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EVOLUTIONARY EXISTENCE PROOFS FOR THE PENDANT DROP AND n-DIMENSIONAL CATENARY PROBLEMS

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Two problems about surfaces, both involving a gravitational forcing term, are studied from an evolutionary perspective. It is shown that, in each case, the existence of a unique solution to the associated Boundary Value Problem (BVP) may be established using a suitable mean curvature type flow. By considering two different flows for one of the problems it is illustrated that the best choice of flow, for use in the evolutionary construction of solutions to such mean curvature type BVPs, may often be determined more by geometric considerations than by analytic ones.

0. Basic notation and conventions. Throughout the following let Ω denote an open, bounded domain in $\mathbf{R}^n$, with $C^{2,\alpha}$-boundary $\partial\Omega$, and let A denote the *minimal surface operator*, so that A acts on functions $u \in C^2(\Omega)$ via

$$Au = -D_i(a^i(Du))$$

where the functions $a^i\colon \mathbf{R}^n \to \mathbf{R}$ are defined by

$$a^i(p) = \frac{p_i}{\sqrt{1+|p|^2}}.$$

Here, as in the following, we are using the convention of summing over repeated indices.

Also, for convenience, set $a^i \equiv a^i(Du)$ and $v = \sqrt{1+|Du|^2}$, and introduce the additional notation

$$a^{ij}(p) = \frac{\partial a^i}{\partial p_j} \equiv \frac{1}{\sqrt{1+|p|^2}}\left(\delta_{ij} - \frac{p_i p_j}{1+|p|^2}\right)$$

so that then we may write

$$Au = -a^{ij} D_i D_j u \quad \forall u \in C^2(\Omega)$$

where a^{ij} stands for $a^{ij}(Du)$. Note that the least eigenvalue of the matrix (a^{ij}) is v^{-3}.

Furthermore, for $u \in C^2(\Omega)$, let H denote the mean curvature of the surface graph(u). Observe that the well-known relation $H = -Au$

then allows replacement of the minimal surface operator in equations by the quantity H, and this will be frequently done in the sequel (notwithstanding that the symbol H is sometimes reserved in the literature for a prescribed curvature function), so as to emphasize the *geometrical* character of much of the working.

Finally let $C(\cdot, \dots, \cdot)$ denote any constant determined by the quantities listed, and let $\mathbf{e}_1, \dots, \mathbf{e}_{n+1}$ denote the standard basis for $\mathbf{R}^{n+1}$, with $\langle \cdot, \cdot \rangle$ the usual inner product.

1. Introduction. Consider the following three problems:

(i) (*The "Minimal Surface Problem".*) Here we seek to know if graphical surfaces exist which minimise area, while spanning some prescribed boundary. This models mathematically a soap film spanning a given wire frame, and leads to consideration of the Boundary Value Problem (BVP):

$$\begin{aligned} -Au &= 0 \quad \text{on } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where $\varphi \in C^{2,\alpha}(\overline{\Omega})$ is some specified function.

(ii) (*The "Hanging Drop Problem".*) Here we model the existence of pendant drops hanging from a fixed boundary, such as the end of a pipette. In this context it is notationally convenient to take the $x^{n+1} = 0$ level as the zero reference level for gravitational potential energy, and then to reverse the gravitational field so as to make it point *upwards*, rather than *downwards* (see **[Hu1]**). This means that "hanging drops" would hang upwards, above the zero gravitational potential energy level, rather than downwards, and is done so that functions, u, describing such drops graphically, would typically be positive, rather than negative, and would increase if the drop began to fall, not decrease. Then in seeking 'hanging drops' whose surfaces are graphical we are led to search for functions, u, which are critical points of the energy functional

$$E(u) = \int_\Omega \left(v - \frac{\kappa}{2}u^2\right) dx$$

where $\kappa > 0$ is a constant determined by the density of the liquid, its surface tension constant, and the acceleration due to gravity. This gives rise to the BVP:

$$\begin{aligned} -Au + \kappa u &= 0 \quad \text{on } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where, again, $\varphi \in C^{2,\alpha}(\overline{\Omega})$ is some fixed function.

(iii) (*The "Hanging Roof Problem"*.) Here we model a surface M, of unit mass density, hanging under its own weight from some $(n-1)$-dimensional boundary $\Gamma \subset \partial\Omega \times \mathbf{R}_{>0}$. If M is graphical, and lies everywhere above 'ground level', $u = 0$, then it will be the graph of some C^2-function $u \colon \Omega \to \mathbf{R}_{>0}$ which is a stationary point for the energy functional

$$E(u) = \int_\Omega u\sqrt{1 + |Du|^2}\, dx\,.$$

This leads to consideration of the Dirichlet BVP:

$$(1.3) \qquad \begin{aligned} -vAu - \frac{1}{u} &= 0 \quad \text{on } \Omega\,, \\ u &= \varphi \quad \text{on } \partial\Omega \end{aligned}$$

where $\varphi \in C^{2,\alpha}(\overline{\Omega})$ is some *positive* function on $\overline{\Omega}$.

The above are three examples of a large class of stationary BVPs of *mean curvature type*, arising primarily from physical problems involving surface tension, which have been extensively studied in recent times (see, for instance, **[CF]**, **[Gi]**, **[Hu1]**, **[DH]**; for further examples of such problems, including in particular a thorough treatment of the phenomenon of capillarity, see the book by Finn, **[Fi]**). Existence results for these BVPs, under suitable conditions on Ω and φ, have been derived. However these have often been obtained by means of fixed point theorems, and so have yielded little direct information as to what such solutions look like (see, say, **[Hu1**, §5**]** in relation to Problem (ii), and **[DH]** on Problem (iii)).

An alternative and more geometrically satisfying approach to these problems is to view them from an *evolutionary* perspective; that is, we try to show that, by choosing an appropriate evolution equation, solutions to such problems may be 'constructed' from any suitable initial function by evolving this function forward in time under the flow, and having it converge to a solution of the relevant BVP. This approach has the further advantage that such flows are of considerable geometrical interest in their own right, even leaving aside their usefulness in the construction of solutions to stationary problems.

In this evolutionary setting Problem (i) has already been widely treated. Most authors here (see for example **[Ec]**, **[Ge1]**, **[LT]**) have chosen to study the behaviour of graphical surfaces under the *pure* mean curvature evolution

$$(1.4) \qquad \dot{u} = -Au \quad \text{on } \Omega \times [0\,, T)$$

in which the surfaces $M_t = \text{graph}(u(\cdot\,,\,t))$ move in the x^{n+1}-direction with speed given by their mean curvature $H = -Au$.

However as noted in **[Hu2]**, an alternative and more geometrically natural flow (see also **[Br]**, **[EH]**, **[Hu3]**) is given by the evolution equation

$$\dot{u} = -vAu \quad \text{on } \Omega \times [0\,,\,T) \tag{1.5}$$

in which all points on the surfaces M_t move with *component of speed*, in the unit normal direction, equal to the mean curvature of the surface at that point. Note that flows (1.4) and (1.5) both have, of course, the same set of stationary functions.

Using flow (1.5) Huisken has been able (see **[Hu2]**) to supply a very short proof of the existence of graphical solutions to BVP (1.1), for suitable domains Ω.

The aim of this paper is to demonstrate that Problems (ii) and (iii) may also be solved readily via similar evolutionary means. The details of the methods used in each case will be supplied later, along with precise statements of results. First, however, we make two further observations.

The first is that there may often be more than one flow which could be used to treat a particular problem. Results for all such flows are naturally of independent interest. Nevertheless we will see that, for strongly geometrical problems such as (ii) and (iii), the more *geometrically natural* the flow we use the easier it will be to obtain suitable results, particularly regarding the existence of solutions to the related stationary BVP. This will be most obvious from our treatment of Problem (ii), which we will analyse, in §§2 and 3, using two different flows, one of which is more geometrical and turns out to yield the desired results with much less effort.

Our final remark concerns the use, alluded to earlier, of the Leray-Schauder Fixed Point Theorem in the stationary treatment of Problems (ii) and (iii) (see **[Hu1]** and **[DH]**). In §3 of this paper we show also that many of the estimates required for the application of this theorem in the stationary setting are mirrored closely in the evolutionary context. As such the use of flows provides, in some sense, the more natural setting for such calculations, without the need for the introduction of any class of auxiliary BVPs, or such like.

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2. The Hanging Drop Problem—Method 1. Here we give the first (and simpler) of two evolutionary treatments of Problem (ii). Our aim is to prove the following main theorem:

THEOREM 2.1. *Let $\Omega \subset \mathbf{R}^n$ be open and bounded, with $C^{2,\alpha}$-boundary $\partial\Omega$, and suppose further that the mean curvature, H', of $\partial\Omega$ satisfies*

$$H'(y) \geq \mu > 0 \quad \forall y \in \partial\Omega \tag{2.1}$$

where μ is a constant. Suppose also that $u_0, \varphi \in C^{2,\alpha}(\overline{\Omega})$ satisfy the compatibility condition $u_0 = \varphi$ on $\partial\Omega$. Then there exists a constant

$$\kappa_0 = \kappa_0\left(\sup_\Omega(|u_0|), \sup_\Omega(v(Du_0)), \Omega, \partial\Omega, n, \mu, |\varphi|_{2;\Omega}\right) \tag{2.2}$$

such that for all $0 < \kappa \leq \kappa_0$ the parabolic Initial Value Problem (IVP)

$$\begin{aligned} \dot{u} &= v(H + \kappa u) && \text{on } \Omega \times [0, \infty), \\ u &= \varphi && \text{on } \partial\Omega \times [0, \infty), \\ u(x, 0) &= u_0(x) && \text{on } \Omega \end{aligned} \tag{2.3}$$

has a $C^{2,\alpha,\alpha/2}(\overline{\Omega} \times (0, \infty))$-solution, u, smooth on the interior $\Omega \times (0, \infty)$; and moreover the functions $u_t(x) = u(x, t)$ converge exponentially fast as $t \to \infty$, in any C^k-norm, to a $C^{2,\alpha}(\overline{\Omega})$-solution of the stationary BVP (1.2).

Note that the factor v has been included in the evolution equation of (2.3) to make the mean curvature component of the flow more geometrical, as discussed in §1. Note also, for later reference, that we do *not* need here the existence of a positive lower bound, m_0, for our initial function $u_0(x)$ on $\overline{\Omega}$. By contrast, in §3, when we again treat the "Hanging Drop Problem", but there with the less geometrical evolution equation in which the factor v has been omitted, the existence of such an m_0 will in fact prove to be essential to obtain an a priori global gradient estimate, and hence to ensure the long-time existence of a graphical solution to the flow.

Proof of Theorem. The short-time existence of a solution to (2.3) may be established, via the Implicit Function Theorem, in the usual way. Here, as in all subsequent cases in this paper, we omit the details relating to this aspect of the proof.

The next step is to establish the long-time existence of a solution to (2.3). Since the evolution equation of (2.3) is uniformly parabolic if we have a bound for $|Du|$, this will follow by standard theory provided we can derive an a priori estimate for $|u_t|_{C^1}$ which does not fail in finite time. In fact we will derive a *time-independent a priori* estimate for $|u_t|_{C^1}$. In this regard a time-independent *interior* gradient estimate will prove to be the chief obstacle.

We begin, however, by obtaining time-independent sup and *boundary* gradient estimates. To derive these there are only two simple constraints which we require κ_0 to satisfy, namely: first define $d = \operatorname{diam}(\Omega)$ and $\sigma = \sup_\Omega(|u_0|)$, and then set $C_0 = \sigma + d$. Then require κ_0 to be small enough that

$$\kappa_0 C_0 < n/d \tag{2.4}$$

and

$$\kappa_0 C_0 < \mu . \tag{2.5}$$

We can now prove the following lemmas:

LEMMA 2.1. *Suppose κ_0 satisfies condition* (2.4). *Then for any $\kappa \le \kappa_0$, if u is a solution of IVP* (2.3) *we have*

$$\sup_{\overline{\Omega}\times[0,\infty)} |u(x,t)| \le C_0 .$$

Proof. The proof is by stationary barriers. Define upper and lower barrier functions ψ^+ and ψ^- by

$$\psi^+(x,t) = \sigma + (d^2 - |x - x_0|^2)^{1/2}$$

and

$$\psi^-(x,t) = -\psi^+(x,t)$$

where $x_0 \in \Omega$ is arbitrary. Geometrically, graph(ψ^+) is a portion of spherical cap, namely that part of the upper hemisphere, centre (x_0, σ), radius d, lying over $\overline{\Omega}$.

Then to see that ψ^+ is indeed an upper barrier for u, observe first that graph(ψ^+) evidently lies everywhere above graph(u) at $t = 0$, and also remains everywhere strictly above graph(u) on $\partial\Omega\times[0,\infty)$. Moreover, since it has constant negative mean curvature of $-n/d$ on all of Ω, it would also, in view of (2.4), initially move at all points in

Ω strictly downwards, under the flow in (2.3). Thus ψ^+ must be an upper barrier for u, or we would have an immediate contradiction at the first point and instant at which the graphs of the functions touched.

The proof that ψ^- is a lower barrier for u is identical. □

Note here that, if our initial function u_0 did in fact satisfy $\min_{\overline{\Omega}} u_0 = m_0$ for some *positive* constant m_0, then an alternative lower barrier would simply be the function

$$\tilde{\psi}^-(x, t) = m_0 .$$

We mention this because, from a practical viewpoint, we might typically want the $u = 0$ level in our formulation to correspond to an obstacle, such as the base of a tube. Since the physical character of the problem would change if the surface were ever to strike such an obstacle, we would therefore prefer it if we could guarantee that our final stationary solution of BVP (1.2) does not touch this level. By the above observation this will certainly hold provided the *initial* surface lies everywhere above the zero level.

LEMMA 2.2. *Suppose now* κ_0 *satisfies both conditions* (2.4) *and* (2.5). *Then for any* $\kappa \leq \kappa_0$, *if* u *is a solution of IVP* (2.3) *we have*

$$\sup_{\partial\Omega\times[0,\infty)} v \leq C_1$$

where $C_1 = C_1(n, C_0, \mu, \Omega, \partial\Omega, |\varphi|_{2;\Omega})$ *is some constant.*

Proof. Again this is via stationary barriers, this time constructed from a pair of related BVPs. Note first that, since (2.4) holds, so Lemma 2.1 and (2.5) yield that

$$\kappa\left(\sup_{\Omega\times[0,\infty)} |u|\right) < \mu . \tag{2.6}$$

Now define a pair of related stationary BVPs by

$$\begin{aligned} H &= \pm\mu \quad \text{on } \Omega, \\ u(x) &= \varphi(x) \quad \text{on } \partial\Omega. \end{aligned} \tag{2.7$\pm$}$$

Then it is well known (see **[GT**, Chapter 14**]**) that, by using a construction involving the function $\operatorname{dist}(\cdot, \partial\Omega)$ and choosing a number of arbitrary parameters sufficiently large, we can find, for each $y \in \partial\Omega$, a *local upper barrier* $\delta_y^+(x)$ for BVP (2.7−) such that, in some neighbourhood N_y of y,

$$\delta_y^+(x) = \varphi(x) \quad \forall x \in \partial\Omega \cap N_y \tag{2.8}$$

and

(2.9) $$\delta_y^+(x) > u_0(x) \quad \forall x \in \Omega \cap N_y$$

and

(2.10) $$H^+(x) \leq -\mu \quad \forall x \in \Omega \cap N_y$$

and

(2.11) $$\delta_y^+(x) > M_0 \quad \forall x \in (\Omega \cap N_y) \backslash N_{\varepsilon'}(\partial\Omega)$$

where $N_{\varepsilon'}(\partial\Omega)$ is some fixed ε'-neighbourhood of $\partial\Omega$, and $H^+(x)$ denotes the mean curvature of $\delta_y^+(x)$.

To see that $\delta_y^+(x)$ is then, in fact, also a local upper barrier for $u(x, t)$ on $\Omega \cap N_y$, note first that property (2.9) guarantees that graph(δ_y^+) lies initially everywhere above graph(u) on $\Omega \cap N_y$. Property (2.11) then ensures that it must remain so on $(\Omega \cap N_y) \backslash N_{\varepsilon'}(\partial\Omega)$, while (2.6) along with property (2.10) allows us to demonstrate, by a similar contradiction procedure to that used in the proof of Lemma 2.1, that the same must be true on $(\Omega \cap N_y) \cap N_{\varepsilon'}(\partial\Omega)$.

Thus $\delta_y^+(x)$ is indeed a local upper barrier for $u(x, t)$ on $\Omega \cap N_y$.

Similarly, using BVP (2.7+), we can also construct local lower barriers, $\delta_y^-(x)$, for $u(x, t)$, for all $y \in \partial\Omega$.

Therefore, by property (2.8), we get, at each $y \in \partial\Omega$, a *time-independent sup bound* for $|Du(y, t)|$ by $\max(|D\delta_y^+(y)|, |D\delta_y^-(y)|)$. But this, in view of the method of construction of the functions δ_y^+ and δ_y^-, gives us precisely a uniform estimate for v of the desired form. □

We turn now to the derivation of a uniform *a priori global* bound for v. Because of the strongly geometrical nature of the flow in (2.3) this could be most easily achieved by working locally on the surfaces graph(u), in the manner employed in, for example, **[DH]** or **[EH]**. However we shall not adopt this approach here, for two reasons.

The first is to illustrate that the necessary calculations may still be carried out, and the estimate obtained, while working on the base domain Ω.

The second is to allow more ready comparison of the (relatively few) difficulties that arise in obtaining this estimate when using the more geometrical flow employed in this section, vis-a-vis those that arise at the corresponding point in the next section, where we are again treating the "Hanging Drop" problem, but there using the less geometrical evolution equation $\dot{u} = H + \kappa u$ in which the factor v is not included.

Nevertheless we remark, at this point, that the technique of exploiting the geometry of appropriate mean curvature flows by working locally on the evolving surfaces themselves will be employed later in this paper, when we treat the "Hanging Roof" problem, Problem (iii), in §4. A few more detailed comments on its particular advantages will be made at that time.

Returning to the derivation of our interior bound for v, we aim to obtain this by bounding instead a suitable auxiliary function, χ, which involves v. Precisely, define

$$K = \max\left(C_1\,,\ \sup_{\Omega}(v(Du_0))\right)\,, \qquad C_2 = 4K \geq 4\,.$$

Also require now further that κ_0 be small enough that the condition

$$\kappa_0(v^2 + 2v - 1) < \frac{2(v^2-1)}{C_0^2 v^2} \quad \text{for all } (C_2 - 1) \leq v \leq C_2 \tag{2.12}$$

holds. Then we have

LEMMA 2.3. *Suppose κ_0 satisfies conditions* (2.4), (2.5) *and* (2.12). *Define an auxiliary function $\chi(x\,,t)$ by*

$$\chi = v + \frac{u^2}{C_0^2}\,.$$

Then for any $\kappa \leq \kappa_0$, if u is a solution of IVP (2.3) *we have*

$$\sup_{\overline{\Omega}\times[0\,,\infty)} |\chi(x\,,t)| \leq C_2 \tag{2.13}$$

whence also, obviously,

$$\sup_{\overline{\Omega}\times[0\,,\infty)} v \leq C_2\,. \tag{2.14}$$

Proof. To establish estimate (2.13) without working locally on the surfaces graph$(u(x\,,t))$ we need a pair of preliminary lemmas.

LEMMA 2.4. *We have the identities*

$$\dot{v} = a^l D_l(\dot{u}) = a^l D_l(vH + \kappa v u) \tag{2.15}$$

and

$$a^l D_l(vH) = vD_i(a^{ij}D_j v) + Ha^i D_i v - va^{ij}a^{kl}D_iD_k u D_j D_l u\,. \tag{2.16}$$

Proof. Equation (2.5) is immediate from the definition of v. As for (2.16), this is proven by direct computation, a sketch of which is as follows. By the definitions of v and a^l we have

$$D_j v = v^{-1} D_l u D_j D_l u \quad \text{and} \quad D_i a^l = a^{kl} D_i D_k u .$$

Hence, since $D_i H = D_l(D_i a^l)$, we obtain that $D_i H = D_l(a^{kl} D_i D_k u)$, and thence we may derive that

$$D_i(a^{ij} D_j v) = a^{ij} a^{kl} D_j D_l u D_i D_k u + a^l D_l H .$$

But then, combining this identity with the fact that $a^l D_l(vH) = v a^l D_l H + H a^l D_l v$, equation (2.16) follows. □

LEMMA 2.5. *The quantity* $a^{ij} a^{kl} D_k u D_j D_l u$ *is positive; indeed*

$$a^{ij} a^{kl} D_i D_k u D_j D_l u = v^{-2} |\nabla^2 u|^2$$

where $\nabla^2 u$ *denotes the matrix of second tangential derivatives of* u.

Proof. A proof of this may be found in Claus Gerhardt's paper **[Ge2**, Lemma 1.3]. □

Returning to the proof of Lemma 2.3, we can now calculate directly from Lemmas 2.4 and 2.5 that

$$\begin{aligned} \dot{v} &\le v D_i(a^{ij} D_j v) + H a^i D_i v + \kappa a^l D_l(uv) \\ &= v a^{ij} D_i D_j v + v D_i(a^{ij}) D_j v + H a^i D_i v \\ &\quad + u\kappa a^l D_l v + \kappa |Du|^2 . \end{aligned} \tag{2.17}$$

But also, by explicit computation, we have that

$$a^i = v^2 a^{il} D_l u \tag{2.18}$$

and

$$a^i D_i(a^j) = a^{kj} D_k v \tag{2.19}$$

whence in turn

$$\begin{aligned} D_i(a^{ij}) D_j v &= D_i(v^{-1}\{\delta_{ij} - a^i a^j\}) D_j v \\ &= (-v^{-1}\{H a^j + a^i D_i a^j\} - v^{-2}\{\delta_{ij} - a^i a^j\} D_i v) D_j v \\ &= -v^{-1}(H a^j + a^{jk} D_k v + a^{ij} D_i v) D_j v \\ &= -v^{-1}(v^2 H a^{ij} D_i u D_j v + 2 a^{ij} D_i v D_j v) . \end{aligned} \tag{2.20}$$

Thus by (2.18) and (2.20) in (2.17), and using the ellipticity of (a^{ij}), we obtain

$$(2.21)\quad \begin{aligned}\dot{v} &\le va^{ij}D_iD_jv - v^2Ha^{ij}D_iuD_jv - 2a^{ij}D_ivD_jv \\ &\quad + v^2Ha^{il}D_luD_iv + \kappa ua^lD_l\left(\chi - \frac{u^2}{C_0^2}\right) + \kappa|Du|^2 \\ &\le va^{ij}D_iD_jv + \kappa ua^lD_l\chi + \kappa|Du|^2 .\end{aligned}$$

Yet also, by Lemma 2.1, and noting that $a^{ij}D_iuD_ju = v^{-3}|Du|^2$, we have

$$(2.22)\quad \begin{aligned}\frac{d}{dt}\left(\frac{u^2}{C_0^2}\right) &= \frac{2u}{C_0^2}(va^{ij}D_iD_ju + \kappa vu) \\ &= va^{ij}D_iD_j\left(\frac{u^2}{C_0^2}\right) - \frac{2v}{C_0^2}a^{ij}D_iuD_ju + 2\kappa v\left(\frac{u^2}{C_0^2}\right) \\ &\le va^{ij}D_iD_j\left(\frac{u^2}{C_0^2}\right) - \frac{2|Du|^2}{C_0^2v^2} + 2\kappa v .\end{aligned}$$

Combining (2.21) and (2.22) we therefore have that

$$(2.23)\quad \dot{\chi} \le va^{ij}D_iD_j\chi + \kappa ua^lD_l\chi + \left(\kappa(v^2 + 2v - 1) - \frac{2(v^2-1)}{C_0^2v^2}\right) .$$

But then it follows that

(i) By the definition of C_2, and Lemma 2.2, χ is initially less than C_2 on all of $\overline{\Omega}$ and never reaches C_2 on $\partial\Omega \times [0, \infty)$; and

(ii) By condition (2.12), and noting that of course $\chi - 1 \le v \le \chi$, χ can also never reach C_2 on $\Omega \times [0, \infty)$ or we would have an immediate contradiction at the first point and instant at which it did so. □

This completes the proof of the long-time existence claim of Theorem 2.1.

As for regularity, our uniform bound on $v(Du)$ means that $a^{ij}(Du)$ is now *uniformly* elliptic, and this, by standard theory, guarantees uniform estimates also on all higher derivatives of u in time and space on $\Omega \times [0, \infty)$.

Moreover, regarding the convergence claim of Theorem 2.1, the uniformity of our bound on $|u_t|_{C^1}$ implies we can also find a subsequence of times $t_k \to \infty$ such that the functions u_{t_k} converge to a $C^{2,\alpha}(\overline{\Omega})$-function, u_∞. Then to see first that u_∞ will be a stationary point of the flow, that is, a solution of BVP (1.2), we consider the time-derivative of the energy functional describing this situation, and use Lemma 2.3. This yields

$$\frac{d}{dt}E(u) \equiv \frac{d}{dt}\int_\Omega \left(v - \frac{\kappa}{2}u^2\right)\, dx = -\int_\Omega \frac{\dot{u}^2}{v}\, dx \le -\frac{1}{C_2}\int_\Omega \dot{u}^2\, dx$$

whence we obtain

$$\int_0^T \int_\Omega \dot{u}^2\, dx\, dt \le C(C_0\,,\, C_2\,,\, v(Du_0)\,,\, \Omega)$$

independent of T. Hence $\dot{u}$ goes uniformly to zero as $t \to \infty$, and the claim follows.

It remains then only to prove that we have not just convergence of the subsequence u_{t_k}, but in fact have *exponential* convergence, in any C^k-norm, of all the u_t, as $t \to \infty$, to u_∞. For this we invoke the following interpolation inequality (see [**GT**, Theorem 7.27]):

LEMMA 2.6. *If* $\psi \in W_0^{k,p}(\Omega)$, *then for any* $\varepsilon > 0$ *and multiindex* β *with* $0 < |\beta| < k$, *we have*

$$\|D^\beta \psi\|_{p;\Omega} \le \varepsilon \|\psi\|_{k,p;\Omega} + C(k)\varepsilon^{|\beta|/(|\beta|-k)}\|\psi\|_{p;\Omega}\,.$$

In view of this it will evidently suffice to show exponential convergence of the u_t to u_∞ in the L^2-norm.

LEMMA 2.7. *Suppose* u_∞ *is as above and* $\kappa \le \kappa_0$, *where* κ_0 *now satisfies* (2.4), (2.5), (2.12) *and the further constraint*

$$\kappa_0 \le \frac{\lambda C_p^{-1}}{3} \tag{2.24}$$

where $C_p(n\,,\, |\Omega|)$ *is the constant in Poincaré's inequality. Here* λ *is defined to be the smallest eigenvalue of any of the matrices* $a^{ij}(sDu + (1-s)Du_\infty)$, $s \in [0\,,\, 1]$; *that is*

$$\lambda = \min_{s\in[0,1]} \frac{1}{(1 + |sDu + (1-s)Du_\infty|^2)^{3/2}} \ge C(C_2) > 0\,.$$

Then there exists a positive constant $\xi = \xi(n\,,\, \Omega\,,\, C_2)$ *such that*

$$\|u_t - u_\infty\|_2^2 \le C_2\|u_T - u_\infty\|_2^2 e^{-\xi(t-T)} \quad \forall t \ge T$$

where T *is some sufficiently large but finite time.*

Proof. Noting $\dot{u}_\infty \equiv 0$ and $\dot{v} = a^l D_l(\dot{u})$, we may compute that

$$\frac{d}{dt}\int_\Omega \frac{1}{v}(u-u_\infty)^2\,dx = 2\int_\Omega (u-u_\infty)(D_l a^l + \kappa u)\,dx - \int_\Omega \frac{(u-u_\infty)^2}{v^2} a^l D_l \dot{u}\,dx\,.$$

Thus since $v \geq 1$, and since $\dot{u}_\infty \equiv 0$ implies $D_l a^l_\infty + \kappa u_\infty \equiv 0$ on Ω, where a^l_∞ denotes $a^l(Du_\infty)$, we get

$$\frac{d}{dt}\int_\Omega v^{-1}(u-u_\infty)^2\,dx \leq 2\kappa\int_\Omega (u-u_\infty)^2\,dx + 2\int_\Omega (u-u_\infty)D_l(a^l - a^l_\infty)\,dx + \int_\Omega (u-u_\infty)^2 |a^l D_l \dot{u}|\,dx\,. \tag{2.25}$$

But now recall that $\dot{u} = 0$ on $\partial\Omega$, and $\dot{u} \to 0$ uniformly as $t \to \infty$, and all higher derivatives of u in time and space are bounded. Thus by Lemma 2.6 we must have that also $D_l \dot{u} \to 0$ as $t \to \infty$, and hence $|a^l D_l \dot{u}| \to 0$ as $t \to \infty$. Thus, using this and integration by parts in (2.25), there exists some time T such that for all $t \geq T$, we have

$$\frac{d}{dt}\int_\Omega v^{-1}(u-u_\infty)^2\,dx \leq 3\kappa\int_\Omega (u-u_\infty)^2\,dx - 2\int_\Omega (a^l - a^l_\infty)D_l(u-u_\infty)\,dx\,. \tag{2.26}$$

Yet also now, following an idea from [**Hu1**, §4], observe that by the Fundamental Theorem of Calculus and Poincaré's inequality, and noting that $\frac{d}{ds}a^l(p) = a^{jl}(p)\frac{dp_j}{ds}$, we may deduce that

$$\begin{aligned}\int_\Omega (a^l - a^l_\infty)D_l(u-u_\infty)\,dx &= \int_\Omega \left(\int_0^1 \frac{d}{ds}[a^l(sDu + (1-s)Du_\infty)]\,ds\right) D_l(u-u_\infty)\,dx \\ &\geq \lambda C_p^{-1}\int_\Omega (u-u_\infty)^2\,dx\,.\end{aligned} \tag{2.27}$$

Thus, noting condition (2.24), we obtain from (2.27) in (2.26) that, for all $t \geq T$,

$$\begin{aligned}\frac{d}{dt}\int_\Omega v^{-1}(u-u_\infty)^2\,dx &\leq -3\lambda C_p^{-1}\int_\Omega (u-u_\infty)^2\,dx \\ &\leq -3\lambda C_p^{-1}\int_\Omega v^{-1}(u-u_\infty)^2\,dx\,.\end{aligned}$$

The result now follows in view of Lemma 2.3. □

This completes the proof of Theorem 2.1. □

3. The Hanging Drop Problem—Method 2. Here we again treat the "hanging drop problem", but now using the less geometrically natural flow of the two discussed in §1. That is, we study now the IVP

$$\begin{aligned} \dot{u} &= H + \kappa u && \text{on } \Omega \times [0, \infty), \\ u &= \varphi && \text{on } \partial\Omega \times [0, \infty), \\ u(x, 0) &= u_0(x) && \text{on } \Omega. \end{aligned} \tag{3.1}$$

Our aim in this section is to illustrate that, while we can derive a similar result to Theorem 2.1 for this flow also, it is in fact significantly harder to establish such existence and convergence results in this case. This is so despite the fact that the operator $-A$, unlike $-vA$, is in divergence form. Our main theorem in this section is as follows:

THEOREM 3.1. *Let Ω, $\partial\Omega$, μ, u_0 and φ be as in Theorem 2.1, and suppose now further that the initial function u_0 satisfies*

$$\min_{\overline{\Omega}} u_0 = m_0 > 0.$$

Then there exists a constant

$$\kappa_0 = \kappa_0\left(\sup_{\Omega}(u_0), \sup_{\Omega}(v(Du_0)), \sup_{\Omega}(|H(x, 0)|), \Omega, \partial\Omega, m_0, n, \mu, |\varphi|_{2;\Omega}\right)$$

such that for all $0 < \kappa \leq \kappa_0$ the parabolic IVP (3.1) has a $C^{2,\alpha,\alpha/2}(\overline{\Omega} \times (0, \infty))$-solution, u, smooth on the interior $\Omega \times (0, \infty)$; and moreover the functions $u_t(x) = u(x, t)$ converge exponentially fast as $t \to \infty$, in any C^k-norm, to a $C^{2,\alpha}(\overline{\Omega})$-solution of the stationary BVP (1.2).

Note, vis-a-vis Theorem 2.1, the added requirement of the existence of the positive constant m_0, and also the added dependence now of κ_0 both on this constant m_0 and on the mean curvature of the initial surface.

Proof of Theorem. As with the proof of Theorem 2.1, the main step is to establish a *time-independent a priori* estimate for $|u_t|_{C^1}$. Once again we do this by obtaining, in turn, appropriate sup, boundary gradient, and finally global gradient estimates.

In this regard, the latter of these again proves to be the one which presents all the difficulties. Indeed, the sup and boundary gradient estimates turn out to be immediate, in this setting, without the need for any further work. This is because, provided conditions (2.4) and (2.5) on κ_0 again hold, exactly the same stationary barriers as were used to obtain these estimates in §2 actually work again here, to yield identical bounds to those of Lemmas 2.1 and 2.2.

For consistency let us again denote these bounds by C_0 and C_1 respectively. As the same time let us also note that here, unlike in §2, we now have, in addition, an *a priori lower* bound for solutions of IVP (3.1), of the sort discussed after the proof of Lemma 2.1, namely that

$$\sup_{\overline{\Omega}\times[0,\infty)} u \geq m_0 > 0 . \tag{3.2}$$

We mention this because, while in §2 such a result was not required, this bound will prove to be essential at one point in our later analysis of IVP (3.1)—see also our earlier remarks following Theorem 2.1.

Turning to the derivation of the *a priori global* gradient estimate, this by contrast turns out to be considerably more complex here than it was in §2. The reason for this is as follows.

Recall that, for the flow in §2, we had for the evolution of the quantity $v = \sqrt{1+|Du|^2}$ that (see inequality (2.21))

$$\dot{v} \leq v a^{ij} D_i D_j v + \kappa u a^l D_l \chi + \kappa |Du|^2 .$$

Importantly the "bad" term, $Ha^i D_i v$, present in inequality (2.17), cancelled out, leaving no terms still involving the mean curvature function, H. The only remaining "bad" term in this inequality, $\kappa|Du|^2$, was able to be handled through consideration of the auxiliary function χ, in which a multiple of u^2 was added to v.

If, however, we try to repeat these calculations for the flow in IVP (3.1), then the absence of the factor v in the evolution equation here means that what we end up with is the inequality

$$\begin{aligned} \dot{v} &\leq v a^{ij} D_i D_j v + D_i(a^{ij}) D_j v + \kappa \frac{|Du|^2}{v} \\ &\leq v a^{ij} D_i D_j v - v H a^{ij} D_i u D_j v + \kappa v . \end{aligned} \tag{3.3}$$

While the term κv here may once again be dealt with via the use of a suitable auxiliary function, the really awkward component of (3.3) is the term $-vHa^{ij}D_i u D_j v$, involving the mean curvature, H, of the evolving surface. Since we have as yet no control over the behaviour of this term under evolution, we cannot for the present deal with this component of inequality (3.3).

To obtain our global gradient estimate we therefore need first of all to derive some form of at least conditional control over the evolution of this quantity, H, under the flow in IVP (3.1).

Having observed this, it is appropriate however, before attempting to derive such control, to outline in detail the overall strategy that we will be employing for the deduction of our global gradient estimate.

Essentially it entails using *simultaneous* contradiction arguments involving estimates for the two quantities v and (via studying $\dot{u}$) H. So as to be able to state precisely the bounds we will obtain, as well as the order in which we will be deriving them, let us introduce immediately the following further notation. First define

$$K = \max\left(C_1, \sup_{\Omega} v(Du_0)\right), \qquad k_0 = \sup_{\Omega} |\dot{u}(x, 0)| \tag{3.4}$$

and then set

$$C_2 = 4K, \qquad C_3 = 2k_0. \tag{3.5}$$

Also then define two separate *conditions* that a time T may or may not satisfy, the first by the specification that the estimate

$$\sup_{\overline{\Omega}\times[0,T)} v \le C_2 \tag{3.6}$$

should hold, and the second by the specification that the estimate

$$\sup_{\overline{\Omega}\times[0,T)} |\dot{u}| \le C_3 \tag{3.7}$$

should hold.

We can now state precisely that in the following we will show that, provided κ_0 is suitably small, the quantities v and $\dot{u}$ are bounded, for all time and over all of $\overline{\Omega}$, by C_2 and C_3 respectively; in other words that the class of times T such that conditions (3.6) and (3.7) *both* hold is in fact all of $\mathbf{R}_{\ge 0}$. This will be done in three steps:

Step 1. Show (Lemma 3.1) that if T is any time such that conditions (3.6) and (3.7) both hold, then in fact on $\overline{\Omega} \times [0, T)$ the improved estimate

$$\sup_{\overline{\Omega}\times[0,T)} |\dot{u}| \le \frac{3C_3}{4}$$

holds. This, along with our a priori sup bound, will allow us (Corollaries 3.2 and 3.3) to deduce a uniform bound for $\dot{u}$, and thence for the mean curvature H, on $\overline{\Omega} \times [0, T)$, provided now only that T is such that condition (3.6) holds.

Step 2. Establish some necessary preliminary results (Lemmas 3.4 and 3.5) about a suitable auxiliary function χ, needed in Step 3.

Step 3. Finally use this function χ, along with the results of Steps 1 and 2, to derive that also (Lemma 3.6), if T is any time such that condition (3.6) alone is satisfied, then we must in fact have the improved estimate for v on $\overline{\Omega} \times [0, T)$ that

$$\sup_{\overline{\Omega} \times [0, T)} v \leq \frac{3C_2}{4}.$$

This will allow us to conclude finally, by a contradiction argument, that in fact condition (3.6) must be satisfied by all $T \geq 0$; in other words that we must have precisely the *uniform global* estimate for v on $\overline{\Omega} \times [0, \infty)$ that we claimed, namely

$$\sup_{\overline{\Omega} \times [0, \infty)} v \leq C_2.$$

Turning at last to the implementation of this strategy, as far as Step 1 is concerned we have the following key lemma:

LEMMA 3.1. *Suppose that, in addition to constraints* (2.4) *and* (2.5), κ_0 *also now satisfies three additional conditions,* (3.14), (3.20) *and* (3.25), *which for convenience are listed at the points in the following working where they are first needed. Next suppose* $T \geq 0$ *is any time such that conditions* (3.6) *and* (3.7) *both hold. Then for any* $\kappa \leq \kappa_0$, *if* u *is a solution of IVP* (3.1) *on* $\overline{\Omega} \times [0, T)$ *we must in fact have the improved estimate for* $\dot{u}$ *on* $\overline{\Omega} \times [0, T)$ *that*

$$\sup_{\overline{\Omega} \times [0, T)} |\dot{u}| \leq \frac{3C_3}{4}. \tag{3.8}$$

Proof. Observe first that, by direct computation, we have

$$\frac{d}{dt} \int_\Omega \left(v - \frac{\kappa}{2} u^2\right) dx = - \int_\Omega \dot{u}^2 \, dx. \tag{3.9}$$

If we now integrate over time, and use our time-independent sup bound, C_0, we thus obtain that there is a constant

$$C_4 = C_4(C_0, v(Du_0), \Omega)$$

such that, for any $\widetilde{T} \geq 0$, we have

$$\int_0^{\widetilde{T}} \int_\Omega \dot{u}^2 \, dx \, dt \leq C_4 . \tag{3.10}$$

To obtain the improved bound for $\dot{u}$, (3.8), we now aim to use this estimate, along with the iteration technique of De Giorgi and Stampacchia. To this end we define $\eta = \dot{u}$ and then set

$$\eta_k = \max(\eta - k, 0), \qquad A(k) = \{x \in \Omega : \eta(x) > k\}.$$

Also for convenience we write

$$|A(k)| \equiv \int_{A(k)} dx, \qquad \|A(k)\|_T \equiv \int_0^T |A(k)| \, dt .$$

Integration by parts then yields that

$$\begin{aligned} \frac{d}{dt} \int_\Omega \eta_k^2 \, dx &= 2 \int_\Omega \eta_k (\dot{H} + \kappa \dot{u}) \, dx \\ &= -2 \int_{A(k)} \dot{a}^l D_l \eta \, dx + 2\kappa \int_{A(k)} \eta_k \eta \, dx . \end{aligned} \tag{3.11}$$

But now observe that, by explicit calculation, we have

$$\dot{a}^l D_l \eta = \left(\frac{v D_l \eta - D_l u (a^i D_i \eta)}{v^2} \right) D_l \eta \geq v^{-3} |D\eta|^2 . \tag{3.12}$$

Hence, by (3.12) in (3.11), and an application of Cauchy's inequality, we may deduce, for all $t \leq T$, that

$$\frac{d}{dt} \int_\Omega \eta_k^2 \, dx \leq \frac{-2}{C_2^3} \int_{A(k)} |D\eta|^2 \, dx + \kappa \int_{A(k)} \eta_k^2 \, dx + \kappa C_3^2 |A(k)| \tag{3.13}$$

whence, provided κ_0 is small enough to ensure that

$$\kappa_0 < \frac{2}{C_2^3} \tag{3.14}$$

we may conclude that

$$\frac{d}{dt} \int_\Omega \eta_k^2 \, dx \leq -\kappa \int_{A(k)} |D\eta_k|^2 \, dx + \kappa \int_{A(k)} \eta_k^2 \, dx + \kappa C_3^2 |A(k)| . \tag{3.15}$$

To proceed further we now make use of the Sobolev inequality. Putting this in (3.15), and integrating over time, we thence obtain that for all $k \geq k_0$

$$\begin{aligned} \sup_{[0,T]} \int_{A(k)} \eta_k^2 \, dx &+ \kappa C_5 \int_0^T \left(\int_{A(k)} \eta_k^{2q} \, dx \right)^{1/q} dt \\ &\leq 2\kappa C_3^2 \|A(k)\|_T + 2\kappa \int_0^T \int_{A(k)} \eta_k^2 \, dx \, dt . \end{aligned} \tag{3.16}$$

Here $C_5 = C_5(n)$ is a constant from the Sobolev inequality, while $q = n/(n-2)$ (or $q < \infty$ if $n = 2$) is half the Sobolev exponent.

Next we employ an interpolation inequality for L^p-norms (see [GT, p. 146, inequality (7.9)]), which in this context gives that

$$\|\eta_k^2\|_{q_0\,;A(k)} \le \|\eta_k^2\|^a_{q\,;A(k)} \|\eta_k^2\|^{1-a}_{1\,;A(k)} \tag{3.17}$$

where q_0 and a are defined by

$$q_0 = 2 - \frac{1}{q}, \qquad a = \frac{1}{q_0}.$$

This result, along with Hölder's inequality, and a careful application of Young's inequality, then yields that we can find q_0, with $1 < q_0 < q$, such that

$$\begin{aligned}\kappa^{(q_0-1)/q_0} C_6 &\left(\int_0^T \int_{A(k)} \eta_k^{2q_0}\, dx\, dt\right)^{1/q_0} \\ &\le 2\kappa C_3^2 \|A(k)\|_T + 2\kappa \left(\int_0^T \int_{A(k)} \eta_k^{2q_0}\, dx\, dt\right)^{1/q_0} \|A(k)\|_T^{1-1/q_0}\end{aligned}$$

where $C_6 = C_6(n)$. Rearranging we thus obtain

$$\begin{aligned}\left(\int_0^T \int_{A(k)} \eta_k^{2q_0}\, dx\, dt\right)^{1/q_0} &(1 - 2\kappa^{1/q_0} C_6^{-1} \|A(k)\|_T^{1-1/q_0}) \\ &\le 2\kappa^{1/q_0} C_3^2 C_6^{-1} \|A(k)\|_T.\end{aligned} \tag{3.18}$$

But now observe that, for all $k \ge k_0$, inequality (3.10) implies that

$$\|A(k)\|_T \le C_4/k_0^2. \tag{3.19}$$

Therefore, provided κ_0 satisfies, say,

$$\kappa_0^{1/q_0} \left(\frac{2}{C_6}\right) \left(\frac{C_4}{k_0^2}\right)^{1-1/q_0} \le 1/2 \tag{3.20}$$

then we have immediately from (3.19) in (3.18) that

$$\left(\int_0^T \int_{A(k)} \eta_k^{2q_0}\, dx\, dt\right)^{1/q_0} \le 4\kappa^{1/q_0} C_3^2 C_6^{-1} \|A(k)\|_T. \tag{3.21}$$

We are now in a position to derive the desired estimate via Stampacchia's Lemma (see [KS, p. 63]). To complete the argument, observe finally that for all $h > k \geq k_0$ we have that, on $A(h)$, $\eta_k = \eta - k > h - k$. Hence, for any $h > k \geq k_0$, it follows that

$$(3.22)\qquad (h-k)^2\|A(h)\|_T \leq \int_0^T \int_{A(k)} \eta_k^2\,dx\,dt < 4\kappa^{1/q_0} C_3^2 C_6^{-1} \|A(k)\|_T^{2-1/q_0}.$$

Here we have used Hölder's inequality and the fact that $A(k) \supseteq A(h)$, along with result (3.21). Stampacchia's Lemma now immediately yields the estimate

$$(3.23)\qquad \sup_{\overline{\Omega}\times[0,T)} \eta \leq k_0 + \delta$$

where, by (3.19),

$$(3.24)\qquad \delta^2 \leq \kappa^{1/q_0}(4C_3^2 C_6^{-1} 2^{2\omega/(\omega-1)}) \left(\frac{C_4}{k_0^2}\right)^{1-1/q_0}, \qquad \omega = 2 - 1/q_0,$$
$$= \kappa^{1/q_0} C_3^2 C_7(n, k_0, C_4), \quad \text{say}.$$

Finally therefore, provided κ_0 is also small enough that

$$(3.25)\qquad \kappa_0^{1/2q_0} C_7^{1/2} \leq 1/4$$

then (3.23) gives the estimate

$$\sup_{\overline{\Omega}\times[0,T)} \eta \leq \frac{3C_3}{4}.$$

But this, along with the corresponding bound from below, amounts to precisely the improved estimate we wanted. □

To complete Step 1 we can now write down the following two corollaries to this lemma.

COROLLARY 3.2. *Suppose κ_0 again satisfies conditions* (2.4), (2.5), (3.14), (3.20) *and* (3.25), *and suppose $T \geq 0$ is any time for which condition* (3.6) *holds. Then for any $\kappa \leq \kappa_0$, if u is a solution to IVP* (3.1) *on $\overline{\Omega} \times [0, T)$ we have the uniform estimate for $\dot{u}$ that*

$$(3.26)\qquad \sup_{\overline{\Omega}\times[0,T)} |\dot{u}| < C_3.$$

Proof. By the definition of C_3 (in (3.5)), the function $|\dot{u}(x, t)|$ is *initially* bounded everywhere by C_3. Lemma 3.1 then shows that it must remain so. □

COROLLARY 3.3. *Let κ_0, T be as in Corollary 3.2, and suppose $\kappa \le \kappa_0$. Then there is a constant $C_8 = C_8(C_3, n, \Omega)$ such that, if u is a solution to IVP (3.1) on $\overline{\Omega} \times [0, T)$, we have the uniform a priori estimate for the mean curvature of* graph(u) *that*

$$\sup_{\overline{\Omega}\times[0,T)} |H(x, t)| \le C_8. \tag{3.27}$$

Proof. This follows immediately from (2.4) and the relationship $H = \dot{u} - \kappa u$. □

Having obtained this *qualified* uniform mean curvature bound, we can now pass on to Step 2 of our strategy for deriving a full *a priori* gradient estimate for solutions of IVP (3.1). This step is devoted to introducing an auxiliary function, χ, suitable for use in that derivation, and to establishing some preliminary results about it. In this regard we have two lemmas.

LEMMA 3.4. *Define*

$$\chi = v + \frac{u^\varrho}{C_0^\varrho} \tag{3.28}$$

where ϱ is, for the present, any positive integer. Then χ satisfies

$$\dot{\chi} \le a^{ij} D_i D_j \chi + \Theta - \Psi \tag{3.29}$$

where Θ and Ψ are given by

$$\Theta = \kappa\left(v + \varrho\frac{u^\varrho}{C_0^\varrho}\right) - \frac{\varrho(\varrho-1)}{2C_0^\varrho} u^{\varrho-2} a^{ij} D_i u D_j u, \tag{3.30}$$

$$\Psi = \frac{\varrho(\varrho-1)}{2C_0^\varrho} u^{\varrho-2} a^{ij} D_i u D_j u + \frac{2}{v} a^{ij} D_i v D_j v + vHa^{ij} D_i u D_j v. \tag{3.31}$$

Proof. This is proven via direct computation, proceeding in almost identical fashion to the derivation of results (2.17) to (2.23) in §2. For this reason we omit all the details here. □

LEMMA 3.5. *Suppose that at time t_0 the function χ, defined in* (3.28), *has an interior maximum at $x_0 \in \Omega$. Then the behaviour of the function Ψ at (x_0, t_0) is given by*

$$\Psi(x_0, t_0) = \frac{\varrho u^{\varrho-2}|Du|^2}{2C_0^\varrho v^3}\left((\varrho-1) + \frac{4\varrho u^\varrho}{vC_0^\varrho} - 2vHu\right)\Bigg|_{(x_0,t_0)}. \tag{3.32}$$

Proof. Interior maximality implies that at (x_0, t_0) we have $D\chi = 0$, whence $Dv = -\varrho u^{\varrho-1} C_0^{-\varrho} Du$. Putting this in (3.31) the result then follows on noting that $a^{ij} D_i u D_j u = v^{-3}|Du|^2$. □

Turning to Step 3 of our strategy, we are now in a position at last to be able to extend our a priori boundary gradient estimate to a uniform gradient estimate for all time on the *interior* of Ω as well. The chief result we require is the following:

LEMMA 3.6. *Let ϱ now be any fixed positive integer such that*

$$\varrho > 1 + 2C_0C_2C_8 \tag{3.33}$$

and suppose that, in addition to conditions (2.4), (2.5), (3.14), (3.20) *and* (3.25), κ_0 *now also satisfies the further constraint that*
(3.34)

$$\kappa_0(v + \varrho) < m_0^{\varrho-2}\left(\frac{\varrho(\varrho-1)}{2C_0^{\varrho}}\right)\left(\frac{v^2-1}{v^3}\right) \quad \text{for } 2 \le v \le 3C_2/4.$$

Next let $T \ge 0$ be any time for which condition (3.6) *holds. Then for any $\kappa \le \kappa_0$, if u is a solution of IVP* (3.1) *on $\overline{\Omega} \times [0, T)$ we in fact have the improved estimate for v on $\overline{\Omega} \times [0, T)$ that*

$$\sup_{\overline{\Omega}\times[0,T)} v \le \frac{3C_2}{4}. \tag{3.35}$$

Proof. Let χ be as in (3.28), but with ϱ now fixed in the above way. The idea here is now to bound this auxiliary function χ suitably, and deduce from this the improved upper bound on v. To be explicit we assert that, with this choice of ϱ, χ must satisfy

$$\sup_{\overline{\Omega}\times[0,T)} \chi \le \frac{3C_2}{4}. \tag{3.36}$$

To see this note first that, by the definition of C_2, χ is certainly everywhere less than $3C_2/4$ at $t = 0$, and moreover must remain so on $\partial\Omega \times [0, T)$. Thus if, on $\overline{\Omega} \times [0, T)$ χ ever attained the value $3C_2/4$, then it would have to do so at some point in Ω itself. But suppose it did so *for the first time* at t_0, at some point $x_0 \in \Omega$. Then clearly this would imply that

$$\dot{\chi}(x_0, t_0) \ge 0. \tag{3.37}$$

However by condition (3.33) in Lemma 3.5 we would also have that

$$\Psi(x_0, t_0) > 0 \tag{3.38}$$

where Ψ is as defined in (3.31). And furthermore, since $C_2 \geq 4$ and $\chi \leq v+1$, so at (x_0, t_0) we would have to have $v \geq 2$, whence by condition (3.34) and result (3.2) in (3.30) we would also have that

$$\Theta(x_0, t_0) < 0 \tag{3.39}$$

on noting once again that $a^{ij}D_iuD_ju = (v^2-1)/v^3$.

But then by (3.29) we would obtain that

$$\dot{\chi}(x_0, t_0) < a^{ij}D_iD_j\chi|_{(x_0, t_0)} \leq 0 \tag{3.40}$$

contradicting (3.37).

This contradiction shows that χ can also never reach the value $3C_2/4$ on $\Omega \times [0, T)$, and so completes the proof of (3.36).

Finally (3.35) then follows immediately from (3.2) and our definition of χ. □

Before continuing, note, in relation to our earlier remarks following Theorems 2.1 and 3.1, that it is precisely for the derivation of (3.39) via condition (3.34) that we critically require the existence (see (3.2)) of the positive lower bound, m_0, for u under the evolution in IVP (3.1). As yet we have been unable to overcome the need for such a lower bound here.

Returning to the derivation of our time independent global gradient bound, Lemma 3.6 now yields directly the desired estimate:

COROLLARY 3.7. *Suppose κ_0 is as in Lemma 3.6. Then for any $\kappa \leq \kappa_0$, if u is a solution of IVP (3.1) we have the uniform estimate*

$$\sup_{\overline{\Omega}\times[0, T)} v < C_2. \tag{3.41}$$

Proof. This is identical in form to the proof of Corollary 3.2. □

We are now nearly finished with the proof of Theorem 3.1. For Corollary 3.7 completes the derivation of a uniform a priori estimate for $|u_t|_{C^1}$, and so establishes the long-time existence claim of the theorem, while the regularity and convergence aspects of Theorem 3.1 now follow in virtually identical fashion to those of Theorem 2.1. Indeed, unlike the global gradient estimate, the desired convergence properties actually turn out to be slightly easier to obtain for IVP (3.1) than they were for IVP (2.3). This is because

(i) We already have, in estimate (3.10), a uniform bound for $\int_0^T \int_\Omega \dot{u}^2 \, dx \, dt$; and

(ii) In deriving the analogue of Lemma 2.7 for IVP (3.1) we can now consider $\frac{d}{dt}\int_\Omega (u-u_\infty)^2\,dx$, rather than $\frac{d}{dt}\int_\Omega v^{-1}(u-u_\infty)^2\,dx$, and in this way can arrive at the corresponding result with rather *less* working, and also with T equal to zero.

The only feature in the above that perhaps ought to be explicitly pointed out is that, just as a further condition on κ_0, (2.24), was required to be able to derive Lemma 2.7, so also an additional condition on κ_0 is needed to be able to prove the analogous result for IVP (3.1). This condition (very similar to (2.24)) is that κ_0 satisfy

$$\kappa_0 \le \frac{\lambda C_p^{-1}}{2} \tag{3.42}$$

where C_p and λ are again as defined in Lemma 2.7. With this further constraint the convergence claim of Theorem 3.1 goes through exactly as stated above.

This completes the proof of Theorem 3.1. □

4. The Hanging Roof Problem. Here we turn finally to the evolutionary treatment of Problem (iii). Our main theorem in this regard is:

THEOREM 4.1. *Let Ω, $\partial\Omega$, m_0, μ, u_0 and φ again be as in Theorem* 3.1, *and suppose that m_0 now also satisfies the two conditions*

$$m_0 > \frac{2\operatorname{diam}(\Omega)}{\sqrt{2n-1}} \tag{4.1}$$

and

$$m_0 > \frac{2}{\mu} \tag{4.2}$$

(that is to say, our surface starts at a sufficient height). Then the parabolic IVP

$$\begin{aligned} \dot u &= vH - \frac{1}{u} && \text{on } \Omega\times[0,\infty),\\ u &= \varphi && \text{on } \partial\Omega\times[0,\infty),\\ u(x,0) &= u_0(x) && \text{on } \Omega \end{aligned} \tag{4.3}$$

has a $C^{2,\alpha,\alpha/2}(\overline{\Omega}\times(0,\infty))$-solution, u, smooth on the interior $\Omega\times(0,\infty)$, and moreover some subsequence at least of the functions $u_t(x) = u(x,t)$ converge as $t\to\infty$ to a $C^{2,\alpha}(\overline{\Omega})$-solution, u_∞, of the *stationary BVP* (1.3). *Furthermore if, in fact,* m_0 *also satisfies the additional conditions*

$$m_0 > \frac{2}{\lambda} \tag{4.4}$$

and

$$\frac{8}{m_0^2} + \frac{3}{m_0} < \frac{\lambda}{2C_p} \tag{4.5}$$

where λ *and* C_p *are as defined in Lemma* 2.7, *then all of the functions* $u_t(x)$, *and not just some subsequence of them, converge exponentially fast, as* $t \to \infty$, *to* u_∞, *in any* C^k*-norm.*

Proof of theorem. Once again we seek a uniform a priori estimate for $|u_t|_{C^1}$, with the bulk of the work lying in the derivation of a time-independent *interior* gradient bound. However we begin, as usual, with the sup and boundary gradient estimates.

LEMMA 4.1. *Define* $C_0 = \sup_{\overline{\Omega}} u_0$. *Then if* u *is a solution of IVP* (4.3) *we have*

$$\frac{m_0}{2} \le u(x, t) \le C_0 \quad \text{on } \overline{\Omega} \times [0, \infty).$$

Proof. The estimate from above is obvious. As for the estimate from below, first let $x_0 \in \Omega$ be arbitrary, and then define

$$d = \operatorname{diam}(\Omega), \qquad r = \frac{1}{m_0}\left(d^2 + \left(\frac{m_0}{2}\right)^2\right) \ge d.$$

Now let $\psi \colon \overline{\Omega} \times [0, \infty) \to \mathbf{R}$ be the stationary function defined by

$$\psi(x, t) = \left(r + \frac{m_0}{2}\right) - \sqrt{r^2 - |x - x_0|^2}.$$

That is, graph(ψ) is here a portion of lower hemisphere, centre $(x_0, r + \frac{m_0}{2})$, radius r.

Then clearly, for all $x \in \overline{\Omega}$, we have

$$\frac{m_0}{2} \le \psi(x) < m_0 \tag{4.6}$$

and so certainly graph(ψ) lies everywhere below graph(u) initially, and also remains so on $\partial\Omega$ for all time. Thus since graph(ψ) has constant *positive* mean curvature over all of Ω equal to

$$\frac{n}{r} \equiv \frac{4nm_0}{4d^2 + m_0^2}$$

it is easily checked, using also (4.1), that ψ must be a lower barrier for $u(x, t)$, and the result follows from (4.6). □

LEMMA 4.2. *If u is a solution of IVP* (4.3) *then there exists a constant*

$$C_1 = C_1(n, C_0, \mu, \Omega, \partial\Omega, |\varphi|_{2;\Omega})$$

such that we have the estimate

$$\sup_{\partial\Omega\times[0,\infty)} v \leq C_1 .$$

Proof. Once again this is established via stationary local barriers, using the constructions of [**GT**, Chapter 14] and condition (4.2) on m_0. As the approach is identical to that employed in Lemma 2.2 we omit all details here. □

Turning now to the *global a priori* gradient estimate, we this time obtain this via an approach, as noted in §2, of considering a modified flow and working locally on the surfaces produced, as follows.

Observe first that, if ν denotes the upward unit normal, $v^{-1}(-Du, 1)$, to the surfaces $M_t = \operatorname{graph}(u_t)$, then the component of the flow in IVP (4.3) in direction ν is exactly $v^{-1}\dot{u}$.

Thus, if a solution existed to IVP (4.3), then clearly we would be able to find a one-parameter family of immersions $\mathbf{F}_t = \mathbf{F}(\cdot, t)\colon M_0 \to \mathbf{R}^{n+1}$, $t \in [0, \infty)$, which satisfied

$$(4.7)\quad \begin{aligned} \frac{d}{dt}\mathbf{F}(p, t) &= v^{-1}\left(vH - \frac{1}{\tilde{u}}\right)\nu(p, t) && \text{on } M_0 \times [0, \infty), \\ \mathbf{F}(p, t) &= \mathbf{F}(p, 0) && \text{on } \partial M_0 \times [0, \infty), \\ \mathbf{F}(p, 0) &= p && \text{on } M_0 \end{aligned}$$

where the function $\tilde{u}$, analogous to u, is defined by

$$\tilde{u}(p, t) = \langle \mathbf{F}(p, t), \mathbf{e}_{n+1} \rangle$$

and where v is now viewed as a function on $M_0 \times [0, \infty)$ via the relation $v = \langle \nu, \mathbf{e}_{n+1} \rangle^{-1}$. Moreover the image surfaces of this new flow, $\widetilde{M}_t \equiv \{\mathbf{F}(p, t) : p \in M_0\}$, would obviously be identical to the surfaces $M_t = \operatorname{graph}(u_t)$ up to tangential diffeomorphisms.

Therefore to obtain a uniform a priori global gradient estimate for solutions of IVP (4.3) it will clearly suffice to derive such a uniform a priori bound for the quantity v under the flow in (4.7). To this end we have the following lemma:

LEMMA 4.3. *Suppose $\mathbf{F}\colon M_0 \times [0, \infty) \to \mathbf{R}^{n+1}$ is a one-parameter family of immersions satisfying* (4.7). *Then there exists a constant*

$C_2 = C_2(m_0, C_0, C_1)$ *such that* v *satisfies the estimate*

$$\sup_{M_0\times[0,\infty)} v \le C_2 .$$

Proof. Let $\Delta = \Delta_t$ denote the Laplace-Beltrami operator on the surfaces $\widetilde{M}_t$, pulled back to $\widetilde{M}_0$. Then for any $p \in \widetilde{M}_t$ let $\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_n$ be a local framing corresponding to nearly flat co-ordinates (with respect to the induced metric) on a neighbourhood of p, so that

$$\nabla_i \boldsymbol{\nu} = -h_{il}\boldsymbol{\tau}_l \quad \text{and} \quad \nabla_i \boldsymbol{\tau}_j = h_{ij}\boldsymbol{\nu} \tag{4.8}$$

where ∇_i denotes the tangential derivative operator with respect to $\boldsymbol{\tau}_i$, and $\widetilde{A} = h_{il}$ denotes the second fundamental form (see also **[DH]**). Set $\nabla = (\nabla_1, \dots, \nabla_{n+1})$.

Also note that the new "height function" $\tilde{u}\colon M_0 \times [0, \infty) \to \mathbf{R}$ defined earlier differs from u in that it tracks the height above $\Omega \subset \mathbf{R}^n$ of points $\mathbf{F}(p, t)$ for a *fixed* $p \in M_0$, whereas u measures the height of image surfaces above fixed points in Ω (see **[EH]**). This new height function $\tilde{u}$ satisfies the relation $H = v\Delta\tilde{u}$.

Turning now to the proof of the lemma observe that, by proceedings as in **[Hu3]**, we have for the flow in (4.7) that

$$\partial_t \boldsymbol{\nu} = -\nabla\left(\left\langle \frac{d\mathbf{F}}{dt}, \boldsymbol{\nu}\right\rangle\right) = -\nabla H - \frac{1}{\tilde{u}^2 v^2}\nabla\tilde{u}v$$

and so

$$\dot{v} \equiv \frac{d}{dt}(\langle \boldsymbol{\nu}, \mathbf{e}_{n+1}\rangle^{-1}) = v^2\langle \nabla H, \mathbf{e}_{n+1}\rangle + \frac{1}{\tilde{u}^2}\langle \nabla(\tilde{u}v), \mathbf{e}_{n+1}\rangle . \tag{4.9}$$

But also we have the Jacobi-Codazzi equation

$$\begin{aligned} \Delta v &= \nabla_i\nabla_i(\langle \boldsymbol{\nu}, \mathbf{e}_{n+1}\rangle^{-1}) = \nabla_i(v^2\langle h_{il}\boldsymbol{\tau}_l, \mathbf{e}_{n+1}\rangle) \\ &= |\widetilde{A}|^2 v + 2v^{-1}|\nabla v|^2 + v^2\langle \nabla H, \mathbf{e}_{n+1}\rangle \end{aligned} \tag{4.10}$$

where of course $|\widetilde{A}|^2 \equiv h_{il}h^{il}$. Hence from (4.10) in (4.9) we obtain

$$\dot{v} = \Delta v - |\widetilde{A}|^2 v - 2v^{-1}|\nabla v|^2 + \frac{1}{\tilde{u}^2}\langle \nabla(\tilde{u}v), \mathbf{e}_{n+1}\rangle . \tag{4.11}$$

It then follows directly that we must, in turn, have

$$\begin{aligned} \frac{d}{dt}(\ln(v)) &= \Delta(\ln(v)) - |\widetilde{A}|^2 - v^{-2}|\nabla v|^2 \\ &\quad + \frac{1}{\tilde{u}^2}\langle \nabla\tilde{u}, \mathbf{e}_{n+1}\rangle + \frac{1}{\tilde{u}v}\langle \nabla v, \mathbf{e}_{n+1}\rangle . \end{aligned} \tag{4.12}$$

But also now, since $H = v\Delta\tilde{u}$, we have directly from the evolution equation in (4.7) that

$$\dot{\tilde{u}} = \left\langle \frac{d\mathbf{F}}{dt}, \mathbf{e}_{n+1} \right\rangle = \Delta\tilde{u} - \frac{1}{\tilde{u}v^2}. \tag{4.13}$$

Therefore if we now introduce the auxiliary function

$$\chi \equiv \sigma \ln(v) + \tilde{u}$$

where σ is some positive constant, then we obtain immediately from (4.12) and (4.13) that χ satisfies

$$\dot{\chi} \leq \Delta\chi + \frac{\sigma}{\tilde{u}^2}\langle \nabla\tilde{u}, \mathbf{e}_{n+1}\rangle + \frac{\sigma}{\tilde{u}v}\langle \nabla v, \mathbf{e}_{n+1}\rangle - \frac{1}{\tilde{u}v^2}.$$

Then since we clearly have

$$\nabla\chi = \frac{\sigma}{v}\nabla v + \nabla\tilde{u}$$

and

$$\begin{aligned} \nabla\tilde{u} = \nabla u &= (Du, 0)^{\text{tangential}} \\ &= (Du, 0) - \langle (Du, 0), \boldsymbol{\nu}\rangle\boldsymbol{\nu} = (Du, 0) + \left(\frac{|Du|^2}{v}\right)\boldsymbol{\nu} \end{aligned}$$

we may deduce that in fact

$$\begin{aligned} \dot{\chi} &\leq \Delta\chi + \frac{1}{\tilde{u}}\langle \nabla\chi, \mathbf{e}_{n+1}\rangle - \frac{|Du|^2}{\tilde{u}^2v^2}(\tilde{u} - \sigma) - \frac{1}{\tilde{u}v^2} \\ &< \Delta\chi + \frac{1}{\tilde{u}}\langle \nabla\chi, \mathbf{e}_{n+1}\rangle - \frac{1}{\tilde{u}^2}(\tilde{u} - \sigma). \end{aligned}$$

But thus, if we now take $\sigma = m_0/4$ say, then by Lemma 4.1 we must have that, on all of $M_0 \times [0, \infty)$,

$$\dot{\chi} < \Delta\chi + \frac{1}{\tilde{u}}\langle \nabla\chi, \mathbf{e}_{n+1}\rangle$$

which immediately implies that

$$\sup_{M_0\times[0,\infty)} \chi \leq \max\left(\sup_{M_0}\chi, \sup_{\partial M_0\times[0,\infty)} \chi\right) = C(C_0, C_1).$$

The result then follows from the definition of χ. □

This completes the proof of a uniform *a priori* global gradient estimate for solutions of IVP (4.3), and so establishes the long-time existence claim of Theorem 4.1.

The remaining step is now to prove also the regularity and convergence claims of this theorem. Before continuing with this, however, it is worth returning briefly to an earlier remark we made, namely that the above approach of working locally on the surfaces $\widetilde{M}_t$, to obtain a global gradient estimate, is in fact simpler and more direct than doing the working on the base domain Ω, as was done in proving Lemma 2.3.

At first glance this may not appear to be the case. However it should be pointed out that this is only because much of the hard work involved in our proof of Lemma 2.3 was hidden away in Lemmas 2.4 and 2.5.

For example, when we worked on the base domain Ω to establish Lemma 2.3 we required Lemma 2.5, the proof of which is quite long and technical (see **[Ge2]**). This was needed to deal with the term of $a^{ij}a^{kl}D_iD_kuD_jD_lu$ which arose, and to show in particular that it was always non-negative. However, when we work locally on the surfaces $\widetilde{M}_t$ as above, we find that the corresponding term which arises turns out to be simply $|\widetilde{A}|^2v$, which may immediately be seen to be non-negative without any further work.

Returning to the proof of Theorem 4.1, regularity now follows once again from standard theory and the uniformity of our estimate for $|Du|$. As for the convergence claim in the theorem, this is established in very similar fashion to that used in the proofs of Theorems 2.1 and 3.1.

To begin with, the uniformity of our earlier estimates guarantees that we can again find a subsequence of times $t_k \to \infty$ such that the functions u_{t_k} converge to some $C^{2,\alpha}(\overline{\Omega})$-function u_∞. That u_∞ must be a stationary point of the flow, and thus a solution of BVP (1.3), then follows once again from the trick of analysing the time-derivative of the energy functional for this situation, and computing that this yields, using also Lemmas 4.1 and 4.3,

$$\frac{d}{dt}\int_\Omega uv\,dx = -\int_\Omega\left(\frac{u}{v}\right)\dot{u}^2\,dx \le -\frac{m_0}{2C_2}\int_\Omega \dot{u}^2\,dx\,. \tag{4.14}$$

Finally, to prove the exponential convergence, in any C^k-norm, of *all* the u_t to u_∞ as $t\to\infty$, we once again need only prove such convergence in the L^2-norm, and in this regard we have the following lemma:

LEMMA 4.4. *Suppose that u_∞ is as above, and that m_0 now satisfies all of the conditions* (4.1), (4.2), (4.4) *and* (4.5). *Then there exists a* *positive constant* $\xi = \xi(C_2, n, \Omega)$ *such that*

$$\|u_t - u_\infty\|_2^2 \le C_2\|u_T - u_\infty\|_2^2 e^{-\xi(t-T)} \quad \forall t \ge T$$

where T *is some sufficiently large but finite time.*

Proof. Observe first that, just as in the proof of Lemma 2.7, our uniform estimates on all higher derivatives of u in time and space, along with inequality (4.14), ensure that the quantity $|a^l D_l \dot{u}|$ decays to zero as $t \to \infty$. In view of this, let $T > 0$ be any time such that, for all $t \ge T$, $|a^l D_l \dot{u}| \le 1/m_0$ say.

Then since $\dot{u}_\infty \equiv 0$ on $\overline{\Omega} \times [0, \infty)$, we have by direct computation that for all $t \ge T$,

$$\begin{aligned}(4.15)\quad \frac{d}{dt}\int_\Omega \frac{1}{v}(u-u_\infty)^2\,dx &= 2\int_\Omega (u-u_\infty)\left(D_l a^l - \frac{1}{uv}\right)dx \\ &\quad - \int_\Omega (u-u_\infty)^2\left(\frac{a^l D_l \dot{u}}{v^2}\right)dx \\ &\le 2\int_\Omega (u-u_\infty) D_l(a^l - a^l_\infty)\,dx \\ &\quad + 2\int_\Omega (u-u_\infty)\left(\frac{1}{u_\infty v_\infty} - \frac{1}{uv}\right)dx \\ &\quad + \frac{1}{m_0}\int_\Omega (u-u_\infty)^2\,dx\end{aligned}$$

where, of course, a^l_∞ stands for $a^l(Du_\infty)$, and v_∞ for $v(Du_\infty)$.

However, using Lemma 4.1 and the fact that

$$|(v - v_\infty)| \le |D(u-u_\infty)|,$$

we also have by Cauchy's inequality that

$$\begin{aligned}(4.16)\quad (u-u_\infty)\left(\frac{1}{u_\infty v_\infty} - \frac{1}{uv}\right) &\le (u-u_\infty)^2\left(\frac{4}{m_0^2} + \frac{1}{m_0}\right) \\ &\quad + \frac{|D(u-u_\infty)|^2}{m_0}.\end{aligned}$$

Moreover, on integrating by parts and applying the Fundamental Theorem of Calculus, exactly as in the proof of Lemma 2.7, we may derive that (for λ as in that lemma)

$$(4.17)\quad 2\int_\Omega (u-u_\infty) D_l(a^l - a^l_\infty)\,dx \le -2\lambda \int_\Omega |D(u-u_\infty)|^2\,dx.$$

Hence, by (4.16) and (4.17) in (4.15), we obtain that for all $t \geq T$,

$$\frac{d}{dt}\int_\Omega \frac{1}{v}(u-u_\infty)^2\,dx \leq \left(-2\lambda + \frac{2}{m_0}\right)\int_\Omega |D(u-u_\infty)|^2\,dx + \left(\frac{8}{m_0^2}+\frac{3}{m_0}\right)\int_\Omega (u-u_\infty)^2\,dx. \tag{4.18}$$

But then, by the successive use of condition (4.4), Poincaré's inequality, and condition (4.5), equation (4.18) yields that for all $t \geq T$,

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{1}{v}(u-u_\infty)^2\,dx &\leq \frac{-\lambda}{2C_p}\int_\Omega (u-u_\infty)^2\,dx \\ &\leq \frac{-\lambda}{2C_p}\int_\Omega \frac{1}{v}(u-u_\infty)^2\,dx\end{aligned}$$

and the desired decay estimate follows immediately from this and our bound for v in Lemma 4.3 (which thus yields a lower bound for λ in terms of C_2). □

This completes the proof of Theorem 4.1. □

References

[Br] K. A. Brakke, *The motion of a surface by its mean curvature*, Math. Notes, Princeton Univ. Press, Princeton, NJ, 1978.

[CF] P. Concus and R. Finn, *On capillary free surfaces in a gravitational field*, Acta Math., **132** (1974), 207–223.

[DH] U. Dierkes and G. Huisken, *The n-dimensional analogue of the catenary: existence and non-existence*, Pacific J. Math., **141** (1990), 47–54.

[Ec] K. Ecker, *Estimates for evolutionary surfaces of prescribed mean curvature*, Math. Z., **180** (1982), 179–192.

[EH] K. Ecker and G. Huisken, *Mean curvature evolution of entire graphs*, Annals of Math., **130** (1989), 453–471.

[Fi] R. Finn, *Equilibrium Capillary Surfaces*, Springer-Verlag, New York, 1986.

[Ge1] C. Gerhardt, *Evolutionary surfaces of prescribed mean curvature*, J. Differential Equations, **36** (1980), 139–172.

[Ge2] ——, *Global regularity of the solutions to the capillarity problem*, Ann. Scuola Norm. Sup. Pisa Ser. (4), **3** (1976), 157–175.

[Gi] E. Giusti, *The pendant water drop. A direct approach*, Boll. Un. Mat. Ital., **17** (1980), 458–465.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., A series of comprehensive studies in mathematics, vol. 224, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[Hu1] G. Huisken, *Capillary surfaces in negative gravitational fields*, Math. Z., **185** (1984), 449–464.

[Hu2] ——, *Non-parametric mean curvature evolution with boundary conditions*, J. Differential Equations, **77** (1989), 369–378.

[Hu3] ——, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom., **20** (1984), 237–266.

[KS] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York and London, 1980.

[LT] A. Lichnewski and R. Temam, *Surfaces minimales d'évolution: Le concept de pseudosolution*, C. R. Acad. Sci. Paris, **284** (1977), 853–856.

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