BRANCHED COVERINGS OF SURFACES
WITH AMPLE COTANGENT BUNDLE

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Let $f: X \to Y$ be a branched covering of compact complex surfaces, where the ramification set in X consists of smooth curves meeting with at most normal crossings and Y has ample cotangent bundle. We further assume that f is locally of form $(u, v) \to (u^n, v^m)$. We characterize ampleness of T^*X. A class of examples of such X, which are branched covers of degree two, is provided.

1. Introduction. An interesting problem in surface theory is the construction and characterization of surfaces with ample cotangent bundle. They are necessarily algebraic surfaces of general type. Natural examples occur among the complete intersection surfaces of abelian varieties. More subtle examples are those constructed by Hirzebruch [6] using line-arrangements in the plane. The characterization of those of Hirzebruch's line-arrangement surfaces with ample cotangent bundle is due to Sommese [8]. In this article, we will give a characterization of ampleness of the cotangent bundle of a class of surfaces which branch cover another surface with ample cotangent bundle. We will also construct certain branched coverings of explicit line-arrangement surfaces; these constructions will again have ample cotangent bundle.

For any vector bundle E over a base manifold M, the projectivization $\mathbf{P}(E)$ is a fiber bundle over M, with fiber $\mathbf{P}_q(E)$ over $q \in M$ given by $\mathbf{P}_q(E) \approx (E_q^* \backslash 0)/\mathbf{C}^*$. There is a tautological linebundle ξ_E over $\mathbf{P}(E)$ satisfying (i) $\xi_E|_{\mathbf{P}_q(E)} \approx O(1)_{\mathbf{P}_q(E)} \ \forall q \in M$, and (ii) the projection $\rho_E: \mathbf{P}(E) \to M$ gives $\rho_{E_*}(\xi_E) \approx E$. In the case that $E = T^*X$ we will denote $\rho_E = \rho_{T^*X}$ simply by ρ.

DEFINITION. The vector bundle E is *ample* if ξ_E over $\mathbf{P}(E)$ is ample.

In §2 we prove preliminary results along with:

THEOREM 1.1. *Let X and Y be compact complex surfaces, with Y having ample cotangent bundle. Let $f: X \to Y$ be a branched covering which can be locally represented with coordinate charts of form $f: (u, v) \to (u^n, v^m)$. Let f have ramification set $\bigcup B_j$ in X*

consisting of smooth curves meeting in normal crossings. Let $\bigcup C_\alpha$ *be the branch locus in* Y. *Then*:

$$T^*X \text{ is ample } \Leftrightarrow B_j \cdot B_j < 0 \; \forall j \Leftarrow C_\alpha \cdot C_\alpha < 0 \; \forall \alpha.$$

In §3 we give explicit examples of Theorem 1.1 which lie in a class of degree 2 branched covers.

In what follows $e(C)$ will denote the euler number of a curve C.

2. Ample cotangent bundles. Let $f: X \to Y$ be a branched covering of compact complex surfaces, with T^*Y ample and with ramification set $\bigcup B_j$ in X consisting of smooth curves meeting in normal crossings. Note that T^*Y ample gives that the canonical bundle K_Y is ample [4]. Hence Y is projective algebraic, which gives that X is also projective algebraic [1], and in turn that $\mathbf{P}(T^*X)$ is also projective algebraic. Let f be locally represented with coordinate charts of form $f: (u, v) \to (u^n, v^m)$. The differential $f_*: TX \to TY$ induces a meromorphic mapping $F: \mathbf{P}(T^*X) \to \mathbf{P}(T^*Y)$ given by $F(x, [w]) = (f(x), [f_*(w)])$, where $[w]$ denotes the line in the tangent bundle containing the tangent vector w. The indeterminacy set I of F corresponds to $[w]$ such that $f_*(w) = 0$. Blowing up I to resolve the indeterminacy of f (see [5], [10]) one gets $\Pi(T^*X)$ and obtains $b: \Pi(T^*X) \to \mathbf{P}(T^*X)$ and $\Phi: \Pi(T^*X) \to \mathbf{P}(T^*Y)$, holomorphic, with $F \circ b = \Phi$ on $\Pi(T^*X) \backslash b^{-1}(I)$. Let $E = b^{-1}(I)$ be the exceptional set over I in $\Pi(T^*X)$. We need to precisely describe the indeterminacy set I. Before proceeding, we mention that, in the case that f is locally of form $(u, v) \to (u^n, v^m)$, over each curve B_j in the ramification set in X there is a splitting of TX, due to Sommese [8], [9], namely $TX|_{B_j} \approx TB_j \oplus N_{B_j}$. In particular N_{B_j}, the normal bundle to B_j, is a subbundle of $TX|_{B_j}$, and the pair B_j, N_{B_j} gives a curve $\widetilde{B}_j$ in $\mathbf{P}(T^*X)$.

PROPOSITION 2.1. *The indeterminacy set* I *of* $F: \mathbf{P}(T^*X) \to \mathbf{P}(T^*Y)$ *is of form*

$$I = \bigcup_j \widetilde{B}_j \cup \bigcup_\alpha F_\alpha$$

where the F_α *are fibers of* p *and the* $\widetilde{B}_j$ *are the curves in* $\mathbf{P}(T^*X)$ *corresponding to the pairs* B_j *and* N_{B_j} *for* B_j *in the ramification set of* X.

Proof. In local coordinates let $f: X \to Y$ be given by $f(u, v) = (u^n, v^m)$. Then note that the indeterminacy set in $\mathbf{P}(T^*X)$ locally

corresponds to the (non-zero) annihilators in $\operatorname{ann}\langle d(u^n), d(v^m)\rangle :=$ $\{w \in TX | d(u^n)(w) = 0 \text{ and } d(v^m)(w) = 0\}$. By analyzing the Jacobian f_* one sees that $\operatorname{ann}\langle d(u^n), d(v^m)\rangle = \{w \in TX | f_*(w) = 0\}$. Therefore locally $I = \{\operatorname{ann}\langle d(u^n), d(v^m)\rangle \backslash 0\}/\mathbf{C}^*$. If both $n, m > 1$ then I includes the whole fiber $\rho^{-1}(0, 0)$ giving an F_α. If only one of n, m is greater than 1, say $n > 1$ with $m = 1$, then the curve B_1 corresponding to $u = 0$ is in the ramification set and $\operatorname{ann}\langle d(u^n), d(v^m)\rangle$ corresponds to the normal bundle of B_1 via Sommese's splitting lemma [8], [9]. Thus B_1 and N_{B_1} determine $\widetilde{B}_1$. Similarly one obtains the remaining $\widetilde{B}_j$. $\square$

Let ξ_1 be the tautological bundle over $\mathbf{P}(T^*X)$ and let ξ_2 be the tautological bundle over $\mathbf{P}(T^*Y)$, as in the second paragraph of §1. Then $b^*(\xi_1)$ on $\Pi(T^*X)$ relates to $\Phi^*(\xi_2)$ in a key manner via the following:

PROPOSITION 2.2. $b^*(\xi_1^{-1}) + D = \Phi^*(\xi_2^{-1})$ where $D = \sum n_\alpha D_\alpha$ is an effective divisor on $\Pi(T^*X)$ supported on the exceptional set E of $\Pi(T^*X)$.

Proof. $f_*: TX \to TY$ given by $(x, w) \to (f(x), f_*(w))$ induces $F: \mathbf{P}(T^*X) \to \mathbf{P}(T^*Y)$ which is given by $(x, [w]) \to (f(x), [f_*(w)])$. Here $x \in X$, w is a tangent vector at x, and $[w]$ denotes the line in the tangent bundle containing w. F has indeterminacy set I as described in Proposition 2.1. Over $\mathbf{P}(T^*X)\backslash I$, f induces the mapping $f_*: \xi_1^{-1} \to \xi_2^{-1}$ given by $(x, [w], w) \to (f(x), [f_*(w)], f_*(w))$. This in turn yields the globally defined holomorphic mapping over $\Pi(T^*X)$ $\beta: b^*(\xi_1^{-1}) \to \xi_2^{-1}$ given by $(p, w) \to (f(\rho(b(p))), \Phi(p),$ $f_*(w))$ where $p \in \Pi(T^*X)$, $b(p) = [w]$, and $\rho(b(p)) = x$. Furthermore $\Phi(p) = [f_*(w)]$ if $f_*(w) \neq 0$ (i.e. off $b^{-1}(I)$). In turn, β gives the mapping γ over $\Pi(T^*X)$ $\gamma: b^*(\xi_1^{-1}) \to \Phi^*(\xi_2^{-1})$ given by $(p, w) \to (p, \Phi(p), f_*(w))$. There is vanishing of $f_*(w)$ over $E = b^{-1}(I)$, giving D. $\square$

We will prove Theorem 1.1 using the Nakai Criterion for ampleness [7]: the holomorphic line bundle ξ_1 on the projective algebraic manifold $P(T^*X)$ is ample if and only if for every subvariety V_n of dimension $n \leq \dim \mathbf{P}(T^*X)$ one has that $\int_{V_n} c_1^n(\xi_1) > 0$. For brevity we define $\xi_1^n \cdot V_n := \int_{V_n} c_1^n(\xi_1)$.

Proof of Theorem 1.1. Assume that $C_\alpha \cdot C_\alpha < 0$ for each C_α in the branch locus in Y. We show that $B_j \cdot B_j < 0$ for all B_j in the

ramification set. Let $\pi^*(C_\alpha) = \sum_k n_{\alpha k} B_{\alpha k}$. Then

$$B_{\alpha j} \cdot \pi^*(C_\alpha) = B_{\alpha j} \cdot \sum_k n_{\alpha k} B_{\alpha k}$$

$$= n_{\alpha j} B_{\alpha j} \cdot B_{\alpha j} + B_{\alpha j} \cdot \sum_{k \neq j} n_{\alpha k} B_{\alpha k} = \deg(\pi|_{B_{\alpha j}}) C_\alpha \cdot C_\alpha$$

and hence

$$B_{\alpha j} \cdot B_{\alpha j} = n_{\alpha j}^{-1} \deg(\pi|_{B_{\alpha j}}) C_\alpha \cdot C_\alpha - n_{\alpha j}^{-1} B_{\alpha j} \cdot \sum_{k \neq j} n_{\alpha k} B_{\alpha k} < 0$$

giving the implication.

Assume next that T^*X is ample. We show that $B_j \cdot B_j < 0$ for any B_j in the ramification set in X. Now for any B_j in the ramification set in X the splitting lemma of Sommese [8], [9] gives that $TX|_{B_j} \approx TB_j \oplus N_{B_j}$. For $\widetilde{B}_j$ the curve in $\mathbf{P}(T^*X)$ determined by B_j along with N_{B_j}, we have $0 > \xi_1^{-1} \cdot \widetilde{B}_j = N_{B_j} \cdot B_j = B_j \cdot B_j$.

Conversely, assume that $B_j \cdot B_j < 0$ for each B_j in $\bigcup B_j$. We show that T^*X is ample. First note that since T^*Y is ample we have that $e(f(B_j))$ is negative; hence by Riemann-Hurwitz $e(B_j) \leq e(f(B_j)) < 0$. To prove ampleness of T^*X (i.e. of ξ_1) we show that $\xi_1^n \cdot V_n > 0$ for all subvarieties V_n in $\mathbf{P}(T^*X)$ where ξ_1 is the tautological bundle over $\mathbf{P}(T^*X)$. We handle the three cases $n = 1, 2, 3$ separately.

Case (1) $n = 1$. Let $V_n = C$ be an effective irreducible curve in $\mathbf{P}(T^*X)$. We show that $C \cdot \xi_1^{-1} < 0$. This is accomplished in three sub-cases:

Case (1i). Suppose that $\rho^{-1}(\bigcup B_j) \not\supset C$. Let C' be the proper transform of C in $\Pi(T^*X)$. By Proposition 2.2, $\Phi^*(\xi_2^{-1}) \cong b^*(\xi_1^{-1}) + D$ where $D = \sum_\alpha n_\alpha D_\alpha$ is an effective divisor. So

$$C \cdot \xi_1^{-1} = C' \cdot b^*(\xi_1^{-1}) = C' \cdot (\Phi^*(\xi_2^{-1}) - D)$$

$$= \deg(\Phi|_{C'}) \Phi(C') \cdot \xi_2^{-1} - C' \cdot D < 0.$$

The last inequality follows since: $\Phi(C')$ is a curve in $P(T^*Y)$, ξ_2 is ample (which gives that $\Phi(C) \cdot \xi_2^{-1} < 0$), and $D|_{C'}$ is effective on C' as C' is not contained in D.

Case (1ii). Suppose that C is a fiber of ρ (and therefore $C \cong \mathbf{P}^1$). Then $C \cdot \xi_1^{-1} = -1 < 0$, since $\xi_1|_C \cong O(1)$.

Case (1iii). Suppose C is contained in $\rho^{-1}(\bigcup B_j)$ but is not a fiber of ρ. Then $\rho(C) = B_j$ for some j. For $\nu: \eta C \to C$ the normalization of C, one has that $\rho \circ \nu: \eta C \to \rho(C)$. One has the vector bundle maps:

$$0 \to \nu^*(\xi_1^{-1}|_C) \to \nu^* \rho^*(TX|_{\rho(C)})$$

and

$$0 \to \nu^* \rho^* T\rho(C) \to \nu^* \rho^*(TX|_{\rho(C)}) \to \nu^* \rho^* N\rho(C) \to 0.$$

Hence one of the sequences of sheaves (2.1) or (2.2) below must be valid.

$$(2.1) \qquad 0 \to \nu^* \xi_1^{-1} \to \nu^* \rho^* T\rho(C) \to Z_1 \to 0,$$

$$(2.2) \qquad 0 \to \nu^* \xi_1^{-1} \to \nu^* \rho^* N\rho(C) \to Z_2 \to 0,$$

where Z_1 and Z_2 are sheaves with finite support on ηC. By letting $M_1 = \nu^* \rho^* T\rho(C)$ and $M_2 = \nu^* \rho^* N\rho(C)$ we rewrite (2.1) and (2.2) as:

$$(2.3) \qquad 0 \to \nu^* \xi_1^{-1} \to M_i \to Z_i \to 0.$$

By utilizing the long exact sequence associated to (2.3), along with Riemann-Roch, one concludes that:

$$c_1(\xi_1^{-1}) \cdot C = c_1(\nu^* \xi_1^{-1}) \cdot \eta C$$

$$\leq c_1(M_i) \cdot \eta C = \begin{cases} \deg(\rho \circ \nu) c_1(T\rho(C)) \cdot \rho(C) \\ \deg(\rho \circ \nu) c_1(N\rho(C)) \cdot \rho(C). \end{cases}$$

Now $c_1(T\rho(C)) \cdot \rho(C) = e(\rho(C)) = e(B_j) < 0$. We have that

$$c_1(N(\rho(C))) \cdot \rho(C) = \rho(C) \cdot \rho(C) = B_j \cdot B_j < 0$$

by hypothesis. Therefore in all cases $\xi_1^{-1} \cdot C < 0$. So $\xi_1 \cdot C > 0$ and Case 1 is proven.

Case (2) $n = 2$. $V_n = S$ is an effective irreducible surface in $\mathbf{P}(T^*X)$. Let Σ be the proper transform of S in $\Pi(T^*X)$.

Case (2i) $\Phi(\Sigma)$ is 2 dimensional.

$$S \cdot \xi_1^2 = S \cdot (\xi_1^{-1})^2 = \Sigma \cdot (b^*(\xi_1^{-1}))^2 = \Sigma \cdot (\Phi^*(\xi_2^{-1}) - D)^2$$

$$= \Sigma \cdot \Phi^*(\xi_2^{-1}) \cdot (\Phi^*(\xi_2^{-1}) - D) - \Sigma \cdot D \cdot (\Phi^*(\xi_2^{-1}) - D)$$

$$= \Sigma \cdot \Phi^*(\xi_2^{-1})^2 - \Sigma \cdot \Phi^*(\xi_2^{-1}) \cdot D - \Sigma \cdot D \cdot (\Phi^*(\xi_2^{-1}) - D).$$

So

$$(2.4) \qquad S \cdot \xi_1^2 = \mathrm{Deg}(\Phi|_\Sigma)\Phi(\Sigma) \cdot (\xi_2^{-1})^2$$
$$- \Sigma \cdot \Phi^*(\xi_2^{-1}) \cdot D - \Sigma \cdot D \cdot (\Phi^*(\xi_2^{-1}) - D).$$

Since $\Phi(\Sigma)$ is 2 dimensional and ξ_2 is ample, one has that

$$\mathrm{Deg}(\Phi|_\Sigma)\Phi(\Sigma) \cdot (\xi_2^{-1})^2 = \mathrm{Deg}(\Phi|_\Sigma)\Phi(\Sigma) \cdot (\xi_2)^2 > 0.$$

In order to conclude that $S \cdot \xi_1^2 > 0$ we show that (a) $\Sigma \cdot \Phi^*(\xi_2^{-1}) \cdot D \leq 0$ and that (b) $\Sigma \cdot D \cdot (\Phi^*(\xi_2^{-1}) - D) \leq 0$.

(a) Observe that Σ cuts out an effective divisor on D, namely $\Sigma \cdot D$, which we denote by $\Delta =: \sum m_\alpha \Delta_\alpha$. Hence

$$\Sigma \cdot \Phi^*(\xi_2^{-1}) \cdot D = \Delta \cdot \Phi^*(\xi_2^{-1}) =: \sum_\alpha m_\alpha (\deg(\Phi|_{\Delta_\alpha}))\Phi(\Delta_\alpha) \cdot \xi_2^{-1} \leq 0$$

since ξ_2 is ample and $\Phi(\Delta_\alpha)$ is either a curve or a point.

(b) With $\Delta = \Sigma \cdot D =: \sum m_\alpha \Delta_\alpha$ as in (a) we have that

$$(2.5) \quad \Sigma \cdot D \cdot (\Phi^*(\xi_2^{-1}) - D) = \Delta \cdot (\Phi^*(\xi_2^{-1}) - D) = \Delta \cdot (b^*(\xi_1^{-1}))$$
$$= \sum m_\alpha (\deg(b|_{\Delta_\alpha}))b(\Delta_\alpha) \cdot \xi_1^{-1}.$$

Now for each component Δ_α of Δ there are three possibilities: $b(\Delta_\alpha)$ is a point, $b(\Delta_\alpha) = \widetilde{B}_j$ for some j (where $\widetilde{B}_j$ denotes the curve in $\mathbf{P}(T^*X)$ corresponding to the curve B_j and the normal bundle N_{B_j}), or finally $b(\Delta_\alpha) \approx \mathbf{P}^1$ is a fiber of $\mathbf{P}(T^*X)$. If $b(\Delta_\alpha)$ is a point, then $b(\Delta_\alpha) \cdot \xi_1^{-1} = 0$. If $b(\Delta_\alpha) = \widetilde{B}_j$ for some j, then $b(\Delta_\alpha) \cdot \xi_1^{-1} = \widetilde{B}_j \cdot \xi_1^{-1} = B_j \cdot N_{B_j} = B_j \cdot B_j < 0$, by hypothesis. If $b(\Delta_\alpha) \approx \mathbf{P}^1$ is a fiber of $\mathbf{P}(T^*X)$, then $b(\Delta_\alpha) \cdot \xi_1^{-1} = \mathbf{P}^1 \cdot O(-1) = -1$. We conclude that for all α $b(\Delta_\alpha) \cdot \xi_1^{-1} \leq 0$ and hence from (2.5) that $\Sigma \cdot D \cdot (\Phi^*(\xi_2^{-1}) - D) \leq 0$.

Now (2.4) along with (a) and (b) gives that $S \cdot \xi_1^2 > 0$.

Case (2ii) $\Phi(\Sigma)$ is not 2 dimensional. This implies that S is contained in $\rho^{-1}(B_j)$ for some B_j in the ramification set $\bigcup B_k$. Then $S = \rho^{-1}(B_j) = \mathbf{P}(T^*X|_{B_j})$. Therefore:

$$\xi_1^2 \cdot S = \deg(T^*X|_{B_j}) := \deg(\det(T^*X|_{B_j})) \quad \text{as in [3]}$$
$$= \deg(K_X|_{B_j})$$
$$= (K_{B_j} - B_j) \cdot B_j \quad \text{by adjunction}$$
$$= K_{B_j} \cdot B_j - B_j \cdot B_j = -e(B_j) - B_j \cdot B_j > 0.$$

The last inequality follows from the hypothesis that $B_j \cdot B_j < 0$ along with a first remark of the proof (that T^*Y ample implies $e(B_j) < 0$).

Case (3) $n = 3$. $V_n = \mathbf{P}(T^*X)$. We have that

$$
\begin{aligned}
(2.6) \quad -\mathbf{P}(T^*X) \cdot \xi_1^3 &= \mathbf{P}(T^*X) \cdot (\xi_1^{-1})^3 = \Pi(T^*X) \cdot (b^*(\xi_1^{-1}))^3 \\
&= \Pi(T^*X) \cdot (b^*(\xi_1^{-1}))^2 \cdot (\Phi^*(\xi_2^{-1}) - D) \\
&= \Pi(T^*X) \cdot (b^*(\xi_1^{-1}))^2 \cdot \Phi^*(\xi_2^{-1}) \\
&\quad - \Pi(T^*X) \cdot (b^*(\xi_1^{-1}))^2 \cdot D \\
&= \Pi(T^*X) \cdot (b^*(\xi_1^{-1}))^2 \cdot \Phi^*(\xi_2^{-1}) - (b^*(\xi_1^{-1}))^2 \cdot D \\
&= \Pi(T^*X) \cdot (b^*(\xi_1^{-1}))^2 \cdot \Phi^*(\xi_2^{-1}) \\
&\quad - (\xi_1^{-1})^2 \cdot \sum_\alpha n_\alpha \deg(b|_{D_\alpha}) b(D_\alpha) \\
&= \Pi(T^*X) \cdot (b^*(\xi_1^{-1}))^2 \cdot \Phi^*(\xi_2^{-1}).
\end{aligned}
$$

The last equality follows since $b(D_\alpha)$ is one dimensional by Proposition 2.1. Furthermore:

$$
\begin{aligned}
(2.7) \quad \Pi(T^*X) &\cdot (b^*(\xi_1^{-1}))^2 \cdot \Phi^*(\xi_2^{-1}) \\
&= \Pi(T^*X) \cdot (\Phi^*(\xi_2^{-1}) - D) \cdot (b^*(\xi_1^{-1})) \cdot \Phi^*(\xi_2^{-1}) \\
&= \Pi(T^*X) \cdot (\Phi^*(\xi_2^{-1})) \cdot (b^*(\xi_1^{-1})) \cdot \Phi^*(\xi_2^{-1}) \\
&\quad - \Pi(T^*X) \cdot D \cdot (b^*(\xi_1^{-1})) \cdot \Phi^*(\xi_2^{-1}) \\
&= \Pi(T^*X) \cdot (\Phi^*(\xi_2^{-1})) \cdot (\Phi^*(\xi_2^{-1}) - D) \cdot \Phi^*(\xi_2^{-1}) \\
&\quad - D \cdot (b^*(\xi_1^{-1})) \cdot \Phi^*(\xi_2^{-1}) \\
&= \Pi(T^*X) \cdot (\Phi^*(\xi_2^{-1}))^3 - D \cdot (\Phi^*(\xi_2^{-1}))^2 \\
&\quad - D \cdot (b^*(\xi_1^{-1})) \cdot \Phi^*(\xi_2^{-1}).
\end{aligned}
$$

We next show that the above term is negative by analyzing each summand. First

$$
\begin{aligned}
\Pi(T^*X) \cdot (\Phi^*(\xi_2^{-1}))^3 &= (\deg \Phi) \Phi(\Pi(T^*X)) \cdot (\xi_2^{-1})^3 \\
&= (\deg \Phi) \mathbf{P}(T^*Y) \cdot (\xi_2^{-1})^3 < 0
\end{aligned}
$$

since ξ_2 is ample. Next, for each irreducible component D_α of D one has that

$$
\begin{aligned}
D_\alpha \cdot (\Phi^*(\xi_2^{-1}))^2 &= (\deg \Phi|_{D_\alpha}) \Phi(D_\alpha) \cdot (\xi_2^{-1})^2 \\
&= (\deg \Phi|_{D_\alpha}) \Phi(D_\alpha) \cdot (\xi_2)^2 \geq 0
\end{aligned}
$$

with positivity depending on the dimension of $\Phi(D_\alpha)$, since ξ_2 is ample. Therefore

$$-D \cdot (\Phi^*(\xi_2^{-1}))^2 = -\sum n_\alpha D_\alpha \cdot (\Phi^*(\xi_2^{-1}))^2 \leq 0$$

since $n_\alpha \geq 0$. Finally, by Proposition 2.3 below, we have that $D \cdot (b^*(\xi_1^{-1})) \cdot \Phi^*(\xi_2^{-1}) \geq 0$ giving by (2.7) that $\Pi(T^*X) \cdot (b^*(\xi_1^{-1}))^2 \cdot \Phi^*(\xi_2^{-1}) < 0$. By (2.6) one concludes that $\mathbf{P}(T^*X) \cdot \xi_1^3 > 0$. This finishes Case 3 and proves the converse, that ξ_1 is ample.

This completes the proof of Theorem 1.1. $\square$

PROPOSITION 2.3. *In the setting of Theorem* 1.1,

$$D \cdot (b^*(\xi_1^{-1})) \cdot \Phi^*(\xi_2^{-1}) \geq 0.$$

Proof. Let $D = \sum n_\alpha D_\alpha$ with $n_\alpha > 0$ $\forall \alpha$. For each D_α there are three possibilities: $b(D_\alpha)$ in $\mathbf{P}(T^*X)$ is either a point, a fiber of ρ (say $F_\alpha \approx \mathbf{P}^1$), or $\widetilde{B}_j$ for some j. If $b(D_\alpha)$ is a point then $D_\alpha \cdot (b^*(\xi_1^{-1})) \cdot \Phi^*(\xi_2^{-1}) = 0$ as $b^*(\xi_1^{-1})$ is then trivial on D_α.

If $b(D_\alpha)$ is a fiber F_α then on D_α $b^*(\xi_1^{-1}) \approx b^*(O(-1)) \approx b^*(-p_\alpha)$ where p_α is a point on F_α. Letting G_α be the (effective) divisor cut out by $b^*(p_\alpha)$ on D_α, one has that

$$D_\alpha \cdot (b^*(\xi_1^{-1})) \cdot \Phi^*(\xi_2^{-1}) = -G_\alpha \cdot \Phi^*(\xi_2^{-1}) = -\Phi(G_\alpha) \cdot \xi_2^{-1} \geq 0$$

since ξ_2 is ample.

Finally if $b(D_\alpha) = \widetilde{B}_j$ for some j, then on D_α one has that $b^*(\xi_1^{-1}) \approx b^*\rho^*(N_{B_j})$. Furthermore, since B_j is a Riemann surface with $B_j \cdot B_j < 0$ then $c_1(N_{B_j}) = c_1(\sum_k -p_{jk})$ where the p_{jk} are points on B_j. Letting $D_{\alpha jk}$ be the divisor cut out on D_α by $b^*\rho^*(p_{jk})$ we have that

$$D_\alpha \cdot (b^*(\xi_1^{-1})) \cdot \Phi^*(\xi_2^{-1}) = D_\alpha \cdot b^*\rho^*(N_{B_j}) \cdot \Phi^*(\xi_2^{-1})$$

$$= D_\alpha \cdot b^*\rho^* \left(\sum_k -p_{jk} \right) \cdot \Phi^*(\xi_2^{-1}) = -\sum_k D_{\alpha jk} \cdot \Phi^*(\xi_2^{-1})$$

$$= -\sum_k \Phi(D_{\alpha jk}) \cdot \xi_2^{-1} \geq 0$$

since ξ_2 is ample.

Hence $D \cdot (b^*(\xi_1^{-1})) \cdot \Phi^*(\xi_2^{-1}) = \sum_\alpha n_\alpha D_\alpha \cdot (b^*(\xi_1^{-1})) \cdot \Phi^*(\xi_2^{-1}) \geq 0$ as claimed. $\square$

3. Examples. We provide in Example 3.1 a construction of such an X as in Theorem 1.1. Then we expand Example 3.1 into a class of similar examples in Examples 3.2 and 3.3. These are all degree 2 branched covers of a surface Y with ample T^*Y. Good background references for this section are [6], [8], and [2]. We recall that given any $n \geq 2$ and any arrangement Λ of k lines in complex projective 2 space $\mathbf{P}^2$, Hirzebruch [6] constructed the compact complex surface $H(\Lambda, n)$ which is the minimal desingularization of the singular surface associated to the function field:

$$\mathbf{C}(z_1/z_0, z_2/z_0)(\sqrt[n]{l_2/l_1}, \sqrt[n]{l_3/l_1}, \ldots, \sqrt[n]{l_k/l_1})$$

where (z_0, z_1, z_2) are homogeneous coordinates for $\mathbf{P}^2$ and $l_j = 0$ is the equation of the jth line in Λ. $H(\Lambda, n)$ is a branched covering of $\mathbf{BP}^2$, the blowup of $\mathbf{P}^2$ at each point p with $r_p \geq 3$ where $r_p = \#\{L \in \Lambda | p \in L\}$. The branch locus in $\mathbf{BP}^2$ consists of the set of proper transforms of lines in Λ along with the exceptional curves in $\mathbf{BP}^2$. The branching order is n above the branch locus. Sommese [8] characterized the $H(\Lambda, n)$ with ample cotangent bundle as satisfying:
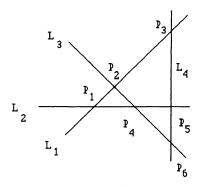
  (a) for any $L \in \Lambda$, $\#\{p \in L | r_p \geq 3\} \geq 2$ and

  (b) if $n = 3$ then $t_3 = 0$, while if $n = 2$ then $t_3 = t_4 = 0$, where $t_r = \#\{p \in \mathbf{P}^2 | r_p = r\}$. In Examples 3.1–3.3 that follow, we will assume $n \geq 4$ and avoid concern with condition (b).

We will make use of the Chern numbers of $H(\Lambda, n)$, which have been computed in [6], [8]. If one defines $f_0 := \sum_{r=2}^k t_r$ and $f_1 := \sum_{r=2}^k r t_r$ then one has:

$$(3.1) \quad c_1^2(H(\Lambda, n)) = n^{k-3}[n^2(-5k + 9 + 3f_1 - 4f_0)$$
$$+ 4n(k + f_0 - f_1) + f_1 - f_0 + k + t_2],$$
$$c_2(H(\Lambda, n)) = n^{k-3}[n^2(3 - 2k + f_1 - f_0)$$
$$+ 2n(k + f_0 - f_1) + f_1 - t_2].$$

EXAMPLE 3.1. Choose six points p_i, $i = 1, \ldots, 6$, in $\mathbf{P}^2$ satisfying four collinearity conditions on the p_i as follows:

  (a) p_1, p_2, and p_3 are collinear along the line L_1.

  (b) p_1, p_4, and p_5 are collinear along the line L_2.

  (c) p_2, p_4, and p_6 are collinear along the line L_3.

  (d) p_3, p_5, and p_6 are collinear along the line L_4.

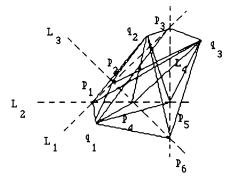Figure 1 represents this pattern succinctly.

FIGURE 1

FIGURE 2

Next we choose an arrangement of lines Λ in $\mathbf{P}^2$, satisfying:

(1) The above lines of collinearity are not in Λ (i.e. $L_j \notin \Lambda$ for $j = 1, \ldots, 4$).

(2) $r_{p_i} \geq 3$ for each point p_i, $i = 1, \ldots, 6$.

(3) For any other point $p \in L_1 \cup L_2 \cup L_3 \cup L_4 \backslash \{p_1, p_2, p_3, p_4, p_5, p_6\}$ one has $r_p \leq 1$.

(4) For each $L \in \Lambda$ at least 2 points p of L satisfy $r_p \geq 3$.

An explicit example of such an arrangement Λ is given in Figure 2. The Λ in Figure 2 is obtained by choosing three generic points, say q_1, q_2, q_3, not on $\bigcup L_j$. To each q_i form the "pencil" of the six lines through q_i containing the p_j, $j = 1, \ldots, 6$. The resulting arrangement of 18 lines Λ will be called a threefold cone on $p_1, p_2, p_3, p_4, p_5, p_6$ and it will satisfy all the above assumptions (1) through (4), provided (as in the generic choice of q_1, q_2, q_3) that there are no points p on both the threefold cone and $\bigcup L_j$ with $r_p \geq 2$, other than $p_1, p_2, p_3, p_4, p_5, p_6$. (One can similarly construct a-fold cones on the $q = 6$ points, where $a \geq 3$: pick a points not among the original q and run lines from each of the a points to each of the q points. This is a line arrangement consisting of a "pencils" with q lines in each "pencil".)

Given such an arrangement Λ satisfying (1)–(4), blowup all points with $r_p \geq 3$ to get $\mathbf{BP}^2$, which is branch covered by $H(\Lambda, n)$. For $n \geq 4$, $H(\Lambda, n)$ has ample cotangent bundle [8]. We pick $Y = H(\Lambda, n)$ as the base of our branched cover X, which we next construct.

Let H denote the pullback of the hyperplane in $\mathbf{P}^2$ to $\mathbf{BP}^2$. Let L' denote the proper transform of L in $\mathbf{BP}^2$. Let E_i denote the exceptional curve in $\mathbf{BP}^2$ over p_i in $\mathbf{P}^2$ for $i = 1, \ldots, 6$. Then we have that on $\mathbf{BP}^2$:

(1) L_1' is linearly equivalent to $H - E_1 - E_2 - E_3$,
(2) L_2' is linearly equivalent to $H - E_1 - E_4 - E_5$,
(3) L_3' is linearly equivalent to $H - E_2 - E_4 - E_6$,
(4) L_4' is linearly equivalent to $H - E_3 - E_5 - E_6$.

Hence $\sum_{i=1}^{4} L_i'$ is linearly equivalent to $4H - 2\sum_{i=1}^{6} E_i = 2(2H - \sum_{i=1}^{6} E_i)$. Therefore the line bundle corresponding to $\sum_{i=1}^{4} L_i'$ has a square root, and this fact provides a smooth branched cover Z of $\mathbf{BP}^2$ of degree 2, which is branched precisely over $\bigcup_{i=1}^{4} L_i'$ (see p. 42 of [1]).

Let $c: H(\Lambda, n) \to \mathbf{BP}^2$ denote the branched covering due to Hirzebruch, and $C: Z \to \mathbf{BP}^2$ denote the above constructed branched covering. Take X to be $C^*(H(\Lambda, n)) = H(\Lambda, n) \times_{\mathbf{BP}^2} Z = \{(r, q) \in H(\Lambda, n) \times Z \,|\, c(r) = C(q)\}$. Then for $i = 1, \ldots, 4$, transversality of L_i' to the branch locus of c in $\mathbf{BP}^2$ (i.e. to $\Lambda \cup$ exceptional curves) gives that X is smooth. Furthermore first factor projection $\pi_1: X \to H(\Lambda, n)$ exhibits X as a branched covering of $H(\Lambda, n)$ with ramification set $\bigcup B_j$ where the union ranges over B_j in $(c \circ \pi_1)^{-1}(\bigcup_{i=1}^{4} L_i')$. This implies that $B_j \cdot B_j < 0$ since $L_i' \cdot L_i' = -2$. By Theorem 1.1, T^*X is ample. Note that $c \circ \pi_1: X \to \mathbf{BP}^2$ cannot be an $f: H(\Lambda^*, n^*) \to \mathbf{BP}^2$ for any Λ^* or any n^*, since the branching order in $H(\Lambda^*, n^*)$ is always n^*, while we have branching orders of 2 along the B_j and $n \geq 4$ along $(c \circ \pi_1)^{-1}(\Lambda' \cup$ exceptional curves$)$.

From formulas (3.4) in Example 3.2 we have the Chern numbers $c_2(X)$ and $c_1^2(X)$ for the double cover X of $H(\Lambda, n)$ where Λ is the a-fold cone on the $q = 6$ points $p_1, \ldots, p_6$. For $a \neq 6$ these are given by:

$$(3.2) \quad c_2(X) = 2n^{6a-1}(15a^2 - 10a - 1)$$
$$- 4n^{6a-2}(15a^2 - 7a - 3) + 6n^{6a-3}(5a^2 - a),$$

$$c_1^2(X) = 2n^{6a-1}(30a^2 - 16a - 5)$$
$$- 8n^{6a-2}(15a^2 - 7a - 3) + 2n^{6a-3}(30a^2 - 13a - 6).$$

In particular for the 3-fold cone on the $q = 6$ points we have:

$$c_2(X) = n^{15}(208n^2 - 444n + 252),$$

$$c_1^2(X) = n^{15}(434n^2 - 888n + 450).$$

When $n = 5$ this reduces to $c_2(X) = 5^{15}(3232) = 5^{15}2^5101$ and $c_1^2(x) = 5^{15}(6860) = 5^{16}2^27^3$. We show that when $n = 5$, $a = 3$, and $q = 6$ (for our 6 points $p_1, \ldots, p_6$) that the double cover X cannot be any one of Hirzebruch's surfaces $H(\Lambda', m)$ (for any m and any line arrangement Λ'). If X were an $H(\Lambda', m)$ then by (3.1)

$$c_1^2(X) = m^{k-3}[m^2(-5k + 9 + 3f_1 - 4f_0)$$
$$+ 4m(k + f_0 - f_1) + f_1 - f_0 + k + t_2]$$

$$c_2(X) = m^{k-3}[m^2(3 - 2k + f_1 - f_0) + 2m(k + f_0 - f_1) + f_1 - t_2]$$

giving that m^{k-3} divides both $c_2(X) = 5^{15}2^5101$ and $c_1^2(X) = 5^{16}2^27^3$. This implies that either $k = 3$ and m is arbitrary, or $k > 3$ and $m = 5^b2^c$ for some b, c (where the restrictions $0 \le b(k-3) \le 15$ and $0 \le c(k-3) \le 2$ must hold). The case $k = 3$ and m arbitrary is easily ruled out as $k = 3$ lines do not produce the appropriate Chern numbers. In the case that $k > 3$ we have that k, b, and c are bounded, and a computer search on the possible cases for k, b, and c on formal line arrangements rules out any case with

$$\frac{c_1^2}{c_2} = \frac{5^{16}2^27^3}{5^{15}2^5101} = 2.122\overline{5247}.$$

A formal line arrangement is taken here to mean a tuple of non-negative integers $(t_2, t_3, \ldots, t_k)$ where

$$\binom{k}{2} = \sum_{r=2}^{k} t_r \binom{r}{2}$$

must hold. Thus we see that in the case that $n = 5$, $a = 3$, and $q = 6$, we have a surface with ample cotangent bundle which cannot be one of Hirzebruch's surfaces $H(\Lambda', m)$.

By computing values for (3.2) and letting $n \ge 4$ and $a \ge 3$ one sees that the various Chern ratios for the double covers (of the $H(\Lambda, n)$ for Λ the a-fold cone over 6 points) range from a high of $c_1^2/c_2 \approx 2.12971$ (when $n = 4$ and $a = 3$) down through $c_1^2/c_2 \approx 2$ asymptotically.

EXAMPLE 3.2. We can generalize the above construction in the following way. Replace the lines L_i, $i = 1, \ldots, 4$ (which we interpet as two "pencils" of two lines each) by $p \geq 2$ "pencils" P_i (for $i = 1, \ldots, p$) in general position, with each "pencil" P_i consisting of an even number, say $2k_i$, of lines through a fixed point p_i. Choose a line arrangement Λ as before satisfying the analogues of (1)-(4) in Example 3.1: each line in a "pencil" is not a line in Λ, if any two lines of the "pencils" intersect at a point p then $r_p \geq 3$ in Λ, away from the intersection points of the lines in the "pencils" at most one line of Λ passes through a point of the "pencils" (and it does so transversely), each line in Λ contains at least 2 points p with $r_p \geq 3$. Again, a generic a-fold cone on the q points of intersection of the lines in the "pencils" will provide an explicit example of such a Λ.

Given such a Λ, let $Y = H(\Lambda, n)$ for $n \geq 4$ and let Z be the double cover of $\mathbf{BP}^2$ corresponding to $\sum L'_j = (\sum_i 2k_i)H - (\sum n_\alpha E_\alpha)$. Here the L'_j are the proper transforms of all the lines in the p "pencils", H is the pullback of the hyperplane to $\mathbf{BP}^2$, and the E_α are the exceptional curves over the intersections of the lines in the "pencils" and the n_α denote the multiplicity of the αth intersection. Since each n_α is even (in fact $n_\alpha = 2$ or $2k_i$) $\sum L'_j$ has a square root, yielding Z a degree two branched cover of $\mathbf{BP}^2$ as in Example 3.1. For $n \geq 4$, take $X = H(\Lambda, n) \times_{\mathbf{BP}^2} Z$, which branch covers $H(\Lambda, n)$, with the branching of $\pi_1 : X \to H(\Lambda, n)$ occurring on $(c \circ \pi_1)^{-1}(\bigcup L'_j)$. Since $L'_j \cdot L'_j < 0$, T^*X is ample by Theorem 1.1.

We begin computation of the Chern numbers of the surface X which double covers $H(\Lambda, n)$ where Λ is a generic a-fold cone on the

$$q = p + \binom{\sum_{i=1}^p 2k_i}{2} - \sum_{i=1}^p \binom{2k_i}{2}$$

intersection points of the p "pencils". To be more explicit we rename $H(\Lambda, n)$ to be $H(\Lambda, a, q, n)$, the Hirzebruch line-arrangement surface constructed with the line-arrangement Λ consisting of an a-fold cone on q points, $a \neq q$, with branching order n. There are $k = aq$ lines in Λ. Let L_{ij} be the jth line in the ith pencil and let B_{ij} be the preimage in $H(\Lambda, a, q, n)$ of the proper transform L'_{ij} of L_{ij}. Then using the Hurwitz formula one has the euler number $e(B_{ij})$ in

$H(\Lambda, a, q, n)$ given by

$$e(B_{ij}) = n^{aq-1}(2) - (n^{aq-1} - n^{aq-2})\left[aq + \left(1 + \sum_{l \neq i} 2k_l\right)(1 - a)\right]$$

for each $j = 1, \ldots, 2k_i$. Since $(L'_{ij})^2 = -2(\sum_{l \neq i} k_l)$ we have that $B^2_{ij} = -2(\sum_{l \neq i} k_l)n^{aq-1}$ for $j = 1, \ldots, 2k_i$.

For any line-arrangement Λ which is an a-fold cone on q points (where $a \neq q$), we have that $k = aq$, $t_q = a$, $t_a = q$, and $t_2 = \binom{k}{2} - \sum_{r=3}^{k} t_r\binom{r}{2}$, from which we have that $f_0 = t_2 + t_a + t_q = \binom{aq}{2} - q\binom{a}{2} - a\binom{q}{2} + q + a$ while $f_1 = 2t_2 + at_a + qt_q = 2(\binom{aq}{2} - q\binom{a}{2} - a\binom{q}{2}) + aq + qa$. Using (3.1) one computes that:

(3.3) $c_2(H(\Lambda, a, q, n))$

$$= n^{aq-3}\{n^2(1/2)[a^2q(q-1) - a(q^2 - q + 2) - 2q + 6]$$

$$- n[a^2q(q-1) - a(q^2 - 3q + 2) - 2q]$$

$$+ (1/2)[aq(a(q-1) - q + 5)]\}$$

$c_1^2(H(\Lambda, a, q, n))$

$$= n^{aq-3}\{n^2[a^2q(q-1) - a(q^2 - 2q + 4) - 4q + 9]$$

$$- 2n[a^2q(q-1) - a(q^2 - 3q + 2) - 2q]$$

$$+ [a^2q(q-1) - a(q^2 - 4q + 1) - q]\}.$$

These allow computation of the Chern numbers of the double cover X of $H(\Lambda, a, q, n)$ via the fact that $c_2(X) = 2c_2(H(\Lambda, a, q, n)) - (2-1)\sum_{i=1}^{p}\sum_{j=1}^{2k_i} e(B_{ij})$ and (letting $\pi_1: X \rightarrow H(\Lambda, a, q, n)$ be our branched covering and K be the canonical bundle of $H(\Lambda, a, q, n)$)

$$c_1^2(X) = \pi_1^*\left(K + \sum_{i=1}^{p}\sum_{j=1}^{2k_i}(1 - (1/2))B_{ij}\right)^2$$

$$= 2\left(K^2 + \sum_{i=1}^{p}\sum_{j=1}^{2k_i} K \cdot B_{ij} + (1/4)\sum_{i=1}^{p}\sum_{j=1}^{2k_i} B_{ij}^2\right).$$

Using $K \cdot B_{ij} = -e(B_{ij}) - B_{ij}^2$ we obtain

(3.4)

$$c_2(X) = 2c_2(H(\Lambda, a, q, n))$$

$$- \sum_{i=1}^{p} 2k_i \left\{ n^{aq-1}(2) - (n^{aq-1} - n^{aq-2}) \left[aq + \left(1 + \sum_{l \neq i} 2k_l\right)(1-a) \right] \right\}$$

$$= n^{aq-1} \left[a^2 q(q-1) - a(q^2 - q + 2) - 2q + 6 - 4\sum_{i=1}^{p} k_i \right.$$

$$\left. + \sum_{i=1}^{p} 2k_i \left[aq + \left(1 + \sum_{l \neq i} 2k_l\right)(1-a) \right] \right]$$

$$+ n^{aq-2} \left[-2a^2 q(q-1) + 2a(q^2 - 3q + 2) \right.$$

$$\left. + 4q - \sum_{i=1}^{p} 2k_i \left[aq + \left(1 + \sum_{l \neq i} 2k_l\right)(1-a) \right] \right]$$

$$+ n^{aq-3}[aq(a(q-1) - q + 5)],$$

$$c_1^2(X) = 2 \left\{ c_1^2(H(\Lambda, a, q, n)) \right.$$

$$+ n^{aq-1} \left[-4\left(\sum_{i=1}^{p} k_i\right) \right.$$

$$\left. + 2\left(\sum_{i=1}^{p} k_i \left[aq + \left(1 + \sum_{l \neq i} 2k_l\right)(1-a) \right]\right) + 3\sum_{i=1}^{p}\sum_{l \neq i} k_i k_l \right]$$

$$\left. + n^{aq-2} \left[-2\left(\sum_{i=1}^{p} k_i \left[aq + \left(1 + \sum_{l \neq i} 2k_l\right)(1-a) \right]\right) \right] \right\}$$

$$= n^{aq-1} 2 \left[a^2 q(q-1) - a(q^2 - 2q + 4) - 4q + 9 - 4\left(\sum_{i=1}^{p} k_i\right) \right.$$

$$\left. + 2\left(\sum_{i=1}^{p} k_i \left[aq + \left(1 + \sum_{l \neq i} 2k_l\right)(1-a) \right]\right) + 3\sum_{i=1}^{p}\sum_{l \neq i} k_i k_i \right]$$

$$+ n^{aq-2} 2 \left[-2a^2 q(q-1) + 2a(q^2 - 3q + 2) + 4q \right.$$

$$\left. - 2\left(\sum_{i=1}^{p} k_i \left[aq + \left(1 + \sum_{l \neq i} 2k_l\right)(1-a) \right]\right) \right]$$

$$+ n^{aq-3} 2[a^2 q(q-1) - a(q^2 - 4q + 1) - q]$$

where

$$q = p + \binom{\sum_{i=1}^{p} 2k_i}{2} - \sum_{i=1}^{p} \binom{2k_i}{2} \text{ and } a \neq q.$$

In particular, if we let $p = 2$, $k_1 = k_2 = 1$, and $q = 6$ then we obtain (3.2) in Example 3.1.

EXAMPLE 3.3. One can make similar constructions by taking higher degree curves as well. Examples include:

(a) A "pencil" of $2k$ lines $L_1, \ldots, L_{2k}$ (through a common point p_1) and a smooth curve C (in general position with respect to the "pencil"), along with a line arrangement Λ, where:

(i) each line L_i in the "pencil" is not in Λ.

(ii) each point of intersection, p, of the $2k$ lines or of the lines and curve, satisfies $r_p \geq 3$ in Λ.

(iii) away from $\{p_1\} \cup \{L_1 \cap C\} \cup \{L_2 \cap C\} \cup \cdots \cup \{L_{2k} \cap C\}$ the lines in Λ meet $L_1 \cup \cdots \cup L_{2k} \cup C$ transversely at points p with $r_p = 1$.

(iv) For each $L \in \Lambda$ at least two points p of L satisfy $r_p \geq 3$.

(v) C is of even degree $2d$ in $|H^{2d}|$ in $\mathbf{P}^2$ where $d < k$.

A generic a-fold cone on $\{p_1\} \cup \{L_1 \cap C\} \cup \{L_2 \cap C\} \cup \cdots \cup \{L_{2k} \cap C\}$ gives an explicit example of such Λ for $a \geq 3$. Then $C' + \sum_{j=1}^{2k} L'_j = 2((k+d)H - kE_1 - \sum_{j>1} E_j)$ where E_1 is the exceptional curve over p_1 and the E_j in the second summation are the exceptional curves over $\{L_1 \cap C\} \cup \{L_2 \cap C\} \cup \cdots \cup \{L_{2k} \cap C\}$. Hence $C' + \sum_{j=1}^{2k} L'_j$ has a square root. Construct the corresponding branched cover Z of $\mathbf{BP}^2$ of degree 2, along with $X = H(\Lambda, n) \times_{\mathbf{BP}^2} Z$ where X branch covers $H(\Lambda, n)$ and $n \geq 4$. Then $d < k$ gives $(C')^2 < 0$, and T^*X is ample by Theorem 1.1.

The Chern numbers of X, the double cover of Hirzebruch's surface $H(\Lambda, a, q, n)$ associated to the a-fold cone on the q points of intersection of the "pencil" and curve, are given by:

$$
\begin{aligned}
c_2(X) = {} & 2(H(\Lambda, a, q, n)) \\
& + n^{aq-1}[4d^2 - 6d + 8kd - 2k] + n^{aq-2}[-2k - 8kd] \\
= {} & n^{aq-1}[a^2q(q-1) - a(q^2 - q + 2) \\
& \qquad - 2q + 6 + 4d^2 - 6d + 8kd - 2k] \\
& + n^{aq-2}[-2a^2q(q-1) + 2a(q^2 - 3q + 2) + 4q - 2k - 8kd] \\
& + n^{aq-3}[aq(a(q-1) - q + 5)],
\end{aligned}
$$

$$
\begin{aligned}
c_1^2(X) = {} & 2\{c_1^2(H(\Lambda, a, q, n)) \\
& + n^{aq-1}[d^2 - 6d + 14kd - 2k] + n^{aq-2}[-2k - 8kd]\} \\
= {} & n^{aq-1}2[a^2q(q-1) - a(q^2 - 2q + 4) \\
& \qquad - 4q + 9 + d^2 - 6d + 14kd - 2k] \\
& + n^{aq-2}2[-2a^2q(q-1) + 2a(q^2 - 3q + 2) + 4q - 2k - 8kd] \\
& + n^{aq-3}2[a^2q(q-1) - a(q^2 - 4q + 1) - q]
\end{aligned}
$$

where $q = 1 + 4kd$, $a \neq q$, and $d < k$.

(b) A "pencil" of $2k$ lines $L_1, \ldots, L_{2k}$ (through a common point) and a "pencil" of $2m$ smooth curves $C_1, \ldots, C_{2m}$ of degree d in $|H^d|$ in $\mathbf{P}^2$ (having base locus d^2 points) in general position with respect to the "pencil", where each point of intersection, p, (of the $2k$ lines, of the lines and curve, or of the curves) satisfies $r_p \geq 3$ in a line-arrangement Λ, and the analogues of (i) through (iv) in (a) are also satisfied. $\sum C_i' + \sum L_j'$ has a square root. Construct the corresponding branched cover Z of $\mathbf{BP}^2$ of degree 2, along with $X = H(\Lambda, n) \times_{\mathbf{BP}^2} Z$ where X branch covers $H(\Lambda, n)$ and $n \geq 4$. Then T^*X is ample by Theorem 1.1.

The Chern numbers for X, the double cover of Hirzebruch's surface $H(\Lambda, a, q, n)$ associated to the a-fold cone on the q points of intersection of the $2k$ lines and $2m$ curves, are:

$$
\begin{aligned}
c_2(X) &= 2(H(\Lambda, a, q, n)) + n^{aq-1}[-2k - 6md + 8kmd + 4md^2] \\
&\quad + n^{aq-2}[-2k - 8kmd - 2md^2] \\
&= n^{aq-1}[a^2 q(q-1) - a(q^2 - q + 2) \\
&\qquad - 2q + 6 - 2k - 6md + 8kmd + 4md^2] \\
&\quad + n^{aq-2}[-2a^2 q(q-1) + 2a(q^2 - 3q + 2) \\
&\qquad\qquad + 4q - 2k - 8kmd - 2md^2] \\
&\quad + n^{aq-3}[aq(a(q-1) - q + 5)] \\
c_1^2(X) &= 2\{c_1^2(H(\Lambda, a, q, n)) + n^{aq-1}[-2k - 6md + 14kmd + 4md^2] \\
&\qquad\qquad\qquad + n^{aq-2}[-2k - 8kmd - 2md^2]\} \\
&= n^{aq-1} 2[a^2 q(q-1) - a(q^2 - 2q + 4) - 4q \\
&\qquad + 9 - 2k - 6md + 14kmd + 4md^2] \\
&\quad + n^{aq-2} 2[-2a^2 q(q-1) + 2a(q^2 - 3q + 2) \\
&\qquad\qquad + 4q - 2k - 8kmd - 2md^2] \\
&\quad + n^{aq-3} 2[a^2 q(q-1) - a(q^2 - 4q + 1) - q]
\end{aligned}
$$

where $q = 1 + d^2 + 4kmd$ and $a \neq q$.

(c) Two or more "pencils" each consisting of an even number of curves will also yield similar examples.

EXAMPLE 3.4. In [8], Sommese constructs branched coverings X of $H(\Lambda, 5)$ where Λ is the $A_1(6)$ arrangement of lines. He uses

the coverings to show that the possible Chern ratios c_1^2/c_2 between 2 and 3 are assumed. However these coverings do not have ample cotangent bundle. This follows from Theorem 1.1 after one observes that: $H(A_1(6), 5)$ fibers over a Riemann surface, the branch locus of each of Sommese's coverings in $H(A_1(6), 5)$ consists of smooth fibers with self-intersection 0, and the ramification set in each covering X also consists of smooth curves with self-intersection 0.

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