3-VALENT GRAPHS AND THE KAUFFMAN BRACKET

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We explicitly determine the tetrahedron coefficient for the one-variable Kauffman bracket, using only Wenzl's recursion formula for the Jones idempotents (or augmentation idempotents) of the Temperley-Lieb algebra.

1. Statement of the main result. In this paper, we consider unoriented knot or tangle diagrams in the plane, up to regular isotopy. Furthermore, we impose the *Kauffman relations* [**Ka1**]

(the first relation refers to three diagrams identical except where shown, and in the second relation, **D** represents any knot or tangle diagram). Coefficients will always be in $\mathbf{Q}(A)$, the field generated by the indeterminate A over the rational numbers. With these relations, (n, n)-tangles (i.e. tangles in the square with n boundary points on the upper edge and n on the lower edge) generate a finite-dimensional associative algebra T_n over $\mathbf{Q}(A)$, called the *Temperley-Lieb algebra* on n strings (see [**Li1**] for more details on what follows). Multiplication in T_n is induced by placing one diagram above another, and the identity element 1_n is given by the identity (n, n)-tangle. Up to isotopy, there is a finite number of (n, n)-tangles without crossings and without closed loops; they form the standard basis of the algebra T_n. Let $\varepsilon\colon T_n \to \mathbf{Q}(A)$ denote the associated augmentation homomorphism, i.e. $\varepsilon(1_n) = 1$, and ε is zero on the other basis elements. The algebra T_n (with coefficients in $\mathbf{Q}(A)$) is semisimple, and in particular there is an *augmentation idempotent* $f_n \in T_n$ with the property

$$f_n x = x f_n = \varepsilon(x) f_n$$

for all $x \in T_n$, and $f_n^2 = f_n$.[1] These idempotents were first discovered by Jones [**J1**]. Up to normalisation, they are equal to the "magic weaving elements" of [**Ka2**]. There is a recursion formula for these idempotents due to Wenzl [**We**], which we will now state graphically.

We make the convention that writing n beneath a component of a knot or tangle diagram means that this component has to be replaced by n parallel ones (n may be zero).[2] We will often say that this component is *colored* by n (a *color* is just a non-negative integer). For graphical reasons, we make the further convention that we may omit indicating the color if it can easily be deduced from the rest of the diagram. Let us denote the augmentation idempotent $f_n \in T_n$ by a little box. Then Wenzl's formula can be stated graphically as follows:

Here, we use the notation $[n] = (A^{2n} - A^{-2n})/(A^2 - A^{-2})$. For a proof of this formula, see [**We**] or [**Li1**]. (Notice that in the last diagram of the formula above, we did not indicate the color of the line joining the two boxes. As explained above, this does not mean that the color is one, but that the color must be calculated form the rest of the diagram. Here clearly the color is $n - 2$, because the little boxes represent elements of the Temperely-Lieb algebra on $n - 1$ strings.)

Following Kauffman [**Ka2**], we define a 3-*valent vertex* as follows. A triple (a, b, c) of colors is called *admissible* if $a + b + c$ is even and $|a - b| \leq c \leq a + b$ (triangle inequality). Given an admissible triple, we define a 3-valent vertex by


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[1] In [**Li1**, **Li2**], our f_n is denoted $f_{n-1} = f^{(n)}$.

[2] This convention is consistent with [**Li1**, **Li2**], but *not* with the graphical conventions of [**BHMV1**].

The numbers $i = (b+c-a)/2$, $j = (a+c-b)/2$, $k = (a+b-c)/2$ of internal lines of the 3-valent vertex will be called its *internal colors*. Notice that a triple (a, b, c) is admissible if and only if one can find corresponding internal colors.

The goal of this paper is to give an explicit determination of the following three coefficients, using only Wenzl's formula for the augmentation idempotents f_n.

trihedron coefficient:

$$= \langle a, b, c \rangle$$

tetrahedron coefficient:

$$= \begin{pmatrix} A & B & E \\ D & C & F \end{pmatrix}$$
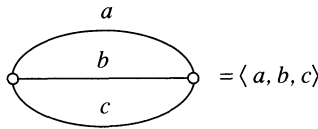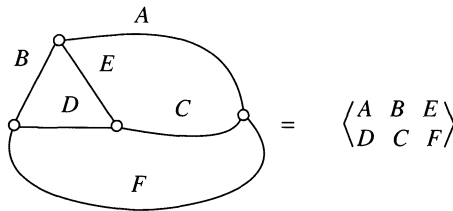
(Of course, the trihedron coefficient is a special case of the tetrahedron coefficient, e.g. $\begin{pmatrix} a & a & 0 \\ b & b & c \end{pmatrix} = \langle a, b, c \rangle$.)

half-twist coefficient:

$$= \delta(c; a, b)$$
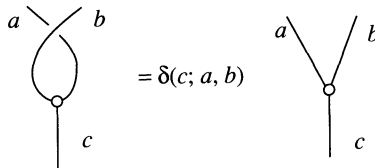
REMARKS. (1) The names trihedron and tetrahedron coefficient are motivated by the fact that one may imagine the corresponding graphs as a regular trihedron or tetrahedron drawn on a sphere.

(2) Knowing the tetrahedron coefficient is equivalent to knowing the (more popular) recoupling coefficients or 6-j-symbols (see §2).

(3) Reversing the sign of the crossing in the half-twist coefficient replaces $\delta(c\,;a\,,b)$ by its conjugate $\overline{\delta(c\,;a\,,b)}$, where the conjugation on $\mathbf{Q}(A)$ is defined by $\overline{A}=A^{-1}$.

Recall $[n]=(A^{2n}-A^{-2n})/(A^2-A^{-2})$. This notation is motivated by the fact that $[n]|_{A=1}=n$. Notice $[0]=0$. We also define $[n]!=[1]\cdots[n]$ for $n\geq 1$, and $[0]!=1$.

THEOREM 1 (*Trihedron coefficient*). *Let* $(a\,,b\,,c)$ *be admissible and let* $i\,,j\,,k$ *be the internal colors of a* 3-*valent vertex* $(a\,,b\,,c)$. *Then*

$$\langle a\,,b\,,c\rangle = (-1)^{i+j+k}\frac{[i+j+k+1]![i]![j]![k]!}{[i+j]![j+k]![i+k]!}.$$

THEOREM 2 (*Tetrahedron coefficient*). *Let* $A\,,B\,,C\,,D\,,E\,,F$ *be colors such that the triples* $(A\,,B\,,E)$, $(B\,,D\,,F)$, $(E\,,D\,,C)$ *and* $(A\,,C\,,F)$ *are admissible. Set* $\Sigma=A+B+C+D+E+F$ *and*

$$\begin{aligned}
a_1 &= (A+B+E)/2\,, & b_1 &= (\Sigma-A-D)/2\,,\\
a_2 &= (B+D+F)/2\,, & b_2 &= (\Sigma-E-F)/2\,,\\
a_3 &= (C+D+E)/2\,, & b_3 &= (\Sigma-B-C)/2\,.\\
a_4 &= (A+C+F)/2\,,
\end{aligned}$$

Then

$$\left\langle\begin{matrix}A & B & E\\ D & C & F\end{matrix}\right\rangle = \frac{\prod_{i=1}^{3}\prod_{j=1}^{4}[b_i-a_j]!}{[A]![B]![C]![D]![E]![F]!}\begin{pmatrix}a_1 & a_2 & a_3 & a_4\\ & b_1 & b_2 & b_3\end{pmatrix}$$

where

$$\begin{pmatrix}a_1 & a_2 & a_3 & a_4\\ & b_1 & b_2 & b_3\end{pmatrix} = \sum_{\max(a_j)\leq\zeta\leq\min(b_i)}\frac{(-1)^{\zeta}[\zeta+1]!}{\prod_{i=1}^{3}[b_i-\zeta]!\prod_{j=1}^{4}[\zeta-a_j]!}.$$

(Notice that the a_j correspond to the four vertices of the tetrahedron, whereas the b_j correspond to the three pairs of opposite edges. Moreover, it is easy to verify that the 12 numbers b_i-a_j are the internal colors of the four 3-valent vertices.)

THEOREM 3 (*Half-twist coefficient*). *Let* $(a\,,b\,,c)$ *be admissible and let* $i=(b+c-a)/2$, $j=(a+c-b)/2$, $k=(a+b-c)/2$ *be the*

interval colors of a 3-*valent vertex* (a, b, c). *Then*

$$\delta(c; a, b) = (-1)^k A^{ij-k(i+j+k+2)}.$$

REMARKS. (1) Similar (but not identical) formulas were deduced by Reshetikhin and Kirillov [KR] from the representation theory of the quantum group $U_q \mathrm{Sl}_2$.[3] They are in fact q-analogues of formulas that have been used by physicists for a long time in the quantum theory of angular momentum (see for example [BL]). Piunikhin [P] has worked out the precise relationship between Kauffman's definition of a 3-valent vertex, which is the one we use here (up to normalisation), and the definition coming form the representation theory of $U_q \mathrm{Sl}_2$ (we will state his formula in an appendix to §2). Using the result of [KR], he thus obtains a formula for the tetrahedron coefficient. Nevertheless, it seems fair to say that the approach we take here is a much simpler way to get our formulas, because the representation theory approach involves much more complicated calculations than our approach through the Kauffman bracket.

(2) The results of this paper are used in [BHMV3], where it is shown how the invariants θ_p of [BHMV1, BHMV2] lead to an elementary construction of a "Topological Quantum Field Theory".

The remainder of this paper is organized as follows. In §2, we will show how the knowledge of the three coefficients above allows one to evaluate any colored link diagram or any colored 3-valent graph. (This is a fact well known to specialists.) The bulk of this paper is §3, where we will give the proof of Theorems 1–3.

The authors wish to acknowledge helpful conversations with O. Viro, who pointed out a great simplification in our original proof of Theorem 3. The proof of Theorem 3 given here is due to him.

2. Evaluating colored link diagrams and colored 3-valent graphs. This section presents a graphical calculus which allows one to evaluate any colored link diagram or any colored 3-valent graph, using the coefficient determined in Theorems 1–3.

We first need to recall some concepts. Let $\mathscr{B}$ be the $\mathbf{Q}(A)$-module generated by link diagrams in an annulus, modulo regular isotopy and the Kauffman relations.[4] Given a k-component link diagram L, and

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[3]In order to make contact with q-formulas and $U_q \mathrm{Sl}_2$, the correct substitution seems to be $A = -q^{-1/4}$. Note that the Kauffman bracket is related to Jones' original V-polynomial [J2] through a similar substitution (see [Ka1]).

[4] $\mathscr{B}$ was called $\mathscr{A}$ in [Li2]. It is also the same as the Jones-Kauffman module of a solid torus as defined in [BHMV1], because of the usual correspondence between planar link diagrams up to regular isotopy, and banded links in 3-space.

$b_1, \ldots, b_k \in \mathscr{B}$, let $\langle b_1, \ldots, b_k \rangle_L$ denote the Kauffman bracket of the linear combination of link diagrams obtained from L by replacing the jth component by b_j. Notice that $\langle, \ldots, \rangle_L$ is a k-linear form on $\mathscr{B}$.[5] There is a "trace map" $T_n \to \mathscr{B}$ given by mapping a tangle in the square to the diagram in the annulus obtained by identifying the upper and lower edges of the square. The image of an element $x \in T_n$ under this map is called the *closure* of x, and will be denoted by $\hat{x}$. Set

$$e_n = \hat{f}_n \in \mathscr{B}.$$

It turns out that the e_n $(n \geq 0)$ form a basis of $\mathscr{B}$, and hence the meta-bracket of L is determined by the values

$$\langle e_{i_1}, \ldots, e_{i_k} \rangle_L.$$

We call this the *colored bracket polynomial* of L. This is up to normalisation the same as the colored Jones polynomial appearing in the construction of 3-manifold invariants in [RT] and [KM][6] (see also [MS]).

The goal of this section is to evaluate $\langle e_{i_1}, \ldots, e_{i_k} \rangle_L$. With our graphical conventions, this is the value of the diagram obtained form L by writing i_j beneath the jth component, and inserting one little box into each component. Such a diagram will be called a *colored link diagram*. (We may place the little box wherever we want on the component. Also, we may insert several boxes since a box represents an idempotent.) Let C denote the diagram of a circle, and set

(This was denoted by $\langle e_k \rangle$ in [BHMV1], and is equal to $(-1)^k[k+1] = (-1)^k(A^{2k+2} - A^{-2k-2})/(A^2 - A^{-2})$.) Gluing two annuli together so as to get a third endows $\mathscr{B}$ with a multiplication for which $e_0 = 1$ and $e_1 e_i = e_{i+1} + e_{i-1}$ [BHMV1]. It follows easily that in $\mathscr{B}$, one has the equation

$$e_i e_j = \sum_k e_k$$

(the sum being over all k such that the triple (i, j, k) is admissible).

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[5]This k-linear form was first considered in [Li2]. We call it the *meta-bracket* of L [BHMV1].

[6]See also a remark in the proof of Prop. 2.2 in [BHMV2].

This implies the following graphical equation in T_{i+j}:

$$\left| \begin{array}{c} i \\ \boxed{} \end{array} \right| \left| \begin{array}{c} j \\ \boxed{} \end{array} \right| = \sum_k \frac{\langle k \rangle}{\langle i, j, k \rangle} \quad \begin{array}{c} i \bigvee j \\ k \\ i \bigwedge j \end{array}$$

We now show how to use this and the coefficients determined in Theorems 1–3 to evaluate colored link diagrams. (The rangle of summation is not indicated explicitly in the formulas below. It is determined precisely by the requirement that the colors meeting at a 3-valent vertex should form an admissible triple.) First of all, we can get rid of crossings using the above formula and the half-twist coefficient, as follows:

$$\begin{array}{c} i \\ \diagup \\ j \end{array} = \sum_k \frac{\langle k \rangle}{\langle i, j, k \rangle} \qquad \begin{array}{c} i \; j \\ k \\ j \bigwedge i \end{array} = \sum_k \delta(k; i, j) \frac{\langle k \rangle}{\langle i, j, k \rangle} \qquad \begin{array}{c} i \bigvee j \\ k \\ j \bigwedge i \end{array}$$

Thus we are left with a colored 3-valent graph. Notice that an edge colored with zero may just as well be removed, since f_0 is represented by the empty $(0, 0)$-tangle. Since

$$(*) \qquad \begin{array}{c} n \\ \left(i \quad j \right) \\ k \end{array} = \delta_k^n \frac{\langle i, j, k \rangle}{\langle k \rangle} \quad \begin{array}{c} k \\ \boxed{} \end{array}$$

any diangular face (that is a face with precisely two vertices) of a graph can be eliminated. Similarly, any triangular face can be simplified using the tetrahedron coefficient as follows:

$$\begin{array}{c} A \\ B \diagup \quad E \\ F \; \overline{\quad D \quad} \; C \end{array} = \frac{\left\langle \begin{array}{ccc} A & B & E \\ D & C & F \end{array} \right\rangle}{\langle A, F, C \rangle} \quad \begin{array}{c} A \\ \diagdown \\ F \qquad C \end{array}$$

More generally, any triangle, square, pentagon, etc. can be reduced to a diangular face (which can then be eliminated by formula (∗) above) using

$$= \sum_i \left\{ \begin{matrix} a & b & i \\ c & d & j \end{matrix} \right\}$$

where $\left\{ \begin{smallmatrix} a & b & i \\ c & d & j \end{smallmatrix} \right\}$ is the *recoupling coefficient* or *6-j-symbol* defined by

$$\left\{ \begin{matrix} a & b & i \\ c & d & j \end{matrix} \right\} = \frac{\langle i \rangle \left\langle \begin{matrix} i & b & c \\ j & d & a \end{matrix} \right\rangle}{\langle i, a, d \rangle \langle i, b, c \rangle}$$

(see [**Ka2**] for a graphical proof of this formula). We now are left with a connected graph each of whose faces has at most one vertex. As explained above, we may assume no edge is colored by zero. Then if the graph has a vertex, the value is zero, because for $k \geq 1$ one has

as follows immediately from the fact that f_k is the augmentation idempotent. If however the graph has no vertex, then it is simply a disjoint union of circles colored by k_i's, and its value is the product of the $\langle k_i \rangle$'s. (By the very definition of the Kauffman bracket, the value of a disjoint union of diagrams is equal to the product of the values of the individual components.)

This completes the evaluation of colored link diagrams.

REMARK. The reader will find two more useful graphical identities in the proof of Theorem 3.

REMARK. In this paper, we have made the convention that writing n beneath a component of a knot or tangle diagram just means that this component has to be replaced by n parallel ones. It is sometimes useful to make the additional convention that writing n beneath a

component also includes insertion of the idempotent f_n . For example, this would be convenient when using the graphical calculus to evaluate the colored bracket polynomial of a link diagram. Indeed, in this context it is sufficient to consider only diagrams where each component contains an f_n , and it would then no longer be necessary to draw the little boxes explicitly. However, for the purpose of this paper, which is to give an elementary proof of Theorems 1–3, it is useful *not* to make this additional convention.

Appendix. *Comparison with the representation theoretical approach.* Here is the relationship of our graphical calculus with the one of [**KR**] (see [**P**] for more details). Let V_n denote the quantum deformation of the standard irreducible representation of sl_2 of spin $n/2$ (i.e. of dimension $n + 1$). The 3-valent vertices of [**KR**] represent certain intertwining homomorphisms $K^c_{ab} \colon V_a \otimes V_b \to V_c$, $K^{ab}_c \colon V_c \to V_a \otimes V_b$ as follows:

As explained in [**P**], the trivalent vertex as defined in §1 may also be interpreted as an intertwiner, and there are coefficients ϕ^{ab}_c , ϕ^c_{ab} such that

Piunikhin has shown that

$$\phi^{ab}_c = \phi^c_{ab} = \pm \sqrt{\frac{\langle a, b, c \rangle}{\langle c \rangle}}$$

(see formula 4.12 of [**P**]). [7],[8]

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[7] Actually, formula 4.12 of [**P**] contains a certain 6-j-symbol which can however be evaluated.

[8] The meaning of the square root is not specified in [**P**].
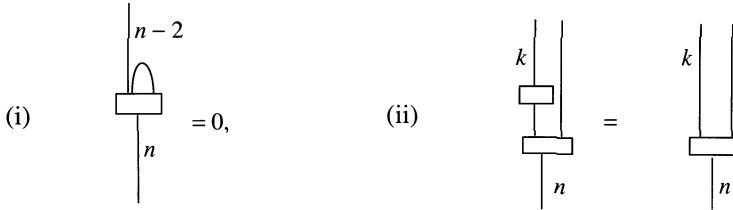
REMARK. In view of formula $(*)$ above, the equation

$$\phi_c^{ab}\,\phi_{ab}^c = \frac{\langle a\,,\,b\,,\,c\rangle}{\langle c\rangle}$$

is equivalent to

$$K_{ab}^c \circ K_c^{ab} = \mathrm{id}_{V_c}$$

(assuming $(a\,,\,b\,,\,c)$ is an admissible triple).

3. Proof of Theorems 1–3. The properties of the f_n that we will need are Wenzl's recursion formula (see §1) and the fact that they are augmentation idempotents, which implies the following two equations:

(Recall that writing n beneath a component of a tangle diagram means that this component has to be replaced by n parallel ones.)

LEMMA 1.
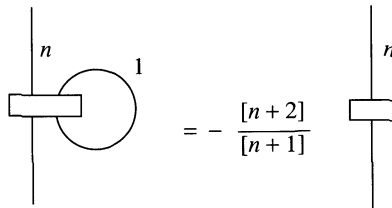
Proof. Let us denote by g_n the element of T_n represented by the diagram on the L.H.S. Using (ii), one sees that $g_n = \lambda f_n$ for some constant λ. The closure $\widehat{g_n} \in \mathscr{B}$ of g_n satisfies

$$\langle \widehat{g_n}\rangle_C = \langle n+1\rangle = \langle e_{n+1}\rangle_C$$

(these notations are defined in §2). Hence

$$\lambda = \frac{\langle e_{n+1}\rangle_C}{\langle e_n\rangle_C} = -\frac{[n+2]}{[n+1]}$$
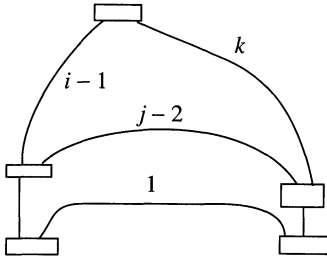
as asserted.

LEMMA 2. *For $j \geq 1$:*

Proof. The proof is by induction on j. For $j = 1$, this follows from (ii). For $j \geq 2$, we apply Wenzl's formula on the L.H.S. to the box representing f_{i+j-1}. The first term of that formula yields zero by (i): hence we get

Using (ii), we can draw this as follows:

Applying the induction hypothesis to the upper part, we get

$$= \frac{[i+j-2]}{[i+j-1]} \frac{[i]}{[i+j-2]}$$

Using (ii) again, the lemma follows.

REMARK. Lemma 2 remains valid for $i = 0$ if we make the convention that a diagram is zero if it has a negative number of strings somewhere.

LEMMA 3. $[n + r][m + r] = [n][m] + [n + m + r][r]$.

Proof. This can be verified by a direct calculation. Alternatively, it also follows easily from the equation
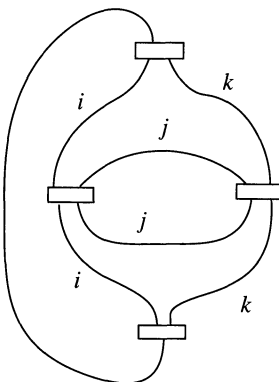
$$e_i e_j = \sum_k e_k$$

(the sum being over all k such that the triple (i, j, k) is admissible) (see §2). Details are left to the reader.

Proof of Theorem 1. It is convenient to set
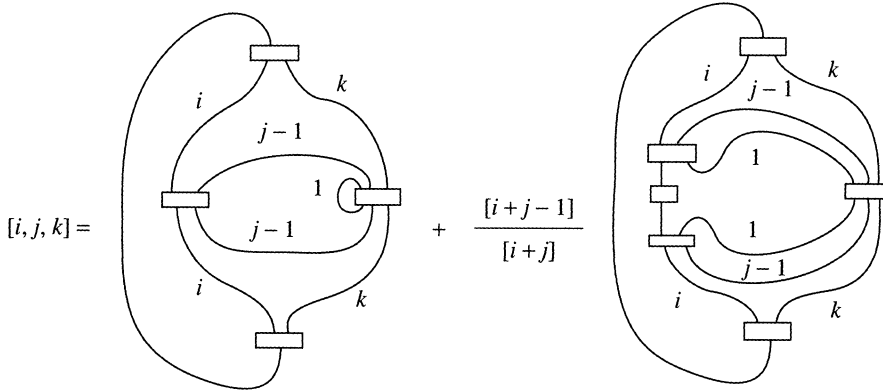
$$\langle a, b, c \rangle = [i, j, k]$$

where the i, j, k are the internal colors of a 3-valent vertex (a, b, c). We have

$$\langle a, b, c \rangle = [i, j, k] =$$

The proof is by induction on j. If $j = 0$, then clearly

$$[i, j, k] = \langle i + k \rangle = (-1)^{i+k}[i + k + 1]$$

and the theorem is true in this case. If $j \geq 1$, we apply Wenzl's formula to get

We can simplify this, using Lemmas 1 and 2. Thus we get the following recursion formula:

$$[i, j, k] = -\frac{[j + k + 1]}{[j + k]}[i, j - 1, k]$$

$$+ \frac{[i]^2}{[i + j][i + j - 1]}[i - 1, j - 1, k + 1].$$

Applying the induction hypothesis to the R.H.S., we get after simplifying

$$[i, j, k]$$

$$= (-1)^{i+j+k}\frac{[i + j + k + 1]![i]![j]![k]!}{[i + j]![j + k]![i + k]!} \frac{[i + j][j + k + 1] - [i][k + 1]}{[j][i + j + k + 1]}.$$

But $[i + j][j + k + 1] - [i][k + 1] = [j][i + j + k + 1]$ because of Lemma 3, whence the result.

LEMMA 4. *For $y \geq 1$*
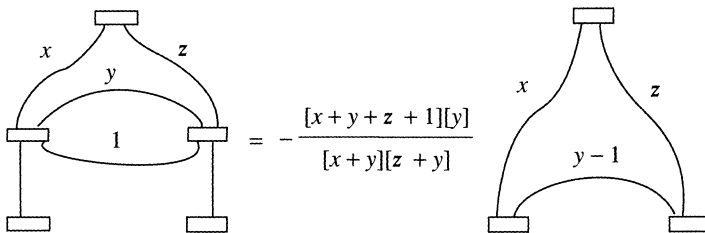
Proof. Wenzl's recursion formula applied to the box representing f_{x+y} yields

The first diagram on the R.H.S. can be simplified using Lemma 1. The second diagram can be simplified using Lemma 2 twice, as indicated below:

Hence

The result now follows from Lemma 3.

Proof of Theorem 2. We start by reformulating Theorem 2 as follows. Notice that the numbers a_j and b_i defined in Theorem 2 satisfy $\sum a_j = \sum b_i = \Sigma$ and $\max(a_j) \leq \min(b_i)$. Conversely, it is not hard to see that whenever $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ are nonnegative integers satisfying these two conditions, then they uniquely determine A, B, C etc. such that there is a well-defined tetrahedron coefficient. With this in mind, we define

$$
\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ & b_1 & b_2 & b_3 & \end{bmatrix}
$$

$$
= \begin{cases} \dfrac{[A]![B]![C]![D]![E]![F]!}{\prod_{i=1}^{3} \prod_{j=1}^{4} [b_i - a_j]!} \left\langle \begin{matrix} A & B & E \\ D & C & F \end{matrix} \right\rangle & \text{if } \max(a_j) \leq \min(b_i) \\ 0 & \text{if } \max(a_j) > \min(b_i) \end{cases}
$$

whenever $\sum a_j = \sum b_i$ (and $a_j \geq 0$, $b_i \geq 0$).

Theorem 2 can now be stated as follows:

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ & b_1 & b_2 & b_3 & \end{bmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ & b_1 & b_2 & b_3 & \end{pmatrix},$$

where the R.H.S. is defined by

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ & b_1 & b_2 & b_3 & \end{pmatrix} = \sum_{\max(a_j) \leq \zeta \leq \min(b_i)} \frac{(-1)^\zeta [\zeta + 1]!}{\prod_{i=1}^3 [b_i - \zeta]! \prod_{j=1}^4 [\zeta - a_j]!}.$$

To obtain a recursion formula for $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ & b_1 & b_2 & b_3 & \end{bmatrix}$, it is convenient to introduce some more notation. Let i, j, k denote the internal colors of the vertex labeled a_1, and similarly α, β, γ those of a_2, and x, y, z those of a_3, as in the following figure:
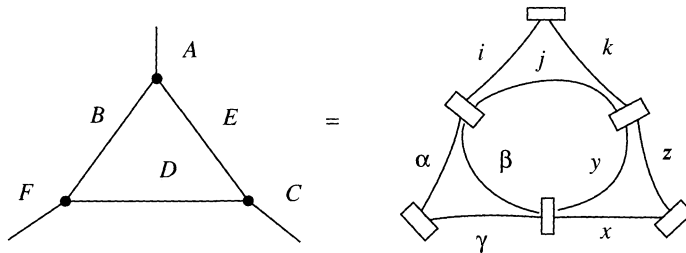
Assume that $j \geq 1$ and $\beta \geq 1$. Applying Wenzl's formula to the edge labeled $B = i + j = \alpha + \beta$, we obtain the above
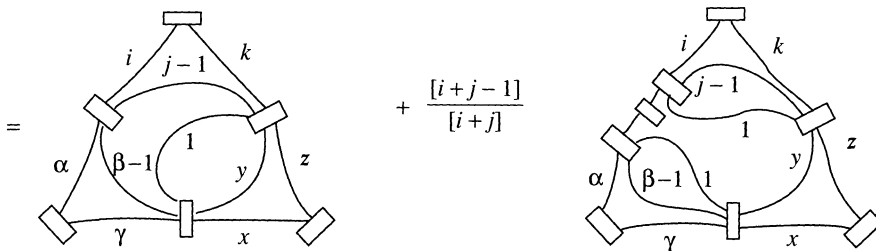
Observe that if $y = 0$, then the first term is zero, and if $i = 0$ or $\alpha = 0$, then the second term is zero (cf. the remark following Lemma 2). We apply Lemmas 4 and 2 and obtain the above

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$$= -\frac{[x+y+z+1][y]}{[x+y][y+z]} \quad \left(\text{diagram with } i,\ j-1,\ k,\ \alpha,\ \beta-1,\ y-1,\ z,\ \gamma,\ x\right) + \frac{[i][\alpha]}{[i+j][i+j-1]} \quad \left(\text{diagram with } i-1,\ j-1,\ k+1,\ \alpha-1,\ \beta-1,\ y,\ z,\ \gamma+1,\ x\right)$$

For this recursion formula to be valid it is sufficient to suppose that $j \geq 1$ and $\beta \geq 1$, but we may allow some of the other seven numbers $i, k, \alpha, \gamma, x, y, z$ to be zero. (We make the convention that a diagram is zero if it has a negative number of strings somewhere. This is consistent with the convention that

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ & b_1 & b_2 & b_3 & \end{bmatrix} = 0$$

if $\max(a_j) > \min(b_i)$, because the 12 numbers $b_i - a_j$ are the internal colors of the four vertices of the tetrahedron.) If $j = 0$ or $\beta = 0$, however, the R.H.S. is zero whereas the L.H.S. may be non-zero. But if we multiply both sides of the recursion formula by $[j][\beta] = [b_1 - a_4][b_2 - a_4]$, the result will be true without restriction. After rewriting things in terms of the a_j and the b_i, we thus obtain:

$$[b_1 - a_4][b_2 - a_4] \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ & b_1 & b_2 & b_3 & \end{bmatrix}$$

$$= -[a_3 + 1][a_1 + a_2 - b_3] \begin{bmatrix} a_1 - 1 & a_2 - 1 & a_3 - 1 & a_4 \\ & b_1 - 1 & b_2 - 1 & b_3 - 1 & \end{bmatrix}$$

$$+ [b_3 - a_1 + 1][b_3 - a_2 + 1] \begin{bmatrix} a_1 - 1 & a_2 - 1 & a_3 & a_4 \\ & b_1 - 1 & b_2 - 1 & b_3 & \end{bmatrix}.$$

We now set out to prove

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ & b_1 & b_2 & b_3 & \end{bmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ & b_1 & b_2 & b_3 & \end{pmatrix}.$$

If $\max(a_j) > \min(b_i)$, this is true by definition.

Next, we consider the case where $\max(a_j) = \min(b_i)$. Then (at least) one of the 12 numbers $b_i - a_j$ is zero, and without loss of generality, we may suppose $y = b_3 - a_4 = 0$. We proceed by induction on $\min(j, \beta) = \min(b_1 - a_4, b_2 - a_4)$. If this is zero, then the tetrahedron coefficient reduces to a trihedron coefficient, and the result

follows easily from Theorem 1. Now suppose $\min(j, \beta) \geq 1$. The first term on the R.H.S. of our recursion formula being zero, we have

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ & b_1 & b_2 & b_3 & \end{bmatrix}$$
$$= \frac{[b_3 - a_1 + 1][b_3 - a_2 + 1]}{[b_1 - a_4][b_2 - a_4]} \begin{bmatrix} a_1 - 1 & a_2 - 1 & a_3 & a_4 \\ & b_1 - 1 & b_2 - 1 & b_3 & \end{bmatrix}.$$

It is easy to verify that the same relation holds for $\left(\begin{smallmatrix} \cdots \\ \cdots \end{smallmatrix} \right)$ in place of $\left[\begin{smallmatrix} \cdots \\ \cdots \end{smallmatrix} \right]$ (the sum reduces to the term for $\zeta = b_3 = a_4$). This yields the induction step. Thus we have shown Theorem 2 in the case $\max(a_j) = \min(b_i)$.

To prove the theorem in the general case, we wish to simplify the recursion formula obtained above. At this point, it is useful to look at symmetries. Observe that the coefficient $\left(\begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{smallmatrix} \right)$ has symmetry $\mathscr{S}_4 \times \mathscr{S}_3$; that is, it is unchanged under any permutation of the a_i or the b_i. We claim the same is true for the coefficient $\left[\begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{smallmatrix} \right]$. Indeed, it follows from the recursion formula above that $\left[\begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{smallmatrix} \right]$ is unchanged under the permutation $a_1 \leftrightarrow a_2$, and also under the permutation $b_1 \leftrightarrow b_2$. But recall that the recursion formula was obtained after choosing a face of the tetrahedron (the face whose vertices are labeled a_1, a_2, a_3) and an edge on the face (the edge joining a_1 and a_2). It is easy to see that the recursion formulas we can get from making the other choices imply that the coefficient $\left[\begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{smallmatrix} \right]$ indeed has symmetry $\mathscr{S}_4 \times \mathscr{S}_3$.

Hence we can exchange the roles of a_2 and a_3 in the recursion formula above. Subtracting the old recursion formula from the new one, we obtain

$$0 = ([a_3 + 1][a_1 + a_2 - b_3] - [a_2 + 1][a_1 + a_3 - b_3])$$
$$\cdot \begin{bmatrix} a_1 - 1 & a_2 - 1 & a_3 - 1 & a_4 \\ & b_1 - 1 & b_2 - 1 & b_3 - 1 & \end{bmatrix}$$
$$+ [b_3 - a_1 + 1][b_3 - a_3 + 1] \begin{bmatrix} a_1 - 1 & a_2 & a_3 - 1 & a_4 \\ & b_1 - 1 & b_2 - 1 & b_3 & \end{bmatrix}$$
$$- [b_3 - a_1 + 1][b_3 - a_2 + 1] \begin{bmatrix} a_1 - 1 & a_2 - 1 & a_3 & a_4 \\ & b_1 - 1 & b_2 - 1 & b_3 & \end{bmatrix}.$$

However, applying Lemma 3 with $n = a_2 - a_3$, $m = a_1 - b_3 - 1$,

$r = a_3 + 1$, we find

$$[a_3 + 1][a_1 + a_2 - b_3] - [a_2 + 1][a_1 + a_3 - b_3]$$
$$= -[a_2 - a_3][a_1 - b_3 - 1].$$

Hence, the equation above has $[b_3 - a_1 + 1]$ (which is equal to $-[a_1 - b_3 - 1]$) as a common factor. If $b_3 \neq a_1 - 1$, we may divide by this common factor, and obtain

$$[a_3 - a_2] \begin{bmatrix} a_1 - 1 & a_2 - 1 & a_3 - 1 & a_4 \\ b_1 - 1 & b_2 - 1 & b_3 - 1 & \end{bmatrix}$$
$$= [b_3 - a_3 + 1] \begin{bmatrix} a_1 - 1 & a_2 & a_3 - 1 & a_4 \\ b_1 - 1 & b_2 - 1 & b_3 & \end{bmatrix}$$
$$- [b_3 - a_2 + 1] \begin{bmatrix} a_1 - 1 & a_2 - 1 & a_3 & a_4 \\ b_1 - 1 & b_2 - 1 & b_3 & \end{bmatrix}.$$

This is the "simpler" recursion formula we sought. Using Lemma 3 as above, it is not hard to verify that the same formula is true with $\left(\begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{smallmatrix} \right)$ in place of $\left[\begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{smallmatrix} \right]$. We leave this to the reader. (It is true "term by term", that is, even before summing over ζ.) Notice that it follows that our formula remains true if $b_3 = a_1 - 1$, because in that case we know already that $\left(\begin{smallmatrix} \cdots \\ \cdots \end{smallmatrix} \right) = \left[\begin{smallmatrix} \cdots \\ \cdots \end{smallmatrix} \right]$ for the three terms appearing in that formula.

Setting

$$\left\{ \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{matrix} \right\} = \left[\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{matrix} \right] - \left(\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{matrix} \right)$$

we thus obtain after a slight change of variables the following final formula:

$$[a_3 - a_2 + 1] \left\{ \begin{matrix} a_1 & a_2 - 1 & a_3 & a_4 \\ b_1 & b_2 & b_3 - 1 \end{matrix} \right\}$$
$$= [b_3 - a_3] \left\{ \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{matrix} \right\}$$
$$- [b_3 - a_2 + 1] \left\{ \begin{matrix} a_1 & a_2 - 1 & a_3 + 1 & a_4 \\ b_1 & b_2 & b_3 \end{matrix} \right\}.$$

We now conclude the proof of Theorem 2 as follows. We must show $\left\{ \begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{smallmatrix} \right\} = 0$. We use double induction on $\Sigma = \sum a_j = \sum b_i$ and on $d = \min(b_i) - \max(a_j)$. For $\Sigma = 0$ there is nothing to show, and

if $d = 0$, then we know already the result. If $\Sigma \geq 1$ and $d \geq 1$, we look at our final formula. By the hypothesis of the induction on Σ, the L.H.S. is zero. Since $\left\{ \begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ & b_1 & b_2 & b_3 \end{smallmatrix} \right\}$ is unchanged under any permutation of the a_j, we may assume that $\max(a_j) = a_3$; hence by the hypothesis of the induction on d, the second term of the R.H.S. is zero. Hence

$$[b_3 - a_3] \left\{ \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ & b_1 & b_2 & b_3 \end{matrix} \right\} = 0.$$

But we assumed $d \geq 1$; hence $[b_3 - a_3]$ is non-zero, and the result follows.

This completes the proof of Theorem 2.

Proof of Theorem 3. We reproduce here an argument of O. Viro [V]. One has the following two equations:
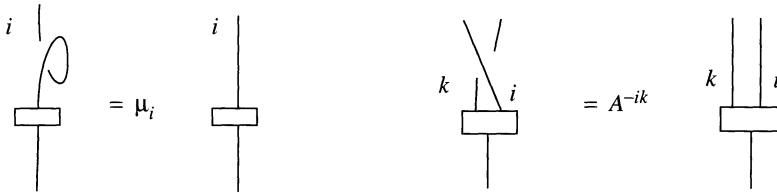
where $\mu_i = (-1)^i A^{i^2 + 2i}$. Indeed, the first equation is an immediate consequence of the equation $te_i = \mu_i e_i$ of [BHMV1] (see also [Li1, Li2]), and the second equation follows from applying the Kauffman relations to the ik crossings of the diagram, together with the fact that the box represents the augmentation idempotent f_{k+i}. Hence
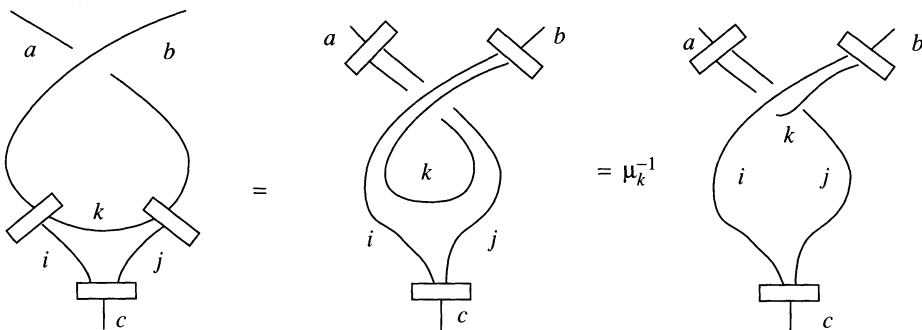
whence

$$\delta(c\,;a\,,b) = \mu_k^{-1}A^{ij-ik-kj} = (-1)^k A^{-k^2-2k+ij-ik-kj}$$

as asserted.

REMARK. One has

$$\delta(c\,;a\,,b)^2 = \frac{\mu_c}{\mu_a\mu_b}.$$

This follows immediately from Theorem 3.

REFERENCES

[BHMV1]   C. Blanchet, N. Habegger, G. Masbaum and P. Vogel, *Three-manifold in-variants derived from the Kauffman bracket*, Topology, **31**, No. 4, (1992), 685–699.

[BHMV2]   ____, *Remarks on the Three-manifold Invariants* θ_p , NATO Advanced Research Workshop and Conference on Operator Algebras, Mathematical Physics, and Low-Dimensional Topology, Istanbul, July 1991 (to appear).

[BHMV3]   ____, *Topological quantum field theories derived from the Kauffman bracket*, preprint May 1992 (in revised from January 1993).

[BL]   L. C. Biedenharn and J. D. Louck, *Angular momentum in quantum physics*, Encyclopedia of Mathematics and its Applications, vol. 8, Addison-Wesley 1981.

[J1]   V. F. R. Jones, *Index of subfactors*, Invent. Math., **72** (1983), 1–25.

[J2]   ____, *A polynomial invariant for links via von Neumann algebras*, Bull. Amer. Math. Soc., **12** (1985), 103–111.

[Ka1]   L. H. Kauffman, *State models and the Jones polynomial*, Topology, **26** (1987), 395–401.

[Ka2]   ____, *Knots, spin networks, and 3-manifold invariants*, (preprint 1990).

[KM]   R. C. Kirby and P. Melvin, *The 3-manifold invariants of Witten and Reshetikhin-Turaev for* sl(2, **C**) , Invent. Math., **105** (1991), 473–545.

[KR]   A. N. Kirillov and N. Yu. Reshetikhin, *Representations of the algebra* $U_q(\mathrm{sl}_2)$, *q-orthogonal polynomials, and invariants of links*, LOMI preprint 1988.

[Li1]   W. B. R. Lickorish, *Three-manifold invariants and the Temperly-Lieb algebra*, Math. Ann., **290** (1991), 657–670.

[Li2]   ———, *Calculations with the Temperley-Lieb algebra*, Comment. Math. Helv., **68** (1992), 571–591.

[MS]   H. R. Morton and P. Strickland, *Jones polynomial invariants for knots and satellites*, Math. Proc. Cambridge Philos. Soc., **109** (1991), 83–103.

[P]   S. Piunikhin, *Turaev-Viro and Kauffman invariants for 3-manifolds coincide*, J. Knot Theory and its Ramifications, vol. I no. 2, (1992), 105–135.

[RT]   N. Yu. Reshetikhin and V. G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math., **103** (1991), 547–597.

[TV]   V. G. Turaev and O. Ya. Viro, *State sum invariants of 3-manifolds and quantum 6-j-symbols*, Topology, **31** (1992), 865–902.

[V]   O. Ya. Viro, private communication.

[We]   H. Wenzl, *On sequences of projections*, C. R. Math. Rep. Acad. Sci. Canada, **IX** (1987), 5–9.

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