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# ASYMPTOTIC RADIAL SYMMETRY FOR SOLUTIONS OF $\Delta u + e^u = 0$ IN A PUNCTURED DISC

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**In this paper a representation formula for solutions of the equation**

$$(*) \qquad \Delta u + 2Ke^u = 0, \quad K \text{ a constant},$$

**in a punctured disc in terms of multi-valued meromorphic functions is found. As application it is deduced that a necessary and sufficient condition for a solution of $(*)$, $K > 0$, being asymptotic radially symmetric is**

$$\int e^u < \infty.$$

**1. Introduction.** In [3], L. A. Caffarelli, B. Gidas, and J. Spruck proved that non-negative smooth solutions of the conformally invariant equation

$$(1) \qquad \Delta u + u^{(n+2)/(n-2)} = 0, \qquad u \geq 0,$$

in a punctured $n$-dimensional ball, $n \geq 3$, with an isolated singularity at the origin, are asymptotically radial. More precisely, if $u$ is a solution of (1), then

$$u(x) = (1 + o(1))\psi(|x|) \quad \text{as } x \to 0,$$

for some radial singular solution $\psi(r)$.

Geometrically speaking, to solve equation (1) is to find locally a conformal metric on a conformally flat $n$-dimensional manifold with constant scalar curvature. Therefore, its two-dimensional analogue is

$$(2) \qquad \Delta u + e^u = 0.$$

In this paper, we shall establish a similar asymptotic radial symmetry result for a smooth solution $u$ of (2) in the punctured disc, $D^* = D\backslash\{0\}$, $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$, with an isolated singularity at the origin, under

$$(3) \qquad \int_{D^*} e^u < +\infty.$$

Unlike the higher dimensional case, as one will see, that the integrability condition (3) is necessary for $u$ being asymptotically radial.

We point out that the isolated singularities or the behaviour at infinity of (2) in a punctured ball $B_1(0)\backslash\{0\} = \{x \in \mathbb{R}^3 : 0 < |x| < 1\}$ in 3-dimensions have been studied by M. Bidaut-Véron and L. Véron [2].

**2. Results.** Our approach to this problem is based on a classical result of Liouville which gives a representation of solutions of equation (2) in a simply-connected domain by analytic functions. We extend this representation to a punctured disc, and then deduce the result from analytic function theory.

Let us first recall Liouville's theorem.

THEOREM 1 (*Liouville* [6]; *see also* [1]). *Let* $\Omega$ *be a simply-connected domain in* $\mathbb{R}^2$. *Then all real solutions of*

$$\Delta u + 2Ke^u = 0 \quad \text{in } \Omega, \qquad K \text{ a constant}, \tag{4}$$

*are of the form*

$$u = \log \frac{|f'|^2}{(1 + (K/4)|f|^2)^2}, \tag{5}$$

*where* $f(z)$ *is a locally univalent meromorphic function in* $\Omega$.

COROLLARY 2. *All solutions of equation* (4) *in* $\Omega = \mathbb{R}^2$ *with* $K > 0$ *and*

$$\int_{\mathbb{R}^2} e^u < \infty$$

*are of the form*

$$u(x) = \log \frac{16\lambda^2}{(4 + \lambda^2 K|x - x_0|^2)^2}, \quad \lambda > 0, \ x_0 \in \mathbb{R}^2.$$

*Proof.* Let $u$ and $f$ be given in (5). Observe that Theorem 1 implies that $e^u|dz|^2 = f^* g_K$, where $g_K$ denotes the standard metric on $\mathbb{S}^2$ with curvature $K$. By the integrability assumption $f$ cannot have an essential singularity at infinity, for otherwise $f$ would cover $\mathbb{S}^2$ (possibly except one point) infinitely many times near infinity, which is impossible. Therefore $\lim_{z\to\infty} f(z) = \infty$ or some $z_0 \in \mathbb{C}$. By compositing with an inversion, we may assume the former case holds. Then $f$ maps $\mathbb{S}^2$ onto $\mathbb{S}^2$. Since $\mathbb{C}$ cannot cover $\mathbb{S}^2$ (notice that $f'(z) \neq 0$ for all $z \in \mathbb{C}$), $f$ does not have poles in $\mathbb{C}$. This means $f: \mathbb{C} \to \mathbb{C}$ is a covering map and therefore it assumes the form $f(z) = \alpha z + \beta$ for some $\alpha \neq 0$ and $\beta$ in $\mathbb{C}$. A substitution into (5) gives the desired conclusion. □

Corollary 2 was previously proved by Chen and Li [4] by the method of moving planes. From (5), one can see that the integrability condition is also necessary for asymptotic radial symmetry. All non-radial solutions, which arise from transcendental functions, satisfy $\int e^u = \infty$.

Theorem 1 is, in general, not true for domains which are not simply-connected. For instance, the function $u = -\log 4r(1 + \frac{K}{4}r)^2$ is a solution of equation (4) in the punctured disc $D^*$, with an isolated singularity at the origin. Yet it is easy to see that this solution is given by a multi-valued analytic function $f(z) = z^{1/2}$ instead of a single-valued analytic function in the punctured disc via the formula (5).

We now give an extension of Liouville's theorem for the punctured disc.

THEOREM 3. *Real solutions of the equation* (4) *are of the form* (5), *with* $f$ *a multi-valued locally univalent meromorphic function satisfying*:

1. *When* $K > 0$, $f(z) = g(z)z^\alpha$, $\alpha \in \mathbb{R}$, *or* $\varphi(\sqrt{z})$,
2. *when* $K = 0$, $f(z) = g(z)z^\alpha$ *or* $g(z) + c\log z$, $\alpha \in \mathbb{R}$, $c \in \mathbb{C}$; *and*
3. *when* $K < 0$, $f(z) = h(z)z^\beta$, $\beta \geq 0$.

*Here* $g$, $\varphi$, *and* $h$ *are single-valued analytic functions in* $D^*$, $D^*$, *and* $D$ *respectively*, $\varphi(z)\varphi(-z) = 1$, $h(0) \neq 0$, *and* $|h(D)| < 1$.

*Proof*. Consider the universal cover $\widetilde{D}^* = (0, 1] \times \mathbb{R}$ of the punctured disc. Let $\pi(r, \theta) = re^{i\theta}$ be the projection and let $\tilde{g} = dr^2 + \frac{1}{r^2}d\theta^2 = \pi^*|dx|^2$. It follows from Theorem 1 that there exists a local univalent meromorphic function $\tilde{h}(z)$ on $\widetilde{D}^*$ such that $e^{\tilde{u}}\tilde{g} = \tilde{h}^* g_K$, where $\tilde{u} = \pi^* u = u \circ \pi$ and now $g_K$ denotes the standard metric on the two dimensional space form $S_K$ with curvature $K$. Let $\tau \colon \widetilde{D}^* \to \widetilde{D}^*$ be the map $\tau(r, \theta) = (r, \theta + 2\pi)$. Then

$$\tau^* \tilde{h}^* g_K = \tau^*(e^{\tilde{u}}\tilde{g}) = e^{\tilde{u}}\tilde{g} = \tilde{h}^* g_K.$$

Therefore, $\tilde{h} \circ \tau \circ \tilde{h}^{-1}$ is a local isometry of $S_K$. By a result in differential geometry (Corollary 6.4, p. 256 in [5]), $\tilde{h} \circ \tau \circ \tilde{h}^{-1}$ can be extended uniquely to a global isometry of $S_K$. Locally

$$\tilde{h} \circ \tau = \rho \circ \tilde{h}, \qquad \rho \in \operatorname{Isom}(S_K).$$

Since $\widetilde{D}^*$ is simply connected, this holds globally. Moreover, $\rho$ is analytic since $\tilde{h}$ and $\tau$ are analytic. Therefore, there exists a locally univalent multi-valued meromorphic function $h(z) = \tilde{h}(\pi^{-1}z)$ satisfying $h(ze^{2\pi i}) = \rho(h(z))$, $\rho \in \text{Isom}(S_K)$, $\rho$ analytic, in $D^*$ such that

$$u = \log \frac{|h'|^2}{(1 + (K/4)|h|^2)^2}.$$

Here $h(ze^{2\pi i})$ denotes the value of $h$ after a turn along the circle centered at the origin with radius $|z|$.

By a change of coordinates, we only need to prove the theorem for $K = 4$, $K = 0$, and $K = -4$, where now $\rho$ is an analytic isometry of the standard unit sphere, the Eucidean plane, and the Poincaré disc respectively.

For $K = 4$, $\rho$ is given by

$$\frac{w - a}{1 + \bar{a}w} = e^{i\theta} \frac{z - a}{1 + \bar{a}z}$$

and

$$\frac{w - a}{1 + \bar{a}w} = e^{i\theta} \frac{1 + \bar{a}z}{z - a}$$

for some $a \in \mathbb{C}$ and $\theta \in [0, 2\pi)$. In the first case, let

$$f(z) = \frac{h(z) - a}{1 + \bar{a}h(z)}.$$

Then $f$ satisfies

$$f(ze^{2\pi i}) = e^{i\theta} f(z), \qquad \forall z \in D^*.$$

Consider the function

$$g(z) = f(z)z^{-\alpha}$$

on $D^*$, where $\alpha = \theta/2\pi$. We have

$$\begin{aligned} g(ze^{2\pi i}) &= f(ze^{2\pi i})(ze^{2\pi i})^{-\alpha} \\ &= f(z)e^{i\theta} z^{-\alpha} e^{-2\pi\alpha i} = g(z) \end{aligned}$$

for all $z \in D^*$. Hence $g(z)$ is a single-valued function and therefore analytic in $D^*$. So $f(z)$ takes the form $g(z)z^{\alpha}$. Using the fact that $w = (z - a)/(1 + \bar{a}z)$ is an isometry of the standard unit sphere,

$$u = \log \frac{|h'|^2}{(1 + |h|^2)^2} = \log \frac{|f'|^2}{(1 + |f|^2)^2},$$

which proves the first case.

In the second case, letting

$$f(z) = \frac{h(z) - a}{1 + \bar{a}h(z)},$$

we have $f(ze^{4\pi i}) = f(z)$. Hence there exists a single-valued analytic function $\varphi$ in the punctured disc satisfying $f(z) = e^{i\theta/2}\varphi(\sqrt{z})$. The condition $f(ze^{2\pi i})f(z) = e^{i\theta}$ implies $\varphi(z)\varphi(-z) = 1$. The proof of the positive case is completed.

For $K = 0$, we notice that analytic isometries of the Euclidean plane are of the form $w = e^{i\theta}z + c$, which can be represented by $w - a = e^{i\theta}(z - a)$ or $w = z + c$. Similar argument as in the positive case gives us the desired result.

Finally, for $K = -4$, analytic isometries of the Poincaré disc are in one of the following forms:

$$\frac{w-a}{1-\bar{a}w} = e^{i\theta}\frac{z-a}{1-\bar{a}z}, \quad \text{with } |a| < 1,$$
$$\frac{w-e^{i\theta_1}}{w-e^{i\theta_2}} = k\frac{z-e^{i\theta_1}}{z-e^{i\theta_2}}, \quad \text{with } k > 1,\ \theta_1 \neq \theta_2 \in \mathbb{R},$$
$$\frac{w-e^{i\theta}}{w+e^{i\theta}} = \frac{z-e^{i\theta}}{z+e^{i\theta}} + c, \quad \text{with } \theta \in \mathbb{R},\ c \in \mathbb{C}.$$

Using the same argument as above one can show that $f$ assumes one of the following forms:

(i) $g(z)z^{\alpha}$,
(ii) $e^{i\theta_1}(e^{i\theta_2} - g(z)z^{i\alpha})/(e^{-i\theta_2} - g(z)z^{i\alpha})$, and
(iii) $e^{i\alpha}(1 + g(z) + \alpha\log z)/(1 - g(z) - \alpha\log z)$,

where $g$ is analytic in $D^*$, and $\alpha$, $\theta_1$, $\theta_2$, $\theta \in \mathbb{R}$. Observe that in (5) ($K = -4$) $u$ becomes singular at $|f| = 1$. Hence, by the analyticity of $f$ and the regularity of $u$, the image of $f$ lies either inside or outside $D$. Replacing $f$ by $1/f$ if $|f| > 1$, we may assume $f(D^*)$ is contained in $D$. This immediately implies that the expression in (i) can be rewritten as $h(z)z^{\beta}$ where $h(0) \neq 0$ and $\beta \geq 0$.

In the following let $h$ stand for an analytic function in $D$ with $h(0) \neq 0$. We shall show that in (ii) and (iii) $\alpha = 0$ and $g(z) = h(z)$, and consequently they are special cases of (i). To see this first observe that in case (ii) the image of $D^*$ under the map $g(z)z^{i\alpha}$ lies in a half plane, which, modulo a rotation, may be taken to be the upper half plane. We have

$$0 < \arg(g(z)z^{i\alpha}) = \arg g(z) + \alpha\log|z| < \pi \pmod{2\pi}.$$

Applying the maximum principle to $\operatorname{Im} g(z)$ in the annulus $r_j < |z| < r_{j_0}$, $r_j = e^{-2j\pi/|\alpha|}$, $j > j_0$, $j_0$ large, we conclude that $\operatorname{Im} g(z) > 0$ for all $z$ in a deleted neighborhood of 0. Hence 0 cannot be an essential singularity of $g$. Now we can write $g(z) = h(z)z^k$, $k \in \mathbb{Z}$. Then the inequality

$$0 < \arg(g(z)z^{i\alpha}) = \arg h(z) + \alpha \log|z| + k \arg z < \pi \pmod{2\pi}$$

implies $\alpha = k = 0$. Similarly one can show that in (iii) $\alpha = 0$ and $g(z) = h(z)$. This completes our proof of the theorem. □

Now we can deduce an asymptotic radial symmetry result for equation (4) from Theorem 3. First we need a lemma from complex analysis.

LEMMA 4. *Suppose that $g(z)$ is a holomorphic function in $D^*$ which has an essential singularity at the origin. Then the multi-valued function $f(z) = z^\alpha g(z)$, $\alpha \in \mathbb{R}$, takes all values infinitely many times except at most one value.*

*Proof*. Consider the single-valued function $\phi(z) = z^{k-\alpha} f(z) = z^k g(z)$, where $k$ is an integer such that $k > \alpha$. Since $g$ has an essential singularity at the origin, so has $\phi$. The sequence

$$\phi_n(z) = \phi\left(\frac{z}{2^n}\right)$$

is not a normal sequence on some annulus $\Gamma$: $r/4 < |z| < 2r$. In particular, the sequence is not a normal sequence on intersection $\Omega$ of $\Gamma$ with any sector: $|\arg z - \arg z_0| < \varepsilon$, in the unit disc. Therefore the sequence

$$f_n(z) = f\left(\frac{z}{2^n}\right)$$

cannot be normal on $\Omega$. Now, applying the Montel theorem [7], we see that for any $a \in \mathbb{C}$, except at most one point, there exist infinitely many $n$ such that $f_n$ takes the value $a$ in $\Omega$. This implies that $f$ takes the value $a$ infinitely many times in the sector. □

THEOREM 5. *Let $u$ be a smooth real solution of the equation* (4) *with $K > 0$ in the punctured disc $D^*$. Then $u$ is asymptotically radial, more precisely,*

$$u(z) = \alpha \log|z| + O(1) \quad \text{as } |z| \to 0,\ \alpha > -2,$$

*if and only if*

$$\int_{D^*} e^u < +\infty.$$

*Proof*. By Theorem 3, the metric $e^u|dz|^2$ is the pull-back of the spherical metric with curvature $K$ via the holomorphic map $f$. Moreover $f$ is a covering map on $D\backslash\{z<0\}$ since $f'\neq 0$ for all $z\in D^*$. If $g$ takes the value $\infty$ infinitely many times, then so does $f$. This implies $e^u|dz|^2$ has infinite volume, i.e. $\int_{D^*} e^u = +\infty$. So we may assume $g$ takes $\infty$ for finitely many times. Then $g$ is holomorphic near the essential singularity and we can apply Lemma 4 (in case $f(z) = g(z)z^\alpha$) to conclude that $f$ covers the image of $f$ in the sphere infinitely many times. Thus $\int_{D^*} e^u = +\infty$. Therefore, the integrability condition implies that $g$ at most has a pole at the origin. Simple calculation now establishes the asymptotic radial symmetry of the solution $u$. □

REMARK. Theorem 5 no longer holds for $K = 0$. In fact, it is straightforward to show that $\int e^u|dz|^2 < \infty$ for some deleted neighborhood of 0 if and only if $f(z) = h(z)z^\alpha$, $h(0)\neq 0$ and $\alpha > 0$. In particular, all radially symmetric solutions corresponding to $f(z) = h(z)z^k + c\log z$, $k\in\mathbb{Z}$, $c\neq 0$, satisfy $\int e^u|dz|^2 = \infty$ in any deleted neighborhood of 0.

On the other hand, Theorem 5 holds for $K < 0$. In fact, all solutions are asymptotic radially symmetric and satisfy $\int e^u|dz|^2 < \infty$.

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