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THE JONES POLYNOMIAL OF PARALLELS AND APPLICATIONS TO CROSSING NUMBER

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In this paper we study the Jones polynomial of the parallels of a link or knot. From the extremal exponents occurring we derive lower bounds on the crossing number of the knot, reproducing in particular a number of results of Thistlethwaite. We apply these techniques to give lower bound on the crossing number of some simple satellites of adequate and semi-adequate knots (including cable satellites) that are usually quadratic in the degree of the satellite.

Introduction. Several recent papers have used the Jones polynomial to study the crossing number of a link. First, Kauffman [**2**], using the Jones polynomial, showed that any two reduced, connected, alternating diagrams for a link have the same crossing number. This result was extended independently by Murasugi [**5**] and Thistlethwaite [**6**] showing that a reduced alternating diagram has the minimal crossing number. Thistlethwaite [7, **8**] has extended these results, using the Kauffman (or semi-oriented) polynomial, to show that the writhe of a reduced alternating diagram of an alternating link is an isotopy invariant of L and to show that an adequate diagram of a link, has minimal crossing number.

In this paper, we will reproduce these results and some other results of Thistlethwaite using instead the Jones polynomial of the parallels of a link. Using this method, we will be able to give lower bounds for the crossing number of the r-fold parallels of an adequate knot, which in most cases are quadratic in r. We will further show that these lower bounds are stable under a class of variations. These variants may be thought of as being the satellites coming from flows that are C^1-close to the parallel flow, in the sense of [**1**].

The Kauffman bracket polynomial of a planar diagram of an unoriented link is an element $\langle D\rangle \in \mathbf{Z}[A, A^{-1}]$ defined by the following procedure. A state for D is defined to be a map s from the crossings of D (which we take to be indexed by $1 \le i \le n$) to $\{-1, 1\}$. Let sD denote the diagram obtained from D by nullifying the crossings of D according to s as in Figure 1. For any s, sD consists of disjoint simple closed curves. Let $|sD|$ denote the number of such simple

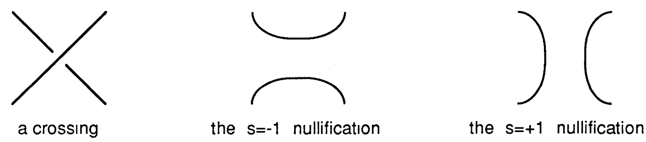


FIGURE 1. Nullifying crossings.

closed curves. One either defines

$$\langle D\rangle = \sum_{s} \langle D|s\rangle\,, \quad \text{where } \langle D|S\rangle = A^{\Sigma s(i)}(-A^{-2}-A^{2})^{|sD|-1}$$

where the second sum is over the vertices of D or derives this formula from a recursive definition of $\langle D\rangle$. If L is an oriented link represented by a diagram D with writhe $w(D)$ then the Kauffman bracket of D is related to the Jones polynonial of L by $V_L(t) = (-A)^{3w(D)}\langle D\rangle$, where $t = A^{-4}$. Good references for the basic properties of the Kauffman bracket are **[2, 3]**.

Let s_+ (respectively s_-) denote the state that assigns 1 (respectively -1) to every crossing of D. Then, following Lickorish and Thistlethwaite **[4]**, we say that D is $+$ adequate if $|s_+D| > |sD|$ for every s for which $\sum s(i) = n-2$. Similarly D is $-$ adequate if $|s_-D| > |sD|$ for every state s for which $\sum s(i) = 2-n$. The diagram D is said to be adequate if it is both $+$ and $-$ adequate, semi-adequate if it is either $+$ or $-$ adequate and inadequate if it is neither $+$ nor $-$ adequate.

Equivalently, a diagram D is $+$ adequate if and only if changing s_+ on a single crossing of D always joins two different components of s_+D. A similar remark applies to $-$ adequacy.

For any Laurent polynomial $\rho \in \mathbf{Z}[A, A^{-1}]$ let $M\rho$ denote the maximum exponent of A in ρ and let $m\rho$ denote the minimum exponent. We view the Jones polynomial as a polynomial in A by setting $t = A^{-4}$. The following proposition, which is essentially taken from **[4]**, will be our central tool.

PROPOSITION 1. (a) *If D be a link diagram with n crossings then $M\langle D\rangle \leq n + 2|s_+D| - 2$ with equality if D is $+$ adequate.*

(b) *If D be a link diagram with n crossings then $m\langle D\rangle \geq -n - 2|s_-D| + 2$ with equality if D is $-$ adequate.*

Proof. (a) Clearly $M\langle D|s_+\rangle = n+2|s_+D|-2$. Suppose s is any other state. Then there is a sequence of states $s_+ = s_0, s_1, \ldots, s_k = s$, such that s_{r-1} and s_r agree on all but one crossing and on that crossing $s_{r-1}=1$ and $s_r=-1$. Therefore $\sum s_r(i)=\sum s_{r-1}(i)-2$ and $|s_rD|$ and $|s_{r-1}D|$ differ by 1; hence $M\langle D|s_{r-1}\rangle \geq M\langle D|s_r\rangle$. This implies the upper bound. If further D is $+$ adequate, then $|s_0D|=|s_1D|+1$. Therefore $M\langle D|s_+\rangle > M\langle D|s_1\rangle$ and hence $M\langle D|s_+\rangle > M\langle D|s\rangle$ for all other s. Thus $M\langle D\rangle = M\langle D|s_+\rangle$.

Part (b) follows similarly or by replacing D with its reflection $\overline{D}$.

It is easy to check that, in particular, a reduced alternating diagram is adequate. Further, Thistlethwaite [**8**] has observed that every knot with crossing number at most eleven is semi-adequate in its minimum crossing number projection. These results illustrate that, at least for easy knots, semi-adequacy is a fairly common condition.

For any component of a link there is a family of parallels indexed by an integer, the framing, which is the linking number of any one parallel with any other parallel. We will use the following notation. For a diagram D let D^r denote the result of replacing every link-component of D by r components all parallel in the plane (henceforth referred to as the r-fold planar parallel of D). If a link L has link-components $1\leq i\leq c$, let $L^r(t_1, t_2, \dots, t_c)$ denote the result of replacing link-component i by the r-fold parallel with framing t_i. Thus if D is a diagram for L, then D^r is a diagram for the link $L^r(w_1, w_2, \dots, w_c)$ where w_i is the writhe of link-component i. In general, we get a diagram for $L^r(t_1, t_2, \dots, t_c)$ from D by adding $|t_i-w_i|$ small kinks (of the appropriate sign) to link-component i and then forming the r-fold planar parallel. Denote this diagram by $D^r(t_1, t_2, \dots, t_c)$. With these definitions, we have the following slight generalization of [**4**, Proposition 2].

PROPOSITION 2. (a) *If D is any diagram, then*

$$|s_+D^r(t_1, t_2, \dots, t_c)| = r\left(|s_+D| + \sum \max(t_i-w_i, 0)\right)$$

and

$$|s_-D^r(t_1, t_2, \dots, t_c)| = r\left(|s_-D| + \sum \max(w_i-t_i, 0)\right).$$

(b) *If D is a $+$ adequate diagram and $t_i\geq w_i$ for all i, $1\leq i\leq c$, then $D^r(t_1, t_2, \dots, t_c)$ is also $+$ adequate.*

(c) *If D is a $-$ adequate diagram and $w_i\geq t_j$ for all i, $1\leq i\leq c$, then $D^r(t_1, t_2, \dots, t_c)$ is also $-$ adequate.*

Proof. Clearly adding loops with a $+1$ crossings does not destroy $+$ adequacy or change $|s_-D|$ but it adds one more component to

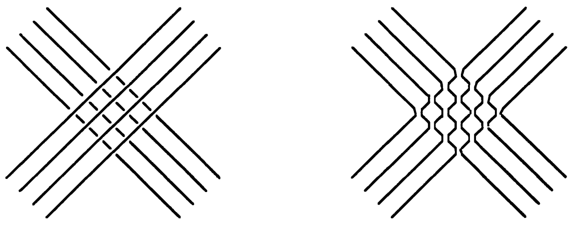


FIGURE 2. Nullification of a planar parallel of a crossing.

s_+D. Similarly adding loops with a -1 crossing does not destroy $-$ adequacy or change $|s_+D|$ but it adds one more component to s_+D. Thus the proposition holds for $r = 1$ and we are reduced to considering the effect of taking r-fold planar parallels.

As indicated in **[4]** and Figure 2, taking r-fold planar parallels commutes with acting by s_+. It follows that $|s_+D^r(t_1, t_2, \ldots, t_c)| = r|s_+D^1(t_1, t_2, \ldots, t_c)| = r(|s_+D| + \sum \max(t_i - w_i, 0))$. If D is $+$ adequate, then $D^r(t_1, t_2, \ldots, t_c)$ is also $+$ adequate since changing the state s_+ on any crossing to -1 either joins two different parallels (in $s_+D^r(t_1, t_2, \ldots, t_c)$) of the same component of $s_+D^1(t_1, t_2, \ldots, t_c)$ or joins parallels of different components. Part (c) and the other half of part (a) follow by replacing D by its reflection.

Combining these two propositions gives the following results.

PROPOSITION 3. (a) *Let L be an oriented link with c link-components and with a diagram D with n crossings and writhe w, if $t_i \geq w_i$ $(1 \leq i \leq c)$ Then*

$$\begin{aligned} &MV(L^r(t_1, t_2, \ldots, t_c)) \\ &\quad \leq (n - w)r^2 + 2(|s_+D| - w)r - 2 \\ &\qquad - 2r(r-1)\sum t_i - 2r(r-1)\left(w - \sum w_i\right). \end{aligned}$$

If further D is $+$ adequate, then we have equality.

(b) *Let L be an oriented link with c link-components and with a diagram D with n crossings and writhe w, if $w_i \geq t_i$ $(1 \leq i \leq c)$. Then*

$$\begin{aligned} &mV(L^r(t_1, t_2, \ldots, t_c)) \\ &\quad \geq -(n + w)r^2 - 2(|s_-D| - w)r + 2 \\ &\qquad - 2r(r-1)\sum t_i + 2r(r-1)\left(w - \sum w_i\right). \end{aligned}$$

If further D is $-$ adequate, then we have equality.

Proof. (a) The diagram $D^r(t_1, t_2, \dots, t_c)$ has $(n + \sum(t_i - w_i))r^2$ crossings and $|s_+D^r(t_1, t_2, \dots, t_c)|$ as given above. Therefore

$$M\langle D^r(t_1, t_2, \dots, t_c)\rangle \\ \leq \left(n + \sum(t_i - w_i)\right) r^2 + 2\left(|s_+D| + \sum(t_i - w_i)\right) r - 2.$$

Also $D^r(t_1, t_2, \dots, t_c)$ has writhe $wr^2 + \sum(t_i - w_i)r^2$. Therefore

$$MV(L^r(t_1, t_2, \dots, t_c)) \\ \leq (n - w)r^2 + 2(|s_+D| - w)r - 2 \\ - 2\left(\sum t_i - \sum w_i + w\right) r(r-1),$$

as claimed. If D is + adequate, then $D^r(t_1, t_2, \dots, t_c)$ is also + adequate and we have equality. Part (b) follows similarly or by replacing D by its reflection.

The quantity $w - \sum w_i$ can be rewritten as

$$w - \sum w_i = 2\sum_{i<j} \operatorname{lk}(L_i, L_j),$$

where L_i denotes the ith link-component of L and is an isotopy invariant of L. Further, $MV(L^r(t_1, t_2, \dots, t_c))$ is an isotopy invariant of L for all r and $\{t_i\}$ and under the hypotheses of the proposition above it has the form $ar^2 + 2br - 2 - 2cr(r-1)$, where c is also an isotopy invariant depending on $\{t_i\}$ and a and b independent of r and $\{t_i\}$. Thus a and b are isotopy invariants of L. Therefore if L admits a + adequate diagram with n crossings and writhe w, then $n - w$ is an isotopy invariant of L. Further, if D' is any other diagram representing L, with say n' crossings and writhe w', then $n' - w' \geq n - w$. To rephrase this let $c_+(L)$ (respectively $c_-(L)$) denote the minimum number of sign $+1$ (respectively -1) crossings in any projection of L. Let $c(L)$ denote the crossing number of L. Then we have the following corollary (which is essentially [**8**, Corollary 3.1] rephrased).

COROLLARY 3.1. (a) *If L is a link that admits a + adequate diagram D with n_- sign -1 crossings, then $c_-(L) = n_-$. In particular, D is not regularly isotopic to a diagram with fewer crossings.*

(b) *If L is a link that admits a − adequate diagram D with n_+ sign $+1$ crossings, then $c_+(L) = n_+$. In particular, D is not regularly isotopic to a diagram with fewer crossings.*

(c) *If* L *is a link that admits an adequate diagram* D *with* n *crossings and writhe* w, *then* $c(L) = n$ *and any other* n *crossing diagram for* L *has writhe* w.

Thistlethwaite actually obtains a slightly stronger statement in **[8]**. In our terminology, he shows that if L admits a $+$ adequate diagram with n_- sign -1 crossings, then $c_-(L) = n_-$ and any other diagram for L with n_- sign -1 crossings is $+$ adequate. The quantities $c_+(L)$ and $c_-(L)$ would seem to be almost completely uncomputable without the Jones polynomial techniques above or the Kauffman polynomial techniques of **[8]**.

This corollary together with the trivial bound $c(L) \geq c_+(L) + c_-(L)$ gives bounds on the crossing number of certain cable satellites of semi-adequate knots. One also gets similar lower bounds for links but with more complicated conditions.

COROLLARY 3.2. (a) *Let* K *be a knot with a* $+$ *adequate diagram* D *with* n *crossings and writhe* w. *If* $q \geq wr$ *and* $K(r, q)$ *denotes the* (r, q)*-cable satellite of* K, *then*

$$c(K(r, q)) \geq c_-(K(r, q)) \geq \tfrac{1}{2}(n - w)r^2.$$

(b) *Let* K *be a knot with a* $-$ *adequate diagram* D *with* n *crossings and writhe* w. *If* $q \leq wr$ *and* $K(r, q)$ *denotes the* (r, q)*-cable satellite of* K, *then*

$$c(K(r, q)) \geq c_+(K(r, q)) \geq \frac{1}{2}(n + w)r^2.$$

This result depends very heavily on the assumption of semi-adequacy. If K is as above and we obtain a link L from $K(r, q)$ by introducing even a single sign -1 crossing the method of proof breaks down completely. In principle, we have lower bounds on $c_+(L)$ and $c_-(L)$ coming from the Jones polynomial of the highly twisted parallels (or using **[8]** the Kauffman polynomial). Unfortunately for an arbitrary knot K it is difficult to calculate the extremal powers in the Jones polynomial of highly twisted parallels of its satellites (or the Kauffman polynomial of its satellites). For a certain restricted class of satellites, which we will now define, this is however possible.

Recall that the r-string braid group B_r is the group with presentation

$$B_r = \langle \sigma_1, \sigma_2, \dots, \sigma_{r-1} : \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i - j| \geq 2,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle.$$

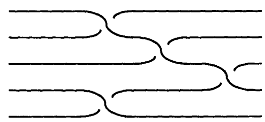


FIGURE 3(a). The element $\alpha = \sigma_1\sigma_4\sigma_2\sigma_3 \in B_r$.

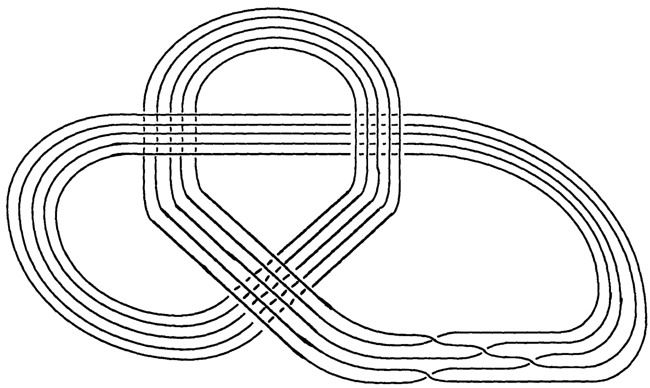


FIGURE 3(b). The braid α glued into the 5-fold planar parallel of the trefoil knot.

Elements of B_r can be viewed geometrically as r strings traversing a box monotonically from left to right as shown in Figure 3(a). For a knot K with diagram D (with writhe w) and $\alpha \in B_r$ let $D(r; \alpha)$ denote the diagram obtained from the r-fold planar parallel D^r by gluing in α as in Figure 3(b) and let $K(r, w; \alpha)$ denote the corresponding knot. Note that $K(r, w; \alpha)$ depends only on the conjugacy class of α. In some sense, if α is not too large, our prior results go through for $K(r, w; \alpha)$.

Let T be the subset of B_r consisting of all elements of the form $\sigma_{i_1}^{\pm 1}\sigma_{i_2}^{\pm 1}\cdots\sigma_{i_k}^{\pm 1}$ where $i_{j+1} > i_j + 1$ for all j. We define the length of α to be the least $n \geq 0$ such that $\alpha \in T^n$, and denote it by $l(\alpha)$. To motivate the use of the word "length", note that since the generators of B_n occurring in any element of T commute any element of T can be drawn unambiguously as a layer one crossing long. A braid α of length $l(\alpha)$ can then be drawn as the union of $l(\alpha)$ layers where each layer is one crossing long. (One can define many other versions of the length for example versions that treat positive and negative crossings differently and derive stronger versions of the results below.)

In some weak sense, if $l(\alpha) < Cr$, then one should regard $K(r, w; \alpha)$ as being derived from a flow C^1-close to the flow producing the r-fold w-twisted parallel $K(r, w; 1)$. Roughly the distance between neighboring strings is $O(r^{-1})$ and strings C^1-close to parallel can angle towards each other only slowly. Therefore the only new crossings introduced occur as a braid α and each new crossing introduced requires a distance that is $O(r^{-1})$. Thus α has length that is at most $O(r)$. The precise context in which $K(r, w; \alpha)$ is a C^1-approximation is at present still a little unclear.

If $\alpha \in B_r$, we let $\alpha(s)$ denote the s-fold planar parallel of α, viewed as an element of B_{rs}. Note that for any α we have $l(\alpha(s)) \leq (2s-1)l(\alpha)$. Using more information about α one can write down stronger linear bounds. If α can be written as $\alpha = t_1 t_2 t_3 \cdots t_l$ where $t_i \in T$ and no σ_j occurs in t_i and t_{i+1} (where $t_{l+1} = t_1$), then $l(\alpha(s)) \leq sl$. Note that full twists have this nice form.

PROPOSITION 4. *Let K be a knot that admits a + adequate diagram D with n crossings and writhe w. Suppose $\beta \in B_r$ contains only positive crossings and $\alpha \in B_r$ satisfies $l(\alpha) \leq nr/2$. Then*

$$\begin{aligned} &MV(K(r, w; \beta\alpha)) \\ &\qquad = (n-w)r^2 + 2(|s_+D| - w)r - 2 - 2r(r-1)w - 2w(\beta\alpha), \end{aligned}$$

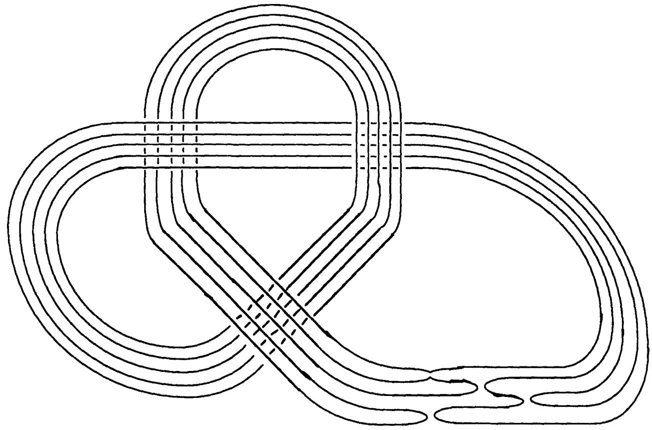
where $w(\beta\alpha)$ denotes the writhe of $\beta\alpha$.

Proof. View $D(r; \beta\alpha)$ as being composed of two sections each of which connects r adjacent entrance points to r adjacent exit points. The first section S_1 consists of the r-fold planar parallel of D with a small section removed. The second section S_2 consists of $\beta\alpha$. Let u be a map from the crossings of S_2 to the set $\{-1, 1\}$, thought of as a partial state. Let $D(u)$ and $S_2(u)$ denote the results of nullifying the crossings of D and S_2 according to u. Then as above we have a state formula for $\langle D(r; \beta\alpha)\rangle$.

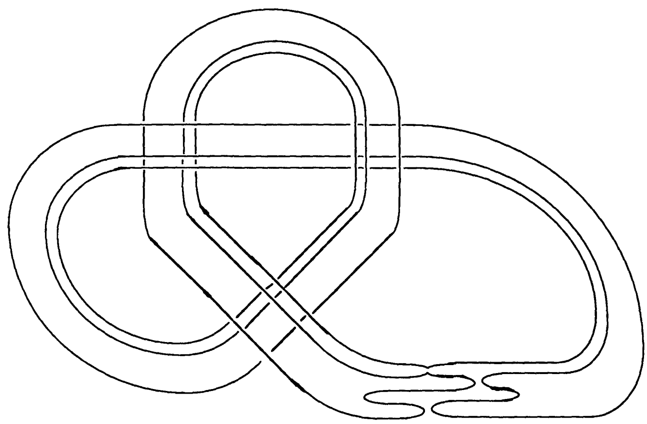
$$\langle D(r; \beta\alpha)\rangle = \sum_u A^{\Sigma u(i)} \langle D(u)\rangle.$$

The diagram $D(u)$ can be simplified using Reidemeister moves of type II (which do not change the Kauffman bracket). The diagram $S_2(u)$ consists of some number, say μ, of simple closed curves and disjoint paths joining pairs of the $2r$ entrance and exist points. If $S_2(u)$ joins two adjacent exit points then the resulting loop may be pushed through S_1 using type II moves (possibly increasing μ) (see Figure 4). Continue pushing through loops as long as possible. Let m

(a) The initial diagram D(u) for K the trefoil, α as in Figure 3 and some u.



(b) The diagram after a first simplification.



(c) The diagram after all simplifications.

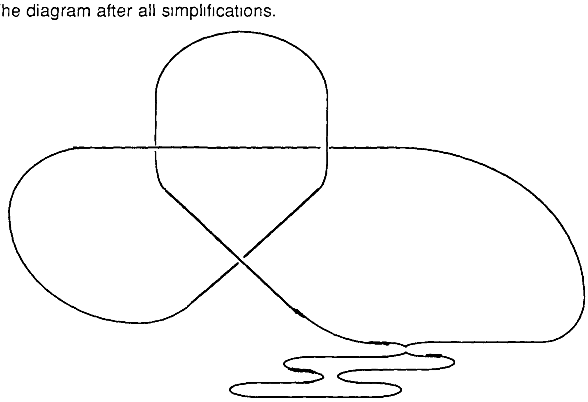


FIGURE 4. Simplification of $D(u)$ by Reidemeister moves of type II.

be the number of loops pushed through. Then the remaining diagram consists of the $r-2m$ fold parallel of D together with μ simple closed curves.

Suppose first that u is the state u_0 that nullifies each crossing of S_2 to produce r parallel lines (i.e. u_0 is 1 on sign $+1$ crossings and -1 on sign -1 crossings). Then $D(u_0)$ is the r-fold planar parallel of D and

$$m(A^{\Sigma u(i)}\langle D(u_0)\rangle) = nr^2 + 2|s_+D|r - 2 + w(\beta\alpha).$$

If u is any other state let M_+ and M_- be the number of $+1$ and -1 sign crossings on which u disagrees with u_0. Then $\sum u(i) = w(\beta\alpha) - 2M_+ + 2M_-$. Also $2M_+ + 2M_-$ is the number of direction changes introduced so $\mu \leq 2M_+ + 2M_-$. Therefore

$$\begin{aligned} M(A^{\Sigma u(i)}\langle D(u)\rangle) \leq\ & n(r-2m)^2 + 2|s_+D|(r-2m) - 2 \\ & + 2\mu + w(\beta\alpha) - 2M_+ + 2M_-, \\ \leq\ & nr^2 + 2|s_+D|r - 2 + w(\beta\alpha) \\ & - 4(rnm - m^2n + m|s_+D| - M_-). \end{aligned}$$

We will show below that because $l(\alpha) \leq nr/2$ we have $M_- \leq \frac{1}{2}rnm$. Therefore the rightmost term is negative (since $m \leq \frac{1}{2}r$) and

$$M\langle D(r;\,\beta\alpha)\rangle = M(A^{\Sigma u(i)}\langle D(u_0)\rangle) = nr^2 + 2|s_+D|r - 2 + w(\beta\alpha).$$

This gives the stated value for $MV(K(r,\,w\,;\,\beta\alpha))$.

To see that $M_- \leq \frac{1}{2}rnm$, view α as being built from $l(\alpha)$ layers which are elements of T. If one of the layers has k crossings on which $u \neq u_0$, then only $r-2k$ paths in $S_2(u)$ go through that layer and hence $k \leq m$. Therefore summing over layers $M_- \leq ml(\alpha) \leq \frac{1}{2}rnm$.

As a result of this proposition we have the following theorem.

THEOREM 5. (a) *Let K be a knot that admits a $+$ adequate diagram D with n crossings and writhe w. Suppose $\alpha \in B_r$ has $l(\alpha(s)) < nrs/2$; then*

$$c(K(r,\,w\,;\,\alpha)) \geq c_-(K(r,\,w\,;\,\alpha)) \geq (n-w)r^2/2.$$

In particular this condition holds if $l(\alpha) \leq nr/4$, or if α consists of fewer than $n/4$ full negative twists.

(b) *Let K be a knot that admits a $-$ adequate diagram D with n crossings and writhe w and suppose $\alpha \in B_r$ has $l(\alpha(s)) < nrs/2$.*

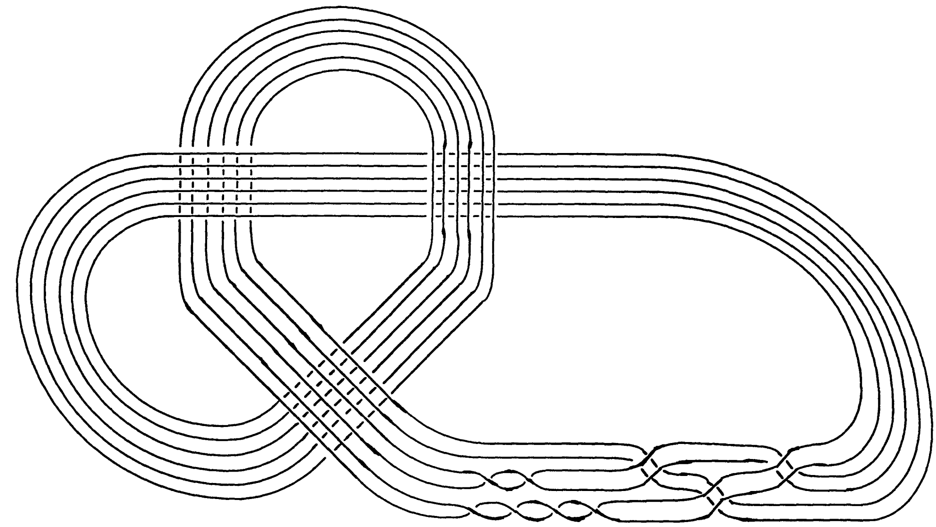


FIGURE 5. The diagram $\mathbf{D}$ for K the trefoil, $\alpha = \sigma_1\sigma_2^{-1}\sigma_1 \in B_3$, $r = 3$, $s = 2$, $a_1 = 0$, $a_2 = 1$ and $a_3 = 2$.

Then

$$c(K(r, w; \alpha)) \geq c_+(K(r, w; \alpha)) \geq (n+w)r^2/2.$$

Proof. It is sufficient to prove part (a). Let $L = K(r, w; \alpha)$. We will use the proposition above to calculate the highest exponent of A in the Jones polynomial of $L^2(t_1, t_2, \ldots, t_c)$ for t_i large. This together with the previous bounds will give the desired result.

A diagram $\mathbf{D}$ for $L^s(t_1, t_2, \ldots, t_c)$ can be built as follows. The diagram $\mathbf{D}$ consists of three sections cyclically ordered, each of which has rs parallel incoming strings and rs parallel outgoing strings (see Figure 5). The first section S_1 consists of the rs-fold planar parallel of D with a small section removed. The second section S_2 consists of the s-fold planar parallel of r parallel segments with a_i full twists added to segment i (the a_i are chosen so that the sum of the a_i's in segments in link-component number k is $t_k - w_k$ and hence $\sum a_i = \sum(t_i - w_i)$). The third section S_3 consists of the s-fold planar parallel of α.

This description shows that $L^s(t_1, t_2, \ldots, t_c)$ has the form $K(rs, w; \beta\alpha(s))$ where β is the braid consisting of the a_i twists (all positive crossings) followed by the s-fold planar parallel of α which satisfies $l(\alpha(s)) < \frac{1}{2}rsn$. Therefore the proposition above applies and

$$
\begin{aligned}
&MV(L^2(t_1, t_2, \dots, t_c)) \\
&\quad = nr^2s^2 + \sum a_i s^2 + 2rs|s_+D| \\
&\qquad + 2\sum a_i s - 2 + w(\alpha)s^2 - 3\left(wr^2s^2 + \sum a_i s^2 + w(\alpha)s^2\right) \\
&\quad = (n-w)r^2s^2 + 2(r|s_+D| - wr^2 - w(\alpha))s \\
&\qquad - 2 - 2\left(\sum t_i + wr^2 + w(\alpha) - \sum w_i\right) s(s-1).
\end{aligned}
$$

From this and the earlier results it follows that $c_-(L) \geq \frac{1}{2}(n-w)r^2$.

The proof above used only the quadratic dependence of the highest power of the Jones polynomial of a cable satellite, thus prompting the following definition.

Definition. A framed knot K, say with framing t, is said to be quadratic (or for definitiveness a-quadratic) if $MV(K(r, t; 1)) = (a-t)r^2 - 2r(r-1)t + O(r)$.

A similar definition could be made for the lowest exponent but to save notation we will omit this definition here. Proposition 3 can be rephrased as saying that if K admits a $+$ adequate diagram D with n crossings and writhe w, then K with the w framing is n-quadratic. It is unclear whether all framed knots are quadratic or whether all knots are quadratic for sufficiently large t. With this definition we have a slightly weaker version of Proposition 4 for quadratic knots.

Proposition 6. *Let K be a framed knot, with framing t, that is a-quadratic. There is an R such that for all $r \geq R$, if $\alpha \in B_r$ satisfies $l(\alpha) < ar/2$, then*

$$MV(K(r, t; \alpha)) = MV(K(r, t; 1)) - 2w(\alpha),$$

where $w(\alpha)$ denotes the writhe of α.

The proof is exactly the same as that of Proposition 4. We have a number of easy corollaries of this proposition including the obvious generalization of Theorem 5.

Corollary 6.1. (a) *Let K be an a-quadratic framed knot with $a \geq 5$, and let K' be the same knot with one lower framing. Then K' is $(a-1)$-quadratic*

(b) *Let K be an a-quadratic framed knot with $a > 0$, and let K' be the same knot with one higher framing. Then K' is $(a+1)$-quadratic.*

Proof. Apply the proposition to α a full negative or full positive twist.

This is not the strongest result possible, a detailed analysis of the effect of gluing in a full negative twist shows that $a \geq 1$ is sufficient. If K is 0-quadratic, then at least in the $+$ adequate case and probably in general, so is the framed knot K' obtained by lowering the framing by one.

COROLLARY 6.2. *Let K be a knot that admits a $+$ adequate diagram with n crossings and writhe d and let (K, t) denote K with the t framing. Then $(K, w - t)$ is $(n - t)$-quadratic for $t \leq n - 4$.*

THEOREM 7. *Let K be a framed knot, say with framing t, that is a-quadratic and suppose $\alpha \in B_r$ has $l(\alpha(s)) < ars/2$. Then*

$$c(K(r, t; \alpha)) \geq c_-(K(r, t; \alpha)) \geq (a - t)r^2/2.$$

In particular this condition holds if $l(\alpha) \leq ar/4$, or if α consists of fewer than $a/4$ full negative twists.

REFERENCES

[1] M. Freedman and Z. X. He, *Incompressible flows: Energy and asymptotic crossing number*, preprint.

[2] L. H. Kauffman, *State models for knot polynomials*, Topology, **26** (1987), 395–407.

[3] W. B. R. Lickorish, *Polynomials for links*, Bull. London Math. Soc., **20** (1988), 558–588.

[4] W. B. R. Lickorish and M. B. Thistlethwaite, *Some links with non-trivial polynomials and their crossing-numbers*, Comment. Math. Helv., **64** (1988), 527–539.

[5] K. Murasugi, *Jones polynomials and classical conjectures in knot theory*, Topology, **26** (1987), 187–194.

[6] M. B. Thistlethwaite, *A spanning tree expansion of the Jones polynomial*, Topology, **26** (1987), 297–309.

[7] ——, *Kauffman's polynomial and alternating links*, Topology, **27** (1988), 311–318.

[8] ——, *Kauffman's polynomial and adequate links*, Invent. Math., **93**, no. 2 (1988), 285–296.

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