PACIFIC JOURNAL OF MATHEMATICS
Vol. 163, No. 2, 1994

ON SIEVED ORTHOGONAL POLYNOMIALS X: GENERAL BLOCKS OF RECURRENCE RELATIONS

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Orthogonal polynomials defined by general blocks of recurrence relations are examined. The connection with polynomial mappings is established, and applications are given to sieved orthogonal polynomials. This work extends earlier work on symmetric sieved polynomials to the case when the polynomials are not necessarily symmetric.

1. Introduction. We study in this paper systems $\{p_n(x)\}$ of orthogonal polynomials defined by general blocks of recurrence relations of the type

$$(1.1)\quad \begin{aligned} (x-b_n^{(0)})p_{nk}(x) &= p_{nk+1}(x)+a_n^{(0)}p_{nk-1}(x),\\ &\vdots \\ (x-b_n^{(j)})p_{nk+j}(x) &= p_{nk+j+1}(x)+a_n^{(j)}p_{nk+j-1}(x),\\ &\vdots \\ (x-b_n^{(k-1)})p_{(n+1)k-1}(x) &= p_{(n+1)k}(x)+a_n^{(k-1)}p_{(n+1)k-2}(x),\end{aligned}$$

$0\le j\le k-1$, $n\ge 0$, and satisfying initial conditions

$$(1.2)\qquad p_{-1}(x)=0,\quad p_0(x)=1.$$

We shall assume $a_n^{(j)}>0$, $j=0,1,\dots,k-1$, $n\ge 0$ and also that $k\ge 2$. Observe that the p_n's do not depend on $a_0^{(0)}$, so we make the convenient choice $a_0^{(0)}=1$. Clearly $\{p_n(x)\}$ is a system of monic orthogonal polynomials.

The case of $b_n^{(j)}=0$, $n\ge 0$, $0\le j\le k-1$, has been treated in a previous paper [9] by Charris and Ismail, where they also assumed that the determinants

$$(1.3)\ \Delta_n(2,k-1)=\begin{vmatrix} x & -1 & 0 & 0 & \cdots & 0 & 0\\ -a_n^{(2)} & x & -1 & 0 & \cdots & 0 & 0\\ 0 & -a_n^{(3)} & x & -1 & \cdots & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot\\ 0 & 0 & 0 & 0 & \cdots & a_n^{(k-1)} & x\end{vmatrix},\qquad n\ge 0,$$

are independent of n, that is $\Delta_n(2, k-1) = \Delta_0(2, k-1)$, $n \geq 0$. These two assumptions were motivated by the desire of the authors of **[9]** to provide a unified approach to symmetric sieved orthogonal polynomials.

Here we remove those two assumptions. Having done this, now (1.1) covers, of course, all monic three-term recurrence relations defining orthogonal polynomials. However, the separation in blocks is again naturally motivated by general sieved orthogonal polynomials and, as we shall see, also arises naturally when considering systems of polynomials obtained via polynomial mappings. In both cases $\Delta_n(2, k-1)$ (with x changed to $x - b_n^{(1)}, x - b_n^{(2)}, \dots, x - b_n^{(k-1)}$ in descending order along the main diagonal) is independent of n. This is clearly the case for sieved polynomials of the first kind where $a_n^{(j)} = 1/4$, $b_n^{(1)} = b_n^{(j)} = 0$, $n \geq 0$, $2 \leq j \leq k-1$, but it is not so clear for polynomials obtained by means of polynomial mappings. In fact to prove that the modified determinant $\Delta_n(2, k-1)$ is independent of n in the case of polynomials obtained via a polynomial mapping, we needed to apply results where $\Delta_n(2, k-1)$ may depend on n. This is done in §4.

This paper not only represents a further contribution to the understanding of general sieved orthogonal polynomials and systems determined by polynomial mappings, but it also covers more general systems which are not determined by polynomial mappings. As a matter of fact, orthogonal polynomials defined through blocks of recurrence relations, which are not necessarily sieved orthogonal polynomials and do not originate—a priori—in conjunction with polynomial mappings, have continued to appear in the literature, mainly in connection with problems in physics and chemistry (see, for example, **[6]**, **[10]**, **[20]**, **[21]**).

The paper is organized as follows. Section 2 contains basic relationships and preliminaries while §3 describes the link polynomials which tie together the different blocks. Section 3 also exhibits the fundamental recurrence relationships satisfied by the link polynomials. These fundamental recurrence relations will enable us to express the polynomials under consideration in terms of the link polynomials. Section 4 studies the connection with polynomial mappings, and §5 deals with sieved polynomials.

The evaluation of the Stieltjes transform of the orthogonality measures of the polynomials $\{p_n(x)\}$ and their associated families are included in §3. Recall that if $\{p_n(x)\}$ is a system of monic polynomials which are orthogonal with respect to a unique measure μ with total mass 1, then the Stieltjes transform of μ is

$$(1.4) \qquad X(x) = \int_{-\infty}^{+\infty} \frac{d\mu(t)}{x-t}, \qquad x \in \mathbb{C} - \mathbb{R},$$

and the literature on the moment problem (see **[4]**, **[11]**, **[19]**) ensures that

$$(1.5) \qquad X(x) = \lim_{n\to\infty} \frac{p_{n-1}^{(1)}(x)}{p_n(x)}, \qquad x \in \mathbb{C} - \mathbb{R},$$

where $\{p_n^{(1)}(x)\}$ is the system of associated polynomials of order 1 of $\{p_n(x)\}$ (see §2 for the definition of $\{p_n^{(1)}(x)\}$). Hence, if $\{p_n(x)\}$ is given a priori by a recurrence relation such as (1.1), and it is known in advance that they are orthogonal with respect to a unique measure μ with total mass 1, then μ can be determined from $X(x)$, as given by (1.5), via the Perron-Stieltjes inversion formula (**[7]**, **[5]**, **[14]**),

$$(1.6) \quad \int_{-\infty}^{+\infty} f\,d\mu = \lim_{\varepsilon\to 0+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \{X(x-i\varepsilon) - X(x+i\varepsilon)\} f(x)\,dx,$$

which holds for any bounded and continuous numerical function f on $\mathbb{R}$ provided that the support of $d\mu$ is contained in a half line. The existence of a unique measure μ as above can be guaranteed from properties of the coefficients $a_n^{(j)}$ in (1.1). This is the case, for example, if there is a constant $M > 0$ such that

$$(1.7) \qquad 0 < a_n^{(j)} < M, \qquad 0 \le j \le k-1,\ n \ge 0.$$

In what follows, we will assume that conditions such as (1.7) are given which guarantee the uniqueness of μ. This is expressed by saying that the Hamburger moment problem for $\{p_n(x)\}$ is determined.

The notation

$$(1.8) \qquad (a)_n = \begin{cases} 1, & n = 0, \\ a(a+1)(a+2)\cdots(a+n-1), & n \ge 1, \end{cases}$$

for shifted factorials will be used throughout. If $a \neq 0, -1, -2, \dots$, then the shifted factorial is

$$(1.9) \qquad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

where Γ stands for the Gamma Function (**[18]**). The series

$$(1.10) \qquad {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, x \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n, \qquad |x| < 1,$$

is the hypergeometric series. We recall the binomial formula ([**18**])

$$(1.11) \qquad (1-x)^{-a} = {}_2F_1\left(\begin{matrix} a,\ 1 \\ 1 \end{matrix}\middle| x\right), \qquad |x| < 1,$$

and the Euler integral representation ([**18**])

$$(1.12) \qquad \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\middle| x\right),$$

which holds for $|x| < 1$, when $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$. Since the right-hand side of (1.12) is meaningful as long as $b > 0$ and c and $c-b$ are not integers ≤ 0, we can define

$$(1.13) \qquad \int_0^{\overline{1|}} t^c(1-xt)^{-A}(1-t)^{-B}\,dt = \frac{\Gamma(c+1)\Gamma(-B+1)}{\Gamma(-B+c+2)}\,{}_2F_1\left(\begin{matrix} A,\ c+1 \\ -B+c+2 \end{matrix}\middle| x\right)$$

whenever $c > -1$, $|x| < 1$ and B is not an integer ≥ 1. The integral in (1.13) is called a Hadamard integral and will be used in §5. Details about the theory of Hadamard singular integrals can be found in [**4**], [**8**], [**17**].

2. Basic results. The results in this section and the next section follow closely those of §§2, 3 in [**9**], so our treatment will be rather sketchy.

The system of equations (1.1) can be written in matrix form as

$$(2.1) \qquad A\begin{bmatrix} p_{nk+1} \\ p_{nk+2} \\ p_{nk+3} \\ \vdots \\ p_{nk+k-1} \\ p_{nk-1} \end{bmatrix} = \begin{bmatrix} (x-b_n^{(0)})p_{nk} \\ a_n^{(1)}p_{nk} \\ 0 \\ \vdots \\ 0 \\ p_{nk+k} \end{bmatrix},$$

where A is the $k \times k$ matrix

(2.2)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_n^{(0)} \\ x-b_n^{(1)} & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -a_n^{(2)} & x-b_n^{(2)} & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -a_n^{(3)} & x-b_n^{(3)} & -1 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -a_n^{(k-2)} & x-b_n^{(k-2)} & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -a_n^{(k-1)} & x-b_n^{(k-1)} & 0 \end{bmatrix}.$$

We will also write

$$\Delta_n(i\,,\,j) = \begin{cases} 0\,, & j < i-2\,, \\ 1\,, & j = i-2\,, \\ x - b_n^{(i-1)}\,, & j = i-1\,, \end{cases} \tag{2.3}$$

and

$$\Delta_n(i\,,\,j) = \begin{vmatrix} x - b_n^{(i-1)} & -1 & 0 & 0 & \cdots & 0 & 0 \\ -a_n^{(i)} & x - b_n^{(i)} & -1 & 0 & \cdots & 0 & 0 \\ 0 & -a_n^{(i+1)} & x - b_n^{(i+1)} & -1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -a_n^{(j)} & x - b_n^{(j)} \end{vmatrix} \tag{2.4}$$

for $n \geq 0$ and $j \geq i \geq 1$.

We now solve (2.1) for p_{nk+j} in terms of p_{nk} and p_{nk+k} by Cramer's rule and obtain the recursion

$$\begin{aligned} \Delta_n(2\,,\,k-1)p_{nk+j} = \Delta_n(2\,,\,j-1)&p_{nk+k} \\ &+ a_n^{(1)}a_n^{(2)}\cdots a_n^{(j)}\Delta_n(j+2\,,\,k-1)p_{nk}\,, \\ &\qquad n \geq 0\,,\, j = 1\,,\,\ldots\,,\,k-1. \end{aligned} \tag{2.5}$$

Furthermore,

$$\begin{aligned} &a_n^{(0)}\Delta_n(2\,,\,k-1)p_{nk-1} \\ &\quad = [(x - b_n^{(0)})\Delta_n(2\,,\,k-1) - a_n^{(1)}\Delta_n(3\,,\,k-1)]p_{nk} - p_{nk+k}\,, \\ &\qquad\qquad n \geq 0. \end{aligned} \tag{2.6}$$

In particular (we assume $p_{-j}(x) = 0\,,\; j = 1\,,\,2\,,\,\ldots)$,

$$p_k(x) = (x - b_0^{(0)})\Delta_0(2\,,\,k-1) - a_0^{(1)}\Delta_0(3\,,\,k-1) = \Delta_0(1\,,\,k-1). \tag{2.7}$$

For $i = 1\,,\,2\,,\,\ldots\,,\,k-1$, the associated polynomials of order i, $\{p_n^{(i)}(x)\}$, of $\{p_n(x)\}$ are defined recursively by

$$(x - b_n^{(j)})p_{nk-i+j}^{(i)} = p_{nk-i+j+1}^{(i)} + a_n^{(j)}p_{nk-i+j-1}^{(i)}\,, \qquad 0 \leq j \leq k-1\,, \tag{2.8}$$

$$p_{-1}^{(i)}(x) = 0, \quad p_0^{(i)}(x) = 1.$$

Writing (2.8) in matrix form and solving for $p_{nk-i+j}^{(i)}$ in terms of $p_{(n+1)k-i}^{(i)}$ and $p_{nk-i}^{(i)}$ gives, exactly as in **[9]**, the following results

$$\text{(2.9)} \quad \Delta_n(2, k-1)p_{nk-i+j}^{(i)} = \Delta_n(2, j-1)p_{(n+1)k-i}^{(i)} + a_n^{(1)} \cdots a_n^{(j)} \Delta_n(j+2, k-1)p_{nk-i}^{(i)},$$

and

$$\text{(2.10)} \quad a_n^{(0)}\Delta_n(2, k-1)p_{nk-i-1}^{(i)} = -p_{(n+1)k-i}^{(i)} + [(x - b_n^{(0)})\Delta_n(2, k-1) - a_n^{(1)}\Delta_n(3, k-1)]p_{nk-i}^{(i)} \quad \text{for } n \geq 1.$$

Let $\widetilde{\Delta}_n(i+1, k-1)$ be the matrix whose determinant is (2.4) with $j = k-1$ (so that $\operatorname{Det}(\widetilde{\Delta}_n(i+1, k-1)) = \Delta_n(i+1, k-1))$. Then the relationship

$$\text{(2.11)} \qquad \widetilde{\Delta}_0(i+1, k-1) \begin{bmatrix} p_0^{(i)} \\ p_1^{(i)} \\ \vdots \\ p_{k-i-2}^{(i)} \\ p_{k-i-1}^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ p_{k-i}^{(i)} \end{bmatrix},$$

the initial condition $p_0^{(i)}(x) = 1$ and Cramer's rule give

$$\text{(2.12)} \qquad p_j^{(i)} = \Delta_0(i+1, j+i-1), \qquad j = 0, 1, \dots, k-i.$$

In particular we find

$$\text{(2.13)} \qquad p_{k-1}^{(1)}(x) = \Delta_0(2, k-1).$$

The associated polynomials of higher order $\{p_n^{(lk+i)}(x)\}$, $l \geq 0$, $i = 0, 1, 2, \dots, k-1$, are defined by

$$\text{(2.14)} \quad (x - b_{n+l}^{(j)})p_{nk-i+j}^{(lk+i)}(x) = p_{nk-i+j+1}^{(lk+i)}(x) + a_{n+l}^{(j)}p_{nk-i+j-1}^{(lk+i)}(x),$$

$$p_{-1}^{(lk+i)}(x) := 0, \quad p_0^{(lk+i)}(x) := 1, \qquad j = 0, 1, \dots, k-1.$$

Thus,

$$\text{(2.15)} \qquad \Delta_{n+l}(2, k-1)p_{nk-i+j}^{(lk+i)} = \Delta_{n+l}(2, j-1)p_{(n+1)k-i}^{(lk+i)} + a_{n+l}^{(1)} \cdots a_{n+l}^{(j)}\Delta_{n+l}(j+2, k-1)p_{nk-i}^{(lk+i)},$$

$j = 0, 1, \ldots, k-1$, $n \geq 1$, and

$$a_{n+1}^{(0)} \Delta_{n+l}(2, k-1) p_{nk-i-1}^{(lk+i)} = -p_{(n+1)k-i}^{(lk+i)} + [(x - b_{n+l}^{(0)}) \Delta_{n+l}(2, k-1) - a_{n+l}^{(1)} \Delta_{n+l}(3, k-1)] p_{nk-i}^{(lk+i)}, \tag{2.16}$$

$n \geq 1$. Also,

$$p_j^{(lk+i)}(x) = \Delta_l(i+1, j+i-1), \qquad 0 \leq j \leq k-i. \tag{2.17}$$

3. The link polynomials. Denote with $\{P_n^{(l)}(x)\}$ the system of polynomials defined for $l \geq 0$ by

$$\begin{aligned} &[(x - b_{n+l}^{(0)}) \Delta_{n+l}(2, k-1) \Delta_{n+l-2}(2, k-1) \\ &\quad - a_{n+l}^{(1)} \Delta_{n+l}(3, k-1) \Delta_{n+l-1}(2, k-1) \\ &\quad - a_{n+l}^{(0)} \Delta_{n+l}(2, k-1) \Delta_{n+l-1}(2, k-2)] P_n^{(l)}(x) \\ &= \Delta_{n+l-1}(2, k-1) P_{n+1}^{(l)}(x) \\ &\quad + a_{n+l}^{(0)} a_{n+l-1}^{(1)} \cdots a_{n+l-1}^{(k-1)} \Delta_{n+l}(2, k-1) P_{n-1}^{(l)}(x), \qquad n \geq 0, \end{aligned} \tag{3.1}$$

and the initial conditions

$$P_{-1}^{(l)}(x) = 0, \quad P_0^{(l)}(x) = 1. \tag{3.2}$$

We adopt the convention

$$\Delta_{-1}(2, k-2) := 0, \quad \Delta_{-1}(2, k-1) := 1. \tag{3.3}$$

In (2.5) replace $n-1$ by n and take $j = k-1$ to find

$$\Delta_{n-1}(2, k-1) p_{nk-1} = \Delta_{n-1}(2, k-2) p_{nk} + a_{n-1}^{(1)} a_{n-1}^{(2)} \cdots a_{n-1}^{(k-1)} p_{(n-1)k}, \qquad n \geq 1. \tag{3.4}$$

This, together with (2.6) and (3.3), shows that if $P_n(x) = p_{nk}(x)$, $n \geq 0$, then $\{P_n(x)\}$ satisfies (3.1) and (3.2) with $l = 0$. Hence,

$$P_n(x) = P_n^{(0)}(x), \qquad n \geq 0. \tag{3.5}$$

The polynomials $\{P_n(x)\}$ are called the link polynomials of the blocks (1.1) defining $\{p_n(x)\}$.

Let

$$W((P_n^{(l)}(x), P_{n-1}^{(l+1)}(x)) = \Delta_l(2, k-1) \begin{vmatrix} P_n^{(l)}(x), & P_{n-1}^{(l+1)}(x) \\ P_{n+1}^{(l)}(x), & P_n^{(l+1)}(x) \end{vmatrix}, \qquad n \geq 0, \tag{3.6}$$

be the Casorati determinants of $\{P_n^{(l)}(x)\}$. Then

$$W((P_0^{(l)}(x), P_{-1}^{(l+1)}(x)) = \Delta_l(2, k-1)$$

and

$$(3.7)\quad \begin{aligned} &W(P_n^{(l)}(x), P_{n-1}^{(l+1)}(x)) \\ &\quad = \Delta_{n+l-1}(2, k-1) \prod_{j=1}^{n} a_{j+l}^{(0)} a_{j+l-1}^{(1)} \cdots a_{j+l-1}^{(k-1)}, \qquad n \geq 1. \end{aligned}$$

Since $W(P_n^{(l)}(x), P_{n-1}^{(l+1)}(x))$ is not identically zero, $\{P_n^{(l)}(x)\}$ and $\{P_{n-1}^{(l+1)}(x)\}$ are linearly independent solutions of (3.1).

Let $\{Q_n(x)\}$ be a system of polynomials satisfying (3.1) for $n \geq 1$. Then

$$(3.8)\qquad Q_n(x) = AP_n^{(l)}(x) + BP_{n-1}^{(l+1)}(x), \qquad n \geq 0,$$

where

$$(3.9)\qquad A_0 = Q_0(x), \quad B = Q_1(x) - Q_0(x)P_1^{(l)}(x).$$

This follows from $\{P_n^{(l)}(x)\}$, $\{P_{n-1}^{(l+1)}(x)\}$ being a basis of solutions of (3.1).

For example, it is readily seen that $\{p_{nk}^{(lk)}(x)\}$ satisfies the recurrence relation (3.1) for $n \geq 1$, and a calculation based on (3.8) and (3.9) gives

$$(3.10)\quad p_{nk}^{(lk)}(x) = P_n^{(l)}(x) + a_l^{(0)}\Delta_l(2, k-1)\frac{\Delta_{l-1}(2, k-2)}{\Delta_{l-1}(2, k-1)}P_{n-1}^{(l+1)}(x),$$

which holds for $l, n \geq 0$. Observe that $p_{nk}(x) = p_{nk}^{(0)}(x) = P_n^{(0)}(x) = P_n(x)$. On the other hand, if $i = 1, 2, \dots, k-1$ and $Q_n(x) = p_{(n+1)k-i}^{(lk+j)}(x)$, then $\{Q_n(x)\}$ satisfies (3.1), with $l+1$ in the place of l, for $n \geq 1$. A calculation based on (2.15), (2.16), (2.17) and (3.9) then gives

$$(3.11)\quad \begin{aligned} p_{(n+1)k-i}^{(lk+i)}(x) &= \Delta_l(i+1, k-1)P_n^{(l+1)}(x) + a_{l+1}^{(0)}\frac{\Delta_{l+1}(2, k-1)}{\Delta_l(2, k-1)} \\ &\quad \times [p_{k-2}^{(lk+1)}p_{k-i}^{(lk+i)} - p_{k-1}^{(lk+1)}p_{k-i-1}^{(lk+i)}]P_{n-1}^{(l+2)} \end{aligned}$$

for $n \geq 0$, and it is easily verified that

$$(3.12)\quad \begin{aligned} W(p_{k-2}^{(lk+1)}, p_{k-i-1}^{(lk+i)}) &= [p_{k-2}^{(lk+1)}p_{k-i}^{(lk+i)} - p_{k-1}^{(lk+1)}p_{k-i-1}^{(lk+i)}] \\ &= a_l^{(i)} \cdots a_l^{(k-1)}p_{i-2}^{(lk+1)} \\ &= a_l^{(i)} \cdots a_l^{(k-1)}\Delta_l(2, i-2). \end{aligned}$$

Thus we have established the explicit representation

$$\begin{aligned}(3.13)\qquad p_{(n+1)k-i}^{(lk+i)}(x) &= \Delta_l(i+1, k-1)P_n^{(l+1)}(x)\\ &\quad + a_{l+1}^{(0)}a_l^{(i)}\cdots a_l^{(k-1)}\Delta_l(2, i-2)\\ &\quad \times \frac{\Delta_{l+1}(2, k-1)}{\Delta_l(2, k-1)}P_{n-1}^{(l+2)}(x)\end{aligned}$$

which holds for $n \geq 0$, $l \geq 0$, $i = 1, 2, \dots, k-1$. When $i = 1$ we have

$$(3.14)\qquad p_{(n+1)k-1}^{(lk+1)}(x) = \Delta_l(2, k-1)P_n^{(l+1)}(x), \qquad n \geq 0,$$

and when $\Delta_n(2, k-1)$ is independent of n,

$$(3.15)\qquad p_{nk}^{(lk)}(x) = P_n^{(l)}(x) + a_l^{(0)}\Delta_{l-1}(2, k-2)P_{n-1}^{(l+1)}(x),$$

$$(3.16)\qquad p_{(n+1)k-1}^{(lk+1)}(x) = \Delta_l(2, k-1)P_n^{(l+1)}(x),$$

$$\begin{aligned}(3.17)\quad p_{(n+1)k-i}^{(lk+i)}(x) &= \Delta_l(i+1, k-1)P_n^{(l+1)}(x)\\ &\quad + a_{l+1}^{(0)}a_l^{(i)}\cdots a_l^{(k-1)}\Delta_l(2, i-2)P_{n-1}^{(l+2)}(x),\\ &\qquad n \geq 0,\ i = 2, 3, \dots, k-1.\end{aligned}$$

Let

$$(3.18)\qquad P^{(l)}(x) = \lim_{n\to\infty}\frac{P_{n-1}^{(l+1)}(x)}{P_n^{(l)}(x)}, \qquad x \in \mathbb{C} - \mathbb{R}.$$

The Stieltjes transform of the measure of orthogonality of $\{p_n^{(lk+i)}(x)\}$ is

$$(3.19)\qquad X_{i,l}(x) = \lim_{n\to\infty}\frac{p_{n-1}^{(lk+i+1)}(x)}{p_n^{(lk+i)}(x)}, \qquad x \in \mathbb{C} - \mathbb{R},$$

$l \geq 0$, $i = 0, 1, 2, \dots, k-1$. From (3.10), (3.13), and (3.3), we obtain the following formulae

$$(3.20)\qquad X_{0,0}(x) = \Delta_0(2, k-1)P^{(0)}(x), \qquad i, l = 0,$$

$$\begin{aligned}(3.21)\quad X_{0,l}(x)&\\ &= \frac{\Delta_{l-1}(2, k-1)\Delta_l(2, k-1)P^{(l)}(x)}{\Delta_{l-1}(2, k-1) + a_l^{(0)}\Delta_l(2, k-1)\Delta_{l-1}(2, k-2)P^{(l)}(x)},\\ &\qquad l > 0,\end{aligned}$$

and

$$X_{i,l}(x) = \frac{N_{i,l}}{D_{i,l}}, \qquad l \geq 0, \; i = 1, 2, \ldots, k-1, \tag{3.22}$$

where

$$\begin{aligned} N_{i,l} = {} & \Delta_l(2, k-1)\Delta_l(i+2, k-1) \\ & + a_{l+1}^{(0)} a_l^{(i+1)} \cdots a_l^{(k-1)} \Delta_{l+1}(2, k-1)\Delta_l(2, i-1)P^{(l+1)}(x) \end{aligned}$$

and

$$\begin{aligned} D_{i,l} = {} & \Delta_l(2, k-1)\Delta_l(i+1, k-1) \\ & + a_{l+1}^{(0)} a_l^{(i)} \cdots a_l^{(k-1)} \Delta_{l+1}(2, k-1)\Delta_l(2, i-2)P^{(l+1)}(x), \end{aligned}$$

for $l \geq 0$, $i = 1, 2, \ldots, k-1$. When $\Delta_n(2, k-1)$ is independent of n, the above relationships simplify to

$$X_{0,0}(x) = \Delta_0(2, k-1)P^{(0)}(x), \tag{3.23}$$

$$X_{0,l}(x) = \frac{\Delta_l(2, k-1)P^{(l)}(x)}{1 + a_l^{(0)}\Delta_{l-1}(2, k-2)P^{(l)}(x)}, \qquad l > 0, \; i = 0, \tag{3.24}$$

and

$$\begin{aligned} & X_{i,l}(x) \\ & = \frac{\Delta_l(i+2, k-1) + a_{l+1}^{(0)} a_l^{(i+1)} \cdots a_l^{(k-1)} \Delta_l(2, i-1)P^{(l+1)}(x)}{\Delta_l(i+1, k-1) + a_{l+1}^{(0)} a_l^{(i)} \cdots a_l^{(k-1)} \Delta_l(2, i-2)P^{(l+1)}(x)}, \end{aligned} \tag{3.25}$$

$l \geq 0$, $i = 1, 2, \ldots, k-1$.

REMARK 3.1. When $\Delta_n(2, k-1)$ is independent of n, (3.1) becomes

$$\begin{aligned} & [(x - b_{n+l}^0 \Delta_{n+l}(2, k-1) - a_{n+l}^{(1)}\Delta_{n+l}(3, k-1) \\ & \qquad - a_{n+l}^{(0)}\Delta_{n+l-1}(2, k-2)]P_n^{(l)}(x) \\ & = P_{n+1}^{(l)}(x) + a_{n+l}^{(0)} a_{n+l-1}^{(1)} \cdots a_{n+l-1}^{(k-1)} P_{n-1}^{(l)}(x), \qquad n \geq 1, \end{aligned} \tag{3.26}$$

and (3.2) continues to hold.

4. Connection with polynomial mappings. Let $\{q_n(x)\}$ be a system of polynomials such that $q_0(x) = 1$ and for every n, $q_n(x)$ has degree n and positive leading coefficient. In addition, assume that the polynomial set $\{q_n(x)\}$ is orthonormal with respect to a probability measure μ whose support is contained in $[-s, s]$, $0 < s < +\infty$. Let

$T(x)$ be a polynomial of degree $k \geq 2$ with simple zeros such that $T(x) \geq s$ whenever $T'(x) = 0$. We say that $T(x)$ is a polynomial mapping for $\{q_n(x)\}$. Choose $W(x) = k^{-1}T'(x)$, and let $\{p_n(x)\}$ be the system of monic orthogonal polynomials obtained from $\{q_n(x)\}$ via the polynomial mapping $T(x)$ (with $W(x)$ as above) in the sense of Geronimo and Van Assche **[12]**. Assume $\{p_n(x)\}$ is given by (1.1) and (1.2). It follows from (2.3) of **[12]** that

$$p_{nk}(x) = c^{-n}\sqrt{\lambda_n}q_n(T(x)), \qquad n \geq 0, \tag{4.1}$$

where c is the leading coefficient of $T(x)$ and

$$\lambda_n = \frac{a_n^{(0)}}{a_0^{(0)}} \prod_{j=0}^{n-1} a_j^{(0)} a_j^{(1)} \cdots a_j^{(k-1)}, \qquad n \geq 0. \tag{4.2}$$

More explicitly, let $T(x)$ and $W(x)$ be as above, and assume that a system of polynomials $\{Q_n(x)\}$ is given by

$$\begin{gathered}(x - C_n)Q_n(x) = A_nQ_{n+1}(x) + B_nQ_{n-1}(x), \qquad n \geq 0, \\ Q_{-1}(x) = 0, \quad Q_0(x) = 1.\end{gathered} \tag{4.3}$$

Let $\{q_n(x)\}$ be the corresponding system of orthonormal polynomials; i.e.,

$$q_n(x) = \frac{Q_n(x)}{\sqrt{\Lambda_n}}, \qquad n \geq 0,$$

where

$$\Lambda_n = \int_{-s}^{s} Q_n^2(x)\, d\mu(x).$$

If $T(x) = c\widehat{T}(x)$ with $\widehat{T}(x)$ monic, then

$$\begin{gathered}(\widehat{T}(x) - c^{-1}C_n)p_{nk}(x) \\ = p_{nk+k}(x) + c^{-2}A_{n-1}B_np_{(n-1)k}(x), \qquad n \geq 1, \\ p_0(x) = 1, \qquad p_k(x) = \widehat{T}(x) - c^{-1}C_0,\end{gathered} \tag{4.4}$$

so that

$$p_{nk}(x) = c^{-n}A_0 \cdots A_{n-1}Q_n(T(x)), \qquad n \geq 0. \tag{4.5}$$

We also say that $\{p_n(x)\}$ is obtained from $\{Q_n(x)\}$ via the polynomial mapping $T(x)$. Our next result gives a sufficient condition for $\Delta_n(2, k-1)$ to be independent of n.

THEOREM 4.1. *Let* $\{p_n(x)\}$, *as in* (1.1) *and* (1.2), *be obtained from* $\{Q_n(x)\}$, *given by* (4.3), *via the polynomial mapping* $T(x)$. *Then* $\Delta_n(2, k-1)$ *must be independent of* n.

Proof. Let $T(x) = c\widehat{T}(x)$ and $\widehat{T}(x)$ monic. Then (4.4) and (4.5) hold, and from (3.1) with $1 = 0$ and (3.5) we obtain

$$\begin{aligned}
&[(x - b_n^{(0)})\Delta_n(2, k-1)\Delta_{n-1}(2, k-1) \\
&\qquad - a_n^{(1)}\Delta_n(3, k-1)\Delta_{n-1}(2, k-1) \\
&\qquad - a_n^{(0)}\Delta_n(2, k-1)\Delta_{n-1}(2, k-2) \\
&\qquad\qquad\qquad - \Delta_{n-1}(2, k-1)(\widehat{T}(x) - c^{-1}C_n)]p_{nk} \\
&\quad = [\Delta_n(2, k-1)a_n^{(0)}a_{n-1}^{(1)}\cdots a_{n-1}^{(k-1)} \\
&\qquad\qquad - \Delta_{n-1}(2, k-1)c^{-2}A_{n-1}B_n]p_{(n-1)k}, \qquad n \geq 1.
\end{aligned}$$

Since the left-hand side is either 0 or a polynomial of degree at least nk, whereas the right-hand side has degree $nk - 1$ at the most, both sides must vanish. Thus,

$$\Delta_n(2, k-1) = \Delta_{n-1}(2, k-1) = \Delta_0(2, k-1),$$
$$a_n^{(0)}a_{n-1}^{(1)}\cdots a_{n-1}^{(k-1)} = c^{-2}A_{n-1}B_n, \qquad n \geq 1.$$

REMARK 4.1. The preceding results also imply

$$\widehat{T}(x) = (x - b_0^{(0)})\Delta_0(2, k-1) - a_0^{(1)}\Delta_0(3, k-1) - c^{-1}C_0, \tag{4.6}$$

$$\begin{aligned}
C_n = C_0 + c(a_n^{(0)}&\Delta_{n-1}(2, k-2) \\
&+ a_n^{(1)}\Delta_n(3, k-1) - a_0^{(1)}\Delta_0(3, k-1)), \qquad n \geq 1,
\end{aligned} \tag{4.7}$$

and

$$a_n^{(0)} + a_n^{(1)} = a_0^{(1)}, \qquad n \geq 1. \tag{4.8}$$

REMARK 4.2. We shall see in §5 that the condition on $\Delta_n(2, k-1)$ being independent of n is not sufficient for $\{p_n(x)\}$ to be obtained by means of a polynomial mapping. Assume, however, that (1.7) holds and that

$$\Delta_n(x) := a_n^{(0)}\Delta_{n-1}(2, k-2) + a_n^{(1)}\Delta_n(3, k-1) - a_0^{(1)}\Delta_0(3, k-1) \tag{4.9}$$

is independent of x (which implies that (4.8) holds). Let $0 < s < +\infty$ be such that the inverse image of $[-M, M]$ under $\Delta_0(1, k-1)$ is contained in $[-s, s]$, and choose c such that $c\Delta_0(1, k-1) \geq M$ at all points where $\Delta_0'(1, k-1) = 0$. Let

$$T(x) = c\Delta_0(1, k-1). \tag{4.10}$$

Since $\Delta_0(1, k-1) = p_k(x)$, $\Delta_0(1, k-1)$ must have real and simple zeros. Let $\{Q_n(x)\}$ be defined by

$$(x - C_n)Q_n(x) = Q_{n+1} + a_n^{(0)}a_{n-1}^{(1)} \cdots a_{n-1}^{(k-1)}Q_{n-1}(x), \quad n \geq 1, \tag{4.11}$$

$$Q_0(x) = 1, \quad Q_1(x) = x,$$

where $C_0 = 0$, $C_n = c\Delta_n(x)$, $n \geq 1$. Then

$$p_{nk}(x) = c^{-n}Q_n(T(x)), \qquad n \geq 0, \tag{4.12}$$

and $T(x)$ is a polynomial mapping for $\{Q_n(x)\}$. Hence $\{p_n(x)\}$ can be obtained via a polynomial mapping.

5. Sieved orthogonal polynomials. Let $\{p_n(x)\}$ be given by

$$(x - b_n^{(j)})p_{nk+j}(x) = p_{nk+j+1}(x) + a_n^{(j)}p_{nk+j-1}(x), \qquad n \geq 1, \tag{5.1}$$
$$p_{-1}(x) = 0, \qquad p_0(x) = 1,$$

with

$$b_n^{(j)} = 0, \quad 1 \leq j \leq k-1; \quad a_n^{(j)} = \frac{1}{4}, \quad 2 \leq j \leq k-1; \quad n \geq 0. \tag{5.2}$$

Then $\{p_n^{(i)}(x)\}$, $i = 0, 1, 2, \ldots$, is called a system of sieved orthogonal polynomials. When $k > 2$, $\{p_n(x)\}$ is called a system of sieved orthogonal polynomials of the first kind, and $\{p_n^{(1)}(x)\}$ a system of the second kind. Curiously, because of historical reasons (see **[2]**) $\{p_n^{(1)}(x)\}$ is not the system of sieved polynomials of the second kind of the system $\{p_n(x)\}$. Instead, the system of sieved polynomials of the second kind of $\{p_n(x)\}$ is the system of polynomials $\{q_n^{(1)}(x)\}$ with $\{q_n(x)\}$ determined by

$$(x - \tilde{b}_n^{(j)})q_{nk+j}(x) = q_{nk+j+1}(x) + \tilde{a}_n^{(j)}q_{nk+j-1}(x), \quad n \geq 0, \ 0 \leq j \leq k-1, \tag{5.3}$$

where

$$\tilde{a}_n^{(0)} = a_n^{(1)}, \quad \tilde{a}_n^{(1)} = a_n^{(0)}, \quad \tilde{a}_n^{(j)} = \frac{1}{4}, \qquad 2 \leq j \leq k-1, \quad n \geq 0, \tag{5.4}$$

$$\tilde{b}_n^{(0)} = b_n^{(0)}, \quad \tilde{b}_n^{(j)} = 0, \qquad 1 \leq j \leq k-1, \ n \geq 0.$$

When $k = 2$, the above definition is applicable provided that we choose $\tilde{a}_1^{(0)} = 1/4$ instead of $\tilde{a}_1^{(0)} = a_0^{(1)}$ in (5.4).

If $\{p_n(x)\}$ is a system of sieved polynomials of the first kind, then $\Delta_n(2, k-1) = \widehat{U}_{k-1}(x)$, $\Delta_n(2, k-2) = \Delta_n(3, k-1) = \widehat{U}_{k-2}(x)$, and their monic link polynomials $\{P_n(x)\}$ satisfy

$$\begin{aligned}(5.5)\qquad &[(x - b_n^{(0)})\widehat{U}_{k-1}(x) - (a_n^{(0)} + a_n^{(1)})\widehat{U}_{k-2}(x)]P_n(x) \\ &\qquad = P_{n+1}(x) + 4^{2-k}a_n^{(0)}a_{n-1}^{(1)}P_{n-1}(x), \qquad n \geq 1, \\ &P_0(x) = 1, \qquad P_1(x) = (x - b_0^{(0)})\widehat{U}_{k-1}(x) - a_0^{(1)}\widehat{U}_{k-2}(x),\end{aligned}$$

where $\{\widehat{U}_n(x)\}$ (see [18]) is the system of monic Chebyshev polynomials of the second kind: $\widehat{U}_{-1}(x) = 0$, $\widehat{U}_0(x) = 1$; $x\widehat{U}_n(x) = \widehat{U}_{n+1} + \frac{1}{4}\widehat{U}_{n-1}(x)$, $n \geq 0$. This follows from (3.26). Relation (5.5) can also be written in the form

$$\begin{aligned}(5.6)\qquad &2^{1-k}[T_k(x) - b_n^{(0)}U_{k-1}(x) + (1 - 2(a_n^{(0)} + a_n^{(1)}))]P_n(x) \\ &\qquad = P_{n+1}(x) + 4^{2-k}a_n^{(0)}a_{n-1}^{(1)}P_{n-1}(x), \qquad n \geq 1, \\ &P_0(x) = 1, \\ &P_1(x) = 2^{1-k}[T_k(x) - b_0^{(0)}U_{k-1}(x) + (1 - 2a_0^{(1)})U_{k-2}(x)],\end{aligned}$$

where $U_n(x) = 2^n\widehat{U}_n(x) = \sin(n+1)\theta/\sin\theta$, if $x = \cos\theta$, and $T_0(x) = 1$, $T_n(x) = \frac{1}{2}((U_n(x) - U_{n-2}(x))$, $n \geq 0$, are Chebyshev polynomials of the second and first kinds, respectively. It follows that if

$$(5.7)\qquad a_0^{(1)} = \tfrac{1}{2}; \quad b_n^{(0)} = 0, \quad a_{n+1}^{(0)} + a_{n+1}^{(1)} = \tfrac{1}{2}, \qquad n \geq 0,$$

in which case $\{p_n(x)\}$ is called a system of sieved random walk polynomials of the first kind (see [7], [9]), then

$$(5.8)\qquad p_{nk}(x) = P_n(x) = \frac{1}{2^{n(k-1)}}\widehat{Q}_n(T_k(x))$$

where $\{\widehat{Q}_n(x)\}$ is the system of orthogonal polynomials determined by

$$\begin{aligned}(5.9)\qquad &x\widehat{Q}_n(x) = \widehat{Q}_{n+1}(x) + 4a_n^{(0)}a_{n-1}^{(1)}\widehat{Q}_{n-1}(x), \qquad n \geq 0, \\ &\widehat{Q}_{-1}(x) = 0, \quad \widehat{Q}_0(x) = 1.\end{aligned}$$

In other words, $\{\widehat{Q}_n(x)\}$ is the system of monic polynomials of the system $\{Q_n(x)\}$ given by

$$\begin{aligned}(5.10)\qquad &xQ_n(x) = A_nQ_{n+1}(x) + B_nQ_{n-1}(x), \\ &Q_{-1}(x) = 0, \quad Q_0(x) = 1\end{aligned}$$

with $A_n = 2a_n^{(1)}$, $B_n = 2a_n^{(0)}$, $n \geq 1$, $A_0 = 1$, $B_0 = 0$. Relation (5.8), which can also be written

$$p_{nk}(x) = \frac{A_0 \cdots A_{n-1}}{2^{n(k-1)}} Q_n(T_k(x)), \tag{5.11}$$

means that $\{p_n(x)\}$ is obtained from $\{Q_n(x)\}$ (or $\{\widehat{Q}_n(x)\}$) via the polynomial mapping $T(x) = T_k(x)$. Since $A_n + B_n = 1$, $n \geq 0$, $\{Q_n(x)\}$ is a system of random walk polynomials ([7], [9]). The converse is a consequence of the following theorem.

THEOREM 5.1. *Let $\{p_n(x)\}$ be a system of sieved polynomials of the first kind, and assume that $\{p_n(x)\}$ is obtained from the system of orthogonal polynomials $\{Q_n(x)\}$,*

$$(x - C_n)Q_n(x) = A_n Q_{n+1}(x) + B_n Q_{n-1}(x), \qquad n \geq 0, \tag{5.12}$$
$$Q_{-1}(x) := 0, \quad Q_1(x) := 1,$$

by means of the polynomial mapping $T(x)$. If $k > 2$, then

$$b_n^{(0)} = b_0^{(0)}, \quad a_{n+1}^{(0)} + a_{n+1}^{(1)} = a_0^{(1)}, \qquad n \geq 0, \tag{5.13}$$

and $Q_n(x) = R_n(x - C_0)$, where $\{R_n(x)\}$ is a system of symmetric polynomials.

Proof. Assume $\{p_n(x)\}$ is obtained from (5.12) by means of the polynomial mapping $T(x) = c\widehat{T}(x)$, with $\widehat{T}(x)$ a monic polynomial of degree k. It follows at once that

$$p_{nk}(x) = c^{-n} A_0 \cdots A_{n-1} Q_n(T(x)), \qquad n \geq 1; p_0(x) = 1, \tag{5.14}$$

so that

$$\begin{aligned}(\widehat{T}(x) - c^{-1}C_n)p_{nk}(x) \\ = p_{nk+k}(x) + c^{-2}A_{n-1}B_n p_{(n-1)k}(x), \qquad n \geq 1,\end{aligned} \tag{5.15}$$
$$p_0(x) := 1, \quad p_k(x) = \widehat{T}(x) - c^{-1}C_0.$$

Thus,

$$\widehat{T}(x) - c^{-1}C_0 = \widehat{T}_k(x) - b_0^{(0)}\widehat{U}_{k-1}(x) + \tfrac{1}{2}[1 - 2a_0^{(1)}]\widehat{U}_{k-2}(x) \tag{5.16}$$

and

$$\begin{aligned}\widehat{T}(x) - c^{-1}C_n = \widehat{T}_k(x) + \tfrac{1}{2}[1 - (a_n^{(0)} + a_n^{(1)})]\widehat{U}_{k-2}(x) \\ - b_n^{(0)}\widehat{U}_{k-1}(x), \qquad n \geq 1.\end{aligned} \tag{5.17}$$

Therefore,

$$(5.18) \quad c^{-1}(C_n - C_0) = [a_0^{(1)} - (a_n^{(0)} + a_n^{(1)})]\widehat{U}_{k-2}(x) + (b_0^{(0)} - b_n^{(0)})\widehat{U}_{k-1}(x), \qquad n \geq 1,$$

so that

$$(5.19) \quad b_n^{(0)} = b_0^{(0)}, \quad a_{n+1}^{(0)} + a_{n+1}^{(1)} = a_0^{(1)}, \quad C_n = C_0, \qquad n \geq 0.$$

Hence (5.18) holds, and if $R_n(x) = Q_n(x+C_0)$, $n \geq 0$, then $\{R_n(x)\}$ is a system of symmetric orthogonal polynomials and $Q_n(x) = R_n(x - C_0)$. □

COROLLARY 5.1. *Assume the polynomial mapping of Theorem* 5.1 *is* $T(x) = c\widehat{T}_k(x)$, $c \geq 2^{k-1}$, *and that* $\{p_n(x)\}$ *is obtained from the system* $\{Q_n(x)\}$ *by means of the mapping* $T(x)$. *If* $k > 2$, *then* $\{p_n(x)\}$ *is a system of sieved random walk polynomials of the first kind. If, in addition,* $c = 2^{k-1}$, *then* $Q_n(x)$ *is a system of random walk polynomials.*

Proof. From (5.15),

$$\widehat{T}_k(x) - c^{-1}C_0 = \widehat{T}_k(x) - b_0^{(0)}\widehat{U}_{k-1}(x) + \tfrac{1}{2}(1 - 2a_0^{(1)})\widehat{U}_{k-2}(x).$$

It follows that $C_0 = b_0^{(0)} = 0$ and $a_0^{(1)} = \frac{1}{2}$. Also, from (5.17),

$$\widehat{T}_k(x) - c^{-1}C_n = \widehat{T}_k(x) - b_n^{(0)}U_{k-1}(x) + \tfrac{1}{2}[1 - 2(a_n^{(0)} + a_n^{(1)})],$$

so that $b_n^{(0)} = C_n = 0$ and $a_n^{(0)} + a_n^{(1)} = \frac{1}{2}$, $n \geq 1$. On the other hand, if $c = 2^{k-1}$,

$$(5.20) \quad T_k(x)Q_n(T_k(x)) = A_nQ_{n+1}(T_k(x)) + B_nQ_{n-1}(T_k(x)), \qquad n \geq 1.$$

Also,

$$(5.21) \quad T_k(x)Q_n'(T_k(x)) = 2a_n^{(1)}Q_{n+1}'(T_k(x)) + 2a_n^{(0)}Q_{n-1}'(T_k(x)), \qquad n \geq 1,$$

with

$$(5.22) \quad Q_0'(T_k(x)) = 1; \quad Q_n'(T_k(x)) = \frac{2^{n(k-2)}}{a_0^{(1)}a_1^{(1)}\cdots a_{n-1}^{(1)}}p_{nk}(x), \qquad n \geq 1,$$

as follows from (5.10) and (5.11). Hence, from (5.20) and (5.22), $Q_n(x) = Q'_n(x)$, and then $A_n = 2a_n^{(1)}$, $B_n = 2a_n^{(0)}$, $n \geq 1$. Thus, $A_n + B_n = 1$, $n \geq 1$, and, since $A_0 = 2a_0^{(1)} = 1$, it follows that $\{Q_n(x)\}$ is as random walk polynomial system.

Theorem 5.1 and Corollary 5.1 generalize results in **[9]** from the case of symmetric polynomials to general polynomials which are not necessarily symmetric.

REMARK 5.1. The system $\{p_n(x)\}$ of sieved Pollaczek polynomials of the first kind (see **[8]**) has the recurrence coefficients

$$(5.23)\quad a_n^{(0)} = \frac{n}{4(n+a+\lambda)}, \quad a_n^{(1)} = \frac{n+2\lambda}{4(n+a+\lambda)}, \quad a_k^{(j)} = \frac{1}{4}, \quad 2 \leq j \leq k-1, \quad n \geq 0,$$

$$b_n^{(0)} = \frac{b}{n+a+\lambda}, \quad b_n^{(j)} = 0, \quad 1 \leq j \leq k-1, \quad n \geq 0.$$

It follows from Theorem 5.1 that if $k > 2$ and $a \neq 0$, it cannot be obtained from any system of orthogonal polynomials via a polynomial mapping. On the other hand, if $a = b = 0$, then $\{p_n(x)\}$ is a system of sieved random walk polynomials, namely, the sieved ultraspherical polynomials of the first kind of Al-Salam, Allaway and Askey **[2]**, and

$$(5.24)\qquad p_{nk}(x) = \frac{2(\lambda)_n}{(\lambda)_n 2^{nk}} Q_n(T_k(x)), \qquad n \geq 0,$$

where

$$(5.25)\quad xQ_n(x) = \frac{n+2\lambda}{2(n+\lambda)} Q_{n+1}(x) + \frac{n}{2(n+\lambda)} Q_{n-1}(x), \quad n \geq 1,$$
$$Q_0(x) = 1, \quad Q_1(x) = 2x.$$

This follows from (5.11). It is readily seen that

$$(5.26)\qquad Q_n(x) = \frac{n!}{(2\lambda)_n} C_n(x, \lambda), \qquad n \geq 0,$$

where (see **[18]**)

$$(5.27)\quad 2(n+\lambda)C_n(x, \lambda) = (n+1)C_{n+1}(x, \lambda) + (n+2\lambda-1)C_{n-1}(x, \lambda), \qquad n \geq 0,$$

$$C_{-1}(x, \lambda) = 0, \quad C_0(x, \lambda) = 1$$

is the system of ultraspherical polynomials. Thus,

$$(5.28)\qquad p_{nk}(x) = \frac{n!}{2^{nk}(\lambda)_n} C_n(T_k(x), \lambda), \qquad n \geq 0.$$

We also observe that if under the remaining assumptions of the sieved ultraspherical polynomials of the first kind, i.e.,

$$(5.29)\qquad \begin{aligned} &a_n^{(0)} = \frac{n}{4(n+\lambda)}, \quad a_n^{(1)} = \frac{n+2\lambda}{4(n+\lambda)}, \qquad n \geq 1, \\ &a_n^{(j)} = \frac{1}{4}, \quad b_n^{(0)} = b_n^{(1)} = b_n^{(j)} = 0, \\ &\qquad\qquad 2 \leq j \leq k-1, \qquad n \geq 0, \end{aligned}$$

we change $a_0^{(1)}$ from $1/2$ to $\alpha/2$, $\alpha \neq 1$, then, if $k > 2$, $\{p_{nk}(x)\}$ cannot be obtained from any system of orthogonal polynomials by means of polynomial mappings (because $a_n^{(0)} + a_n^{(1)} = 1/2 \neq \alpha/2 = a_0^{(1)}$). However, it easily follows that

$$(5.30)\quad \begin{aligned} p_{nk}(x) = \frac{n!}{2^{nk}(\lambda)_n}[C_n(T_k(x), \lambda) &+ 2\lambda(1-\alpha)U_{k-2}(x)C_{n-1}^{(1)}(T_k(x), \lambda) \\ &+ 2\lambda(1-\alpha)C_{n-1}^{(2)}(T_k(x), \lambda)], \\ &\qquad n \geq 0, \end{aligned}$$

or equivalently,

$$(5.31)\quad \begin{aligned} p_{nk}(x) = \frac{n!}{2^{nk}(\lambda)_n}[\alpha C_n(T_k(x), \lambda) &+ 2\lambda(1-\alpha)xU_{k-1}(x)C_{k-1}^{(1)}(T_k(x), \lambda)], \\ &\qquad n \geq 1, \end{aligned}$$

where $\{C_n^{(i)}(x, \lambda)\}$ denotes the system of ith-associated polynomials of $\{C_n(x, \lambda)\}$. Note that if $k = 2$, (5.30) shows that $\{p_{nk}(x)\}$ originates via a polynomial mapping.

REMARK 5.2. Let $\{p_{nk}(x)\}$ be given by (5.1) and (5.2), and assume that $\{p_{nk}(x)\}$ is obtained from the system (5.12) by means of a polynomial mapping $T(x)$. It follows from the proof of Theorem 3.1 that if $k \geq 2$ then $b_n^{(0)} = b_0^{(0)}$, $n \geq 0$, i.e., $b_n^{(0)}$ is independent of n. The general (non-symmetric) sieved Pollaczek polynomials do not satisfy this condition (as $b \neq 0$). Hence, they cannot be obtained via polynomial mappings, even if $k = 2$. However, the symmetric sieved Pollaczek polynomials ($b = 0$ in (5.23)) can be obtained via a polynomial mapping when $k = 2$. In fact,

$$(5.32)\qquad p_{2n}(x) = \frac{n!}{4^n(\lambda)_n}P_n(T_2(x)), \qquad n \geq 0,$$

where $P_n(x) = P_n(x, \lambda, a, a)$, $n \geq 0$, is the system of the Pollaczek polynomials

$$(5.33)\quad 2[(n+\lambda+a)x + a]P_n(x) = (n+1)P_{n+1}(x) + (n+2\lambda-1)P_{n-1}(x), \qquad n \geq 0,$$
$$p_{-1}(x) = 0, \quad P_0(x) = 1.$$

Thus, Theorem 5.1 cannot be extended to the case $k = 2$.

REMARK 5.3. It is usually assumed that $a_0^{(1)} = 1/2$ for sieved polynomials of the first kind (perhaps for historical reasons, because this was indeed the case for the sieved ultra-spherical and random walk polynomials in [2], [7]). Here we drop this assumption, and some interesting results will come about. For example, the sieved ultra-spherical polynomials of the first kind in [2] (i.e., $a_n^{(j)}$ given by (5.29) with $a_0^{(1)} = 1/2$ and $\lambda > 0$) are orthogonal with respect to an absolutely continuous measure whose support is $[-1, 1]$. However, if $a_0^{(1)}$ is changed to $\alpha/2$ where $\alpha = \frac{2\lambda k}{2\lambda(k-1)+1}$, and $\lambda > 3/2$, the absolutely continuous part of the orthogonality measure of the resulting polynomials $\{p_{nk}(x)\}$ still has $[-1, 1]$ as its support, but now the measure carries masses at the end points ± 1 when k is even. To see this, observe that, from (5.31),

$$p_{nk}(1) = \frac{n!}{2^{nk}(\lambda)_n}[\alpha C_n(1, \lambda) + 2\lambda(1-\alpha)kC_{n-1}^{(1)}(1, \lambda)], \qquad n \geq 1.$$

But

$$C_n(1, \lambda) = \frac{(2\lambda)_n}{n!}, \quad C_{n-1}^{(1)}(1, \lambda) = \frac{1}{2\lambda-1}\left[\frac{(2\lambda)_n}{n!} - 1\right], \qquad n \geq 0,$$

as follows from (5.27). Hence

$$p_{nk}(x) = \frac{n!}{2^{nk}(\lambda)_n}\left[\frac{2\lambda k - \alpha(2\lambda(k-1)+1)}{2\lambda-1}\frac{(2\lambda)_n}{n!} - \frac{2\lambda k(1-\alpha)}{2\lambda-1}\right], \qquad n \geq 1,$$

and, if $\alpha = \frac{2\lambda k}{2\lambda(k-1)+1}$, then

$$p_{nk}(1) = \frac{n!}{2^{nk}(\lambda)_n} \cdot \frac{2\lambda k(\alpha-1)}{2\lambda-1}, \qquad n \geq 1.$$

Let μ denote the orthogonality measure of $\{p_n(x)\}$. The measure μ is compactly supported and its absolutely continuous part has support $[-1, 1]$. Furthermore,

$$\int_{-\infty}^{+\infty} p_n(x)p_m(x)\,d\mu(x) = \lambda_n\delta_{mn}, \qquad m, n \geq 0,$$

where

$$\lambda_0 = 1; \quad \lambda_{kn} = \frac{\alpha}{4^{kn}} \cdot \frac{(2\lambda)_n \cdot n!}{(\lambda)_n^2(\lambda+n)},$$
$$\lambda_{nk+j} = \frac{\alpha}{4^{nk+j}} \cdot \frac{(2\lambda)_{n+1} \cdot n!}{(\lambda)_{n+1}^2},$$
$$n \geq 1, \quad 1 \leq j \leq k-1.$$

It follows that

$$\frac{p_{nk}^2(1)}{\lambda_{kn}} = \frac{n!(\lambda+k)}{(2\lambda)_n \alpha}\left[\frac{2\lambda k(1-\alpha)}{2\lambda-1}\right]^2, \qquad n \geq 1,$$

and, since $\alpha \neq 1$, that

$$\frac{p_{nk}^2(1)}{\lambda_{kn}} \sim \left[\frac{2\lambda k(1-\alpha)}{2\lambda-1}\right]^2 \cdot \frac{\Gamma(2\lambda)}{\alpha} \cdot n^{2-2\lambda}, \qquad n \geq 1.$$

Since $\lambda > 3/2$, $\sum_{n=0}^{\infty} p_{nk}^2(1)/\lambda_{kn}$ converges. Moreover, it follows from (2.5) that

$$U_{k-1}(x)p_{nk+j}(x) = 2^{k-j}U_{j-1}(x)p_{nk+k}(x) + 2^{-j}\frac{n+2\lambda}{n+\lambda}U_{k-j-1}(x), \qquad 1 \leq j \leq k-1.$$

Thus

$$k\frac{p_{nk+j}(1)}{\sqrt{\lambda_{kn+j}}} = j\sqrt{\frac{n+1}{n+\lambda+1}} \cdot \frac{p_{(n+1)k}(1)}{\sqrt{\lambda_{(n+1)k}}} + (k-j)\sqrt{\frac{n+2\lambda}{n+\lambda}}\frac{p_{nk}(1)}{\sqrt{\lambda_{nk}}}$$

from which we deduce (using the inequality $(a+b)^2 \leq 2(a^2+b^2)$) that

$$\sum_{k=0}^{\infty} \frac{p_{nk+j}^2(1)}{\lambda_{nk+j}} < +\infty, \qquad j = 1, 2, \ldots, k-1.$$

Hence $\sum_{n=0}^{\infty} p_n^2(1)/\lambda_n < +\infty$, and from [**4**, p. 13], we conclude that μ has a mass at $x = 1$ and, thus, also at $x = -1$. We observe that if $k > 2$, this conclusion cannot be obtained from the theory of polynomial mappings as presented in [**12**].

REMARK 5.4. Under the circumstances above it can be shown that

$$\mu(\{-1\}) = \mu(\{1\}) = \frac{2\lambda-3}{2(2\lambda-1)} \tag{5.34}$$

when $k = 2$ (see [**10**]).

We finally give an example of how our procedure can be advantageous over other treatments of sieved orthogonal polynomials. To this purpose we shall consider an example of sieved orthogonal polynomials recently dealt with by Al-Salam and Ismail [**1**]: the sieved associated Pollaczek polynomials. Contrary to ours, their treatment is historical, and the polynomials are obtained from the associated q-Pollaczek polynomials (see [**3**]) by the same limit process as in [**8**], [**13**]. Then, the limit process is used to establish generating functions for the polynomials, a very delicate matter, and the asymptotic behavior and the Stieltjes transform of the orthogonality measure are determined via Darboux's method ([**15**], Chap. VIII). We follow a more direct approach.

We recall that the system of associated Pollaczek polynomials $\{R_n(x)\}$ is determined (see [**16**]) by the recurrence relations

$$\begin{aligned}(5.35)\quad & 2[(\lambda+n+a+c)x+b]R_n(x)\\ &\quad = (n+c+1)R_{n+1}(x)+(n+c+2\lambda-1)R_{n-1}(x),\\ &\qquad\qquad n\geq 0, \quad R_{-1}(x)=0, \quad R_0(x)=1.\end{aligned}$$

The notation $R_n(x) = P_n(x;\lambda, a, b, c)$ is also used. We observe that if $P_n(x;\lambda, a, b) = P_n(x;\lambda, a, b, 0)$ and $c = 1, 2, \dots,$ then $\{P_n(x;\lambda, a, b, c)\}$ is the system of cth-associated polynomials of $\{P_n(x;\lambda, a, b)\}$. The latter system is simply called the system of Pollaczek polynomials.

If $\lambda > 0$ and $a, c \geq 0$, $\{R_n(x)\}$ is a system of orthogonal polynomials (other cases of orthogonality are possible). Let

$$(5.36)\qquad R(x, t) = \sum_{n=0}^{\infty} R_n(x)t^{n+c}.$$

By showing from the recurrence relation (5.35) that

$$(5.37)\qquad \frac{\partial R(x, t)}{\partial t} - \frac{2((a+\lambda)x-\lambda t+b)}{t^2-2xt+1}R(x, t) = \frac{t^{c-1}}{t^2-2xt+1}$$

and

$$(5.38)\qquad R(x, 0) = 1, \quad c=0; \qquad R(x, 0)=0, \quad c>0,$$

it follows that

$$\begin{aligned}(5.39)\quad R(x, t) &= c(1-\beta t)^A(1-\alpha t)^B\\ &\quad\times\int_0^t u^{c-1}(1-\beta u)^{-A-1}(1-\alpha u)^{-B-1}\,du,\\ &\qquad\qquad c>0,\end{aligned}$$

and

$$R(x, t) = (1 - \beta t)^A (1 - \alpha t)^B, \qquad c = 0, \tag{5.40}$$

where

$$\alpha = \alpha(x) = x + \sqrt{x^2 - 1}, \quad \beta = \beta(x) = x - \sqrt{x^2 - 1} \tag{5.41}$$

and

$$A = -\lambda + 2\frac{ax + b}{\alpha - \beta}, \quad B = -\lambda - 2\frac{ax + b}{\alpha - \beta}. \tag{5.42}$$

From (5.39) and (5.40), and observing that $R_n^{(1)}(x)$ is $R_n(x)$ with $c + 1$ instead of c, it can be deduced (via Darboux's method, for example) that the Stieltjes transform of the orthogonality measure μ of $\{R_n(x)\}$:

$$R(x) = \lim_{n\to\infty} \frac{R_{n-1}^{(1)}(x)}{R_n(x)} = \int_{-\infty}^{+\infty} \frac{d\mu(t)}{t - x}, \qquad x \in \mathbb{C} - \mathbb{R}, \tag{5.43}$$

is

$$R(x) = \beta \int_0^{1\rceil} (1 - \beta^2 u)^{-A-1} (1 - u)^{-B-1}\, du, \qquad c = 0 \tag{5.44}$$

and

$$R(x) = \frac{c+1}{c} \frac{\int_0^{1\rceil} u^c (1 - \beta^2 u)^{-A-1} (1 - u)^{-B-1}\, du}{\int_0^{1\rceil} u^{c-1} (1 - \beta^2 u)^{-A-1} (1 - u)^{-B-1}\, du}, \qquad c > 0. \tag{5.45}$$

We observe that the integrals in (5.44) and (5.45) are Hadamard integrals. As as matter of fact

$$\begin{aligned} &\int_0^{1\rceil} u^c (1 - zu)^{-A-1} (1 - u)^{-B-1}\, du \\ &\qquad = \frac{\Gamma(c+1)\Gamma(-B)}{\Gamma(-B + c + 1)} {}_2F_1\left(\begin{matrix} A + 1,\ c + 1 \\ -B + c + 1 \end{matrix} \middle| z \right), \\ &\qquad\qquad |z| < 1,\ c > -1, \end{aligned} \tag{5.46}$$

and the integral makes sense as long as B is not an integer ≥ 0 (and not only when $\mathrm{Re}(B) < 0$). This was discussed in §1.

The branch $\sqrt{x^2 - 1}$ of the square root of $x^2 - 1$ in (5.41) is so chosen that $\sqrt{x^2 - 1} \sim x$ as $x \to \infty$.

Relation (5.44) can be obtained from (5.45) by taking

$$c\int_0^1 u^{c-1}\left(1-\frac{\beta}{\alpha}u\right)^{-A-1}(1-u)^{-B-1}\,du = 1 \tag{5.47}$$

when $c=0$.

We begin by considering the system $\{q_n(x)\}$ determined by

$$\begin{aligned}(x-b_n^{(j)})q_{nk+j}(x) &= q_{nk+j+1}(x)+a_n^{(j)}q_{nk+j-1}(x), \qquad n\geq 0,\\ &0\leq j\leq k-1,\ q_{-1}(x)=0, \quad q_0(x)=1.\end{aligned} \tag{5.48}$$

We assume $k\geq 2$ and

$$a_n^{(0)}=\frac{n+2\lambda+c}{4(n+\lambda+a+c)}, \quad a_n^{(1)}=\frac{n+c}{4(n+\lambda+a+c)}, \qquad n\geq 0. \tag{5.49}$$

$$b_n^{(0)}=\frac{b}{n+\lambda+a+c}, \quad b_n^{(j)}=0,\ 1\leq j\leq k-1, \qquad n\geq 0. \tag{5.50}$$

and

$$a_n^{(j)}=\frac{1}{4}, \qquad n\geq 0,\ 2\leq j\leq k-1. \tag{5.51}$$

Thus, the system $p_n(x) = q_n^{(1)}(x)$, $n\geq 0$, will be the system of sieved associated Pollaczek polynomials of the second kind. Clearly $\{p_n^{(r)}(x)\}$, their system of associated polynomials of order r, is the system of monic polynomials of the orthogonal polynomials $\{Q_n^{(\lambda,r)}(x)\}$ in **[1]**, for $0\leq r<k$.

Let $\{P_n(x)\}$ denote the link polynomials of $\{q_n(x)\}$. Then $\{p_n^{(r)}(x)\}$ can be represented in terms of the polynomials $\{P_n^{(1)}(x)\}$ and $\{P_n^{(2)}(x)\}$ via (3.15)–(3.17). Now, $\{P_n^{(1)}(x)\}$ satisfies

$$\begin{aligned}&\frac{1}{2^{k-1}}\left[T_k(x)+\frac{b}{n+\lambda+a+c+1}U_{k-1}(x)\right.\\ &\qquad\qquad\left.+\frac{a}{n+\lambda+a+c+1}U_{k-2}(x)\right]P_n^{(1)}(x)\\ &=P_{n+1}^{(1)}(x)+4^{1-k}\frac{n+2\lambda+c+1}{n+\lambda+a+c+1}\cdot\frac{n+c}{n+\lambda+a+c}P_{n-1}^{(1)}(x),\\ &\qquad\qquad n\geq 1\end{aligned} \tag{5.52}$$

and the initial conditions

$$(5.53)\quad \begin{aligned} P_0^{(1)}(x) &= 1, \\ P_1^{(1)}(x) &= \frac{1}{2^{k-1}}\left[T_k(x) + \frac{b}{\lambda+a+c+1}U_{k-1}(x)\right. \\ &\qquad\qquad \left. + \frac{c}{\lambda+a+c+1}U_{k-1}(x)\right]. \end{aligned}$$

If

$$(5.54)\qquad \widetilde{R}_n(x) = \frac{2^{nk}(\lambda+a+c+1)_n}{(c+1)_n}P_n^{(1)}(x), \qquad n \geq 0,$$

then

$$(5.55)\quad \begin{aligned} 2[(n+\lambda+a+c+1)T_k(x) + bU_{k-1}(x) + aU_{k-2}(x)]\widetilde{R}_n(x) \\ = (n+c+1)\widetilde{R}_{n+1}(x) + (n+2\lambda+c+1)\widetilde{R}_{n-1}(x), \\ n \geq 0, \end{aligned}$$

and

$$(5.56)\quad \begin{aligned} \widetilde{R}_0(x) &= 1, \\ \widetilde{R}_1(x) &= \frac{2}{c+1}[(\lambda+a+c+1)T_k(x) + bU_{k-1}(x) + cU_{k-2}(x)]. \end{aligned}$$

As in the case of the Pollaczek polynomials, it can be shown that

$$(5.57)\quad \begin{aligned} \sum_{n=0}^{\infty} \widetilde{R}_n(x)t^{n+c} &= c(1-\beta^k t)^{A-1}(1-\alpha^k t)^{B-1} \\ &\quad \cdot \int_0^t u^{c-1}(1-\beta^k u)^{-A}(1-\alpha^k u)^{-B}\,du \end{aligned}$$

where $\alpha = \alpha(x)$, $\beta = \beta(x)$ and

$$(5.58)\quad \begin{aligned} A &= -\lambda + 2\frac{2T_k(x) + bU_{k-1}(x) + aU_{k-2}(x)}{\beta^k - \alpha^k} \\ &= -\lambda + 2\frac{ax+b}{\alpha-\beta}, \\ B &= -\lambda - 2\frac{aT_k(x) + bU_{k-1}(x) + aU_{k-2}(x)}{\beta^k - \alpha^k} \\ &= -\lambda - 2\frac{ax+b}{\alpha-\beta}. \end{aligned}$$

We note that $\alpha^k(x) = \alpha(T_k(x))$, $\beta^k(x) = \beta(T_k(x))$. From (5.57) it can be deduced (via Darboux's method, for example, in the same manner as it is done for the polynomials $\{R_n(x)\}$), that

$$\tag{5.59} \begin{aligned} \widetilde{R}(x) &= \lim_{n\to\infty} \frac{\widetilde{R}^{(1)}_{n-1}(x)}{\widetilde{R}_n(x)} \\ &= \frac{c+1}{c}\beta^k \frac{\int_0^1 u^c(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du}{\int_0^1 u^{c-1}(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du}. \end{aligned}$$

On the other hand,

$$\tag{5.60} P^{(2)}_n(x) = \frac{(c+2)_n}{2^{nk}(\lambda+a+c+2)_n}\widetilde{R}^{(1)}_n(x).$$

Therefore,

$$\tag{5.61} \begin{aligned} P^{(1)}(x) &= \lim_{n\to\infty} \frac{P^{(2)}_{n-1}(x)}{P^{(1)}_n(x)} \\ &= 2^k\frac{(\lambda+a+c+1)}{c+1}R(x), \qquad x\in\mathbb{C}-\mathbb{R}. \end{aligned}$$

Now, it follows from (3.25) that the continued fraction $X_r(x)$ of $\{p^{(r)}_n(x)\}$, i.e., of $\{q^{(r+1)}_n(x)\}$, and thus of $\{Q^{(\lambda,r)}_n(x)\}$, is, for $0\le r<k-1$,

$$\tag{5.62} \begin{aligned} X_r(x) &= 2\frac{2^{k-2}U_{k-r-2}(x)+a^{(0)}_1U_r(x)P^{(1)}(x)}{2^{k-1}U_{k-r-1}(x)+a^{(0)}_1U_{r-1}(x)P^{(1)}(x)} \\ &= 2A/B \end{aligned}$$

where

$$\begin{aligned} A = {}& cU_{k-r-2}(x)\int_0^1 u^{c-1}(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du \\ &+(2\lambda+c+1)U_r(x)\beta^k\int_0^1 u^c(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du \end{aligned}$$

and

$$\begin{aligned} B = {}& cU_{k-r-1}(x)\int_0^1 u^{c-1}(1-\beta^{2}u)^{-A}(1-u)^{B}\,du \\ &+(2\lambda+c+1)U_{r-1}(x)\beta^k\int_0^1 u^c(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du. \end{aligned}$$

which, after using the identities

$$(5.63)\quad \begin{aligned}(2\lambda+c+1)&\int_0^1 u^c(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du\\ &=(-A)\int_0^1 u^c(1-\beta^{2k}u)^{-A-1}(1-u)^{-B}\,du\\ &\quad+(-B)\int_0^1 u^c(1-\beta^{2k}u)^{-A}(1-u)^{-B-1}\,du\\ &=c\int_0^1 u^{c-1}(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du\\ &\quad-A(1-\beta^{2k})\int_0^1 u^c(1-\beta^{2k}u)^{-A-1}(1-u)^{-B}\,du\end{aligned}$$

and

$$(5.64)\qquad U_j(x)=\frac{\alpha^{j+1}-\beta^{j+1}}{\alpha-\beta},$$

becomes

$$(5.65)\qquad X_r(x)=2\beta C/E$$

where

$$\begin{aligned}C=c&\int_0^1 u^{c-1}(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du\\ &+A\beta^{2k}(1-\alpha^{2r+2})\int_0^1 u^c(1-\beta^{2k}u)^{-A-1}(1-u)^{-B}\,du\end{aligned}$$

and

$$\begin{aligned}E=c&\int_0^1 u^{c-1}(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du\\ &+A\beta^{2k}(1-\alpha^{2r})\int_0^1 u^c(1-\beta^{2k}u)^{-A-1}(1-u)^{-B}\,du\end{aligned}$$

which is (3.5) of **[1]**. Observe that when $c=r=0$, we obtain (using (5.47)) that

$$(5.66)\, X_0(x)=2\left[\beta+(\beta-\alpha)\beta^{2k}A\int_0^1(1-\beta^{2k}u)^{-A-1}(1-u)^{-B}\,du\right]$$

which is (3.39) of **[8]**.

As for the case $r = k-1$, we need to calculate the continued fraction of $\{p_n^{(k-1)}(x)\}$, or the same, of $\{q_n^{(k)}(x)\}$. According to (3.24), this is

$$X_{k-1}(x) = \frac{2^{1-k} U_{k-1}(x) P^{(1)}(x)}{1 + 2^{2-k} a_1^{(0)} U_{k-2}(x) P^{(1)}(x)}. \tag{5.67}$$

A calculation as above readily gives
(5.68)

$$X_{k-1}(x) = \frac{2\beta(\lambda + a + c + 1) \int_0^1 u^c (1 - \beta^{2k} u)^{-A} (1-u)^{-B}\, du}{D}$$

where

$$\begin{aligned} D = {} & c \int_0^1 u^{c-1} (1 - \beta^{2k} u)^{-A} (1-u)^{-B}\, du \\ & + A\beta^{2k}(1 - \alpha^{2k-2}) \int_0^1 u^c (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B}\, du. \end{aligned}$$

The above procedure can also be applied to the k-sieved associated Pollaczek polynomials of the first kind $P_n^{(\lambda, r)}(x) = P_n^{(\lambda, r)}(x; a, b, c)$, $k \geq 2$, $n \geq 0$, $r = 0, 1, 2, \dots, k-1$. These are given by the recurrence relation

$$\begin{aligned} & 2xP_n^{(\lambda,r)}(x) = P_{n+1}^{(\lambda,r)}(x) + P_{n-1}^{(\lambda,r)}(x), \ k|n+r, \qquad n \geq 0, \\ & 2[(m + a + c + \lambda)x + b] P_{mk-r}^{(\lambda,r)}(x) \\ & \quad = (m + c + 2\lambda) P_{mk-r+1}^{(\lambda,r)}(x) + m P_{mk-r-1}^{(\lambda,r)}(x), \ n + r = mk, \\ & \qquad m \geq 0, \end{aligned} \tag{5.69}$$

and the initial conditions

$$P_{-1}^{(\lambda,r)}(x) = 0, \quad P_0^{(\lambda,r)}(x) = 1. \tag{5.70}$$

For simplicity we will assume that b is a real number and $\lambda > 0$, $a, c \geq 0$, but other cases of orthogonality can be similarly handled.

It is readily verified that the system of monic polynomials of $\{p_n^{(\lambda,r)}(x)\}$ is the associated system $\{p_n^{(r)}(x)\}$ of order r of the orthogonal polynomial set $\{p_n(x)\}$ given by the blocks

$$(x - b_n^{(j)}) p_{nk+j}(x) = p_{nk+j+1}(x) + a_n^{(j)} p_{nk+j-1}(x) \tag{5.71}$$

for $n \geq 0$, $j = 0, 1, 2, \dots, k-1$, and the initial conditions

$$p_{-1}(x) = 0, \quad p_0(x) = 1, \tag{5.72}$$

where

$$
\begin{aligned}
(5.73)\quad b_n^{(0)} &= \frac{-b}{n+a+c+\lambda}; \quad b_n^{(j)} = 0, \qquad j = 1, 2, \ldots, k-1, \\
a_n^{(0)} &= \frac{n+c}{4(n+a+c+\lambda)}, \quad a_n^{(1)} = \frac{n+c+2\lambda}{4(n+a+c+\lambda)} \\
a_n^{(j)} &= \frac{1}{4}, \qquad j = 2, 3, \ldots, k-1, n \geq 0.
\end{aligned}
$$

The link polynomials of $\{p_n(x)\}$ satisfy

(5.74)

$$
\begin{aligned}
&2^{1-k}\left[T_k(x) + \frac{a}{n+\lambda+a+c}U_{k-2}(x) + \frac{b}{n+\lambda+a+c}U_{k-1}(x)\right] P_n(x) \\
&= P_{n+1}(x) + 2^{-2k}\frac{n+c}{n+\lambda+a+c-1}\cdot\frac{n+c+2\lambda-1}{n+\lambda+a+c}P_{n-1}(x), \\
&\qquad n \geq 0,
\end{aligned}
$$

and the initial conditions

$$
(5.75)\qquad P_{-1}(x) = 0, \quad P_0(x) = 1.
$$

If we let

$$
(5.76)\qquad Q_n(x) = \frac{2^{nk}(\lambda+a+c)_n}{(c+1)_n}P_n(x), \qquad n \geq 0,
$$

then $Q_{-1}(x) = 0$, $Q_0(x) = 1$ and

$$
\begin{aligned}
(5.77)\quad & 2[(n+\lambda+a+c)T_k(x) + aU_{k-2}(x) + bU_{k-1}(x)]Q_n(x) \\
&= (n+c+1)Q_{n+1}(x) + (n+c+2\lambda-1)Q_{n-1}(x), \\
&\qquad n \geq 0.
\end{aligned}
$$

Also

$$
(5.78)\qquad Q_n^{(1)}(x) = \frac{2^{nk}(\lambda+a+c)_n}{(c+2)_n}P_n^{(1)}(x), \qquad n \geq 0,
$$

and, as before, we obtain

$$
\begin{aligned}
(5.79)\quad & \lim_{n\to\infty}\frac{Q_{n-1}^{(1)}(x)}{Q_n(x)} \\
&= \frac{c+1}{c}\beta^k\frac{\int_0^1 u^c(1-\beta^{2k}u)^{-A-1}(1-u)^{-B-1}\,du}{\int_0^1 u^{c-1}(1-\beta^{2k}u)^{-A-1}(1-u)^{-B-1}\,du}
\end{aligned}
$$

for $x \in \mathbb{C} - \mathbb{R}$, where $\alpha = \alpha(x)$, $\beta = \beta(x)$ are given by (5.41) and $A = A(x)$, $B = B(x)$ by (5.42) or (5.58). Observe that $\alpha(T_k(x)) = \alpha^k(x)$, $\beta(T_k(x)) = \beta^k(x)$. Thus,

$$P^{(0)}(x) = \lim_{n\to\infty} \frac{P_{n-1}^{(1)}(x)}{P_n(x)} = 2^k \frac{(\lambda + a + c)}{c+1} \lim_{n\to\infty} \frac{Q_{n-1}^{(1)}(x)}{Q_n(x)} \tag{5.80}$$

and therefore, if $x \in \mathbb{C} - \mathbb{R}$, then

(5.81)
$$P^{(0)}(x) = 2^k \frac{\lambda + a + c}{c} \beta^k \frac{\int_0^1 u^c (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du}{\int_0^1 u^{c-1} (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du}$$

where as before

$$c \int_0^1 u^{c-1} (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du = 1, \qquad c = 0.$$

Also,

(5.82)
$$\begin{aligned} P^{(1)}(x) &= \lim_{n\to\infty} \frac{P_{n-1}^{(2)}(x)}{P_n^{(1)}(x)} \\ &= 2^k \frac{\lambda + a + c + 1}{c+1} \beta^k \frac{\int_0^1 u^{c+1} (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du}{\int_0^1 u^{c} (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du}. \end{aligned}$$

From (5.81) and (5.82) and from relations (4.23) and (3.25), we obtain for the continued fraction $X_r(x)$ of $\{P_n^{(\lambda, r)}(x)\}$ the following evaluation

$$\begin{aligned} X_0(x) &= \frac{1}{2^{k-1}} U_{k-1}(x) P^{(0)}(x) \\ &= 2 \frac{\lambda + a + c}{c} \beta^k \\ &\quad \times U_{k-1}(x) \frac{\int_0^1 u^c (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du}{\int_0^1 u^{c-1} (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du} \end{aligned} \tag{5.83}$$

which reduces to

$$X_0(x) = 2(\lambda + a) \beta^k U_{k-1}(x) \int_0^1 (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du \tag{5.84}$$

when $c = 0$, and

$$X_r(x) = 2\frac{2^{k-2}U_{k-r-1}(x) + a_1^{(0)}U_{r-1}(x)P^{(1)}(x)}{2^{k-2}U_{k-r}(x) + a_1^{(0)}U_{r-2}(x)P^{(1)}(x)} = 2\frac{A+B}{C+D} \tag{5.85}$$

where

$$A = U_{k-r-1}(x)\int_0^1 u^c(1-\beta^{2k}u)^{-A-1}(1-u)^{-B-1}\,du,$$

$$C = U_{k-r}(x)\int_0^1 u^{c+1}(1-\beta^{2k}u)^{-A-1}(1-u)^{-B-1}\,du,$$

$$B = U_{r-1}(x)\beta^k\int_0^1 u^{c+1}(1-\beta^{2k}u)^{-A-1}(1-u)^{-B-1}\,du,$$

$$D = U_{r-2}(x)\beta^k\int_0^1 u^{c+1}(1-\beta^{2k}u)^{-A-1}(1-u)^{-B-1}\,du$$

for $c \geq 0$ and $r = 1, 2, \ldots, k-1$.

Acknowledgments. The authors thank G. Rodriguez of the Universidad de los Andes, Bogotá, for his help. The first author also thanks Colciencias, Bogotá, for partial travel support to attend the First U.S.A.–U.S.S.R. Conference on Approximation Theory, Tampa, Florida, March, 1990, where some of the results in this paper were presented.

References

[1] N. A. Al-Salam and M. E. H. Ismail, *On sieved orthogonal polynomials* VIII: *Sieved associated Pollaczek polynomials*, J. Approximation Theory, (1991), to appear.

[2] W. Al-Salam, W. R. Allaway and R. Askey, *Sieved ultraspherical polynomials*, Trans. Amer. Math. Soc., **234** (1984), 39–55.

[3] R. Askey and M. E. H. Ismail, *A generalization of ultraspherical polynomials*, in *Studies in Pure Mathematics* (P. Erdös, Ed.), Birkhäuser, Basel, 1983, 55–78.

[4] ——, *Recurrence relations, continued fractions and orthogonal polynomials*, Mem. Amer. Math. Soc., **300** (1984), 110 pp.

[5] H. Bremerman, *Distributions, Complex Variables and Fourier Transforms*, Addison-Wesley, Reading, Mass., 1965.

[6] J. A. Charris, C. P. Gomez and G. Rodriguez, *Two systems of orthogonal polynomials related to the Pollaczek polynomials*, to appear, Rev. Col. de Mat.

[7] J. A. Charris and M. E. H. Ismail, *On sieved orthogonal polynomials*, II: *Sieved random walk polynomials*, Canad. J. Math., **38** (1986), 397–415.

[8] ——, *On sieved orthogonal polynomials*, V: *Sieved Pollaczek polynomials*, SIAM J. Math. Anal., **18** (1987), 1177–1218.

[9] ——, *On sieved orthogonal polynomials*, VII: *Generalized polynomial mappings*, to appear, Trans. Amer. Math. Soc.

[10] J. A. Charris and G. Rodriguez-Blanco, *On systems of orthogonal polynomials with inner and end-point masses*, Rev. Col. de Mat., **24** (1990), 153–177.

[11] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[12] J. Geronimo and W. Van Assche, *Orthogonal polynomials on several intervals via a polynomial mapping*, Trans. Amer. Math. Soc., **308** (1986), 559–581.

[13] M. E. H. Ismail, *On sieved orthogonal polynomials*, I: *Symmetric Pollaczek analogues*, SIAM J. Math. Anal., **16** (1985), 89–111.

[14] S. Lang, *Real Analysis*, Addison-Wesley, Reading, Mass., 1984.

[15] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974.

[16] F. Pollaczek, *Sur une famille de polynomes orthogonaux a quatre parametres*, C.R. Acad. Sci. Paris, **230** (1950), 2254–2256.

[17] ——, *Sur une generalization des polynomes de Jacobi*, Memor. Sci. Mathematiques, **131** (1956), Gauthier-Villars, Paris.

[18] E. D. Rainville, *Special Functions*, Macmillan, New York, 1974.

[19] J. Shohat and J. D. Tamarkin, *The Problem of Moments*, Math. Surveys, Vol. 1, Amer. Math. Soc., Providence, R.I., 1963.

[20] H. A. Slim, *On co-recursive orthogonal polynomials and their applications to potential scattering*, J. Math. Anal. Appl., **136** (1988), 1–19.

[21] J. C. Wheeler, *Modified moments and continued fraction coefficients for the diatomic linear chain*, J. Chem. Phys., **80** (1984), 472–476.

Received April 2, 1991 and in revised form August 8, 1991. Research partially supported by NSF grants DMS8814026, DMS8912423 and INT8803099.

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