PACIFIC JOURNAL OF MATHEMATICS
Vol. 163, No. 2, 1994

KNOTS WITH ALGEBRAIC UNKNOTTING NUMBER ONE

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Every knot, K, in S^3 has associated to it an equivalence class of matrices based on S-equivalence of Seifert matrices. When the knot is altered by changing a crossing, the S-equivalence class of the new knot is related to that of the original knot in a very specific way. This change in the Seifert matrices can be studied without regard to the underlying geometric situation, leading to a theory of *algebraic crossing changes*. Thus, the *algebraic unknotting number* may be defined as the smallest number of these algebraic crossing changes necessary to convert a Seifert matrix for the knot into a matrix for the unknot. A straightforward test of some well-known knot invariants will reveal that the algebraic unknotting number is one.

In [**4**], Murakami defined an operation on Seifert matrices that he called an *algebraic unknotting operation.* He showed that any geometric crossing change induced an algebraic unknotting operation on a suitably chosen Seifert matrix. Since any knot could be changed into any other knot by a sequence of crossing changes, any Seifert matrix could be transformed into any other Seifert matrix by a sequence of algebraic unknotting operations and S-equivalences. For knots K_1 and K_2 the *algebraic Gordian distance* from K_1 to K_2 is the minimum number of algebraic unknotting operations needed in such a sequence. The *algebraic unknotting number*, $u_a(K)$, is then the algebraic Gordian distance of K from the unknot, i.e. the minimum number of algebraic unknotting operations needed to reduce a Seifert matrix for K to a matrix S-equivalent to the zero matrix.

Since every crossing change induces an algebraic unknotting operation, there is the inequality $u_a(K) \leq u(K)$ where $u(K)$ is the regular geometric unknotting number of the knot. And in many cases $u_a(K)$ is the appropriate object of study rather than $u(K)$ because only the algebraic information contained in a Seifert matrix is used. Such is the case in Murasugi's result on signatures [**5**] and Nakanishi's theorem about minor indices [**6**]. Also, results depending only on the abelian invariants (notably Lickorish [**3**] as generalized by Cochran and Lickorish [**1**]) apply to $u_a(K)$ since all of the homology information about the cyclic covers is contained in the Seifert matrix (see, for instance, [**2**, §§8 and 9]).

In the case $u_a(K) = 1$, Murakami was able to prove that the Alexander module $H_1(C_K)$ (where C_K is the infinite cyclic cover of the knot exterior and t acts by covering translations) is a cyclic $Z[t, t^{-1}]$-module. In addition, there is a generator, g, of this module with $\beta(g, g) = \pm 1/\Delta$, where $\beta(\cdot, \cdot)$ is the Blanchfield pairing and Δ is the Alexander polynomial of the knot. This paper contains a proof of the converse, providing a complete algebraic characterization of knots with $u_a(K) = 1$:

THEOREM. *A knot K with Alexander polynomial Δ_K can be changed by a single crossing change into a knot K' with trivial Alexander polynomial if and only if the Alexander module is cyclic and has a generator g with $\beta(g, g) = \pm 1/\Delta$.*

In addition, the proof often allows direct calculation of the necessary crossing change. A somewhat unfortunate application of the theorem will be made to the knot 8_{10}.

1. Some special surgery curves. Crossing changes will be examined via surgery on a well controlled class of curves in S^3. All knots, curves, and disks will be tame, and oriented when convenient. The orientations chosen will be noted, but they are only used for calculation and will be irrelevant to the outcome. The notation z will be used to denote $t^{1/2} - t^{-1/2}$ and for a matrix M, the calligraphic letter $\mathscr{M}$ will be used to represent the skew-Hermitianized form $t^{1/2}M - t^{-1/2}M^T$ (t^{-1} being considered the conjugate of t).

A disk D in S^3 will be said to be *nice with respect to K* if D and K intersect in two points and $\operatorname{lk}(\partial D, K) = 0$. A simple closed curve γ in $S^3 - K$ is a *nice surgery curve for K* if it bounds a nice disk. Any knot and nice surgery curve pair can be isotoped to look like Figure 1. Clearly ± 1 surgery on a nice surgery curve yields a single crossing change in the knot, and any single crossing change can be effected by ± 1 surgery along a suitably chosen curve.

In Figure 1, a Seifert surface can be chosen for the knot so that the two strands of the knot cobound a band in the surface, and the surface does not meet the curve γ. Generators for the homology of this surface can be chosen so that one of them (to be called g_0) runs over this band from right to left, and the rest of the generators do not

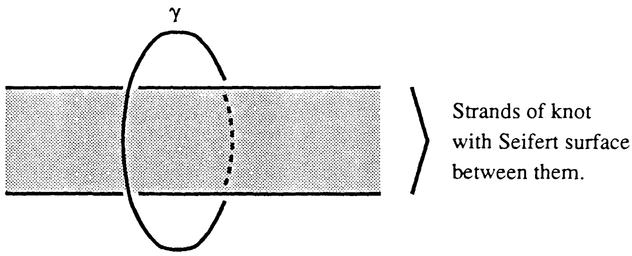

FIGURE 1

go over the band at all. A Seifert matrix for K has the form

$$M = \begin{pmatrix} & V & & * \\ & & & \vdots \\ & & & * \\ * & \cdots & * & x \end{pmatrix}, \qquad \mathscr{M} = \begin{pmatrix} & \mathscr{V} & & * \\ & & & \vdots \\ & & & * \\ * & \cdots & * & xz \end{pmatrix}$$

where $x = \operatorname{lk}(g_0, g_0^+)$, g_0^+ being the pushoff of g_0 from the Seifert surface in the (arbitrarily) chosen positive direction. V is the linking matrix for all the generators that don't go over the shown band, and their pushoffs.

If instead of the mundane curve shown in Figure 1, a nice surgery curve which has n full twists is used, the resulting knot has a Seifert surface which looks like Figure 2 (next page) (for -1 surgery). Note that n could be zero. This operation adds two generators to the homology of the Seifert surface, and the new Seifert matrix is

$$M' = \left(\begin{array}{cccc|cc} & & & & 0 & 0 \\ & M & & & \vdots & \vdots \\ & & & & 0 & 0 \\ & & & & -1 & 0 \\ \hline 0 & \cdots & 0 & 0 & n & 0 \\ 0 & \cdots & 0 & 0 & 1 & \mp 1 \end{array}\right),$$

$$\mathscr{M}' = \left(\begin{array}{cccc|cc} & & & & 0 & 0 \\ & \mathscr{M} & & & \vdots & \vdots \\ & & & & 0 & 0 \\ & & & & -t^{1/2} & 0 \\ \hline 0 & \cdots & 0 & t^{-1/2} & nz & t^{-1/2} \\ 0 & \cdots & 0 & 0 & t^{1/2} & \mp z \end{array}\right).$$

It is then simple to calculate the Alexander polynomial of the new knot, which is given by $\Delta_{K'} = \det(\mathscr{M}')$, by expanding along the bottom

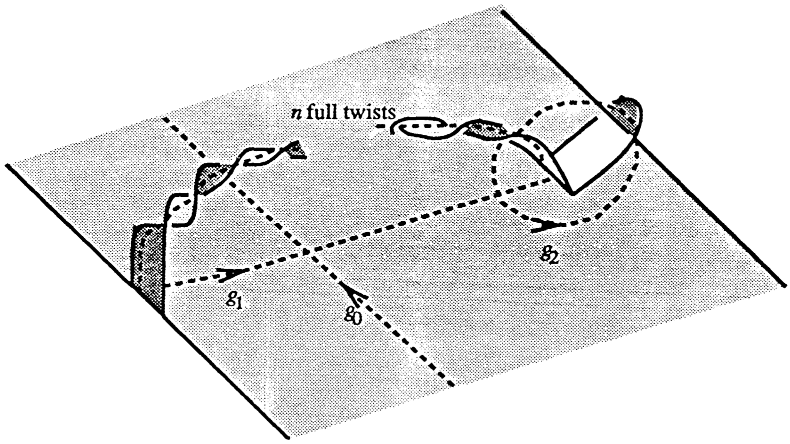

FIGURE 2

row of the matrix. The result is $(1 \mp nz^2)\det(\mathscr{M}) \mp z\det(\mathscr{V})$. Denote by L the knot obtained from K when $n = 0$ in the above, so that $\Delta_L = \det(\mathscr{M}) \mp z\det(\mathscr{V})$. Then noting that $\det(\mathscr{M}) = \Delta_K$, the result is

$$\Delta_{K'} = \Delta_L \mp nz^2\Delta_K .$$

Since n may be any integer, ± 1 surgery along a properly chosen curve can add *any* integral multiple of $z^2\Delta_K$ to Δ_L.

The key fact is that with proper choice of (nice) surgery curve, any *polynomial* multiple of Δ_K can be added to Δ_L, as long as the result satisfies the well known conditions $\Delta(1) = 1$ and $\Delta(t) = \Delta(t^{-1})$ for an Alexander polynomial. This is done by examining curves that wrap around the knot as in Figure 3. Each a_i is a nonzero integer, and if $a_i < 0$ then all the crossings in the magnified view of box i are reversed.

LEMMA 1. *The knot obtained from* ± 1 *surgery on the curve shown in Figure* 3 *has Alexander polynomial*

$$(1) \quad \Delta_{K'} = \Delta_L \mp \Delta_K \left[z^2\left(n + \sum a_i\right) - \sum \operatorname{sign}(a_i)(t^{|a_i|}(t-1) + t^{-|a_i|}(t^{-1}-1))\right] .$$

Proof. Figure 4 shows box i after ± 1 surgery is performed along the curve. The figure shows part of a Seifert surface for the new knot K' along with a set of homology generators for the new handles. Note

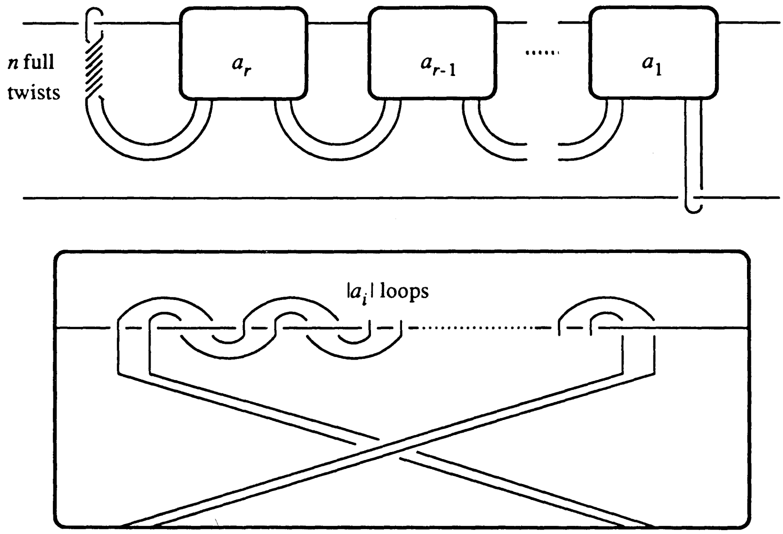

FIGURE 3

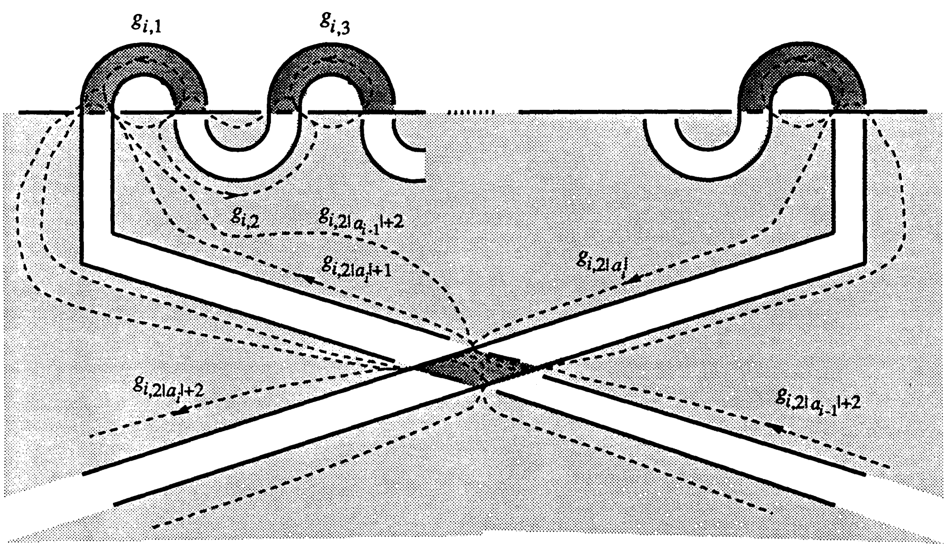

FIGURE 4

that the generators for one box do not interact with those of another box except that $\mathrm{lk}(g^+_{i,2|a_i|+2}, g_{i+1,1}) = -1$ or $\mathrm{lk}(g_{i,2|a_i|+2}, g^+_{i+1,1}) = 1$ depending on whether a_{i+1} is positive or negative.

Thus the Seifert matrix coming from this set of generators has the form

$$M' = \left(\begin{array}{ccccccccccccccc}
 & & & & 0 & & & & & & & & & & \\
 & M & & & \vdots & & & & & & & & & & \\
 & & & & 0 & & & & & & & & & & \\
 & & & & c_1 & 0 & \cdots & 0 & & & & & & & \\
0 & \cdots & 0 & d_1 & & & & & 0 & & & & & & \\
 & & & 0 & \text{Block for} & & & & \vdots & & & & & & \\
 & & & \vdots & \text{box } 1 & & & & 0 & & & & & & \\
 & & & 0 & & & & & c_2 & & & & & & \\
 & & & & 0 & \cdots & 0 & d_2 & & & & & & & \\
 & & & & & & & & \ddots & & & & & & \\
 & & & & & & & & & c_r & 0 & \cdots & 0 & & \\
 & & & & & & & & d_r & & & & & 0 & \\
 & & & & & & & & 0 & \text{Block for} & & & & \vdots & \\
 & & & & & & & & \vdots & \text{box } r & & & & 0 & \\
 & & & & & & & & 0 & & & & & -1 & 0 \\
 & & & & & & & & & 0 & \cdots & 0 & 0 & n & 0 \\
 & & & & & & & & & & & & 0 & -1 & \mp 1
\end{array}\right)$$

where $(c_i, d_i) = (-1, 0)$ if $a_i > 0$, and $(0, 1)$ if $a_i < 0$.

Each block is square of size $2|a_i| + 2$ and has the form

$$M_r = \begin{pmatrix}
1 & -1 & & & & & 0 & 0 \\
0 & 0 & -1 & & & & \vdots & \vdots \\
 & 0 & 1 & -1 & & & & \\
 & & 0 & \ddots & \ddots & & & \\
 & & & \ddots & 1 & -1 & \vdots & \vdots \\
 & & & & 0 & 0 & -1 & 0 \\
-1 & 0 & \cdots & & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & & \cdots & 0 & 1 & 0
\end{pmatrix}$$

for $a_r > 0$, while if $a_r < 0$ then this is replaced with the negative of its transpose. When this block is skew-Hermitianized in order to calculate the Alexander polynomial, it becomes

(2)

$$\mathcal{M}_r = \begin{pmatrix} z & -t^{1/2} & & & & & t^{-1/2} & 0 \\ t^{-1/2} & 0 & -t^{1/2} & & & & \vdots & \vdots \\ & t^{-1/2} & z & -t^{1/2} & & & & \\ & & t^{-1/2} & \ddots & \ddots & & & \\ & & & \ddots & z & -t^{1/2} & \vdots & \vdots \\ & & & & t^{-1/2} & 0 & -t^{1/2} & 0 \\ -t^{1/2} & 0 & \cdots & & \cdots & t^{-1/2} & 0 & -t^{-1/2} \\ 0 & 0 & \cdots & & \cdots & 0 & t^{1/2} & 0 \end{pmatrix}$$

or the negative of the transpose if $a_i < 0$.

Now the calculation of $\Delta_{K'}$ is routine. The determinant of $\mathcal{M}'$ can be expanded along the bottom row and last column. The result is

$$t^{1/2}\det\begin{pmatrix} \mathcal{M} & * & & & \\ * & \boxed{\mathcal{M}_1} & * & & \\ & * & \ddots & * & \\ & & * & \boxed{\mathcal{M}_r} & 0 \\ & & & * & t^{-1/2} \end{pmatrix} \pm z\det\begin{pmatrix} \mathcal{M} & * & & & \\ * & \boxed{\mathcal{M}_1} & * & & \\ & * & \ddots & * & \\ & & * & \boxed{\mathcal{M}_r} & * \\ & & & * & nz \end{pmatrix}$$

$$= (1 \mp nz^2)\det\begin{pmatrix} \mathcal{M} & * & & \\ * & \boxed{\mathcal{M}_1} & * & \\ & * & \ddots & * \\ & & * & \boxed{\mathcal{M}_r} \end{pmatrix} \mp z\det\begin{pmatrix} \mathcal{M} & * & & \\ * & \boxed{\mathcal{M}_1} & * & \\ & * & \ddots & * \\ & & * & \boxed{\widetilde{\mathcal{M}_r}} \end{pmatrix}$$

where $\widetilde{\mathcal{M}_r}$ means $\mathcal{M}_r$ with the last row and column removed.

The determinant in the first term can be computed easily by using the form of the blocks $\mathscr{M}_i$ given in formula (2). The determinant can be expanded repeatedly along the last row and column until all that is left is $\det(\mathscr{M}) = \Delta_K$.

The second term is a bit more complex. It can be expanded along the row and column corresponding to the top row and column of $\widetilde{\mathscr{M}_r}$. The result is

$$\det(\widetilde{\mathscr{M}_r}) \cdot \det \begin{pmatrix} \mathscr{M} & * & & & \\ * & \boxed{\mathscr{M}_1} & * & & \\ & * & \ddots & & \\ & & & & * \\ & & & * & \mathscr{M}_{r-1} \end{pmatrix}$$

$$+ \det \begin{pmatrix} \mathscr{M} & * & & & \\ * & \boxed{\mathscr{M}_1} & * & & \\ & * & \ddots & & \\ & & & & * \\ & & & * & \widetilde{\mathscr{M}_{r-1}} \end{pmatrix} \cdot \det \begin{pmatrix} \widetilde{\mathscr{M}_r} \text{ with first} \\ \text{row and column} \\ \text{removed} \end{pmatrix}.$$

The large determinant in the first term reduces to Δ_K as before. And the smaller matrix in the second term can be expanded along the first row and column repeatedly until it reduces to 1. So inductively, the result is (note that $\widetilde{\mathscr{M}} = \mathscr{V}$)

$$\Delta_{K'} = (1 \mp nz^2)\Delta_K \mp z\left(\Delta_K \sum \det \widetilde{M_i} + \det \mathscr{V}\right) \tag{3}$$

It remains to calculate $\det(\widetilde{\mathscr{M}_i})$. This is done by taking the matrix $\mathscr{M}_i$ as given in (2) and deleting its last row and column. This leaves a matrix whose last row and column have two nonzero entries each. Expanding along the last row and column gives four terms, as follows (assume $a_i > 0$):

$$
\det\begin{pmatrix}
0 & -t^{1/2} & & & & \\
t^{-1/2} & z & -t^{1/2} & & & \\
 & t^{-1/2} & 0 & \ddots & & \\
 & & \ddots & \ddots & -t^{1/2} & \\
 & & & t^{-1/2} & z & -t^{1/2} \\
 & & & & t^{-1/2} & 0
\end{pmatrix}
$$

$$
+\, t\det\begin{pmatrix}
-t^{1/2} & & & & & \\
z & -t^{1/2} & & & & \\
t^{-1/2} & 0 & -t^{1/2} & & & \\
 & \ddots & \ddots & \cdots & & \\
 & & t^{-1/2} & z & -t^{1/2} & \\
 & & & t^{-1/2} & 0 & -t^{1/2}
\end{pmatrix}
$$

$$
+\, t^{-1}\det\begin{pmatrix}
t^{-1/2} & 0 & -t^{1/2} & & & \\
 & t^{-1/2} & z & \ddots & & \\
 & & t^{-1/2} & \ddots & -t^{1/2} & \\
 & & & \ddots & 0 & -t^{1/2} \\
 & & & & t^{-1/2} & z \\
 & & & & & t^{-1/2}
\end{pmatrix}
$$

$$
+\det\begin{pmatrix}
z & -t^{1/2} & & & & \\
t^{-1/2} & 0 & -t^{1/2} & & & \\
 & t^{-1/2} & z & \ddots & & \\
 & & \ddots & \ddots & -t^{1/2} & \\
 & & & t^{-1/2} & 0 & -t^{1/2} \\
 & & & & t^{1/2} & z
\end{pmatrix}
$$

whose values are 0, $-t^{a_i+1/2}$, $t^{-a_i-1/2}$, and $a_i z$ respectively. If $a_i < 0$ then since $\widetilde{\mathscr{M}_i}$ is of odd size, the determinant of the negative of its transpose is opposite in sign. Therefore, for any a_i, $\det(\widetilde{\mathscr{M}_i}) = a_i z - \operatorname{sign}(a_i)(t^{|a_i|+1/2} - t^{-|a_i|-1/2})$. Inserting this in (3) and noting that $\Delta_L = \Delta_K \mp z\det(\mathscr{V})$ gives the desired result.

COROLLARY 2. *For knots, K, appearing as in Figure 3, with Alexander polynomial Δ_K, it is possible to make one crossing change to obtain* *a knot, K', with*

$$\Delta_{K'} = \Delta_L + \Delta_K b(t)$$

where $b(t)$ is any polynomial subject to the constraint that $\Delta_{K'}$ is a knot polynomial.

Proof. Since Δ_L is the Alexander polynomial for some knot, the requirements on $b(t)$ will be that $b(1) = 0$ and $b(t^{-1}) = b(t)$. But looking at formula (1) it is clear that n and a series of a_i may be chosen to give any b of this form. Therefore, there is a nice surgery curve on which ± 1 surgery—which changes exactly one crossing in the knot—gives a knot with the desired polynomial.

2. How nice are nice surgery curves? In this section some properties of nice surgery curves are developed to show that they are useful for more than just making large Seifert matrices.

LEMMA 3. *All of the different surgery curves, γ, given by different choices of n and series of a_i are homotopic in $S^3 - K$.*

Proof. Obvious, since homotopy in $S^3 - K$ allows γ to pass through itself.

Denote by C_K the infinite cyclic cover of the exterior of K. The Alexander module of K is $H_1(C_K)$ and is presented as a $Z[t, t^{-1}]$-module by the matrix $t^{1/2}\mathscr{M}$. Fix a strand of the knot and consider a small loop going around this strand. A lift of this loop represents t in the Alexander module (see Figure 5).

Any nice surgery curve γ can be isotoped to appear as in Figure 5. The can be seen by sliding the top half of Figure 1 around the knot to the left until it comes near the bottom half. Since the nice surgery curve bounds a nice disk, and both intersections of this disk with K

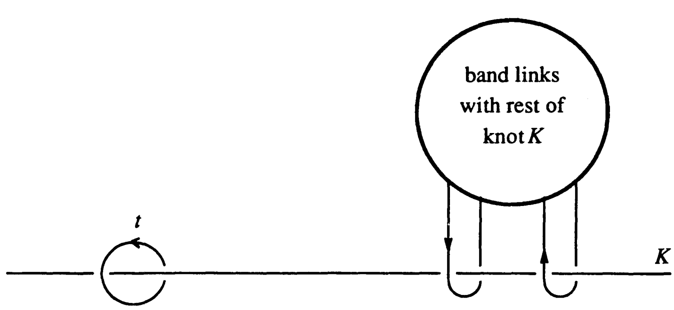

FIGURE 5

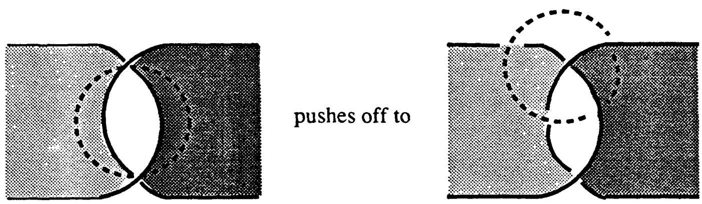

FIGURE 6

are clearly visible in Figure 5, the rest of the disk forms a band that meanders through the knot but never meets it. Let u be a simple closed curve that starts near the strand of K on the band, follows the band through the knot back to K, and then loops around the strand of K enough times so that $\operatorname{lk}(u, K) = 0$. The original surgery curve is homotopic to the product curve $tut^{-1}u^{-1}$. This procedure works in reverse as well; any simple closed curve u with $\operatorname{lk}(u, K) = 0$ gives rise to a nice surgery curve (in fact, a whole family of them) homotopic to $tut^{-1}u^{-1}$.

Both u and γ have linking number zero with the knot K, so both lift to C_K. Call these lifts $\tilde{u}$, $\tilde{\gamma}$. Homologically there is the relation $[\tilde{\gamma}] = (t-1)[\tilde{u}]$. Now note that $t-1$ is invertible mod Δ in $Z[t, t^{-1}]$. Thus for any homology class $\tilde{v}$ in the Alexander module, $(t-1)[\tilde{u}] = [\tilde{v}]$ can be solved for $[\tilde{u}]$. Choosing a representative of the class $[\tilde{u}]$ and projecting it down into S^3 yields a simple closed u with $\operatorname{lk}(u, K) = 0$. Using the construction at the end of the previous paragraph completes the proof of

LEMMA 4. *Any class of curves in the Alexander module can be represented by the lift of a nice surgery curve.*

In light of Lemma 3 and the fact that homotopic curves have homotopic (hence homologous) lifts, it is possible to choose a nice surgery curve representing any element of the Alexander module, yet still have the full power of the previous section available to alter the knot polynomial.

LEMMA 5. *Any nice surgery curve can be taken to be the pushoff of curve on a Seifert surface of the type shown in Figure 6.*

Proof. The moves in Figure 7 show how this can be done.

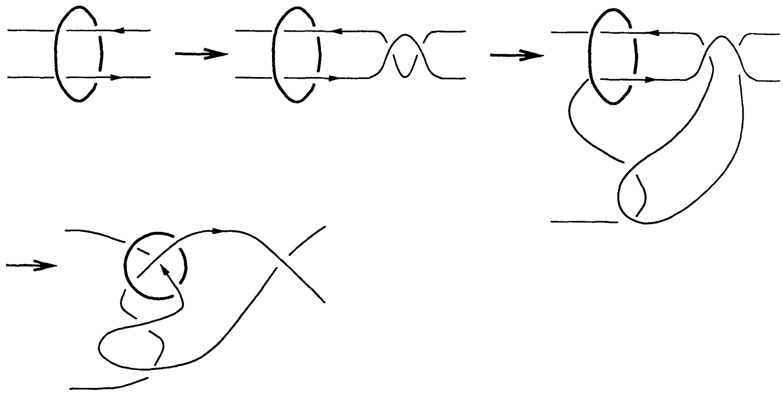

FIGURE 7

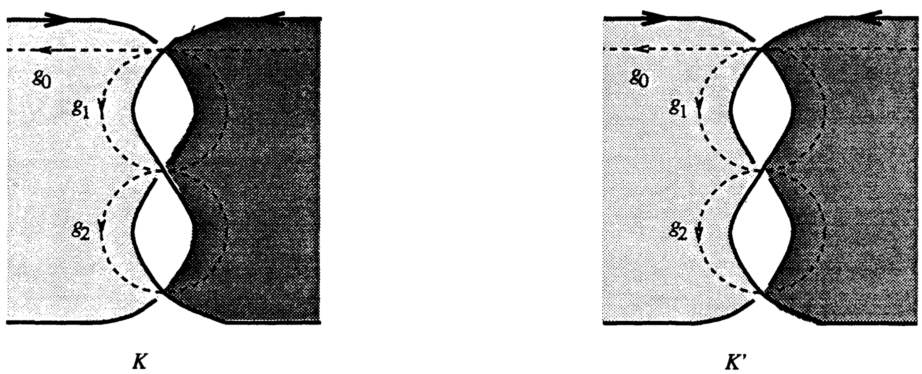

FIGURE 8

3. The main result.

THEOREM. *A knot K with polynomial Δ_K can be changed into K' with $\Delta_{K'} = 1$ by a single crossing change if and only if the Alexander module $H_1(C_K)$ is a cyclic $Z[t, t^{-1}]$-module of order Δ_K and has a generator g such that $\beta(g, g) = \pm 1/\Delta$.*

Proof. Assume that K can be changed into K' with $\Delta_{K'} = 1$ by a single crossing change and that K, K' are as shown in Figure 8, where pieces of Seifert surfaces are also shown. This situation can always be obtained by using Reidemeister moves of type II and III to get a local picture like Figure 8 near the crossing to be changed and then moving it to the top or bottom of a knot diagram, finally using the Seifert circle method to choose the Seifert surface. A simple calculation yields

(4)

$$M = \left(\begin{array}{cccc|cc} & & & * & 0 & 0 \\ & V & & \vdots & \vdots & \vdots \\ & & & * & 0 & 0 \\ * & \cdots & * & x & -1 & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 1 \end{array}\right), \quad M' = \left(\begin{array}{cccc|cc} & & & * & 0 & 0 \\ & V & & \vdots & \vdots & \vdots \\ & & & * & 0 & 0 \\ * & \cdots & * & x & -1 & 0 \\ \hline 0 & \cdots & 0 & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{array}\right)$$

as the Seifert matrices with skew-Hermitianized forms

$$\mathscr{M} = \left(\begin{array}{cccc|cc} & & & * & 0 & 0 \\ & \mathscr{V} & & \vdots & \vdots & \vdots \\ & & & * & 0 & 0 \\ * & \cdots & * & xz & -t^{1/2} & 0 \\ \hline 0 & \cdots & 0 & t^{-1/2} & 0 & t^{-1/2} \\ 0 & \cdots & 0 & 0 & -t^{1/2} & z \end{array}\right)$$

$$\mathscr{M}' = \left(\begin{array}{cccc|cc} & & & * & 0 & 0 \\ & \mathscr{V} & & \vdots & \vdots & \vdots \\ & & & * & 0 & 0 \\ * & \cdots & * & xz & -t^{1/2} & 0 \\ \hline 0 & \cdots & 0 & t^{-1/2} & z & t^{-1/2} \\ 0 & \cdots & 0 & 0 & -t^{1/2} & 0 \end{array}\right).$$

Taking the determinants of these gives

$$\Delta_{K'} = \det \begin{pmatrix} & & & * \\ & \mathscr{V} & & \vdots \\ & & & * \\ * & \cdots & * & xz \end{pmatrix}$$

so that this matrix has determinant one. Thus $\Delta_K = 1 + z\det(\mathscr{V})$. Note also that the Alexander dual to g_2^+ in the picture of K is a nice surgery curve, $+1$-surgery around which gives the knot K'.

Now $t^{1/2}\mathscr{M}$ is a presentation matrix for the Alexander module of K. Since the determinant of the upper left corner of the matrix is a power of t, which is a unit of $Z[t, t^{-1}]$, the first bunch of generators can all be expressed in terms of the generator corresponding to g_1, and the last column of $\mathscr{M}$ shows that this can in turn be expressed in terms of the generator corresponding to g_2. The presentation matrix has determinant Δ_K, proving the assertion that the Alexander module is cyclic of order Δ_K. It is generated by the lift g of the Alexander dual of g_2.

The Blanchfield pairing is given by $\beta(a, b) = z\bar{a}\mathscr{M}^{-1}b$ where a, b are vectors in terms of the spanning set used for calculation of $\mathscr{M}$ (see, for instance [**2**, §8]). Since $g = (0, \dots, 0, 1)$ in these coordinates, $\beta(g, g)$ is simply the bottom right entry in $z\mathscr{M}^{-1}$. Direct computation using the cofactor expansion of the inverse of a matrix gives $\beta(g, g) = z\det(\mathscr{Y})/\Delta_K$. Noting that the Blanchfield pairing takes values in $Q(t)/Z[t, t^{-1}]$ and that $\Delta_K = 1 + z\det(\mathscr{Y})$ gives the desired result $\beta(g, g) = -1/\Delta_K$.

Had a right crossing been made into a left crossing, the calculations would all be the same by switching all the crossings in Figure 8. Everything is the same except that Δ_K is now $1 - z\det(\mathscr{Y})$, which changes the end result to $\beta(g, g) = +1/\Delta_K$.

(The foregoing is essentially Murakami's proof. It should be noted that if $\beta(g, g) = \pm 1/\Delta_K$ then $a \mapsto \Delta_K\beta(g, a)$ is an epimorphism from $H_1(C_K)$ to $Z[t, t^{-1}]/\Delta_K Z[t, t^{-1}]$, and a simple argument based on the fact that $Q[t, t^{-1}]$ is a PID proves the kernel of this map to be zero. Hence, the Blanchfield pairing condition alone is enough to imply the generating condition.)

The converse is a matter of applying the lemmas in the correct order. Assume that the Alexander module for K is cyclic with generator g such that $\beta(g, g) = \pm 1/\Delta_K$. Use Lemma 4 to find a nice surgery curve γ whose lift to C_K is in the homology class g. Use Lemma 5 to arrange the knot and surgery curve to look like Figure 8; if $\beta(g, g) = +1/\Delta_K$ we first change all the crossings in Figure 8.

Once the knot is in this position, its Seifert matrix is given by formula (4). From the above calculation, $\beta(g, g) = z\det(\mathscr{Y})/\Delta_K$. Therefore $z\det(\mathscr{Y}) = \pm 1 + b(t)\Delta_K$ (recall that the Blanchfield pairing takes values in the quotient ring $Q(t)/Z[t, t^{-1}]$ so that values of the numerator are only determined up to adding a multiple of the denominator), where if the plus sign is chosen, the crossings are reversed in Figure 8. But since this is the case, a quick calculation yields $\Delta_{K'} = \Delta_K \pm z\det(\mathscr{Y})$, where again the plus sign is taken if the crossings in Figure 8 have been reversed. Substituting yields

$$\Delta_{K'} = 1 + b(t)\Delta_K.$$

Now since "1" is a knot polynomial, the condition on $b(t)$ to make $\Delta_{K'}$ a knot polynomial are exactly those necessary to apply Corollary 2. And because of Lemma 3, we may alter the surgery curve as necessary in Corollary 2 and still have a curve that will lift to g. Therefore surgery on this new nice surgery curve—which changes exactly one crossing—yields a knot with polynomial 1.

On occasion, the necessary surgery curve can be found explicitly. Suppose, for instance, that a Seifert surface is chosen and the corresponding Seifert matrix calculated. When this matrix is skew-Hermitianized, it becomes a presentation matrix for the Alexander module, with generators the lifts of the Alexander duals to generators of the homology of the Seifert surface.

If this matrix can be column-reduced to a matrix whose only nonzero entries are on the diagonal and any one row, and the diagonal elements in the other columns are $\pm t^n$, then a single generator for the Alexander module has been found—namely the generator g corresponding to the given row. If the matrix cannot be so reduced, then some basis change in the homology of the Seifert surface allows it to be reduced. However, discovering the necessary basis change may not be a simple problem. But if such a generator can be found then $\beta(g, g)$ can be easily calculated from the Seifert matrix. The problem then becomes whether or not a multiple of this generator can be found with $\beta(f(t)g, f(t)g) = \pm 1/\Delta$. This is a problem in Hermitian residues, which again may be difficult to solve.

Assuming both the difficulties in the previous paragraph can be overcome, and some multiple of the known generator has been found, it is now simple to extract the required surgery curve. For tg projects down to the loop whose lift is g, conjugated by the loop whose lift represents t. Thus, the projection of $f(t)g$ can be found in $S^3 - K$, and that will be the needed surgery curve.

4. An application to the knot 8_{10}. Figure 9 (next page) shows two pictures of the knot 8_{10}, the standard picture that appears in knot tables and one for which a Seifert surface is more obvious. The second picture is shown with generators for the homology of the surface. This knot has proven to be a stumbling block in the determination of the unknotting numbers of prime knots with small crossing number. This is because it can easily be unknotted with two crossing changes, yet all of the lower bounds (four-ball genus, minor index, half the signature) are one. The knot is thought to have unknotting number two.

8_{10}

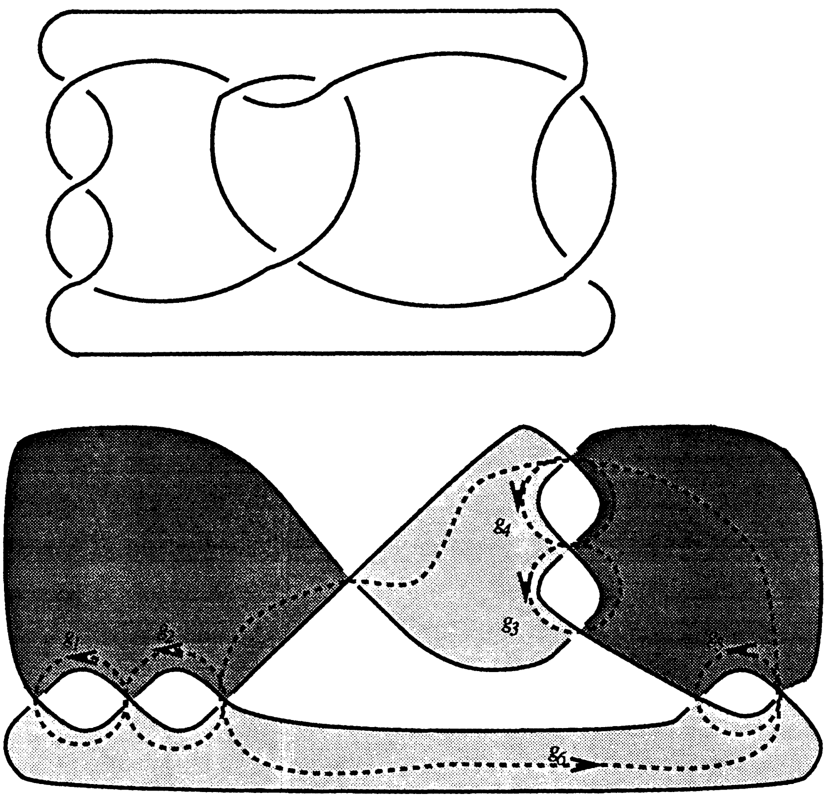

FIGURE 9

A Seifert matrix can be read from Figure 9 as

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 & -1 \end{pmatrix}.$$

Thus a presentation matrix for the Alexander module is given by

$$\mathscr{M} = \begin{pmatrix} -z & -t^{-1/2} & 0 & 0 & 0 & 0 \\ t^{1/2} & -z & 0 & 0 & 0 & -t^{-1/2} \\ 0 & 0 & z & t^{-1/2} & 0 & 0 \\ 0 & 0 & -t^{1/2} & z & 0 & -t^{-1/2} \\ 0 & 0 & 0 & 0 & -z & t^{-1/2} \\ 0 & t^{1/2} & 0 & t^{1/2} & -t^{1/2} & -z \end{pmatrix}.$$

With a lot of tedium, this matrix can be column reduced to one which is upper triangular with ones on the main diagonal except for the first entry, which is $\Delta = t^3 - 3t^2 + 6t - 7 + 6t^{-1} - 3t^{-2} + t^{-3}$. So the homology of the infinite cyclic cover is a cyclic $Z[t, t^{-1}]$ module, generated by the first generator in the presentation. Call this generator g.

Now consider $\beta(f(t)g, f(t)g)$ for some $f(t)$. Using the definition $\beta(a, b) = z\bar{a}\mathscr{M}^{-1}b$ gives

$$\beta(f(t)g, f(t)g) = zf(t^{-1})f(t)\mathscr{M}_{1,1}/\Delta$$

where $\mathscr{M}_{1,1}$ is the cofactor of the $(1, 1)$-entry of $\mathscr{M}$. A little calculation yields $z\mathscr{M}_{1,1} = -(z^6 + 2z^4 + 2z^2)$. Since the Blanchfield pairing takes values in $Q(t)/Z[t, t^{-1}]$ and the denominator here is Δ, we can reduce $z\mathscr{M}_{1,1}$ modulo Δ to arrive at

$$\beta(f(t)g, f(t)g) = (z^4 + z^2 + 1)f(t^{-1})f(t)/\Delta.$$

Choosing $f(t) = t^3 - t^2 + 2t - 1$ and substituting into the above formula yields $\beta(f(t)g, f(t)g) = 1/\Delta$. Furthermore, $f(t)$ inverts modulo Δ (its inverse is $t^4 - 3t^3 + 4t^2 - 2t + 1$ up to multiplication by units). Therefore, $f(t)g$ also generates the homology of the infinite cyclic cover of 8_{10}, so that applying the theorem proves

Corollary 6. *The knot* 8_{10} *has algebraic unknotting number one.*

In this case, a further simplification exists in finding explicitly the crossing change, namely that the lifts of the Alexander duals to the homology generators of the Seifert surface actually form a Z-basis for the Alexander module, so that the surgery curve can actually be calculated in terms of the Alexander duals themselves. When this is done and the curve is suitably modified by the moves in Lemma 1 to trivialize the Alexander polynomial, the resulting curve can be simplified by Reidemeister moves to appear as in Figure 10 (next page). When the surgery is performed, the resulting knot can be reduced to the (at most) 14-crossing knot shown in Figure 11 (next page), which is $6^{**}1.(3, 2)\bar{1}.1.1.\bar{2}\,\bar{1}.2$ in the Conway notation. The Alexander polynomial of this knot can indeed be calculated to be trivial.

In The Introduction this was cited as an unfortunate result. This is because this corollary shows that abelian methods, or any other methods dealing with the Seifert matrix, cannot be used to show that the unknotting number of 8_{10} is not one. More delicated procedures must be found.

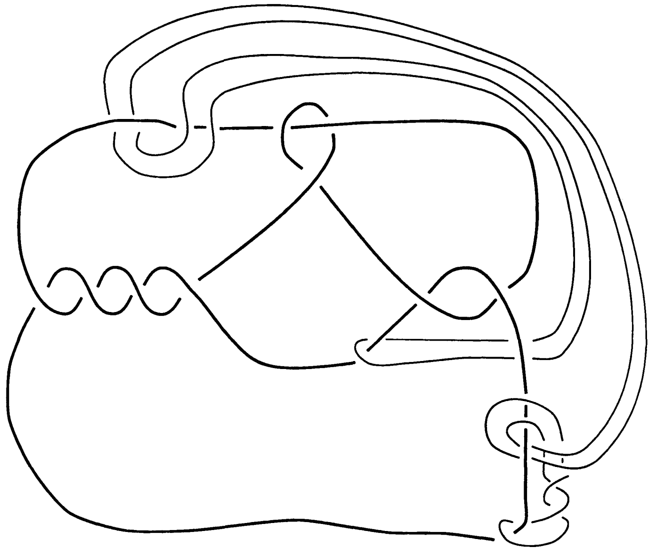

FIGURE 10

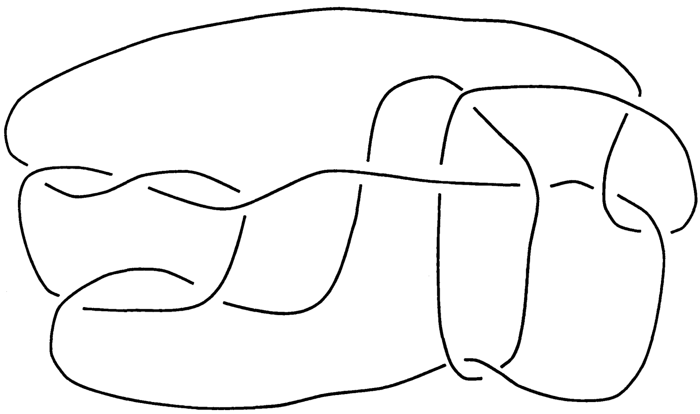

FIGURE 11

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Received December 14, 1991 and in revised form March 3, 1993.

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