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A NOTE ON MURASUGI SUMS

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We give two examples to show that the genus of knots is neither sub- nor super-additive under the Murasugi sum operation.

A number of "addition" operations can be defined on pairs of knots in S^3; the connected sum is the most obvious of these, but there are several other more complicated possibilities. A general question one can ask is: which properties of knots behave "nicely" under these operations? It has long been known that the genus of a knot is additive under connect sum. Schubert [Sc] showed that bridge number is additive minus one under connect sum.

Outstanding questions are how crossing number, unknotting number and tunnel number behave under connect sum. Only the most obvious inequalities are currently available, and they are quite weak—for example, the crossing number is obviously sub-additive, as is the unknotting number, and it is easy to show that the tunnel number of the connect sum of K_1 and K_2 is less than or equal to the sum of their tunnel numbers plus one.

A more complicated operation on pairs of knots is the band-connect sum. This operation is not well defined, since it depends on how the band is chosen. Gabai and Scharlemann simultaneously established the superadditivity of genus under band-connect sum [G1], [S].

Yet another operation combining knots is the Murasugi sum of two knots (see [G2] for a definition); this depends on a choice of Seifert surfaces for the knots as well as a choice of disks along which to do the sum. Gabai [G2] nevertheless has shown that under reasonable conditions many geometric properties of the Seifert surfaces are retained under the Murasugi sum. In particular, he has shown that the Murasugi sum of K_1 and K_2 along minimal genus Seifert surfaces R_1 and R_2 yields a minimal genus Seifert surface R for the resulting knot K, so genus is additive under Murasugi sum provided the addition is done along minimal genus surfaces. Taking the Murasugi sum of two knots can thus be considered a "natural" operation on pairs consisting of knots together with minimal genus Seifert surfaces. However, the

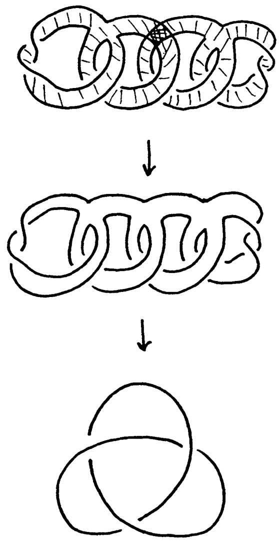

FIGURE 1

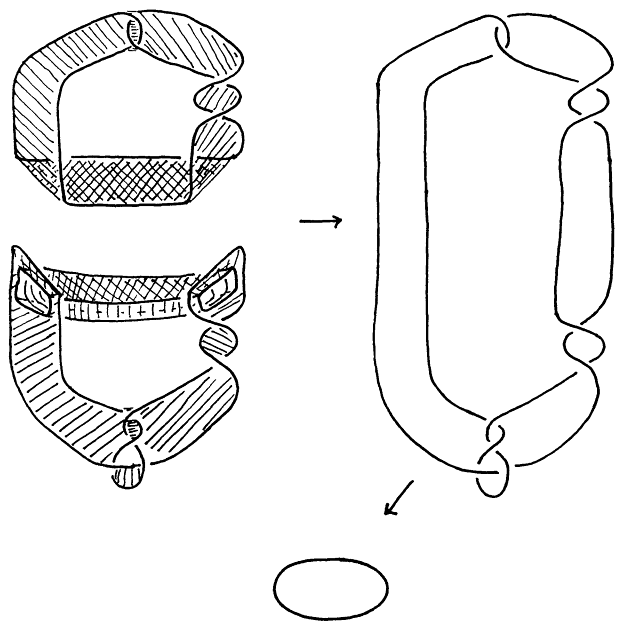

FIGURE 2

operation of constructing a Murasugi sum is not confined to minimal genus or even incompressible Seifert surfaces; we give two examples to illustrate that the genus does not behave in a predictable way in this larger category. The first [Figure 1] is an example of two trivial knots, each bounding a (compressible) genus one surface, summed along a square to yield a trefoil. The second example [Figure 2] is two figure eight knots, one bounding a genus one surface and the other bounding a (compressible) genus two surface, summed along a square to yield the trivial knot.

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