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WEIGHTED MAXIMAL FUNCTIONS AND DERIVATIVES OF INVARIANT POISSON INTEGRALS OF POTENTIALS

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In this paper we prove L^p estimates for weighted maximal functions of invariant Poisson integrals of potentials. From this it follows that the exceptional sets of the Poisson integrals of potentials are sets of zero Hausdorff capacity.

Let S denote the boundary of B_n, the unit ball in C^n, and let $d\sigma$ be the unusual rotation invariant measure defined on S. If g is a function belonging to the usual Lebesgue space $L^1(d\sigma)$ of functions defined on the sphere then by $P[g]$ we will mean the invariant Poisson integral of g defined by the equation

$$P[g](z) = \int_S g(\eta)\frac{(1-|z|^2)^n}{|1-\langle z, \eta\rangle|^{2n}}\, d\sigma(\eta),$$

where $z \in B_n$.

In this paper we will continue the work of Ahern and Cascante [**ACa**] and study invariant Poisson integrals of potentials of distributions in the atomic Hardy spaces H^p_{at} where $0 < p \leq 1$. Precisely, if v denotes a distribution in the space H^p_{at} defined by Garnett and Latter and if $0 < \beta < n$ and $\zeta \in S$ define the (non-isotopic) potential of v by

$$I_\beta v(\zeta) = \int_S v(\eta)\frac{d\sigma(\eta)}{|1-\langle \zeta, \eta\rangle|^{n-\beta}}.$$

Let $f(z) = P[I_\beta v](z)$ and denote by f^*_α the admissible maximal function of f defined on the sphere S associated with the admissible approach region of aperture α. Thus, for each fixed $\alpha > 1$

$$f^*_\alpha(\zeta) = \sup_{w \in \Gamma_\alpha(\zeta)} |f(w)|,$$

where $\Gamma_a(\zeta)$ is the admissible approach region

$$\Gamma_\alpha(\zeta) = \{w \in B_n : |1-\langle w, \zeta\rangle| < \tfrac{\alpha}{2}(1-|w|^2)\}.$$

Suppose that μ is a positive measure on S satisfying the condition

$$\mu(B(\zeta; \delta)) \leq C\delta^{n-\beta p} \tag{1}$$

for every Koranyi ball

$$B(\zeta;\delta)=\{\eta\in S:|1-\langle\eta,\zeta\rangle|\le\delta\}$$

centered at ζ of radius δ contained in S. In **[ACa]** the following result is proved.

THEOREM 1. *Suppose that β is an integer between* 0 *and* $n-1$. *Let* μ *be a positive measure satisfying condition* (1). *Then with* v *and* f *related as above, there is a positive constant* C, *depending on* α *but independent of* v, *such that*

$$\int (f_\alpha^*)^p\,d\mu\le C\|v\|^p_{H^p_{at}}.$$

In this paper we will remove from Theorem 1 the restriction that β be an integer. In order to explain the method we pursue we first recall the basic idea used to establish Theorem 1.

For $z\in B_n$ let R be the operator given by

$$Rf(z)=\sum_{j=1}^n z_jD_jf(z),$$

where $D_j=\frac{\partial}{\partial z_j}$ and let $\overline{R}$ be the operator given by

$$\overline{R}f(z)=\sum_{j=1}^n \overline{z}_j\overline{D}_jf(z),$$

where $\overline{D}_j=\frac{\partial}{\partial\overline{z}_j}$. If $z=r\zeta$ where $\zeta\in S$ then it is easily verified that

$$\frac{\partial}{\partial r}(rf(r\zeta))=(R+\overline{R}+\mathrm{id})f(z).$$

From this it follows that

$$(k-1)!f(z)=\int_0^1\log^{k-1}\left(\frac{1}{t}\right)(R+\overline{R}+\mathrm{id})^kf(tz)\,dt. \tag{2}$$

In **[ACa]** it is shown that if $v\in H^p_{at}$ and $f=P[I_kv]$, then the admissible maximal function of $(R+\overline{R}+\mathrm{id})^kf(z)$ belongs to L^p. The argument used in **[A]** then can be applied to derive the conclusion of Theorem 1. For the case we are considering, that is, $f=P[I_\beta v]$ where β is not an integer, in order to use an argument patterned on the one above, we must find a suitable replacement for equation (2). The difficulty we face is that if we tailor the definition of $(R+\overline{R}+\mathrm{id})^kf(z)$ for non-integral k in such a way that equation (2) still holds then the methods of **[ACa]** are no longer sufficient by themselves to show the other fact that is needed, namely that the admissible maximal function of $(R+\overline{R}+\mathrm{id})^k f(z)$ is in L^p. (This problem does not occur if v belongs to the Hardy space H^p of holomorphic functions; see [A].) We circumvent this obstacle in the following fashion. With $f(z) = P[I_\beta](z)$ let

$$u(z) = (1-|z|)^{k-\beta}(R+\overline{R}+\mathrm{id})^k f(z),$$

where k is an integer greater than β but less than n. It can be verified that

$$(3) \qquad (k-1)!|z|f(z) = \int_0^{|z|} \left(\log\left(\frac{|z|}{t}\right)\right)^{k-1} (1-t)^{\beta-k} u(t\zeta)\,dt,$$

where $z = r\zeta$, and $\zeta \in S$. The main result of this paper will be the following theorem.

THEOREM 2. *Let* $v \in H^p_{at}$, $0 < \beta < n-1$, *and* $f = P[I_\beta v]$. *If* k *is an integer greater than* β *but less than* n, *then the function* $u(z) = (1-|z|)^{k-\beta}(R+\overline{R}+\mathrm{id})^k f(z)$ *has admissible maximal function in* L^p.

Theorem 2 and the representation given by equation (3) can be used to apply the method of [A] to estimate f^*_α; the idea is that the factor $(\log(\frac{|z|}{t}))^{k-1}(1-t)^{\beta-k}$ will serve just as well as the factor $(\log(\frac{1}{t}))^{\beta-1}$ appearing in (2). We thus obtain the following corollary.

COROLLARY 1. *Theorem* 1 *remains true for all values of* β *between* 0 *and* $n-1$.

We will need to make use of the following objects. Let $\zeta \in S$ and for $1 \le j, k \le n$ define the complex tangential vector field

$$T_{j,k} = \overline{\zeta}_j \frac{\partial}{\partial \zeta_k} - \overline{\zeta}_k \frac{\partial}{\partial \zeta_j}$$

and let $\overline{T}_{j,k}$ be the conjugate of $T_{j,k}$. Furthermore, let

$$L = \sum_{j<k} \overline{T}_{j,k} T_{j,k}$$

and

$$\overline{L} = \sum_{j<k} T_{j,k} \overline{T}_{j,k}.$$

If f is a function defined on B_n then for $z \in B_n$ with $z = r\zeta$ and $\zeta \in S$ we define

$$T_{j,k}f(z) = T_{j,k}f(r\zeta),$$

where the right-hand side is computed by holding r fixed and interpreting $f(r\zeta)$ as a function defined on the sphere. Then other operators above are also extended to act on functions on the ball in a similar fashion. We will need the following observations. Suppose that g is a smooth function of one complex variable. Let ζ and η range over the sphere S. Then

$$L_\zeta g(\langle \zeta, \eta\rangle) = \overline{L}_\eta g(\langle \zeta, \eta\rangle), \tag{4}$$

where the subscripts on the operators denote which variable the derivatives are taken with respect to. Furthermore, there is a second function, h, of one complex variable, such that

$$L_\zeta g(\langle \zeta, \eta\rangle) = h(\langle \zeta, \eta\rangle). \tag{5}$$

In fact, direct calculation shows that formula (4) is valid and that both expressions are equal to

$$(1 - |\langle \zeta, \eta\rangle|^2) D\overline{D} g(\langle \zeta, \eta\rangle) - (n-1)\langle \zeta, \eta\rangle D g(\langle \zeta, \eta\rangle),$$

where D and $\overline{D}$ denote the usual operators $D = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\overline{D} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. This proves the second assertion as well.

The following variants of the Poisson kernel used by Geller in [G] will also be of importance. For integers j and l let $P_{j,l}$ be the kernel

$$P_{j,l}(z, \eta) = \frac{(1 - |z|^2)^{n+j+l}}{(1 - \langle z, \eta\rangle)^{n+j}(1 - \langle \eta, z\rangle)^{n+l}}.$$

These kernels will concern us when j and l are non-positive integers whose sum is greater than $-n$. Notice that $P_{0,0}$ is the usual Poisson kernel.

Before proceeding to the proof of Theorem 2 we will need some preliminary results. We remark that in what follows we will follow the custom of using the letter C to stand for a positive constant which changes its value from one appearance to another while remaining independent of the important variables.

LEMMA 1. *Let g and h be bounded functions defined on the unit ball in C^1 and suppose ζ and η are points on the sphere in C^n. Then*

$$\int_S g(\langle \zeta, \omega\rangle) h(\langle \omega, \eta\rangle)\, d\sigma(\omega) = \int_S h(\langle \zeta, \omega\rangle) g(\langle \omega, \eta\rangle)\, d\sigma(\omega).$$

Proof. Denote by $L(\zeta, \eta)$ the left-hand integral in the statement of the theorem and by $R(\zeta, \eta)$ the right-hand integral. For each ζ both expressions are continuous functions in the variable η. The desired conclusion will therefore follow if we show that for any smooth function ϕ defined on the sphere we have the equality

$$\int_S \phi(\eta)L(\zeta, \eta)\, d\sigma(\eta) = \int_S \phi(\eta)R(\zeta, \eta)\, d\sigma(\eta).$$

This in turn will follow if we show it to be true for all functions ϕ belonging to the space $H(p, q)$ of restrictions to S to homogeneous harmonic polynomials of bidegree (p, q) for all p and q. Now,

$$\begin{aligned}&\int_S \phi(\eta)L(\zeta, \eta)\, d\sigma(\eta)\\ &\qquad = \int_S g(\langle\zeta, \omega\rangle) \int_S h(\langle\omega, \eta\rangle)\phi(\eta)\, d\sigma(\eta)\, d\sigma(\omega).\end{aligned}$$

Let the inside integral of the right-hand side of the last equation define the operator

$$T(\phi)(\omega) = \int_S h(\langle\omega, \eta\rangle)\phi(\eta)\, d\sigma(\eta).$$

It is easily checked that T commutes with the usual action of the group of unitary operators on S. By Theorem 12.3.8 in **[R]** it follows that for all $\phi \in H(p, q)$

$$T(\phi) = C_h\phi,$$

where C_h is a constant depending only on h, p and q. Therefore

$$\int_S \phi(\eta)L(\zeta, \eta)\, d\sigma(\eta) = C_h \int_S g(\langle\zeta, \omega\rangle)\phi(\omega)\, d\sigma(\omega)$$

and by the same reasoning it follows that

$$\int_S \phi(\eta)L(\zeta, \eta)\, d\sigma(\eta) = C_h C_g \phi(\zeta).$$

An identical argument gives the formula

$$\int_S \phi(\eta)R(\zeta, \eta)\, d\sigma(\eta) = C_g C_h \phi(\zeta)$$

for all $\phi \in H(p, q)$, where the constants C_g and C_h are the same as before. This completes the proof. □

Remark. The hypothesis that h and g be bounded is clearly not the weakest on h and g which allows some version of the conclusion of Lemma 1 to hold. If, for example, we assume only that the functions used in the proof are integrable on the sphere and therefore permit the application of Fubini's theorem, the argument will show that the equality of Lemma 1 holds almost everywhere $d\sigma(\zeta)\,d\sigma(\eta)$. In what follows, we will use this version of Lemma 1 whenever the hypotheses on g and h satisfy these less restrictive conditions.

While there is no natural group structure that allows us to define convolution, the Hermitian inner product provides a well-known substitute. If g is a function defined on the unit ball in C^1 and $\zeta \in S$ for a function F defined on the sphere let $F * g$ be given by

$$F * g(\zeta) = \int_S F(\eta) g(\langle \zeta, \eta \rangle)\, d\sigma(\eta).$$

The integral will be well-defined whenever $F \in L^1(d\sigma)$ and $g(\zeta_1) \in L^1(d\sigma)$. Here, of course, by ζ_1 we mean the first coordinate of the variable $\zeta \in C^n$. As a corollary of Lemma 1 we have the following result.

COROLLARY 2. *Let g and h be functions defined on the unit ball in C^1 such that both $g(\zeta_1)$ and $h(\zeta_1)$ are in $L^1(d\sigma)$. Let $F \in L^1(d\sigma)$. Then*

$$(F * g) * h = (F * h) * g.$$

Proof. The proof is accomplished through Fubini's theorem and the remark following Lemma 1. □

We will also need to notice that "convolution" commutes with the operators L and $\overline{L}$.

LEMMA 2. *Let F be a smooth function on S and g a smooth function of one complex variable. Let X be either L or $\overline{L}$. Then*

$$X(F * g)(\zeta) = (XF * g)(\zeta).$$

Proof. Use integration by parts and formula (4) to compute that

$$\begin{aligned} X(F * g)(\zeta) &= X_\zeta \int_S F(\eta) g(\langle \zeta, \eta \rangle)\, d\sigma(\eta) \\ &= \int_S F(\eta) X_\zeta g(\langle \zeta, \eta \rangle)\, d\sigma(\eta) \\ &= \int_S F(\eta) \overline{X}_\eta g(\langle \zeta, \eta \rangle)\, d\sigma(\eta) \\ &= \int_S XF(\eta) g(\langle \zeta, \eta \rangle)\, d\sigma(\eta) \\ &= (XF * g)(\zeta), \end{aligned}$$

as claimed. □

We will also need pointwise estimates on the derivatives of an invariant harmonic functions. See Theorem 1.2 of **[G]** for the analogous estimates associated with the Heisenberg group. Let $a \in B_n$ and for $\varepsilon > 0$ define

$$Q(a;\varepsilon) = \{w \in B_n : |1 - \langle w, a\rangle| < \varepsilon\}.$$

LEMMA 3. *Let U be an invariant harmonic function defined on B_n. If $a \in B_n$ let*

$$U^+(a) = \sup\left\{|U(w)| : w \in Q\left(a;\frac{1-|a|}{2}\right)\right\}.$$

Then for each pair of non-negative integers j and l there is a constant $C = C(j, l)$ independent of a or U such that

$$|\overline{R}^j R^l U(a)| \le C(1-|a|)^{-j-l}U^*(a).$$

Proof. The proof is based on the same idea as the proof of Theorem 1.2 in **[G]**. For each $a \in B_n$ let ϕ_a be the automorphism of the ball given on page 25 of **[R]**. Let ψ be a smooth nonnegative function of a real variable supported on the interval $[0, s]$. We may choose s so small that for all a ϕ_a maps the ball in C^n centered at the origin of radius s into $Q(a;\frac{1-|a|}{2})$. Next, let $\Psi(w) = \psi(|w|)$ for $w \in B_n$. The argument used in **[G**, p. 130] (see also **[ACa**, equation 1.2]) shows that there is a constant C independent of U or a such that

$$U(a) = C\int_{B_n} U(w)\Psi(\phi_a(w))\,d\nu(w),$$

where $d\nu$ is the invariant measure

$$d\nu(w) = \frac{dV(w)}{(1-|w|^2)^{n+1}},$$

and dV is Lebesgue measure on C^n. The desired estimate follows now by first differentiating under the integral sign, then using the fact that $\Psi(\phi_a(w))$ is supported on the set $Q(a;\frac{1-|a|}{2})$ together with the formula for $\phi_a(w)$ to bound the resulting expressions by $C(1-|a|)^{-j-l}$, and finally observing that the invariant measure of $Q(a;\frac{1-|a|}{2})$ is bounded by a constant independent of a. This completes the proof. □

We are now ready to give the proof of Theorem 2.

Proof of Theorem 2. At times we will simplify the notation by suppressing the dependence of the admissible maximal function on the parameter α determining its aperture. The atomic decomposition of Garnett and Latter shows that Theorem 2 is a consequence of the following assertion.

Claim. Let a be a (p, ∞) atom in H^p_{at}. Suppose $f = P[I_\beta a]$ and $u(z) = (1 - |z|)^{k-\beta}(\overline{R} + R)^k f(z)$. Then there is a constant C depending only on α, k, and p, but not a such that

$$\int_S (u^*_\alpha)^p \, d\sigma \leq C.$$

We first give a detailed proof of the claim for the case where $0 < \beta < 1$ and $k = 1$. Since all the ideas necessary to establish the claim in full are present in this situation we will only sketch how the argument goes in general. Assume then that $0 < \beta < 1$, $k = 1$, and a is a (p, ∞) atom in H^p_{at}. We may assume that a is an atom centered at e_1 supported in the Koranyi ball

$$B(e_1; \delta) = \{\eta \in S : |1 - \eta_1| \leq \delta\},$$

where $e_1 = (1, 0, \dots, 0)$. Recall that

$$|a| \leq \delta^{-n/p},$$

and that a has vanishing moments up to a certain order depending on p; see **[GL]** for details. We note for later use that the construction of the atomic decomposition given in **[GL]** shows that this order may actually be taken to be arbitrarily large. Let

$$u(z) = (1 - |z|)^{1-\beta}(\overline{R} + R)P[I_\beta a](z).$$

For λ a complex number in the unit disk and $r < 1$ define

$$P_r(\lambda) = \frac{(1 - r^2)^n}{|1 - r\lambda|^{2n}}.$$

If $z = r\zeta$ with $\zeta \in S$ and $F \in L^1(d\sigma)$ then we may write

$$P[F](z) = F * P_r(\zeta).$$

By Corollary 2 it follows that

$$P[I_\beta a](z) = (a * I_\beta) * P_r(\zeta) = (a * P_r) * I_\beta(\zeta).$$

Let V be the invariant harmonic function given by $V = P[a]$. Since $\overline{R} + R = r\frac{\partial}{\partial r}$ it follows that

$$(\overline{R} + R)P[I_\beta a](z) = \left(r\frac{\partial}{\partial r}a * P_r\right) * I_\beta(\zeta). \tag{6}$$

The right-hand side may be rewritten as

$$\int_S \frac{1}{|1-\langle \zeta , \eta \rangle|^{n-\beta}} (\overline{R} + R) V(r\zeta)\, d\sigma(\eta) .$$

Notice that the operator $\overline{R} + R$ now acts on the variable $r\eta$. We may therefore write u as the sum of

$$u_1(z) = (1-|z|)^{1-\beta} \int_S \frac{1}{|1-\langle \zeta , \eta \rangle|^{n-\beta}} RV(r\eta)\, d\sigma(\eta) \tag{7}$$

and a similar expression, $u_2(z)$, which is obtained from the formula for u_1 by replacing R by $\overline{R}$. We proceed to show that there is a constant C independent of a such that

$$\int_S (u_1^*)^p \, d\sigma \leq C .$$

The same argument will establish the same inequality for u_2, and complete the proof for the case we are considering.

We first split u_1 into two parts. Let ψ be a non-negative $\mathscr{C}^\infty$ function supported on the disk in the complex plane centered at the origin of radius $\frac{1}{2}$ which is identically 1 on the disk centered at the origin of radius $\frac{1}{4}$. For $0 \leq r < 1$ and ζ and η in S let

$$\psi_r(\zeta , \eta) = \psi\left(\frac{1-\langle \zeta , \eta \rangle}{1-r}\right) .$$

Then

$$u_1(z) = J_1(z) + J_2(z) ,$$

where

$$J_1(z) = (1-|z|)^{1-\beta} \int_S \frac{\psi_r(\zeta , \eta)}{|1-\langle \zeta , \eta \rangle|^{n-\beta}} RV(r\eta)\, d\sigma(\eta) \tag{8}$$

and

$$J_2(z) = (1-|z|)^{1-\beta} \int_S \frac{1-\psi_r(\zeta , \eta)}{|1-\langle \zeta , \eta \rangle|^{n-\beta}} RV(r\eta)\, d\sigma(\eta) . \tag{9}$$

Consider first J_1. Let $\xi \in S$ and suppose that $z = r\zeta \in \Gamma_\alpha(\xi)$ for some aperture α. Since the integrand in J_1 vanishes for $|1-\langle \zeta , \eta \rangle| > \frac{1}{2}(1-r)$ it is easy to see that on the support of $\psi_r(\zeta , \eta)$ we may apply Lemma 3 to $RV(r\eta)$ to get the estimate

$$|RV(r\eta)| \leq C(1-r)^{-1} V^*(\xi) , \tag{10}$$

provided that the maximal function V^* is taken with respect to an aperture equal to a fixed constant c times α, where c is independent of the atom a. It follows that

$$|J_1(z)| \leq C(1-r)^{-\beta} V^*_{c\alpha}(\xi) \int_S \frac{\psi_r(\zeta, \eta)}{|1-\langle \zeta, \eta\rangle|^{n-\beta}}\, d\sigma .$$

If we use again the fact that $\psi_r(\zeta, \eta)$ is supported on a Koranyi ball centered at ζ of radius $1-r$ then the integral in the last inequality may be estimated as in [**ACo**, p. 427] to yield the conclusion that

$$|J_1(z)| \leq CV^*(\xi).$$

From this it follows easily from the fact that a is a (p, ∞) atom that

$$\int_S (J_1^*)^p\, d\sigma \leq C \int_S (V^*)^p\, d\sigma \leq C ;$$

see [**GL**].

The analysis necessary to handle J_2 will be more complicated. We first make use of Theorem 1 and Lemma 1.4 from [**ACa**] together with Lemma 2 above to write

$$-(n-1)RP[a](r\eta) = P_{0,-1}[La](r\eta) = LP_{0,-1}[a](r\eta). \tag{11}$$

We remark that the equality of the first and last terms above may be verified directly by showing that

$$-(n-1)R_z P_{0,0}(z, \eta) = L_\zeta P_{0,-1}(z, \eta).$$

In any event, let

$$G_r(\lambda) = \frac{(1-r^2)^{n-1}}{(1-r\lambda)^n(1-r\overline{\lambda})^{n-1}},$$

so

$$-(n-1)RV(r\eta) = RP[a](r\eta) = LP_{0,-1}[a](r\eta) = L(a * G_r)(\eta).$$

Next let

$$-(n-1)K_r(\lambda) = \left(1-\psi\left(\frac{1-\lambda}{1-r}\right)\right)|1-\lambda|^{\beta-n}.$$

Then we may write

$$J_2(z) = (1-r)^{1-\beta}(L(a * G_r)) * K_r(\zeta).$$

Integration by parts shows that

$$\begin{aligned}(L(a * G_r)) * K_r(\zeta) &= \int_S L(a * G_r(\eta))K_r(\langle \zeta, \eta\rangle)\, d\sigma(\eta) \\ &= \int_S (a * G_r(\eta))\overline{L}K_r(\langle \zeta, \eta\rangle)\, d\sigma(\eta).\end{aligned}$$

By formula (5) $\overline{L}K_r(\langle\zeta, \eta\rangle)$ is, for fixed r, a function of $\langle\zeta, \eta\rangle$. In fact, a calculation shows that $\overline{L}K_r(\langle\zeta, \eta\rangle)$ is the sum of

$$\left(1 - \psi\left(\frac{1-\langle\zeta, \eta\rangle}{1-r}\right)\right)\overline{L}|1 - \langle\zeta, \eta\rangle|^{\beta-n}$$

and three other terms each of which has a factor which is a derivative of ψ. Now, differentiating ψ yields a function which is supported on the region

$$\left\{\eta \in S : \frac{1-r}{4} < |1 - \langle\zeta, \eta\rangle| < \frac{1-r}{2}\right\}.$$

These terms can then be handled in the same fashion as J_1 above. We therefore are left with the final task of estimating the admissible maximal function of

$$J_3(z) = (1-|z|)^{1-\beta}\int_S \left(1 - \psi\left(\frac{1-\langle\zeta, \eta\rangle}{1-r}\right)\right) \cdot \overline{L}|1 - \langle\zeta, \eta\rangle|^{\beta-n}(a * G_r(\eta))\,d\sigma.$$

To simplify the notation, let

$$Q_r(\langle\zeta, \eta\rangle) = (1-r)^{1-\beta}\left(1 - \psi\left(\frac{1-\langle\zeta, \eta\rangle}{1-r}\right)\right)\overline{L}|1 - \langle\zeta, \eta\rangle|^{\beta-n},$$

where Q_r is a function of one complex variable; equation (5) shows that this is possible. We therefore obtain the formula

$$J_3(z) = (a * G_r) * Q_r(\zeta).$$

Recall that the atom a is supported on the Koranyi ball

$$B(e_1; \delta) = \{\eta \in S : |1 - \eta_1| \le \delta\},$$

where $e_1 = (1, 0, \ldots, 0)$. We will need to partition unity in a manner that lets us take advantage of the support of a. It is possible to find smooth functions ϕ_0 and ϕ defined on the complex plane such that ϕ_0 is supported on the unit disk, ϕ is supported on the annulus $\{\lambda \in C^1 : 1/2 \le |\lambda| \le 2\}$ and

$$1 = \phi_0(\lambda) + \sum_{j=0}^{\infty}\phi\left(\frac{\lambda}{2^j}\right).$$

For η and $\tau \in S$ and $r\eta \in B_n$ let

$$\Phi_0(r\eta, \tau) = \phi_0\left(\frac{1-\langle r\eta, \tau\rangle}{8\delta}\right)$$

and for $j = 1, 2, \ldots, N$ let

$$\Phi_j(r\eta, \tau) = \phi\left(\frac{1 - \langle r\eta, \tau\rangle}{2^{j-1}8\delta}\right).$$

It follows that

$$1 = \sum_{j=0}^{N} \Phi_j(r\eta, \tau),$$

where N is a sufficiently large integer which depends only on δ.

We now write

$$a * G_r(\eta) = P_{0,-1}[a](r\eta) = \sum_{j=0}^{N} A_j(r\eta),$$

where

$$A_j(r\eta) = \int_S P_{0,-1}(r\eta, \tau)\Phi_j(r\eta, \tau)a(\tau)\,d\sigma(\tau).$$

We claim that there is an integer m that we may choose to be arbitrarily large (and whose choice will depend on p) such that

(i) $A_0(r\eta)$ is supported on the set

$$\{r\eta : |1 - r\eta_1| < 32\delta\};$$

(ii) $|A_0(r\eta)| \le C\delta^{-n/p}$;

(iii) For $j = 1, \ldots, N$, $A_j(r\eta)$ is supported on the set

$$\{r\eta : 2^{j-1}\delta < |1 - r\eta_1| < 32 \cdot 2^j g\}d;$$

(iv) For $j = 1, \ldots, N$

$$|A_j(r\eta)| \le C(2^j)^{-n-m+n/p}(2^j\delta)^{-n/p}.$$

Properties (i), (ii) and (iii) follow immediately from the definition of Φ_j, the support and size of the atom a, and the triangle inequality for the pseudometric $d(z, w) = |1 - \langle z, w\rangle|^{1/2}$ proved in **[R]**, Proposition 5.1.2. To verify property (iv) we must use the cancellation properties of the atom a in the usual way. For $2^{j-1}\delta < |1 - r\eta_1|$ estimate that

$$|A_j(r\eta)| \le \left|\int a(\tau)[P_{0,-1}(r\eta, \tau)\Phi_j(r\eta, \tau) - T_m(\tau, e_1)]\,d\sigma(\tau)\right|,$$

where for each fixed $r\eta$, $T_m(\tau, e_1)$ is the non-isotropic Taylor polynomial for $P_{0,-1}(r\eta, \tau)\Phi_j(r\eta, \tau)$ expanded about e_1 of degree m; see **[GL]** for the precise details. Since

$$2^{j-1}\delta < |1 - r\eta_1| < 32 \cdot 2^j\delta$$

we may estimate that

$$|P_{0,-1}(r\eta, \tau)\Phi_j(r\eta, \tau) - T_m(\tau, e_1)| \le C\frac{\delta^m}{(2^j\delta)^{n+m}}.$$

It follows that

$$|A_j(r\eta)| \le C\delta^{n-n/p}\frac{\delta^m}{(2^j\delta)^{n+m}},$$

as claimed.

We now write

$$J_3(z) = \sum_{j=0}^{N} F_j(r\zeta),$$

where

$$F_j(r\zeta) = \int_S A_j(r\eta)Q_r(\zeta, \eta)\, d\sigma(\eta)$$

and proceed to estimate F_j^* for each j. From the formula for $Q_r(\zeta, \eta)$ it is not hard to see that

$$\int_S |Q_r(\zeta, \eta)|\, d\sigma(\eta) \le C$$

for a constant C independent of r; we have used the fact that $Q_r(\zeta, \eta)$ vanishes identically on the Koranyi ball centered at ζ of radius $1-r$ as well as the estimates found in **[R]** Proposition 1.4.10. It follows therefore that for each j

$$\|F_j\|_\infty \le C\|A_j\|_\infty$$

and therefore

$$\|F_j^*\|_\infty \le C(2^j)^{-n-m+n/p}(2^j\delta)^{-n/p}. \tag{12}$$

Recall that the admissible maximal region depends on the parameter α which controls its aperture. Set $M = 1000\alpha$. We will use inequality (12) above to estimate F_j^* on the set $\{\xi \in S : |1-\xi_1| \le M2^j\delta\}$.

Assume then that $|1-\xi_1| > M2^j\delta$, and let $r\zeta \in \Gamma_\alpha(\xi)$. From properties (i) and (iii) it follows that $F_j(r\zeta)$ vanishes unless $1-r < 32\cdot 2^j\delta$ so we may as well assume that $1-r < 32\cdot 2^j\delta$. Let

$$U_{j,r}(\langle\eta, \tau\rangle) = P_{0,-1}(r\eta, \tau)\Phi_j(r\eta, \tau),$$

where, for each fixed r, $U_{j,r}$ is a function of one complex variable; notice that the definition of $\Phi_j(r\eta, \tau)$ makes this possible. Then by Corollary 2

$$\begin{aligned} F_j(r\zeta) &= (a * U_{j,r}) * Q_r(\zeta) \\ &= (a * Q_r) * U_{j,r}(\zeta). \end{aligned}$$

Thus

$$F_j(r\zeta) = \int_S a * Q_r(\eta) U_{j,r}(\langle \zeta, \eta \rangle)\, d\sigma(\eta).$$

Notice that, since $U_{j,r}(\langle \zeta, \eta \rangle)$ vanishes if $|1 - r\langle \zeta, \eta \rangle| > 16 \cdot 2^j \delta$, and since $r\zeta \in \Gamma_\alpha(\xi)$ with $1 - r < 32 \cdot 2^j \delta$, it follows from the triangle inequality of **[R]**, Proposition 5.1.2 that $U_{j,r}(\langle \zeta, \eta \rangle) = 0$ unless $\eta \in B(\xi, 128\alpha 2^j \delta)$. For each such η use the cancellation properties of a to write

$$\begin{aligned} a * Q_r(\eta) &= \int_S a(\tau) Q_r(\langle \eta, \tau \rangle)\, d\sigma(\tau) \\ &= \int_S a(\tau)[Q_r(\langle \eta, \tau \rangle) - T_m(\tau; e_1)]\, d\sigma(\tau), \end{aligned}$$

where for each fixed η, $T_m(\tau, e_1)$ is the non-isotropic Taylor polynomial for $Q_r(\langle \eta, \tau \rangle)$ expanded about e_1 of degree m. From the formula $Q_r(\langle \eta, \tau \rangle)$ and the facts that $\tau \in B(e_1; \delta)$, $\eta \in B(\xi; 128\alpha 2^j \delta)$ and $|1 - \xi_1| > M 2^j \delta$ it can be seen that

$$|Q_r(\langle \eta, \tau \rangle) - T_m(\tau; e_1)| \le C \frac{\delta^m (1 - r)^{1-\beta}}{|1 - \xi_1|^{n+1-\beta+m}}.$$

Therefore with η as above

$$|a * Q_r(\eta)| \le C \frac{\delta^{m+n-n/p} (2^j \delta)^{1-\beta}}{|1 - \xi_1|^{n+1-\beta+m}}.$$

From this it follows that

$$|F_j(r\zeta)| \le C \frac{\delta^{m+n-n/p}}{|1 - \xi_1|^{n+m}},$$

and therefore, if $|1 - \xi_1| > M 2^j \delta$, then

$$F_j^*(\xi) \le C (2^j)^{-n-m+n/p} \frac{(2^j \delta)^{n+m-n/p}}{|1 - \xi_1|^{n+m}}. \tag{13}$$

We now specify that $m > n/p - n$. Then the estimates in (12) and (13) show that

$$\int_S (F_j^*)^p\, d\sigma = \int_{B(e_1; M 2^j \delta)} (F_j^*)^p\, d\sigma + \int_{S - B(e_1; M 2^j \delta)} (F_j^*)^p\, d\sigma,$$

where the first integral on the right-hand side is dominated by

$$C (2^j \delta)^n \|F_j\|_\infty^p \le C (2^j \delta)^n (2^j)^{-np-mp+n} (2^j \delta)^{-n} = C (2^j)^{-np-mp+n},$$

and the second integral is less than

$$C (2^j)^{-np-mp+n} \int_{S - B(e_1; M 2^j \delta)} \frac{(2^j \delta)^{mp+np-n}}{|1 - \xi_1|^{np+mp}}\, d\sigma \le C (2^j)^{-np-mp+n}.$$

Since $0 < p \leq 1$ we may use the triangle inequality to conclude that

$$\int_S (J_3^*)^p \, d\sigma \leq \sum_{j=0}^{N} \int_S (F_j^*)^p \, d\sigma \leq C \sum_{j=0}^{\infty} (2^j)^{-np-mp+n} .$$

This completes the proof of the claim for the special case where $k = 1$.

The proof of the claim for the arbitrary case where $0 < \beta < k \leq n - 1$ proceeds in an analogous fashion; we point out some of the minor differences. Since equation (6) will be replaced by

$$(\overline{R} + R)^k P[I_\beta a](z) = \left(\left(r\frac{\partial}{\partial r} \right)^k a * P_r \right) (I_\beta(\zeta) ,$$

instead of u_1 as given by equation (7), we will have to consider a sum of terms of the form

$$(1 - |z|)^{k-\beta} \int_S \frac{1}{|1 - \langle \zeta , \eta \rangle|^{n-\beta}} \overline{R}^j R^{k-j} V(r\eta) \, d\sigma(\eta) .$$

We split each such item into two pieces J_1 and J_2 as given by equations (8) and (9) with $1 - \beta$ replaced by $k - \beta$ and RV replaced by $\overline{R}^j R^{k-j} V$. To handle J_1 we use the pointwise estimates of Lemma 3 in place of inequality (10). To handle J_2, in place of equation (11) we use Theorem 1 from **[ACa]** and Lemma 2 to get the fact that

$$\overline{R}^j R^{k-j} P[a] = \sum_{|l|+|m| \leq k} Q_{l,m}(L , \overline{L}) P_{l,m}[a] ,$$

where $Q_{l,m}$ is a polynomial in two variables of degree no greater than k. This lets us write J_2 as a sum of terms of the form

$$(1 - r)^{k-\beta} (\overline{L}^j L^l a * G_r) * K_r(\zeta) ,$$

where

$$G_r(\lambda) = \frac{(1 - r^2)^{n-t-s}}{(1 - r\lambda)^{n-t} (1 - r\overline{\lambda})^{n-s}} ,$$

where $j + l \leq n - 1$ and t and s are non-positive integers such that $|t| + |s| \leq n - 1$. The remainder of the argument proceeds without difficulty. This completes the proof of Theorem 2. □

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