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UNIT INDICES OF SOME IMAGINARY COMPOSITE QUADRATIC FIELDS

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Let K be an imaginary abelian number field of type $(2, 2, 2, 2)$ not containing the 8th cyclotomic field. Using the fundamental units of real quadratic subfields of K, we give a necessary and sufficient condition for the unit index Q_K of K to be equal to 2.

1. Introduction and results. Let K be an imaginary abelian number field and K_0 the maximal real subfield of K. Let E and E_0 be the groups of units of K and K_0, respectively, and let W be the group of roots of unity in K. Then we call the group index

$$Q_K = [E : WE_0]$$

the unit index of K.

Using the character group of K, H. Hasse [**2**] gave sufficient conditions for Q_K to be equal to 1 or 2, by which we can determine Q_K for some types of fields K. However by his method we cannot always determine Q_K for arbitrary K, even if K is an imaginary composite quadratic field. (We call a field K a composite quadratic field if K is a composite of quadratic fields.) K. Yoshino and the author [**3**, **4**] gave criteria to determine Q_K of K with Galois group $\mathrm{Gal}(K/\mathbf{Q})$ of type $(2, 2)$ and $(2, 2, 2)$.

In this paper we extend our previous results [**3**, **4**] to the case that K has Galois group $\mathrm{Gal}(K/\mathbf{Q})$ of type $(2, 2, 2, 2)$ and does not contain the 8th cyclotomic field, and then, we give a necessary and sufficient condition for the unit index Q_K to be equal to 2.

NOTATION. $\mathbf{N}, \mathbf{Z}, \mathbf{Q}$: the sets of natural numbers, rational integers and rational numbers, respectively,

$\underset{2}{=}$: the equality except rational quadratic factors,

$d_0, d_1, d_2, \dots, d_7$: square-free positive integers such that $d_4 \underset{2}{=} d_2 d_3$, $d_5 \underset{2}{=} d_3 d_1$, $d_6 \underset{2}{=} d_1 d_2$, $d_7 \underset{2}{=} d_1 d_2 d_3$ and $d_0 \neq d_i$ $(i = 1, 2, \dots, 7)$,

$K = \mathbf{Q}(\sqrt{-d_0}, \sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$: an imaginary composite quadratic field of degree 16,

$K_0 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$,
E_0^+: the group of totally positive units of K_0,
$\overline{E}_0$: the group of units η of E_0^+ such that $K_0(\sqrt{\eta})$ is a composite quadratic field,

$$\begin{aligned} &K_1 = \mathbf{Q}(\sqrt{d_2}, \sqrt{d_3}), \quad K_2 = \mathbf{Q}(\sqrt{d_3}, \sqrt{d_1}), \\ &K_3 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2}), \quad K_4 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2 d_3}), \\ &K_5 = \mathbf{Q}(\sqrt{d_2}, \sqrt{d_3 d_1}), \quad K_6 = \mathbf{Q}(\sqrt{d_3}, \sqrt{d_1 d_2}), \\ &K_7 = \mathbf{Q}(\sqrt{d_2 d_3}, \sqrt{d_3 d_1}), \\ &k_i = \mathbf{Q}(\sqrt{d_i}) \ (i = 1, 2, \dots, 7), \\ &\langle \sigma_i \rangle = \mathrm{Gal}(K_0/K_i) \ (i = 1, 2, \dots, 7), \end{aligned}$$

$N(x)$, $\mathrm{Sp}(x)$: the absolute norm and the absolute trace of x, respectively,

$$A = A(e_1, e_2, e_3) = \begin{cases} 2d_1^{e_1} d_2^{e_2} d_3^{e_3} & \text{if } d_0 = 1, \\ d_0 d_1^{e_1} d_2^{e_2} d_3^{e_3} & \text{otherwise}, \end{cases}$$

ε_i: the fundamental unit of $\mathbf{Q}(\sqrt{d_i})$, $\varepsilon_i > 1$ $(i = 1, 2, \dots, 7)$.

When $N(\varepsilon_i) = +1$, we denote by Δ_i, Δ_i^* the square-free parts of $\mathrm{Sp}(\varepsilon_i + 1)$, $\mathrm{Sp}(\varepsilon_i - 1)$, respectively, and by m_i, n_i the natural numbers such that $\mathrm{Sp}(\varepsilon_i + 1) = \Delta_i m_i^2$, $\mathrm{Sp}(\varepsilon_i - 1) = \Delta_i^* n_i^2$. Then we have

$$\sqrt{\varepsilon_i} = \frac{1}{2}(m_i\sqrt{\Delta_i} + n_i\sqrt{\Delta_i^*}). \tag{1}$$

When $d_i d_j \underset{2}{=} d_k$ with $N(\varepsilon_i) = N(\varepsilon_j) = N(\varepsilon_k) = -1$, we denote by $\Delta_{ij} = \Delta_{ji}$ the square-free integer such that

$$\Delta_{ij} \underset{2}{=} \mathrm{Sp}_{\mathbf{Q}(\sqrt{d_i}, \sqrt{d_j})/\mathbf{Q}}(\varepsilon_i\varepsilon_j\varepsilon_k - \varepsilon_i - \varepsilon_j - \varepsilon_k).$$

(We take $(i, j) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 5), (3, 6)$ and $(4, 5)$.)

When $d_i d_j d_k \underset{2}{=} d_l$ with $N(\varepsilon_i) = N(\varepsilon_j) = N(\varepsilon_k) = N(\varepsilon_l) = -1$ and when $\mathbf{Q}(\sqrt{d_i}, \sqrt{d_j}, \sqrt{d_k}) = K_0$, we denote by Δ_{ijk} the square-free integer such that

$$\Delta_{ijk} \underset{2}{=} \mathrm{Sp}_{K_0/\mathbf{Q}}\left(\varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_l + 1 - \sum_{\alpha<\beta} \varepsilon_\alpha\varepsilon_\beta\right)$$

where α, β run through i, j, k and l.

For a totally positive unit η of K_0 let

(2) $\xi^*(\eta) = \eta + \eta^{\sigma_1} + 2(-1)^{s_1}\sqrt{\eta\eta^{\sigma_1}}$,

(3) $\theta^*(\eta) = \xi^*(\eta) + \xi^*(\eta)^{\sigma_2} + 2(-1)^{s_2}\sqrt{\xi^*(\eta)\xi^*(\eta)^{\sigma_2}}$,

(4) $d^*(\eta) = \theta^*(\eta) + \theta^*(\eta)^{\sigma_3} + 2(-1)^{s_3}\sqrt{\theta^*(\eta)\theta^*(\eta)^{\sigma_3}}$ $(s_i = 0$ or $1)$

under the condition that

(5) $\sqrt{\eta\eta^{\sigma_1}} \in K_1$, $\sqrt{\xi^*(\eta)\xi^*(\eta)^{\sigma_2}} \in k_3$ and $\sqrt{\theta^*(\eta)\theta^*(\eta)^{\sigma_3}} \in \mathbf{Q}$.

We remark that for a totally positive unit η of K_0 this condition (5) is satisfied if and only if η is contained in $\overline{E}_0$. This remark can be proved by Lemmas 4 and 5 (cf. proof of Theorem 4).

Throughout this paper we assume that K does not contain the 8th cyclotomic field $\mathbf{Q}(\sqrt{-1}, \sqrt{2})$. Our result is the following

Main Theorem. *Under the above notation and assumption we have that* $Q_K = 2$ *if and only if*

$$\prod_i \Delta_i^{a_i} \cdot \prod_{i,j} \Delta_{ij}^{b_{ij}} \cdot \prod_{i,j,k} \Delta_{ijk}^{c_{ijk}} \cdot d^*(\eta_0)^f \underset{2}{=} A(e_1, e_2, e_3)$$

for some $a_i, b_{ij}, c_{ijk}, f, e_i = 0, 1$ *and* $\eta_0 \in \overline{E}_0$ *represented in the form*

$$\eta_0 = \sqrt{\prod_{N(\varepsilon_i)=+1} \varepsilon_i^{u_i} \cdot \prod_{N(\varepsilon_i)=-1} \varepsilon_i^{v_i}},$$

where $u_i, v_i = 0$ *or* 1. *The number of* i*'s for which* $u_i = 1$ *is neither* 1 *nor* 2.

More precisely we have the following Theorems 1–6.

Theorem 1. *In the case that* $N(\varepsilon_1) = N(\varepsilon_2) = \cdots = N(\varepsilon_7) = -1$, *we have*

$$Q_K = 2 \Leftrightarrow \Delta_{12}^{b_1}\Delta_{23}^{b_2}\Delta_{31}^{b_3}\Delta_{123}^{c} \underset{2}{=} A(e_1, e_2, e_3)$$

for some $b_i, c, e_i = 0, 1$. *Especially, if* $\sqrt{\Delta_{ij}}$ *is contained in* $\mathbf{Q}(\sqrt{d_i}, \sqrt{d_j})$ *for every* (i, j), *then* $Q_K = 1$.

Theorem 2. *In the case that* $N(\varepsilon_1) = N(\varepsilon_2) = \cdots = N(\varepsilon_6) = -1$ *and* $N(\varepsilon_7) = +1$, *we have*

$$Q_K = 2 \Leftrightarrow \Delta_7^{a}\Delta_{12}^{b_1}\Delta_{23}^{b_2}\Delta_{31}^{b_3} \underset{2}{=} A(e_1, e_2, e_3)$$

for some $a, b_i, e_i = 0, 1$.

THEOREM 3. *In the case that* $N(\varepsilon_1) = N(\varepsilon_2) = \cdots = N(\varepsilon_5) = -1$ *and* $N(\varepsilon_6) = N(\varepsilon_7) = +1$, *we have*

$$Q_K = 2 \Leftrightarrow \Delta_6^{a_6}\Delta_7^{a_7}\Delta_{23}^{b_2}\Delta_{31}^{b_3} \underset{2}{=} A(e_1, e_2, e_3)$$

for some $a_i, b_i, e_i = 0, 1$.

THEOREM 4. (1) *In the case that* $N(\varepsilon_1) = \cdots = N(\varepsilon_4) = -1$ *and* $N(\varepsilon_5) = N(\varepsilon_6) = N(\varepsilon_7) = +1$, *we have*

$$Q_K = 2 \Leftrightarrow \Delta_5^{a_5}\Delta_6^{a_6}\Delta_7^{a_7}\Delta_{23}^{b}\, d^*(\eta_0)^f \underset{2}{=} A(e_1, e_2, e_3)$$

for some $a_i, b, f, e_i = 0, 1$ *and* $\eta_0 \in \overline{E}_0$ *such that*

$$\eta_0 = \sqrt{\varepsilon_5\varepsilon_6\varepsilon_7}\prod_{i=1}^{4}\varepsilon_i^{v_i} \qquad (v_i = 0 \text{ or } 1).$$

(2) *In the case that* $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = N(\varepsilon_7) = -1$ *and* $N(\varepsilon_4) = N(\varepsilon_5) = N(\varepsilon_6) = +1$, *we have*

$$Q_K = 2 \Leftrightarrow \Delta_4^{a_4}\Delta_5^{a_5}\Delta_6^{a_6}\Delta_{123}^{c} \underset{2}{=} A(e_1, e_2, e_3)$$

for some $a_i, c, e_i = 0, 1$.

THEOREM 5. (1) *In the case that* $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = -1$ *and* $N(\varepsilon_4) = N(\varepsilon_5) = N(\varepsilon_6) = N(\varepsilon_7) = +1$, *we have*

$$Q_K = 2 \Leftrightarrow \prod_{i=4}^{7}\Delta_i^{a_i}\cdot d^*(\eta_0)^f \underset{2}{=} A(e_1, e_2, e_3)$$

for some $a_i, f, e_i = 0, 1$ *and* $\eta_0 \in \overline{E}_0$ *such that*

$$\frac{\eta_0}{\prod_{i=1}^{3}\varepsilon_i^{v_i}} = \sqrt{\varepsilon_4\varepsilon_5\varepsilon_7}, \sqrt{\varepsilon_5\varepsilon_6\varepsilon_7} \text{ or } \sqrt{\varepsilon_6\varepsilon_4\varepsilon_7} \qquad (v_i = 0 \text{ or } 1).$$

(2) *In the case that* $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_6) = -1$ *and the others* $N(\varepsilon_i) = +1$, *we have*

$$Q_K = 2 \Leftrightarrow \prod_{N(\varepsilon_i)=+1}\Delta_i^{a_i}\cdot\Delta_{12}^{b}\cdot d^*(\eta_0)^f \underset{2}{=} A(e_1, e_2, e_3)$$

for some $a_i, b, f, e_i = 0, 1$ *and* $\eta_0 \in \overline{E}_0$ *such that*

$$\frac{\eta_0}{\prod_{N(\varepsilon_i)=-1}\varepsilon_i^{v_i}} = \sqrt{\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_7}, \sqrt{\varepsilon_3\varepsilon_4\varepsilon_5}, \sqrt{\varepsilon_3\varepsilon_4\varepsilon_7},$$

$$\sqrt{\varepsilon_3\varepsilon_5\varepsilon_7} \text{ or } \sqrt{\varepsilon_4\varepsilon_5\varepsilon_7} \qquad (v_i = 0 \text{ or } 1).$$

THEOREM 6. *In the case that* $N(\varepsilon_3) = N(\varepsilon_4) = \cdots = N(\varepsilon_7) = +1$, *we have*

$$Q_K = 2 \Leftrightarrow \prod_{N(\varepsilon_i)=+1} \Delta_i^{a_i} \cdot d^*(\eta_0)^f \underset{2}{=} A(e_1, e_2, e_3)$$

for some $a_i, f, e_i = 0, 1$ *and* $\eta_0 \in \overline{E}_0$ *such that*

$$\frac{\eta_0}{\sqrt{\prod_{N(\varepsilon_i)=+1} \varepsilon_i^{u_i}}} = \varepsilon_1^{v_1}\varepsilon_2^{v_2}, \ \varepsilon_1^{v_1} \text{ or } 1 \qquad (u_i, v_i = 0 \text{ or } 1)$$

according as $N(\varepsilon_1) = N(\varepsilon_2) = -1$; $N(\varepsilon_1) = -1$ *and* $N(\varepsilon_2) = +1$; *or* $N(\varepsilon_1) = N(\varepsilon_2) = +1$. *The number of* i*'s for which* $u_i = 1$ *is neither* 1 *nor* 2.

REMARK 1. In Main Theorem η_0 is not represented in the form

$$\eta_0 = \sqrt{\varepsilon_i\varepsilon_j\varepsilon_k} \cdot \prod_{N(\varepsilon_l)=-1} \varepsilon_l^{v_l}$$

where $N(\varepsilon_i) = N(\varepsilon_j) = N(\varepsilon_k) = +1$ and $d_i d_j \underset{2}{=} d_k$ (cf. proof of Case (2) of Theorem 4).

REMARK 2. For some $\eta_0 \in \overline{E}_0$ we can actually calculate the rational integers $d^*(\eta_0)$ defined by (4). For example, we can obtain the following: *Suppose that* $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = +1$ *and that* $\eta_0 = \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}$ *is totally positive. Then* $\eta_0 \in \overline{E}_0$ *if and only if*

$$\Delta_1 \underset{2}{=} d_2 d_3, \quad \Delta_2 \underset{2}{=} d_3 d_1, \quad \Delta_3 \underset{2}{=} d_1 d_2. \tag{6}$$

If this condition (6) *is satisfied, we have*

$$\begin{aligned} d^*(\eta_0) = {} & m_1 m_2 m_3 \sqrt{\Delta_1\Delta_2\Delta_3} \\ & + 2\Delta_1^*\{(-1)^{s_1} n_2 n_3 + (-1)^{s_2} n_3 n_1 + (-1)^{s_3} n_1 n_2\} \\ & - 8(-1)^{s_1+s_2+s_3} \quad (s_i = 0 \text{ or } 1) \end{aligned}$$

where Δ_i, Δ_i^*, m_i, n_i *and* s_i *are as in the notation.*

2. Properties of $\overline{E}_0$ and lemmas on (2, 2)-extensions. In this section we give a proposition and some lemmas which will be used in the proofs of theorems.

Let $\langle x, y, \ldots \rangle$ be a group generated by $x, y, \ldots$. Let E_0^* be the subgroup of E_0 generated by the units of $\mathbf{Q}(\sqrt{d_i})$ for $i = 1, 2, \ldots, 7$. Let $(E_0^*)^+$ be the subgroup of E_0 generated by totally positive units of E_0^*, i.e., $(E_0^*)^+ = E_0^* \cap E_0^+$.

PROPOSITION 1. (1) *If* $N(\varepsilon_1) = \cdots = N(\varepsilon_7) = -1$, *then*

$$(E_0^*)^+ = \langle \varepsilon_2\varepsilon_3\varepsilon_4, \varepsilon_3\varepsilon_1\varepsilon_5, \varepsilon_1\varepsilon_2\varepsilon_6, \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_7 \rangle E_0^{*2}.$$

(2) *If* $N(\varepsilon_1) = \cdots = N(\varepsilon_6) = -1$ *and* $N(\varepsilon_7) = +1$, *then*

$$(E_0^*)^+ = \langle \varepsilon_2\varepsilon_3\varepsilon_4, \varepsilon_3\varepsilon_1\varepsilon_5, \varepsilon_1\varepsilon_2\varepsilon_6, \varepsilon_7 \rangle E_0^{*2}.$$

(3) *If* $N(\varepsilon_1) = \cdots = N(\varepsilon_5) = -1$ *and* $N(\varepsilon_6) = N(\varepsilon_7) = +1$, *then*

$$(E_0^*)^+ = \langle \varepsilon_2\varepsilon_3\varepsilon_4, \varepsilon_3\varepsilon_1\varepsilon_5, \varepsilon_6, \varepsilon_7 \rangle E_0^{*2}.$$

(4_1) *If* $N(\varepsilon_1) = \cdots = N(\varepsilon_4) = -1$ *and* $N(\varepsilon_5) = N(\varepsilon_6) = N(\varepsilon_7) = +1$, *then*

$$(E_0^*)^+ = \langle \varepsilon_2\varepsilon_3\varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7 \rangle E_0^{*2}.$$

(4_2) *If* $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = N(\varepsilon_7) = -1$ *and* $N(\varepsilon_4) = N(\varepsilon_5) = N(\varepsilon_6) = +1$, *then*

$$(E_0^*)^+ = \langle \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_7, \varepsilon_4, \varepsilon_5, \varepsilon_6 \rangle E_0^{*2}.$$

(5_1) *If* $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = -1$ *and* $N(\varepsilon_4) = \cdots = N(\varepsilon_7) = +1$, *then*

$$(E_0^*)^+ = \langle \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7 \rangle E_0^{*2}.$$

(5_2) *If* $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_6) = -1$ *and the others* $N(\varepsilon_i) = +1$, *then*

$$(E_0^*)^+ = \langle \varepsilon_1\varepsilon_2\varepsilon_6, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_7 \rangle E_0^{*2}.$$

(6) *If* $N(\varepsilon_1) = N(\varepsilon_2) = -1$ *and* $N(\varepsilon_3) = \cdots = N(\varepsilon_7) = +1$, *then*

$$(E_0^*)^+ = \langle \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7 \rangle E_0^{*2}.$$

(7) *If* $N(\varepsilon_1) = -1$ *and* $N(\varepsilon_2) = \cdots = N(\varepsilon_7) = +1$, *then*

$$(E_0^*)^+ = \langle \varepsilon_2, \varepsilon_3, \dots, \varepsilon_7 \rangle E_0^{*2}.$$

(8) *If* $N(\varepsilon_1) = \cdots = N(\varepsilon_7) + 1$,*then*

$$(E_0^*)^+ = \langle \varepsilon_1, \varepsilon_2, \dots, \varepsilon_7 \rangle E_0^{*2}.$$

Proof. We only prove the case (1), because the other cases are proved in the same way.

For an element $\alpha \neq 0$ of K we define $s(\alpha) = 0$ or 1 by $(-1)^{s(\alpha)} = \alpha/|\alpha|$.

For $\eta \in (E_0^*)^+$, putting $\eta = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7}$ $(x_i \in \mathbf{Z})$, we have a system of simultaneous linear equations

$$\begin{cases} s(\varepsilon_1)x_1 + s(\varepsilon_2)x_2 + \cdots + s(\varepsilon_7)x_7 \equiv 0 \\ s(\varepsilon_1^{\sigma_1})x_1 + s(\varepsilon_2^{\sigma_1})x_2 + \cdots + s(\varepsilon_7^{\sigma_1})x_7 \equiv 0 \\ \qquad \cdots \\ s(\varepsilon_1^{\sigma_7})x_1 + s(\varepsilon_2^{\sigma_7})x_2 + \cdots + s(\varepsilon_7^{\sigma_7})x_7 \equiv 0. \end{cases} \pmod 2$$

By Gauss-Jordan elimination (see, for example, H. Anton, *Elementary Linear Algebra*, John Wiley & Sons (1973), pp. 18–20) we see that this system has the following four linearly independent solutions:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

To these solutions correspond units $\varepsilon_2\varepsilon_3\varepsilon_4$, $\varepsilon_3\varepsilon_1\varepsilon_5$, $\varepsilon_1\varepsilon_2\varepsilon_6$, $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_7$ respectively. Thus we have

$$(E_0^*)^+ = \langle \varepsilon_2\varepsilon_3\varepsilon_4, \varepsilon_3\varepsilon_1\varepsilon_5, \varepsilon_1\varepsilon_2\varepsilon_6, \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_7 \rangle E_0^{*2}. \qquad \square$$

In general, let K/k be a $(2, 2)$-extension with Galois group $\mathrm{Gal}(K/k) = \langle \sigma, \tau \rangle$. Then, as used by H. Wada **[6]**, we have

$$\alpha^2 = \frac{\alpha^{1+\sigma}\alpha^{1+\tau}}{(\alpha^\sigma)^{1+\sigma\tau}}$$

for $\alpha \in K$, $\alpha \neq 0$. By this simple formula we see that $E_0^4 \subseteq E_0^*$. Moreover, we have $\overline{E}_0^2 \subseteq E_0^*$ by the following

LEMMA 1. *Let $\eta \in \overline{E}_0$ and put $\eta^4 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7}$ $(x_i \in \mathbf{Z})$. Then, every x_i is even.*

Proof. Since $K_0(\sqrt{\eta}) = K_0(\sqrt{d})$ for some $d \in \mathbf{N}$, we can put $\eta = d\alpha_0^2$ $(\alpha_0 \in K_0)$. Taking the norm N_{K_0/k_i} of $\varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7} = d^4\alpha_0^8$, we have $\varepsilon_i^{4x_i} = d^{16}N_{K_0/k_i}(\alpha_0)^8$. This implies that x_i is even. $\square$

LEMMA 2. *Let $\eta \in \overline{E}_0$ and put*

$$\eta^2 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7} \qquad (x_i \in \mathbf{Z}). \tag{7}$$

Then, all x_i are even or at least three x_i's are odd.

Proof. For the simplicity we denote by N_i the norm N_{K_0/K_i} for each i.

First, for example, we assume that $x_1 \equiv 1$, $x_i \equiv 0 \pmod 2$ $(i = 2, 3, \dots, 7)$. Taking the norm N_3 of the equation (7), we have $N_3(\eta) = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\varepsilon_6^{x_6} \in K_3$. On the other hand, putting $\eta = d\alpha_0^2$ $(d \in \mathbf{N}$,

$\alpha_0 \in K_0$), we have $N_3(\eta) = d^2 N_3(\alpha_0)^2$. Therefore, $\sqrt{\varepsilon_1}$ is contained in $K_3 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2})$. In the same way, taking the norm N_2 of (7), we see that $\sqrt{\varepsilon_1}$ is contained in $K_2 = \mathbf{Q}(\sqrt{d_3}, \sqrt{d_1})$. Thus $\sqrt{\varepsilon_1}$ is contained in $K_2 \cap K_3 = \mathbf{Q}(\sqrt{d_1})$, which is impossible.

Secondly, for example, we assume that $x_1 \equiv x_2 \equiv 1$, $x_i \equiv 0 \pmod 2$ ($i = 3, 4, \dots, 7$). Taking the norms N_2, N_4 of (7), we see that $\sqrt{\varepsilon_1}$ is contained in $\mathbf{Q}(\sqrt{d_1})$, which is also impossible.

Thus there is no case that exactly one or two of x_i are odd. □

LEMMA 3. *Let* $\eta \in \overline{E}_0$ *and put*

$$\eta^2 = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \cdots \varepsilon_7^{x_7} \qquad (x_i \in \mathbf{Z}). \tag{8}$$

(1) *If there exists an even* x_i, *then* $N(\varepsilon_j) = +1$ *for each odd* x_j.

(2) *If there exists "i" for which* $x_i \equiv 0 \pmod 2$ *or* $N(\varepsilon_i) = +1$, *then* x_j *is even when* $N(\varepsilon_j) = -1$.

(3) *If* $x_1 \equiv x_2 \equiv \cdots \equiv x_7 \equiv 1 \pmod 2$, *then* $N(\varepsilon_1) = N(\varepsilon_2) = \cdots = N(\varepsilon_7)$.

Proof. (1) Suppose that $x_1 \equiv 1$, $x_2 \equiv 0 \pmod 2$. Taking the norm N_3 of (8), we have $N_3(\eta) = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \varepsilon_6^{x_6}$. Again, taking the norms N_1, N_2 of this equation, we have by $\eta \gg 0$ that

$$N_1(N_3(\eta)) = N(\varepsilon_1)^{x_1} \varepsilon_2^{2x_2} N(\varepsilon_6)^{x_6} > 0,$$
$$N_2(N_3(\eta)) = \varepsilon_1^{2x_1} N(\varepsilon_2)^{x_2} N(\varepsilon_6)^{x_6} > 0.$$

Hence $N(\varepsilon_6)^{x_6} = +1$ and then $N(\varepsilon_1) = +1$.

(2) We suppose that $x_1 \equiv 0 \pmod 2$ or $N(\varepsilon_1) = +1$ and that $N(\varepsilon_2) = -1$.

Taking the norm N_3 of (8), we have $N_3(\eta) = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \varepsilon_6^{x_6}$. Again, taking the norm N_6 of this equation, we have

$$N_6(N_3(\eta)) = N(\varepsilon_1)^{x_1} N(\varepsilon_2)^{x_2} \varepsilon_6^{2x_6} > 0,$$

and so $x_2 \equiv 0 \pmod 2$.

(3) Taking the norm N_1 of (8), we have $N_1(\eta) = \varepsilon_2^{x_2} \varepsilon_3^{x_3} \varepsilon_4^{x_4}$. Moreover, taking the norms N_2, N_3 of this equation, we have

$$N_2(N_1(\eta)) = N(\varepsilon_2)^{x_2} \varepsilon_3^{2x_3} N(\varepsilon_4)^{x_4} > 0,$$
$$N_3(N_1(\eta)) = \varepsilon_2^{2x_2} N(\varepsilon_3)^{x_3} N(\varepsilon_4)^{x_4} > 0.$$

Then $N(\varepsilon_2) = N(\varepsilon_3) = N(\varepsilon_4)$.

In the same way, taking the norms N_2, N_3, N_6 of (8), we obtain $N(\varepsilon_3) = N(\varepsilon_1) = N(\varepsilon_5)$, $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_6)$, $N(\varepsilon_3) = N(\varepsilon_6) = N(\varepsilon_7)$. □

For a field k we denote by "$\underset{2}{=}$ in k" the equality except a square of a number of k.

LEMMA 4 (*F. Halter-Koch* [**1**, *Satz* 1]). *Let* K_1 *be a field with* $\overline{\text{char}}(K_1) \neq 2$. *Let* K_0 *be a quadratic extension of* K_1 *and* $K_0(\sqrt{\eta_0})$ $(\eta_0 \in K_0)$ *a biquadratic (quartic) extension of* K_1. *Then* $K_0(\sqrt{\eta_0})/K_1$ *is bicyclic if and only if* $N_{K_0/K_1}(\eta_0) \underset{2}{=} 1$ *in* K_1.

By this Lemma 4 we can easily obtain

LEMMA 5. *Let* K_1 *be an algebraic number field and* K_0 *a quadratic extension of* K_1. *Let* $K_0(\sqrt{\eta_0})$ $(\eta_0 \in K_0,\ \eta_0 \notin K_1)$ *be a biquadratic bicyclic extension of* K_1 *with* $\text{Gal}(K_0(\sqrt{\eta_0})/K_1) = \langle\sigma, \tau\rangle$ *and* $\text{Gal}(K_0(\sqrt{\eta_0})/K_0) = \langle\tau\rangle$. *Let* F *be the intermediate field of* $K_0(\sqrt{\eta_0})/K_1$ *fixed by* σ. *Then we have*

$$F = K_1(\sqrt{\eta_0} + \sqrt{\eta_0}^{\sigma}).$$

3. Proof of theorems. For the proof of Main Theorem, it is enough to prove Theorems 1–6, because the cases of Proposition 1 cover all the possible cases of the combination of $N(\varepsilon_i) = \pm 1$.

Let K' be the quadratic extension of K generated by a primitive 2^{n+1}th root of unity, $2^n \| \#W$, and let K_0' be the maximal real subfield of K'.

When $d_i d_j \underset{2}{=} d_k$ and $N(\varepsilon_i) = N(\varepsilon_j) = N(\varepsilon_k) = -1$, let

$$\eta_{ij} = \varepsilon_i \varepsilon_j \varepsilon_k, \quad \xi_{ij} = \varepsilon_i \varepsilon_j \varepsilon_k - \varepsilon_i - \varepsilon_j - \varepsilon_k.$$

Then it follows from T. Kubota [**5**, §5] that

$$\eta_{ij}\,\text{Sp}(\xi_{ij}) = \xi_{ij}^2. \tag{9}$$

For the multi-quadratic field $K_0 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$, we can prove:

LEMMA 6. *Suppose that* $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = N(\varepsilon_7) = -1$. *Let*

$$\eta = \eta_{123} = \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_7,$$
$$\xi = \xi_{123} = \eta + 1 - (\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_7 + \varepsilon_2\varepsilon_7 + \varepsilon_3\varepsilon_7).$$

Then we have

$$\eta\,\mathrm{Sp}(\xi) = \xi^2. \tag{10}$$

Proof. Since

$$\xi^{\sigma_1} = \varepsilon_1'\varepsilon_2\varepsilon_3\varepsilon_7' + 1 - \varepsilon_1'\varepsilon_2 - \varepsilon_2\varepsilon_3 - \varepsilon_3\varepsilon_1' - \varepsilon_1'\varepsilon_7' - \varepsilon_2\varepsilon_7' - \varepsilon_3\varepsilon_7',$$

it holds that $\varepsilon_1\varepsilon_7\xi^{\sigma_1} = -\xi$, where ε' is the conjugate of ε with respect to $\mathbf{Q}$. In the same way we have

$$\varepsilon_2\varepsilon_7\xi^{\sigma_2} = \varepsilon_3\varepsilon_7\xi^{\sigma_3} = \varepsilon_2\varepsilon_3\xi^{\sigma_4} = \varepsilon_3\varepsilon_1\xi^{\sigma_5} = \varepsilon_1\varepsilon_2\xi^{\sigma_6} = -\xi,$$
$$\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_7\xi^{\sigma_7} = \xi.$$

Therefore

$$\begin{aligned}\mathrm{Sp}_{K_0/\mathbf{Q}}(\xi) &= \xi + \xi^{\sigma_1} + \cdots + \xi^{\sigma_7} \\ &= \xi\left(1 - \sum_{i<j}\frac{1}{\varepsilon_i\varepsilon_j} + \frac{1}{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_7}\right)\end{aligned}$$

where i, j run through 1, 2, 3 and 7. Thus we have $\eta\,\mathrm{Sp}_{K_0/\mathbf{Q}}(\xi) = \xi^2$. □

LEMMA 7. *Suppose that* $N(\varepsilon_1) = N(\varepsilon_2) = \cdots = N(\varepsilon_7) = -1$ *and that* $\sqrt{\Delta_{ij}} \notin \mathbf{Q}(\sqrt{d_i}, \sqrt{d_j})$ *for some* (i, j). *Then we have* $\overline{E}_0 = (E_0^*)^+E_0^2$.

Proof. Let $\eta \in \overline{E}_0$. By Lemma 1 we have

$$\eta^2 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7} \qquad (x_i \in \mathbf{Z}). \tag{11}$$

Assume that every x_i is odd. Taking the norm N_1 of (11), we have by Lemma 4 that $\varepsilon_2^{x_2}\varepsilon_3^{x_3}\varepsilon_4^{x_4} \underset{2}{=} 1$ in K_1, because $K_0(\sqrt{\eta})/K_1$ is a $(2, 2)$-extension or $\sqrt{\eta}$ is contained in K_0. Therefore $\sqrt{\varepsilon_2\varepsilon_3\varepsilon_4} \in K_1$, and then by (9) we have $\sqrt{\Delta_{23}} \in K_1 = \mathbf{Q}(\sqrt{d_2}, \sqrt{d_3})$. Similarly, taking the norms N_2, N_3, N_4, N_5, N_6 and N_7 of (11), we have $\sqrt{\Delta_{ij}} \in \mathbf{Q}(\sqrt{d_i}, \sqrt{d_j})$ for every (i, j). This contradicts the assumption. Hence there is an even integer among x_i's, and it follows from (2) of Lemma 3 that every x_i is even. Therefore, $\eta \in (E_0^*)^+E_0^2$. Thus we have $\overline{E}_0 \subseteq (E_0^*)^+E_0^2$.

The inverse inclusion $(E_0^*)^+E_0^2 \subseteq \overline{E}_0$ is shown by the equations

$$\sqrt{\eta}\sqrt{\mathrm{Sp}(\xi)} = \xi \tag{12}$$

for $(\eta, \xi) = (\eta_{ij}, \xi_{ij})$ and (η_{ijk}, ξ_{ijk}), since $(E_0^*)^+ E_0^2/E_0^2$ is represented by $\eta_{12}, \eta_{23}, \eta_{31}$ and η_{123}.

Proof of Theorem 1. First we assume that $\sqrt{\Delta_{ij}} \notin \mathbf{Q}(\sqrt{d_i}, \sqrt{d_j})$ for some (i, j).

Suppose that $Q_K = 2$. Then there exists a unit $\eta \in \overline{E}_0$ such that $K_0(\sqrt{\eta}) = K_0'$ (Hasse [**2**, Satz 15]). By Lemma 7 we have $\eta = \varepsilon_1^{a_1}\varepsilon_2^{a_2}\cdots\varepsilon_7^{a_7}\varepsilon_0^2$ $(a_i \in \mathbf{Z},\ \varepsilon_0 \in E_0)$ such that $\varepsilon_1^{a_1}\varepsilon_2^{a_2}\cdots\varepsilon_7^{a_7}$ is totally positive, and by (1) of Proposition 1 $\eta = \eta_{12}^{b_1}\eta_{23}^{b_2}\eta_{31}^{b_3}\eta_{123}^{c}\varepsilon^2$ $(b_i, c \in \mathbf{Z},\ \varepsilon \in E_0)$. Therefore it follows from (12) that

$$K_0(\sqrt{\eta}) = K_0\left(\sqrt{\Delta_{12}^{b_1}\Delta_{23}^{b_2}\Delta_{31}^{b_3}\Delta_{123}^{c}}\right).$$

Since $K_0' = K_0(\sqrt{2})$ or $K_0(\sqrt{d_0})$ according as $d_0 = 1$ or not, we have $K_0' = K_0(\sqrt{A'})$ for some $A' = A(e_1', e_2', e_3')$. Therefore

$$K_0\left(\sqrt{\Delta_{12}^{b_1}\Delta_{23}^{b_2}\Delta_{31}^{b_3}\Delta_{123}^{c}}\right) = K_0(\sqrt{A'}).$$

Thus we have

$$\Delta_{12}^{b_1}\Delta_{23}^{b_2}\Delta_{31}^{b_3}\Delta_{123}^{c} \underset{2}{=} A(e_1, e_2, e_3) \tag{13}$$

for some $e_i = 0, 1$. Because, if $K_0(\sqrt{m}) = K_0(\sqrt{A'})$ for a rational integer m and $A' = A(e_1', e_2', e_3')$, then $\mathbf{Q}(\sqrt{m/A'})$ is equal to $\mathbf{Q}$ or $\mathbf{Q}(\sqrt{m/A'})$ is a quadratic subfield of K_0, and so

$$m = A'\, d_1^{e_1''}\, d_2^{e_2''}\, d_3^{e_3''}\, r^2$$

for some $e_1'', e_2'', e_3'' = 0, 1$ and some $r \in \mathbf{Q}$. Therefore, putting $e_i \equiv e_i' + e_i'' \pmod 2$ $(i = 1, 2, 3)$, we have

$$m \underset{2}{=} A(e_1, e_2, e_3).$$

Conversely, if this equation (13) holds, then the square root of $\eta := \eta_{12}^{b_1}\eta_{23}^{b_2}\eta_{31}^{b_3}\eta_{123}^{c}$ generates K_0' over K_0, i.e., $K_0(\sqrt{\eta}) = K_0'$. Thus, by H. Hasse [**2**, Satz 15] we have $Q_K = 2$.

Secondly, we assume that $\sqrt{\Delta_{ij}} \in \mathbf{Q}(\sqrt{d_i}, \sqrt{d_j})$ for every (i, j). Then it does not hold that

$$\Delta_{12}^{b_1}\Delta_{23}^{b_2}\Delta_{31}^{b_3}\Delta_{123}^{c} \underset{2}{=} A(e_1, e_2, e_3)$$

for any $b_i, c, e_i = 0, 1$.

In fact, by the assumption and by $\eta_{123} = \eta_{12}\eta_{36}\varepsilon_6^{-2}$ we have $K_0(\sqrt{\Delta_{ij}}) = K_0$ for every (i, j) and $K_0(\sqrt{\Delta_{123}}) = K_0(\sqrt{\Delta_{12}\Delta_{36}}) = K_0$. Consequently, we have

$$\Delta_{12}^{b_1}\Delta_{23}^{b_2}\Delta_{31}^{b_3}\Delta_{123}^{c} \underset{2}{=} d_1^{\alpha_1} d_2^{\alpha_2} d_3^{\alpha_3} \underset{2}{\neq} A(e_1, e_2, e_3),$$

where $\alpha_i = 0$ or 1.

In this case we can show that $Q_K = 1$ as follows:

Assume that $Q_K = 2$. Then there is a unit $\eta \in \overline{E}_0$ such that $K_0(\sqrt{\eta}) = K_0'$. By Lemma 1 we have $\eta^2 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7}$ $(x_i \in \mathbf{Z})$. It follows from (2) of Lemma 3 that all x_i are even or odd.

If all x_i are even, then $\eta \in (E_0^*)^+$ and we have $\eta = \eta_{12}^{b_1}\eta_{23}^{b_2}\eta_{31}^{b_3}\eta_{123}^{c}\varepsilon_0^2$ for some b_i, $c \in \mathbf{Z}$ and $\varepsilon_0 \in E_0^*$. Since $\eta_{123} = \eta_{12}\eta_{36}\varepsilon_6^{-2}$, we obtain by the assumption that $\sqrt{\eta} \in K_0$, which contradicts that $K_0(\sqrt{\eta})$ is a quadratic extension over K_0. Therefore, all x_i are odd. Then $\eta = \sqrt{\varepsilon_1\varepsilon_1\cdots\varepsilon_7}\prod_{i=1}^{7}\varepsilon_i^{y_i}$ for some $y_i \in \mathbf{Z}$. Since $\varepsilon_1\varepsilon_2\ldots\varepsilon_7 = \eta_{13}\eta_{23}\eta_{36}\varepsilon_3^{-2}$, we have

$$\eta = \sqrt{\eta_{13}}\sqrt{\eta_{23}}\sqrt{\eta_{36}}\varepsilon_3^{-1}\prod_{i=1}^{7}\varepsilon_i^{y_i}.$$

By (9) we have $\sqrt{\eta_{13}}r_{13}\sqrt{\Delta_{13}} = \xi_{13}$ for some $r_{13} \in \mathbf{N}$. And by the assumption we have $\Delta_{13} \underset{2}{=} d_1^{a_1} d_3^{a_3}$ for some $a_1, a_3 = 0, 1$. Hence $\varepsilon_1^{a_1}\varepsilon_3^{a_3}\sqrt{\Delta_{13}}$ is totally positive. Moreover, from $\xi_{13}^{\sigma_1} < 0$, $\xi_{13}^{\sigma_2} > 0$, $\xi_{13}^{\sigma_3} < 0$ it follows that $\varepsilon_1\varepsilon_3\xi_{13}$ is totally positive. Therefore

$$\varepsilon_1\varepsilon_3\varepsilon_1^{a_1}\varepsilon_3^{a_3}\sqrt{\eta_{13}} = \frac{1}{r_{13}}\cdot\frac{\varepsilon_1^{a_1}\varepsilon_3^{a_3}}{\sqrt{\Delta_{13}}}\cdot\varepsilon_1\varepsilon_3\xi_{13}$$

is totally positive, and then this unit is square in $K_2 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_3})$ (M. Hirabayashi and K. Yoshino [**4**, Proposition 2, IV]). So we can put

$$\varepsilon_1\varepsilon_3\varepsilon_1^{a_1}\varepsilon_3^{a_3}\sqrt{\eta_{13}} = \varepsilon_{13}^2$$

where ε_{13} is a unit of K_2. In the same way we obtain

$$\varepsilon_2\varepsilon_3\varepsilon_2^{b_2}\varepsilon_3^{b_3}\sqrt{\eta_{23}} = \varepsilon_{23}^2, \quad \varepsilon_3\varepsilon_6\varepsilon_3^{c_3}\varepsilon_6^{c_6}\sqrt{\eta_{36}} = \varepsilon_{36}^2 \qquad (b_i, c_j = 0, 1)$$

where ε_{23} and ε_{36} are units of K_1 and K_6, respectively. Therefore we have

$$\eta = \varepsilon_{13}^2\varepsilon_{23}^2\varepsilon_{36}^2\prod_{i=1}^{7}\varepsilon_i^{z_i} \qquad (z_i \in \mathbf{Z}).$$

Since $\prod_{i=1}^{7} \varepsilon_i^{z_i}$ is totally positive, we have, as before,

$$\prod_{i=1}^{7} \varepsilon_i^{z_i} = \eta_{12}^{\alpha_1}\eta_{23}^{\alpha_2}\eta_{31}^{\alpha_3}(\eta_{12}\eta_{36})^{\alpha_4}\varepsilon_0^2$$

for some $\alpha_i \in \mathbf{Z}$ and $\varepsilon_0 \in E_0^*$. By the assumption each η_{ij} is square in $\mathbf{Q}(\sqrt{d_i}, \sqrt{d_j})$ and so is η in K_0, which is also contradiction. □

LEMMA 8. *If exactly one or two of $N(\varepsilon_i)$ $(i = 1, 2, \ldots, 7)$ are $+1$, then we have $\overline{E}_0 = (E_0^*)^+ E_0^2$.*

Proof. It is enough to prove the following two Cases (1) and (2).

Case (1): $N(\varepsilon_1) = \cdots = N(\varepsilon_5) = -1$ and $N(\varepsilon_6) = N(\varepsilon_7) = +1$.

Let $\eta \in \overline{E}_0$ and let $\eta^2 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7}$ $(x_i \in \mathbf{Z})$. By (2) of Lemma 3 we see that $x_1, x_2, \ldots, x_5$ are even. Then it follows from Lemma 4 that

$$\eta\eta^{\sigma_4} = \varepsilon_1^{x_1}\varepsilon_4^{x_4}\varepsilon_7^{x_7} \underset{2}{=} 1 \quad \text{in } K_4,$$
$$\eta\eta^{\sigma_5} = \varepsilon_2^{x_2}\varepsilon_5^{x_5}\varepsilon_7^{x_7} \underset{2}{=} 1 \quad \text{in } K_5.$$

Now, we assume that x_7 is odd. Then $\varepsilon_7 \underset{2}{=} 1$ in $K_4 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_4})$ and in $K_5 = \mathbf{Q}(\sqrt{d_2}, \sqrt{d_5})$. Therefore, $\Delta_7 \underset{2}{=} d_1^{e_1} d_4^{e_4}$, $\Delta_7 \underset{2}{=} d_2^{e_2} d_5^{e_5}$ for some $e_1, e_2, e_4, e_5 = 0, 1$. These equations lead that $\Delta_7 \underset{2}{=} (d_1 d_2 d_3)^{e_1} \underset{2}{=} d_7^{e_1}$, which is impossible (Kubota [**5**, Hilfssatz 9]). Thus x_7 is even. Similarly, by the equations

$$\eta\eta^{\sigma_3} = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\varepsilon_6^{x_6} \underset{2}{=} 1 \quad \text{in } K_3,$$
$$\eta\eta^{\sigma_6} = \varepsilon_3^{x_3}\varepsilon_6^{x_6}\varepsilon_7^{x_7} \underset{2}{=} 1 \quad \text{in } K_6,$$

we see that x_6 is even. Therefore all x_i are even and so $\eta \in E_0^*$. Thus $\overline{E}_0 \subseteq (E_0^*)^+ E_0^2$.

The inverse inclusion $(E_0^*)^+ E_0^2 \subseteq \overline{E}_0$ is shown by the equations (1) and (12).

Case (2): $N(\varepsilon_1) = N(\varepsilon_2) = \cdots = N(\varepsilon_6) = -1$ and $N(\varepsilon_7) = +1$.

Let $\eta \in \overline{E}_0$ and let $\eta^2 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7}$ $(x_i \in \mathbf{Z})$. Then, by (2) of Lemma 3 we see that $x_1, x_2, \ldots, x_6$ are even. In the same way as in the proof of Case (1) we can show that x_7 is even and that $\overline{E}_0 = (E_0^*)^+ E_0^2$. □

Proof of Theorems 2 *and* 3. We only prove Theorem 2, because we prove Theorem 3 in a similar way.

Suppose that $Q_K = 2$. Then there exists a unit $\eta \in \overline{E}_0$ such that $K_0(\sqrt{\eta}) = K_0' = K_0(\sqrt{A})$ where $A = A(e_1, e_2, e_3)$. By Lemma 8 and (2) of Proposition 1 we can put $\eta = \varepsilon_7^a \eta_{12}^{b_1} \eta_{23}^{b_2} \eta_{31}^{b_3} \varepsilon^2$ ($a, b_i \in \mathbf{Z}$, $\varepsilon \in E_0$) and we have

$$K_0(\sqrt{\eta}) = K_0\left(\sqrt{\Delta_7^a \Delta_{12}^{b_1} \Delta_{23}^{b_2} \Delta_{31}^{b_3}}\right).$$

Consequently,

$$\Delta_7^a \Delta_{12}^{b_1} \Delta_{23}^{b_2} \Delta_{31}^{b_3} \underset{2}{=} A(e_1, e_2, e_3). \tag{14}$$

Conversely, if this equation (14) holds, then a square root of $\eta := \varepsilon_7^a \eta_{12}^{b_1} \eta_{23}^{b_2} \eta_{31}^{b_3}$ generates K_0' over K_0, i.e., $K_0' = K_0(\sqrt{\eta})$. Therefore we have $Q_K = 2$. □

Proof of Theorem 4.

Case (1): $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = N(\varepsilon_4) = -1$ and $N(\varepsilon_5) = N(\varepsilon_6) = N(\varepsilon_7) = +1$.

Suppose that $Q_K = 2$. Then there is a unit $\eta \in \overline{E}_0$ such that $K_0(\sqrt{\eta}) = K_0'$. By Lemma 1 and (4_1) of Proposition 1 we have

$$\eta^2 = \eta_{23}^{x_2} \varepsilon_5^{x_5} \varepsilon_6^{x_6} \varepsilon_7^{x_7} \prod_{i=1}^{7} \varepsilon_i^{2y_i}$$

where $x_i, y_i \in \mathbf{Z}$. From (2) of Lemma 3 it follows that $x_2 \equiv 0 \pmod 2$. Hence by Lemma 2 we see that $x_5 \equiv x_6 \equiv x_7 \pmod 2$.

In the case that $x_5 \equiv x_6 \equiv x_7 \equiv 0 \pmod 2$, we have

$$\eta = \varepsilon_5^{a_5} \varepsilon_6^{a_6} \varepsilon_7^{a_7} \eta_{23}^{b} \varepsilon_0^2$$

for some $a_i, b = 0, 1$ and $\varepsilon_0 \in E_0^*$. Therefore,

$$K_0' = K_0(\sqrt{\eta}) = K_0\left(\sqrt{\Delta_5^{a_5} \Delta_6^{a_6} \Delta_7^{a_7} \Delta_{23}^{b}}\right)$$

and then

$$\Delta_5^{a_5} \Delta_6^{a_6} \Delta_7^{a_7} \Delta_{23}^{b} \underset{2}{=} A(e_1, e_2, e_3) \tag{15}$$

for some $e_i = 0, 1$.

In the case that $x_5 \equiv x_6 \equiv x_7 \equiv 1 \pmod 2$, let

$$\eta_0 := \sqrt{\varepsilon_5 \varepsilon_6 \varepsilon_7} \prod_{i=1}^{4} \varepsilon_i^{v_i} \quad (v_i = 0 \text{ or } 1)$$

and let η_0 be totally positive. Then we have $\eta = \varepsilon_5^{a_5}\varepsilon_6^{a_6}\varepsilon_7^{a_7}\eta_{23}^b\eta_0\varepsilon_0^2$ where $a_i, b = 0, 1$ and $\varepsilon_0 \in E_0^*$. Since $\varepsilon_5, \varepsilon_6, \varepsilon_7, \eta_{23}, \eta \in \overline{E}_0$, we see $\eta_0 \in \overline{E}_0$. Then it follows from Lemma 5 that

$$K_0(\sqrt{\eta_0}) = K_0(\sqrt{\xi^*(\eta_0)}) = K_0(\sqrt{\theta^*(\eta_0)}) = K_0(\sqrt{d^*(\eta_0)})$$

where $\xi^*(\eta_0)$, $\theta^*(\eta_0)$ and $d^*(\eta_0)$ is defined by (2), (3) and (4), respectively. Here we take $s_i = 0$ or 1 $(i = 1, 2, 3)$ in accordance with

$$\xi^*(\eta_0) = (\sqrt{\eta_0} + \sqrt{\eta_0}^{\sigma_1})^2, \quad \theta^*(\eta_0) = (\sqrt{\xi^*(\eta_0)} + \sqrt{\xi^*(\eta_0)}^{\sigma_2})^2,$$
$$d^*(\eta_0) = (\sqrt{\theta^*(\eta_0)} + \sqrt{\theta^*(\eta_0)}^{\sigma_3})^2,$$

respectively. Therefore

$$K_0' = K_0(\sqrt{\eta}) = K_0(\sqrt{\Delta_5^{a_5}\Delta_6^{a_6}\Delta_7^{a_7}\Delta_{23}^b\, d^*(\eta_0)})$$

and then we have

$$\Delta_5^{a_5}\Delta_6^{a_6}\Delta_7^{a_7}\Delta_{23}^b\, d^*(\eta_0) \underset{2}{=} A(e_1, e_2, e_3) \tag{16}$$

for some $e_i = 0, 1$.

Conversely, if the equation (15) or (16) holds, the square root of $\eta := \varepsilon_5^{a_5}\varepsilon_6^{a_6}\varepsilon_7^{a_7}\eta_{23}^b$ or $\varepsilon_5^{a_5}\varepsilon_6^{a_6}\varepsilon_7^{a_7}\eta_{23}^b\eta_0$ generates K_0' over K_0, respectively, i.e., $K_0' = K_0(\sqrt{\eta})$. Then we have $Q_K = 2$.

Case (2): $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = N(\varepsilon_7) = -1$ and $N(\varepsilon_4) = N(\varepsilon_5) = N(\varepsilon_6) = +1$.

Suppose that $Q_K = 2$. Then by Lemma 1 and (4_2) of Proposition 1 we have

$$\eta^2 = \varepsilon_4^{x_4}\varepsilon_5^{x_5}\varepsilon_6^{x_6}\eta_{123}^z\prod_{i=1}^{7}\varepsilon_i^{2y_i} \tag{17}$$

where $x_i, y_i, z \in \mathbf{Z}$. Then it follows from (2) of Lemma 3 that $z \equiv 0 \pmod 2$, and from Lemma 2 that $x_4 \equiv x_5 \equiv x_6 \pmod 2$.

If $x_4 \equiv x_5 \equiv x_6 \equiv 0 \pmod 2$, then $\eta \in (E_0^*)^+$. By (4_2) of Proposition 1 we have $\eta = \varepsilon_4^{a_4}\varepsilon_5^{a_5}\varepsilon_6^{a_6}\eta_{123}^c\varepsilon_0^2$ for some $a_i, c = 0, 1$ and $\varepsilon_0 \in E_0^*$. Therefore,

$$K_0(\sqrt{\eta}) = K_0(\sqrt{\Delta_4^{a_4}\Delta_5^{b_5}\Delta_6^{b_6}\Delta_{123}^c}). \tag{18}$$

If $x_4 \equiv x_5 \equiv x_6 \equiv 1 \pmod 2$, taking norms N_1 and N_4 of the equation (17), we have by Lemma 4 that

$$\eta^{1+\sigma_1} = \varepsilon_4^{x_4}\varepsilon_2^{2y_2}\varepsilon_3^{2y_3}\varepsilon_4^{2y_4} \underset{2}{=} 1 \quad \text{in } K_1,$$
$$\eta^{1+\sigma_4} = \varepsilon_4^{x_4}\varepsilon_1^{2y_1}\varepsilon_7^{2y_7}\varepsilon_4^{2y_4} \underset{2}{=} 1 \quad \text{in } K_4.$$

Then $\sqrt{\Delta_4}$ is contained in $K_1 \cap K_4 = \mathbf{Q}(\sqrt{d_2 d_3})$, and then $\Delta_4 \underset{2}{=} 1$ or $d_2 d_3$, which is impossible (T. Kubota [**5**, Hilfssatz 9]).

Thus, if $Q_K = 2$ we have the equation (18) and hence

$$\Delta_4^{a_4}\Delta_5^{a_5}\Delta_6^{a_6}\Delta_{123}^{c} \underset{2}{=} A(e_1, e_2, e_3) \tag{19}$$

for some $e_i = 0, 1$.

Conversely, when the equation (19) holds, we can show, as before, that $Q_K = 2$. □

Proof of Theorem 5. (1) Suppose that $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = -1$ and that $N(\varepsilon_4) = \cdots = N(\varepsilon_7) = +1$. By Lemma 1 and (5_1) of Proposition 1 we have

$$\eta^2 = \varepsilon_4^{x_4}\varepsilon_5^{x_5}\varepsilon_6^{x_6}\varepsilon_7^{x_7}\prod_{i=1}^{7}\varepsilon_i^{2y_i} \tag{20}$$

for any $\eta \in \overline{E}_0$ where $x_i, y_i \in \mathbf{Z}$. Then by Lemma 2 we have the following three cases:

(i) $x_4 \equiv x_5 \equiv x_6 \equiv x_7 \equiv 0 \pmod 2$;

(ii) Among x_4, x_5, x_6 and x_7, exactly one x_i is even;

(iii) $x_4 \equiv x_5 \equiv x_6 \equiv x_7 \equiv 1 \pmod 2$.

Case (i). We have $\eta \in (E_0^*)^+$ and we may put $\eta = \varepsilon_4^{a_4}\varepsilon_5^{a_5}\varepsilon_6^{a_6}\varepsilon_7^{a_7}$ ($a_i \in \mathbf{Z}$). Then we obtain, as before,

$$K_0(\sqrt{\eta}) = K_0(\sqrt{\Delta_4^{a_4}\Delta_5^{a_5}\Delta_6^{a_6}\Delta_7^{a_7}}).$$

Case (ii). We first consider the case that $x_4 \equiv x_5 \equiv x_6 \equiv 1$, $x_7 \equiv 0 \pmod 2$. Taking norms N_1 and N_4 of (20), we have

$$\eta^{1+\sigma_1} = \varepsilon_4^{x_4}\varepsilon_2^{2y_2}\varepsilon_3^{2y_3} \underset{2}{=} 1 \quad \text{in } K_1 = \mathbf{Q}(\sqrt{d_2}, \sqrt{d_3}),$$
$$\eta^{1+\sigma_4} = \varepsilon_4^{x_4}\varepsilon_1^{2y_1}\varepsilon_7^{2y_7} \underset{2}{=} 1 \quad \text{in } K_4 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_4}).$$

Then, as before, $\sqrt{\Delta_4}$ is contained in $\mathbf{Q}(\sqrt{d_4})$, which is impossible.

Next we consider the other cases, for example, $x_4 \equiv x_5 \equiv x_7 \equiv 1$, $x_6 \equiv 0 \pmod 2$. Let

$$\eta_0 := \sqrt{\varepsilon_4\varepsilon_5\varepsilon_7}\prod_{i=1}^{3}\varepsilon_i^{v_i} \qquad (v_i = 0 \text{ or } 1)$$

and let η_0 be totally positive. Then we can prove the assertion in the same way as in the proof of Case (1) of Theorem 4.

Case (iii). As before, taking norms N_1, N_2, N_3 and N_7 of (20), we obtain

$$\Delta_4 \underset{2}{=} d_2 \text{ or } d_3; \quad \Delta_5 \underset{2}{=} d_3 \text{ or } d_1; \quad \Delta_6 \underset{2}{=} d_1 \text{ or } d_2;$$
$$\Delta_4\Delta_5\Delta_6 \underset{2}{=} d_2 d_3,\ d_3 d_1 \text{ or } d_1 d_2,$$

which is impossible.

(2) Suppose that $N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_6) = -1$ and the others $N(\varepsilon_i) = +1$. We have by (5_2) of Proposition 1

$$\eta^2 = \varepsilon_3^{x_3}\varepsilon_4^{x_4}\varepsilon_5^{x_5}\varepsilon_7^{x_7}\eta_{12}^{x_1}\prod_{i=1}^{7}\varepsilon_i^{2y_i}$$

for any $\eta \in \overline{E}_0$ where $x_i, y_i \in \mathbf{Z}$. By (2) of Lemma 3 we have $x_1 \equiv 0 \pmod 2$. Therefore we obtain, as before, the following cases:

(i) $x_3 \equiv x_4 \equiv x_5 \equiv x_7 \equiv 0 \pmod 2$;
(ii) Among x_3, x_4, x_5 and x_7, exactly one x_i is even;
(iii) $x_3 \equiv x_4 \equiv x_5 \equiv x_7 \equiv 1 \pmod 2$.

By the same argument in (1) of this proof we can prove the assertion for each case. □

Proof of Theorem 6. In the following we only consider the first case: $N(\varepsilon_1) = N(\varepsilon_2) = -1$, since the other cases are proved in the same way.

Let

$$\eta_0 := \sqrt{\prod_{N(\varepsilon_i)=+1} \varepsilon_i^{u_i} \cdot \prod_{N(\varepsilon_i)=-1} \varepsilon_i^{v_i}} \quad (u_i, v_i = 0 \text{ or } 1)$$

and let η_0 be totally positive.

For any $\eta \in \overline{E}_0$ we may put $\eta = \varepsilon_3^{a_3} \cdots \varepsilon_7^{a_7} \cdot \eta_0^f$ where $a_i, f = 0$ or 1. Then we have, as before,

$$K_0(\sqrt{\eta}) = K_0(\sqrt{\Delta_3^{a_3} \cdots \Delta_7^{a_7}\, d^*(\eta_0)^f}).$$

Thus we obtain that $Q_K = 2$ if and only if

$$\Delta_3^{a_3} \cdots \Delta_7^{a_7}\, d^*(\eta_0)^f \underset{2}{=} A(e_1, e_2, e_3),$$

as desired. □

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