n(x) = (x - c_n)B_{n-1}(x) - \lambda_n B_{n-2}(x)\,, \qquad n = 1\,, 2\,, 3\,, \ldots\,, \\ B_{-1}(x) = 0\,, \quad B_0(x) = 1\,, \\ A_n(x) = (x - c_n)A_{n-1}(x) - \lambda_n A_{n-2}(x)\,, \qquad n = 1\,, 2\,, \ldots\,, \\ A_{-1}(x) = 1\,, \quad A_0(x) = 0\,. \end{gathered}$$

Notice the similarities between these expressions and the recurrence problem (1).

(d) Let $\{\mu_n\}$ be a sequence of complex numbers and let $\mathscr{L}$ be a complex-valued linear function defined on the vector space of all polynomials by the rule

$$\mathscr{L}(x^n) = \mu_n\,, \qquad n = 0\,, 1\,, 2\,, \ldots\,.$$

$\mathscr{L}$ is called *moment functional* determined by the *formal moment sequence* $\{\mu_n\}$. μ_n is called the *moment of order* n.

(e) A moment functional $\mathscr{L}$ is called *positive-definite* if $\mathscr{L}(\pi(x)) > 0$ for every polynomial $\pi(x)$ that is not identically zero and is nonnegative for all real x.

(f) Let $\mathscr{L}$ be a moment functional with moment sequence $\{\mu_n\}$. Define

$$\Delta_n = \det(\mu_{i+j})_{i,j=0}^n .$$

We will say that $\mathscr{L}$ is *quasi-definite* whenever $\Delta_n \neq 0$ for all $0 \leq n$.

(g) A sequence $\{P_n(x)\}$ is called an *Orthogonal Polynomial Sequence* (OPS) with respect to a moment functional $\mathscr{L}$ provided that, for all nonnegative integers m and n

(i) $P_n(x)$ is a polynomial of degree n,
(ii) $\mathscr{L}(P_m(x)P_n(x)) = 0$ for all $m \neq n$,
(iii) $\mathscr{L}(P_n^2(x)) \neq 0$.

It is not hard to notice that OPS are uniquely determined up to the product by a nonvanishing constant. The next theorem shows how partial denominators for a J-fraction can be interpreted as OPS with respect to a certain moment functional.

FAVARD'S THEOREM. *Let* $\{c_n\}$ *and* $\{\lambda_n\}$ *be arbitrary sequences of complex numbers and let* $\{P_n(x)\}$ *be defined by the recurrence formula*

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad n = 1, 2, 3, \ldots, \tag{3.8}$$
$$P_{-1}(x) = 0, \quad P_0(x) = 1.$$

Then, there is a unique moment functional $\mathscr{L}$ *such that*

$$\mathscr{L}(1) = \lambda_1, \quad \mathscr{L}(P_m(x)P_n(x)) = 0$$

for $m \neq n$, $m, n = 0, 1, 2, \ldots$.

$\mathscr{L}$ *is quasi-definite and* $\{P_n(x)\}$ *are the corresponding monic OPS if and only if* $\lambda_n \neq 0$, *while* $\mathscr{L}$ *is positive-definite if and only if* c_n *are real and* $\lambda_n > 0$ $(n \geq 1)$.

Consider the OPS $\{P_n(x)\}$ with recurrence formula as in Favard's theorem, and define $P_n^{(1)}(x)$ to be a monic polynomial of degree n which satisfies the recurrence

$$P_n^{(1)}(x) = (x - c_{n+1})P_{n-1}^{(1)}(x) - \lambda_{n+1}P_{n-2}^{(1)}(x), \qquad n = 1, 2, 3, \ldots,$$
$$P_{-1}^{(1)}(x) = 0, \quad P_0^{(1)}(x) = 1.$$

The polynomials $P_n^{(1)}(x)$ are called the *monic numerator polynomials* (or *associated polynomials*) corresponding to $P_n(x)$.

It is now clear to us that partial denominators and numerators of a J-fraction, B_k and A_{k+1}, are respectively OPS and associated polynomials with respect to a certain moment functional that is a positive-definite if and only if the partial fractions have all real coefficients and the numerators λ_k are all positive. Observe that A_k are not actually monic unless $\lambda_1 = 1$. To be correct, the associated polynomials are $\lambda_1^{-1} A_{k+1}$, $0 \le k$.

DEFINITION. A moment functional is called *symmetric* if all of its moments of odd order are zero. This is equivalent to $c_n = 0$, $n \ge 1$, in the corresponding recurrence formula.

We can easily recognize the recurrence in problem (1) as corresponding to a symmetric problem, a fact that will be crucial for our final result.

Next we will give some definitions and quote without proof some of the results in the theory of OPS, Jacobi fractions and representation theory that will be more relevant in the resolution of our problem.

THEOREM 3.3. *Let $\mathscr{L}$ be a positive-definite moment functional and let $\mu_0 = \mathscr{L}(1)$. Let ψ_n be defined as*

$$\psi_n = \begin{cases} 0 & \text{if } x < x_{n1}, \\ A_{n1} + \cdots + A_{np} & \text{if } x_{np} \le x < x_{n,p+1} \ (1 \le p < n), \\ \mu_0 & \text{if } x \ge x_{nn}, \end{cases}$$

where $x_{n1} < x_{n2} < \cdots < x_{nn}$ are the zeros of $P_n(x)$ (OPS corresponding to $\mathscr{L}$), and $A_{n1}, \ldots, A_{nn}$ are positive numbers given by the Gauss quadrature formula

$$\mathscr{L}(x^k) = \mu_k = \sum_{i=1}^{n} A_{ni} x_{ni}^k, \qquad k = 0, 1, \ldots, 2n-1.$$

Then there is a subsequence in $\{\psi_n\}$ that converges on $(-\infty, +\infty)$ to a distribution function ψ which has an infinite spectrum and such that

$$\mathscr{L}(x^k) = \int_{-\infty}^{+\infty} x^k \, d\psi(x).$$

ψ is called a natural representative of $\mathscr{L}$.

From now on we will consider $\mathscr{L}$ to be positive-definite, and the associated data $\{x_{nm}\}$, $\{A_{nm}\}$, $\{\mu_n\}$, μ, μ_n defined as in the Theorem 3.3.

THEOREM 3.4. *Let $P_n(x)$ and $\lambda_1 P_n^{(1)}(x)$ be the partial denominators and numerators of a J-fraction as above, with c_n real numbers and $\lambda_n > 0$, $n \geq 1$. Let $\mathscr{L}$ be their associated moment functional. Then we have that*

$$\frac{\lambda_1 P_{n-1}^{(1)}(x)}{P_n(x)} = \sum_{k=1}^{n} \frac{A_{nk}}{x - x_{nk}} = \int_{-\infty}^{+\infty} \frac{d\psi_n(t)}{x-t}.$$

Moreover, A_{nk} can be expressed as

$$A_{nk} = \frac{\lambda_1 P_{n-1}^{(1)}(x_{nk})}{P_n'(x_{nk})}.$$

From Theorems 3.3 and 3.4 one can deduce the main result we will use later on, namely

COROLLARY 3.1. *In the conditions and notations of Theorems* 5.2 *and* 5.3, *there exists a subsequence $\{\psi_{n_k}\}$ in $\{\psi_n\}$ such that*

$$\lim_{k\to+\infty} \frac{\lambda_1 P_{n_k-1}^{(1)}(x)}{P_{n_k}(x)} = \int_{-\infty}^{+\infty} \frac{d\psi(t)}{x-t},$$

whenever x is not in the closure of the spectrum of ψ.

Next we will give a result describing the spectrum of distributions corresponding to symmetric problems. For broader information see Chihara [3] or Szegö [18]. Our notation and most of the results are stated as in Chihara's book.

THEOREM 3.5. *If a system is symmetric and $\lim_{n\to+\infty} \lambda_n = 0$ the set of limit points of the spectrum of ψ reduces to 0, and therefore the measure associated to $\mathscr{L}$ is discrete with 0 as the only possible accumulation point.*

Finally, we will quote a theorem that will be useful to actually compute the coefficients of a Taylor expansion for $C^{-2n2n}(\varepsilon)$.

THEOREM 3.6. *With reference to the recurrence formula* (3.9) *the following are valid for $n \geq 1$:*

(a) $\mathscr{L}(P_n^2(x)) = \lambda_1\lambda_2\cdots\lambda_{n+1}$, *provided that we define $\lambda_1 = \mu_0$.*

(b) $\mathscr{L}(\pi(x)P_n(x)) = 0$ *for any polynomial $\pi(x)$ of degree $m < n$, while $\mathscr{L}(\pi(x)P_n(x)) \neq 0$ if $m = n$.*

(c) $\mathscr{L}(x^n P_n(x)) = \mathscr{L}(P_n^2(x))$.

Now we are in condition to find a Taylor expansion for $C^{-2n2n}(\varepsilon)$.

B. *Taylor expansion for* $C^{-2n2n}(\varepsilon)$. Recall the recurrence problem (1)

$$G_{k+1} = XG_k - \frac{(2k-1)^2}{k(k-1)((kn)^2 - Y)(((k-1)n)^2 - Y)} G_{k-1}$$

with initial conditions

$$F_1 = 0, \quad F_2 = 1, \quad H_2 = 0, \quad H_3 = \frac{1}{2(Y - (2n)^2)}.$$

We can now assert that $F_{k+2}(X)$, $k \geq -1$, as in Proposition 3.2, are the set of monic orthogonal polynomials with respect to certain measure $d\psi_Y(X)$ and $\lambda_3^{-1} H_{k+2}$ the associated polynomials.

We also know that the associated moment functional is symmetric (since $c_n = 0$ in the recurrence). On the other hand

$$\lambda_{k+1} = \frac{(2k-1)^2}{k(k-1)((kn)^2 - Y)(((k-1)n)^2 - Y)} \to 0 \qquad \text{whenever } k \to +\infty,$$

so we can apply Theorem 3.5 to deduce that the measure associated to these orthogonal polynomials is absolutely discrete with zero as the only limit point of the spectrum of the natural representative ψ. Summarizing, one gets that, if we denote by $\mathscr{S}(\psi)$ the spectrum of ψ,

$$S(\psi) = \{z_k, -z_k, k \geq 0 | z_k \to 0 \text{ as } k \to +\infty\}$$

and $\{a_m\}$ are the weights of the corresponding measure, then $\mathscr{L}$ is defined as

$$\mathscr{L}(X^m) = \mu_m = 2\sum_{k=0}^{\infty} z_k^m a_m, \qquad m \geq 0.$$

Next, notice that Favard's theorem actually obtains a whole family of moment functionals associated to a fixed set of polynomials, one for each choice of λ_1, ($\{P_n(x)\}$ are independent of λ_1 given the initial condition $P_{-1}(x) = 0$). Due to the shift in the indices that we have, fix the value

$$\lambda_3 = \frac{1}{2(Y - (2n)^2)},$$

so that the pair (H_k, F_k) can be viewed as the kth partial numerator and denominator of the continued J-fraction

$$\frac{\lambda_3|}{|X} - \frac{\lambda_4|}{|X} - \frac{\lambda_5|}{|X} - \cdots .$$

Therefore, $\lambda_3^{-1}H_{k+2}$ are the associated polynomials with respect to problem (1). Now we can easily obtain a first expression for $C^{-2n2n}(\varepsilon)$.

If we apply Corollary 3.1 we obtain

$$\lim_{k\to+\infty}\frac{H_k(X)}{F_k(X)} = \frac{C^{-2n2n}(\varepsilon)}{\beta} = \int_{-\infty}^{+\infty}\frac{d\psi(t)}{x-t}$$
$$= \sum_{m=0}^{+\infty}\frac{a_m}{X-z_m} + \sum_{m=0}^{+\infty}\frac{a_m}{X+z_m}, \qquad X = (\varepsilon_n\varepsilon_{-n})^{\frac{-1}{2}}.$$

Observe that a_m and z_m depend on $Y = 2\varepsilon_0$ for all m.

We do not have much information about either the weights of the measure or the zeros of the polynomials. Even though this expression does not seem to be easy to compute we will give another expansion with coefficients that can be found following an easy algorithm.

MAIN THEOREM 3.7.

$$C^{-2n2n}(\varepsilon) = \frac{2\pi i}{n}\sum_{k=1}^{\infty}\mu_{2k}(\varepsilon_0)(\varepsilon_n\varepsilon_{-n})^k,$$

where μ_k are the moments corresponding to $\mathscr{L}$. Moreover, there exists an algorithm to obtain the moments up to any desired order.

Proof. Applying the result of Theorem 3.4 one gets

$$\frac{H_k(X)}{F_k(X)} = \sum_{m=3}^{k}\frac{A_{km}}{X-x_{km}} = \sum_{m=3}^{k}\frac{H_k(x_{km})}{F'(x_{km})(X-x_{km})}, \qquad k = 3, 4, \ldots,$$

where A_{km} and x_{km} are analogous to the ones in Theorem 3.4. If we Taylor-expand the expressions as a function of $\frac{1}{X}$ we obtain

$$\frac{A_{km}}{X-x_{km}} = \sum_{l=0}^{p}A_{km}\left(\frac{x_{km}^l}{X^{l+1}}\right) + o\left(\frac{1}{X^{p+2}}\right).$$

Substituting above

$$\frac{H_k(X)}{F_k(X)} = \sum_{m=3}^{k}\frac{A_{km}}{X-x_{km}} = \sum_{m=3}^{k}\left[\sum_{l=0}^{p}A_{km}\left(\frac{x_{km}^l}{X^{l+1}}\right) + o\left(\frac{1}{X^{p+2}}\right)\right].$$

On the other hand, if ψ_k is given as in Theorem 3.3, $\mathscr{L}_k$ is the associated functional and μ_l^k is the corresponding 1-moment, then

$$\mu_l^k = \mathscr{L}_k(x^l) = \sum_{m=3}^{k} A_{km} x_{km}^l ,$$

and therefore

$$\frac{H_k(X)}{F_k(X)} = \sum_{l=0}^{p} \frac{\mu_l^k}{X^{l+1}} + o\left(\frac{1}{X^{p+2}}\right) .$$

A priori we know that the sequence converges to an analytic function on ε_n, ε_{-n}, ε_0; therefore, we can take limits without any problem and deduce the result of the theorem. Notice that $\mathscr{L}$ is symmetric and that property implies $\mu_{2k+1} = 0$ for $k \geq 0$. That is the reason to have only even powers of $X = 1/(\varepsilon_n \varepsilon_{-n})^{1/2}$ in the series above.

To finish with the proof of the theorem, we will give the algorithm to find the moments, avoiding the inconvenience of not having information about the explicit form of $d\psi_Y$.

From Theorem 3.6(a), we can deduce

$$\mu_0 = \int_{-\infty}^{+\infty} F_2^2(X)\, d\psi_Y(X) = \lambda_3 = \frac{1}{2(Y-(2n^2)^2)} ,$$

$$\mu_2 = \int_{-\infty}^{+\infty} F_3^2(X)\, d\psi_Y(X) = \lambda_3\lambda_4 = \frac{5^2}{12(Y-(2n^2)^2)(Y-(3n^2)^2)} .$$

In order to find μ_4, notice that $F_4 = XF_3 - \lambda_4 F_2 = X^2 - \lambda_4$, so $X^2 = F_4 + \lambda_4$ and therefore $X^4 = F_4^2 + \lambda_4^2 + 2\lambda_4 F_4$. Applying Theorem 3.6(b), we get zero when integrating the last term of the sum, so that

$$\begin{aligned}\mu_4 &= \int_{-\infty}^{+\infty} X^4\, d\psi_Y(X) = \int_{-\infty}^{+\infty} F_4^2(X)\, d\psi_Y(X) + \int_{-\infty}^{+\infty} \lambda_4^2\, d\psi_Y(X) \\ &= \lambda_3\lambda_4\lambda_5 + \lambda_4^2\mu_0 .\end{aligned}$$

In this way we can always obtain μ_{2k} in terms of μ_{2l}, $l < k$, and the integral of the square of $P_k(X)$ which value we know from Theorem 3.6(a). Repeating this process we can give the expression for moments up to any order we wish. This algorithm is not very fast since it requires us to solve for the orthogonal polynomials in the first place. For example, we obtain

$$\mu_6 = \lambda_3\lambda_4\lambda_5\lambda_6 + (\lambda_4+\lambda_5)^2\lambda_2 ,$$

$$\begin{aligned}\mu_8 = \lambda_3\lambda_4\lambda_5\lambda_6\lambda_7 &+ (\lambda_4+\lambda_5+\lambda_6)^2\mu_4 \\ &- 2\lambda_4\lambda_6(\lambda_4+\lambda_5+\lambda_6)\mu_2 + \lambda_4^2\lambda_6^2\mu_0 ,\end{aligned}$$

$$\ldots$$

□

The author has a faster algorithm and a short computer program to calculate the moments. It involves Favard's path's theory (see Viennot's notes [**19**]), but we will not give further details in this paper.

Notice that, with very few adjustments, we can follow the exact same reasoning to find a Taylor expansion for any entry of the inverse matrix C^{-1}. That is, this is a general technique to find entries for the inverse of an infinite Jacobi matrix.

C. *Another interpretation for a transverse structure.* Let us look at the function $C^{-2n2n}(p)$ from another point of view. The next theorem will show us how to express transverse structures in terms of the solutions of some nonhomogeneous ordinary differential equations. The corresponding homogeneous equation is always given by the coadjoint action along Q_n.

THEOREM 3.8. *Consider the differential equation*

$$\xi''' + 2p\xi' + p'\xi = 2\cos(2n\theta), \tag{3.9}$$

with $p \in Q_n$.

There exists a periodic solution of (3.9), ξ, *whose Taylor expansion is given by* $\xi = \sum_{k=-\infty}^{+\infty} \gamma_k e^{-ik\theta}$, *with* $\gamma_{2n} = C^{-2n2n}(p)$.

Proof. Assume $\xi = \sum_{k=-\infty}^{+\infty} \gamma_k e^{-ik\theta}$. If we make a simple substitution we can observe that the action of the differential operator $-(\frac{d}{d\theta^3} + 2p\frac{\partial}{\partial\theta} + \frac{dp}{d\theta})$ on ξ is equivalent to the one of the matrix C on γ, where $\gamma = \{\gamma_k\} \in l^2$. This is true since

$$\begin{aligned}\xi''' &+ 2p\xi' + p'\xi \\ &= \sum_{k=-\infty}^{+\infty} [\gamma_k k(2\varepsilon_0(p) - k^2) + \gamma_{n+k}(2k+n)\varepsilon_{-n}(p) \\ &\qquad\qquad + \gamma_{k-n}(2k-n)\varepsilon_n(p)]e^{-ik\theta}.\end{aligned}$$

Notice at this point that the matrix C is antisymmetric. Therefore, we can solve the equation $C\gamma = b$, with b having entries all 0's except for the entry in place $-2n$, and obtain that $\gamma_{2n} = C^{-2n2n}(p)$.

But, on the other hand, to solve $C\gamma = b$ is equivalent to solving the differential equation (3.9), in the sense that the solution of $C\gamma = b$ would correspond to the Fourier coefficients of a solution of (3.9). We are done with the proof. □

Notice that we can follow the same strategy in order to find any entry of the inverse matrix for C. That is, C^{kl} *would be given by the* *lth Fourier coefficient of a periodic solution of the equation*

$$\xi''' + 2p\xi' + p'\xi = 2\cos(k\theta)\,.$$

From Proposition 3.1 we know that such a solution exists.

Finally one comment on the uniqueness problem. If we try to prove uniqueness in the same way that it is done in the finite dimensional case, we would have to try to connect two different transverse sections using the flow of a time-dependent Hamiltonian vector field. This flow would be defined on a neighbourhood of the intersection with the symplectic leaves and would automatically preserve the induced transverse structures. The existence of such a flow would automatically imply uniqueness.

In finite dimensions such a Hamiltonian vector field can always be found. In infinite dimensions we can connect two transverse sections Q_1 and Q_2 with a family of transverse sections Q_t with $1 \le t \le 2$. We can possibly fix the variation on the time so that the equations for the Hamiltonian operator are involutive. Nevertheless, that fact would not imply its integrability. This kind of integrability problem in infinite dimensions is quite complicated and not many results are available.

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