PACIFIC JOURNAL OF MATHEMATICS
Vol. 164, No. 2, 1994

FREE PRODUCTS OF COMBINATORIAL STRICT INVERSE SEMIGROUPS

KARL AUINGER

Each combinatorial strict inverse semigroup S is determined by (1) a partially ordered set X which in fact is the partially ordered set of the $\mathcal{J}$-classes of S, (2) pairwise disjoint sets I_α indexed by the elements of X which in fact form the collection of $\mathcal{D}$- (equivalently: $\mathcal{J}$-) related idempotents and (3) structure mappings $f_{\alpha,\beta}: I_\alpha \to I_\beta$ for $\alpha \geq \beta$ satisfying certain compatibility conditions. The multiplication on S can be described in terms of the parameters X, I_α, $f_{\alpha,\beta}$. Conversely, the system $(X; I_\alpha, f_{\alpha,\beta})$ can be characterized abstractly in order that it defines a uniquely determined combinatorial strict inverse semigroup. In this paper, the constituting parameters X, I_α, $f_{\alpha,\beta}$ of the combinatorial strict inverse free product S of a collection of combinatorial strict inverse semigroups S_i are described in terms of the parameters of the semigroups S_i.

As an application it is shown that the word problem for such a free product in general is not decidable.

1. Introduction. The ($\mathcal{V}$-)free product of an arbitrary family $\{S_i | i \in I\}$ of algebras of the same type all of them belonging to the class $\mathcal{V}$ is the coproduct $\prod^* S_i$ in $\mathcal{V}$. There are homomorphisms $\phi_i: S_i \to \prod^* S_i$, $i \in I$, and for any $T \in \mathcal{V}$ and homomorphisms $\psi_i: S_i \to T$, $i \in I$, there is a unique homomorphism $\psi: \prod^* S_i \to T$ such that $\phi_i \psi = \psi_i$ for all $i \in I$.

From purely universal algebraic considerations it follows that the free product exists for any variety $\mathcal{V}$ of inverse semigroups and is generated by isomorphic copies of the members of the given family (see, for instance, Grätzer [5]). Free products have been studied for several classes of semigroups. Semilattice free products and semilattice of groups free products are considered in the book of Petrich [14]. Band, completely simple and completely regular free products have been investigated by Jones [9, 6, 11]. Inverse semigroup free products have been studied by Jones [7, 8, 10] and Jones, Margolis, Meakin and Stephen [12]. The aim of this paper is to describe *combinatorial strict inverse semigroup* free products. A combinatorial strict

A part of the results has been presented at the Conference of Semigroup Theory, Oberwolfach, July 1991.

inverse semigroup is an inverse subdirect product of combinatorial Brandt semigroups and/or the trivial group. This class forms an inverse semigroup variety and plays an important role in the study of the lattice of inverse semigroup varieties (see [**14**]). Each such semigroup S can be described quite efficiently by

(1) a partially ordered set X (which in fact is the partially ordered set of all principal ideals of S),

(2) pairwise disjoint sets I_α indexed by the elements of X,

(3) structure mappings $f_{\alpha,\beta}\colon I_\alpha \to I_\beta$ (for $\alpha \geq \beta$),

(4) a function $\delta\colon I \times I \to X$ where $I = \bigcup_{\alpha\in X} I_\alpha$.

The function δ is determined by the parameters (1)–(3). The semigroup S is realized as the union of the pairwise disjoint sets $I_\alpha \times I_\alpha$. The multiplication in S is described by the structure mappings $f_{\alpha,\beta}$ and the function δ. After having introduced some basic facts about combinatorial strict inverse semigroups in §2, in §3 we shall outline heuristically how the free product of two combinatorial strict inverse semigroups is constructed. In §4, a combinatorial strict inverse semigroup S will be constituted out of a given family of such semigroups S_i, where $i \in I$, according with the ideas of §3. The structure set X of S, the corresponding sets I_α, $\alpha \in X$ and the structure mappings $f_{\alpha,\beta}$ will be described in terms of the ingredients of the respective semigroups S_i and by means of equivalence relations on certain sets. A process which determines the δ-function for S will be provided. In §5 we shall prove that the so constructed semigroup S is the free product of the combinatorial strict inverse semigroups S_i. This also will lead to certain triples which can be interpreted as "canonical forms" for the free product of the semigroups S_i. Finally, in §6, we shall present an example showing that the word problem for free products of combinatorial strict inverse semigroups in general is not decidable. Throughout the paper, the term "free product" will stand for "combinatorial strict inverse free product".

2. Combinatorial strict inverse semigroups. For undefined notions concerning inverse semigroups the reader is referred to the book of Petrich [**14**]. Following [**14**], an inverse semigroup S will be termed *strict* if S is a subdirect product of Brandt semigroups and/or groups. This class forms an inverse semigroup variety. A structure theorem for such semigroups is provided in [**14**, Chapter XIV]; a slightly modified version thereof is in [**1**]. Free objects in certain varieties of strict inverse semigroups have been studied by Reilly [**15**], Margolis, Meakin, Stephen [**13**] and the author [**1, 2**] using the methods of the present paper. In this context, the variety of all *combinatorial* strict inverse semigroups plays an important role. This class is the least inverse semigroup variety which is not Cliffordian (that is, completely regular). Applying the structure theorems of [**14**, Chapter XIV] or of [**1**] to the special case of combinatorial strict inverse semigroups, the following description can be obtained (see [**1**, Corollary 2.6]).

THEOREM 2.1. *Let X be a partially ordered set. For each $\alpha \in X$ let I_α be a non-empty set such that $I_\alpha \cap I_\beta = \varnothing$ if $\alpha \neq \beta$. For each pair $\alpha \geq \beta$ let $f_{\alpha,\beta}: I_\alpha \to I_\beta$ be a mapping subject to the following conditions:*

(1) $f_{\alpha,\alpha} = \mathrm{id}_{I_\alpha}$,

(2) $f_{\alpha,\beta} f_{\beta,\gamma} = f_{\alpha,\gamma}$ *whenever* $\alpha \geq \beta \geq \gamma$,

(3) *for any* $i \in I_\alpha$, $j \in I_\beta$, $\alpha, \beta \in X$, *the set*

$$D(i, j) = \{\gamma \leq \alpha, \beta \mid i f_{\alpha,\gamma} = j f_{\beta,\gamma}\}$$

has a greatest element, to be denoted by $\delta = \delta(i, j)$.

Let $S = \bigcup_{\alpha \in X} I_\alpha \times I_\alpha$ and define a multiplication on S by

$$(i, j)(r, s) = (i f_{\alpha,\delta(j,r)}, s f_{\beta,\delta(j,r)})$$

where $i, j \in I_\alpha$, $r, s \in I_\beta$. Then the groupoid S, to be denoted by $(X; I_\alpha, f_{\alpha,\beta})$ is a combinatorial strict inverse semigroup. Conversely, every combinatorial strict inverse semigroup can be so constructed.

Given $S = (X; I_\alpha, f_{\alpha,\beta})$ then X is isomorphic to the partially ordered set of all principal ideals of S and will be termed the *structure set* of S. The $\mathscr{D}$-classes of S then are precisely the sets $I_\alpha \times I_\alpha$, $\alpha \in X$. Notice that $\mathscr{D} = \mathscr{J}$ in each strict inverse semigroup. The mappings $f_{\alpha,\beta}$ are the *structure mappings* of S. Further, each mapping $f_{\alpha,\beta} \times f_{\alpha,\beta}: I_\alpha \times I_\alpha \to I_\beta \times I_\beta$, defined by $(i, j) \mapsto (i f_{\alpha,\beta}, j f_{\alpha,\beta})$ in fact is a partial homomorphism from the $\mathscr{D}$-class $I_\alpha \times I_\alpha$ to the $\mathscr{D}$-class $I_\beta \times I_\beta$. The function $\delta: I \times I \to X$ (where $I = \bigcup_{\alpha \in X} I_\alpha$) is the *$\delta$-function* of S. Notice that δ is determined by the parameters X, I_α, $f_{\alpha,\beta}$. The structure set X of a (combinatorial) strict inverse semigroup has the following properties (see [**1**, Proposition 2.7]).

PROPOSITION 2.2. *Let X be the structure set of a strict inverse semigroup. Then*

(1) *X is (downwards) directed,*

(2) *for any two elements $\alpha, \beta \in X$ having a common upper bound $\gamma \geq \alpha, \beta$, the greatest lower bound $\alpha \wedge \beta$ exists in X.*

From Theorem 2.1 the following can be easily deduced.

LEMMA 2.3. *Let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a combinatorial strict inverse semigroup. For $i = 1, \ldots, n$ let $k_i, l_i \in I_{\alpha_i}$ for some $\alpha_i \in X$. Let $\gamma \in X$ be such that $(k_1, l_1)(k_2, l_2) \cdots (k_n, l_n) \in I_\gamma \times I_\gamma$. For each i let $\beta_i \in X$ be such that $\alpha_i \geq \beta_i \geq \gamma$. Then*

$$(k_1, l_1) \cdots (k_n, l_n) = (k_1 f_{\alpha_1,\beta_1}, l_1 f_{\alpha_1,\beta_1}) \cdots (k_n f_{\alpha_n,\beta_n}, l_n f_{\alpha_n,\beta_n}).$$

The greatest lower bound of a finite subset $\{\alpha_1, \ldots, \alpha_n\}$ of X will be denoted by $\inf\{\alpha_1, \ldots, \alpha_n\}$ provided it exists. For the two element set $\{\alpha, \beta\}$ instead of $\inf\{\alpha, \beta\}$ also $\alpha \wedge \beta$ will be written. In the following we shall deduce some further results which will be needed in §4.

LEMMA 2.4. *Let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a combinatorial strict inverse semigroup. For $k = 1, \ldots, n$ let $\alpha_k \in X$ and $i_k \in I_{\alpha_k}$. Then*

$$\max\{\gamma \leq \alpha_1, \ldots, \alpha_n | i_1 f_{\alpha_1,\gamma} = \cdots = i_n f_{\alpha_n,\gamma}\}$$

exists in X. Denoting this maximum by $\delta\{i_1, \ldots, i_n\}$ then

$$\delta\{i_1, \ldots, i_n\} = \inf\{\delta(i_1, i_2), \delta(i_2, i_3), \ldots, \delta(i_{n-1}, i_n)\}.$$

Proof. Consider the product

$$w = (i_1, i_1)(i_2, i_2) \cdots (i_n, i_n)$$

and let $\delta \in X$ be such that $w \in I_\delta \times I_\delta$. By induction and the definition of multiplication in S it follows that $\delta = \delta\{i_1, \ldots, i_n\}$ which can be expressed as the mentioned infimum.

Notice that the mentioned element δ can be obtained by computing the product w. For the construction in §4 we shall need the following concepts.

DEFINITION 1. Let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a combinatorial strict inverse semigroup. For $k = 1, \ldots, n$ let $i_k \in I_{\alpha_k}$ for some $\alpha_k \in X$ and $A = \{i_1, \ldots, i_n\}$. Then

$$\delta A = \max\{\gamma \leq \alpha_1, \ldots, \alpha_n | i_1 f_{\alpha_1,\gamma} = \cdots = i_n f_{\alpha_n,\gamma}\}.$$

For singletons $\{i_k\}$ this means $\delta\{i_k\} = \alpha_k$. Further, if $\mathscr{A} = \{A_j | j \in J\}$ is a collection of finite subsets A_j of $\bigcup_{\alpha \in X} I_\alpha$ then put $\delta\mathscr{A} = \{\delta A_j | j \in J\}$.

Definition 2. Let X be the structure set of a combinatorial strict inverse semigroup S. A finite non-empty subset $A \subseteq X$ is *admissible* if any two distinct elements of A do not have a common upper bound in X.

Let X be the structure set of a combinatorial strict inverse semigroup and let $A \subseteq X$ be a finite non-empty subset of X. Consider a finite sequence (π_j) of partitions of A as follows. Let $\pi_0 = \{\{\alpha_1\}, \dots, \{\alpha_n\}\}$ if $A = \{\alpha_1, \dots, \alpha_n\}$. If A is admissible then let (π_j) consist of π_0 only. Otherwise choose elements $\alpha_1, \alpha_2 \in A$ which have a common upper bound in X and put $\pi_1 = \{\{\alpha_1, \alpha_2\}, \{\alpha_3\}, \dots, \{\alpha_n\}\}$. Suppose that $\pi_j = \{A_{j1}, \dots, A_{jk_j}\}$ has already been defined. Put $\inf \pi_j = \{\inf A_{j1}, \dots, \inf A_{jk_j}\}$. If for any $u \neq v$, $\inf A_{ju}$ and $\inf A_{jv}$ do not have a common upper bound then let π_j be the final partition of the sequence. Otherwise choose $A_{ju}, A_{jv} \in \pi_j$ such that $\inf A_{ju}$ and $\inf A_{jv}$ do have a common upper bound in X and let π_{j+1} consist of $A_{ju} \cup A_{jv}$ and the remaining blocks of π_j. (In such a case it may happen that $\inf A_{ju} = \inf A_{jv}$.) If $\inf A_{ju}$ and $\inf A_{jv}$ have a common upper bound then $\inf A_{ju} \wedge \inf A_{jv} = \inf(A_{ju} \cup A_{jv})$ exists. Hence by induction it is justified to assume the existence of $\inf A_{jk}$. Since the number of blocks $|\pi_j|$ is strictly decreasing there is a least n such that $\inf \pi_n$ is admissible and $|\inf \pi_n| = |\pi_n|$.

Definition 3. Let X be the structure set of a combinatorial strict inverse semigroup and $A \subseteq X$ be a finite non-empty subset of X. The sequence (π_j) of partitions as it is constructed above is *an admissible sequence for* A. If π_n is the final partition then $\overline{A} = \inf \pi_n$ is *the admissible set generated by* A.

Lemma 2.5. *The admissible set $\overline{A}$ generated by A is uniquely determined.*

Proof. If $|A| = 1$ then $A = \overline{A}$ is admissible and there is nothing to prove. Let $|A| = n > 1$ and suppose that the assertion be true for all B with $|B| \leq n-1$. If A is admissible then $A = \overline{A}$ and there is nothing to prove. Otherwise there are elements $\alpha_1, \alpha_2 \in A$ which have a common upper bound in X. These elements can be chosen in order that $\alpha_1 \wedge \alpha_2 \notin \{\alpha_3, \dots, \alpha_n\}$ (for instance, if they are minimal in A). Let $B = \{\alpha_1 \wedge \alpha_2, \alpha_3, \dots, \alpha_n\}$; then $|B| = n-1$. Let $\pi_0, \dots, \pi_k$ be an admissible sequence for A. For each j let σ_j be the partition of A which arises from π_j by forming the union of the blocks containing α_1 and α_2 provided these blocks are distinct, or $\sigma_j = \pi_j$ otherwise. In the respective block of σ_j now replace the elements α_1 and α_2 by $\alpha_1 \wedge \alpha_2$, for each j, which yields a sequence (σ'_j) of partitions of B. The transition $\pi_j \to \pi_{j+1}$ is of the form $A_{ju}, A_{jv} \to A_{ju} \cup A_{jv}$. Hence the transition $\sigma_j \to \sigma_{j+1}$ is of the form either $A_{ju}, A_{jv} \to A_{ju} \cup A_{jv}$ or $A_{ju} \cup A_{jx}, A_{jv} \to A_{ju} \cup A_{jx} \cup A_{jv}$ or $A_{ju} \cup A_{jv} \to A_{ju} \cup A_{jv}$, depending on the blocks of α_1 and α_2. Denoting the blocks in the corresponding partitions σ'_j by A'_{jl} then $\sigma'_j \to \sigma'_{j+1}$ is of the form $A'_{ju}, A'_{jv} \to A'_{ju} \cup A'_{jv}$ or $(A_{ju} \cup A_{jx})', A'_{jv} \to (A_{ju} \cup A_{jx})' \cup A'_{jv}$ or $(A_{ju} \cup A_{jv})' \to (A_{ju} \cup A_{jv})'$. The latter case happens precisely once. In this case, $\sigma'_j = \sigma'_{j+1}$ and σ'_{j+1} may be deleted in the sequence (σ'_j) so that it is an admissible sequence for B. Now $\overline{A} = \inf \pi_n = \inf \sigma_n = \inf \sigma'_{n-1} = \overline{B}$. *Each* admissible sequence (π_j) for A therefore can be associated with an admissible sequence (σ'_j) for B and both of them yield the same admissible set $\overline{B}$. Since by hypothesis of induction $\overline{B}$ is uniquely determined, so is $\overline{A}$.

REMARKS. (1) Let A be a finite subset of X. Then each $\alpha \in A$ has (precisely) one lower bound in $\overline{A}$. Conversely, each $\alpha' \in \overline{A}$ has (at least) one upper bound in A.

(2) If for two elements $\alpha, \beta \in X$ a common upper bound γ is known then the meet $\alpha \wedge \beta$ can be calculated as follows: $\alpha \wedge \beta = \delta(if_{\gamma,\alpha}, if_{\gamma,\beta})$ for any $i \in I_\gamma$ (see [**1**, Proof of Proposition 2.7]).

(3) The set $\mathscr{P}(X)$ of all admissible subsets of X forms a $\wedge$-semilattice if $\wedge$ is defined by $A \wedge B = \overline{A \cup B}$. The mapping $\alpha \mapsto \{\alpha\}$ embeds the partially ordered set X isomorphically into $\mathscr{P}(X)$. However, meets will not be respected in general by this embedding.

(4) If a partial product on X is defined by $\alpha \wedge_p \beta = \inf\{\alpha, \beta\}$ if and only if α and β have a common upper bound in X then $(\mathscr{P}(X), \wedge)$ is the free semilattice generated by the partial semilattice $(X, \wedge_p)$.

The partial order on the set of all admissible sets $\mathscr{P}(X)$ which is defined by the above mentioned semilattice structure is characterized as follows.

LEMMA 2.6. *Let $A, B \in \mathscr{P}(X)$ be two admissible subsets of X. Then $A \wedge B = B$ if and only if each $\alpha \in A$ has a* (*unique*) *lower bound β in B.*

Proof. First, if some $\alpha \in X$ has a lower bound β in the admissible set B then by admissibility of B, this lower bound is necessarily unique. If each $\alpha \in A$ has a lower bound $\beta \in B$ then an admissible sequence for $A \cup B$ can be obtained by successively forming the sets $\{\alpha_{i_1}, \alpha_{i_2}, \dots, \beta\}$ where $\beta \in B$ is the lower bound of the elements $\alpha_{i_1}, \alpha_{i_2}, \dots \in A$. Conversely, suppose that $\alpha \in A$ has no lower bound in B. By definition of the algorithm which constructs $C = \overline{A \cup B}$, it follows that each element of $A \cup B$ has a lower bound in C. Consequently, $C \neq B$.

The following result characterizes the admissible set generated by some set.

LEMMA 2.7. *Let X be a structure set of a (combinatorial) strict inverse semigroup. Let $A \subseteq X$ be a finite subset and let $B \in \mathscr{P}(X)$ be an admissible set such that each $\alpha \in A$ has a lower bound $\beta \in B$. Let $\overline{A}$ be the admissible set generated by A. Then each $\alpha' \in \overline{A}$ has a lower bound in B.*

Proof. We use induction on $|A|$. If $|A| = 1$ then the assertion holds trivially. Let $A \subseteq X$, $|A| = n > 1$, $B \in \mathscr{P}(X)$ and suppose that the assertion be true for each $A' \subseteq X$ with $|A'| < n$. Suppose that each $\alpha \in A$ has a lower bound β in B. If $A = \overline{A}$ then there is nothing to prove. Otherwise choose elements $\alpha, \alpha' \in A$ as in the proof of Lemma 2.5 which have a common upper bound in X. Since α and α' have lower bounds β and β' in B and since B is admissible, $\beta = \beta'$. Hence $\alpha \wedge \alpha' \geq \beta$. Therefore, each element of $A' = \{\alpha \wedge \alpha'\} \cup (A \setminus \{\alpha, \alpha'\})$ has a lower bound in B. By hypothesis of induction, each element of $\overline{A'}$ has a lower bound in B. As in the proof of Lemma 2.5, $\overline{A} = \overline{A'}$ so that the assertion follows.

REMARK. Lemma 2.7 in fact states that $\overline{A} \geq B$ in the natural order of the semilattice $(\mathscr{P}(X), \wedge)$.

3. A heuristic consideration. In this section we briefly outline the idea of how the parameters X, I_α, $f_{\alpha,\beta}$ of the free product of two combinatorial strict inverse semigroups $S = (X_S; I_{\alpha_S}, f_{\alpha_S,\beta_S})$ and $T = (X_T; I_{\alpha_T}, f_{\alpha_T,\beta_T})$ can be expressed in terms of the parameters of the latter semigroups. Let $i, j \in I_{\alpha_S}$, $k, l \in I_{\alpha_T}$ and consider the product $(i, j)(k, l) \in S * T$. The partially ordered sets X_S, X_T can be assumed to be disjoint order filters in X. Let $\alpha \in X$ correspond to the $\mathscr{D}$-class of $(i, j)(k, l)$, that is,

$$(i, j)(k, l) = (i f_{\alpha_S,\alpha}, l f_{\alpha_T,\alpha}) \in I_\alpha \times I_\alpha .$$

Then $\alpha \notin X_S \cup X_T$. All we know is that $\alpha_S \geq \alpha$ and $\alpha_T \geq \alpha$. By the universal property of the free product it seems likely that for $\beta \in X_S \cup X_T$, $\beta \geq \alpha$ if and only if $\beta \geq \alpha_S$ or $\beta \geq \alpha_T$. Considering the mappings $f_{\alpha_S, \alpha}$ and $f_{\alpha_T, \alpha}$, all we know is that $if_{\alpha_S, \alpha} = kf_{\alpha_T, \alpha}$. Again by the universal property of the free product, it is reasonable that $f_{\alpha_S, \alpha}$ is injective on I_{α_S} and so is $f_{\alpha_T, \alpha}$ on I_{α_T}. Also, $uf_{\alpha_S, \alpha} \neq vf_{\alpha_T, \alpha}$ whenever $(u, v) \neq (j, k)$. (If the mentioned assertions were not true then one could construct an example being in contradiction to the universal property of $S * T$.) Finally, $I_\alpha = I_{\alpha_S} f_{\alpha_S, \alpha} \cup I_{\alpha_T} f_{\alpha_T, \alpha}$ since otherwise $S * T$ would not be generated by S and T. We therefore are motivated to *identify* the element α with the equivalence relation on $I_{\alpha_S} \cup I_{\alpha_T}$ identifying j and k and all other equivalence classes being singletons. The set I_α then is an isomorphic copy of $(I_{\alpha_S} \cup I_{\alpha_T})/\alpha$. In this way, each $\alpha \in X$ can be associated with a certain equivalence relation on some set $I_{\alpha_1} \cup \cdots \cup I_{\alpha_n}$ for a suitable finite set $\{\alpha_1, \ldots, \alpha_n\} \subseteq X_S \cup X_T$. The question arises which finite sets $\{\alpha_1, \ldots, \alpha_n\} \subseteq X_S \cup X_T$ and which equivalence relations α on $I_{\alpha_1} \cup \cdots \cup I_{\alpha_n}$ appear in this description of the elements of X. We consider two examples.

Let $(i, j) \in I_{\alpha_S} \times I_{\alpha_S}$, $(u, v) \in I_{\beta_S} \times I_{\beta_S}$ and $(k, l) \in I_{\alpha_T} \times I_{\alpha_T}$ and suppose that α_S and β_S have a common upper bound γ_S in X_S. Then by Proposition 2.2, their greatest lower bound $\delta_S = \alpha_S \wedge \beta_S$ exists in X_S. It can be shown that δ_S also is the greatest lower bound of α_S and β_S in X. Therefore, if $(x, y) \in S * T$ represents any product containing a factor of $I_{\alpha_S} \times I_{\alpha_S}$ and of $I_{\beta_S} \times I_{\beta_S}$ and $(x, y) \in I_\delta \times I_\delta$ then $\delta \leq \delta_S$. We therefore have by Lemma 2.3:

$$(i, j)(k, l)(u, v) = (if_{\alpha_S, \delta_S}, jf_{\alpha_S, \delta_S})(k, l)(uf_{\beta_S, \delta_S}, uf_{\beta_S, \delta_S}).$$

The corresponding α is the equivalence relation on $I_{\delta_S} \cup I_{\alpha_T}$ identifying jf_{α_S, δ_S} and k as well as l and uf_{β_S, δ_S} (rather than an equivalence relation on $I_{\alpha_S} \cup I_{\beta_S} \cup I_{\alpha_T}$). Thus two distinct elements of the above mentioned set $\{\alpha_1, \ldots \alpha_n\}$ belonging to the same structure set cannot have a common upper bound (within their structure set). Now take $(i, j) \in I_{\alpha_S} \times I_{\alpha_S}$, $(u, v) \in I_{\beta_S} \times I_{\beta_S}$ and $(k, k) \in I_{\alpha_T} \times I_{\alpha_T}$. Let $\alpha \in X$ be such that $(i, j)(k, k)(u, v) \in I_\alpha \times I_\alpha$. Then

$$jf_{\alpha_S, \alpha} = kf_{\alpha_T, \alpha} = uf_{\beta_S, \alpha}.$$

In particular, $\alpha \leq \delta_S(j, u)$ where $\delta_S(\cdot, \cdot)$ is the δ-function of S and

$$(i, j)(k, k)(u, v) = (if_{\alpha_S, \delta_S}, jf_{\alpha_S, \delta_S})(k, k)(uf_{\beta_S, \delta_S}, vf_{\beta_S, \delta_S}),$$

where $\delta_S = \delta_S(j, u)$. In this case, the "correct" domain of α is $I_{\delta_S} \cup I_{\alpha_T}$ rather than $I_{\alpha_S} \cup I_{\beta_S} \cup I_{\alpha_T}$. This condition implies that the equivalence relation α cannot identify two distinct elements of $\bigcup_{\alpha_U \in X_U} I_{\alpha_U}$ for $U = S, T$. It turns out that these two observations in fact are sufficient in order to describe the parameters X, I_α, $f_{\alpha,\beta}$ of the free product of a given collection of combinatorial strict inverse semigroups. The next section gives precise definitions and shows that indeed a combinatorial strict inverse semigroup is obtained in the outlined way. Section 5 then proves that the so constructed semigroup in fact is the free product of the given semigroups.

4. The construction. Let I be an index set and

$$\{S_i = (X_i; I_{\alpha^i}, f_{\alpha^i,\beta^i}) | i \in I\}$$

be a set of pairwise disjoint combinatorial strict inverse semigroups S_i whose structure sets X_i are also pairwise disjoint. The upper indices in α^i and f_{α^i,β^i} indicate to which S_i the element α^i and thus the mapping f_{α^i,β^i} "belong". This upper index sometimes will be omitted. The partial order on X_i will be denoted by $\leq_i$ and the δ-function of S_i by δ_i.

NOTATION. Put $I_i = \bigcup_{\alpha^i \in X_i} I_{\alpha^i}$. For a subset $A \subseteq \bigcup_{i \in I} X_i$, $A \cap X_i$ is the i-component of A, to be denoted by iA. Further, put $I_A = \bigcup_{\alpha \in A} I_\alpha$. Then $I_A \cap I_i = \bigcup_{\alpha \in iA} I_\alpha$ is the i-component of I_A, to be denoted by iI_A. For any set J, the identical relation on J will be denoted by ε_J.

Recall the definition of an admissible subset of X_i.

DEFINITION 4. A non-empty subset $A \subseteq \bigcup_{i \in I} X_i$ is *admissible* if there are $i_1, \dots, i_n \in I$ such that

(1) $A = i_1 A \cup i_2 A \cup \cdots \cup i_n A$,

(2) for each $k = 1, \dots, n$ the i_k-component $i_k A$ of A is admissible in X_{i_k}.

That is, a set $A \subseteq \bigcup_{i \in I} X_i$ is admissible if and only if it is finite, non-empty and each non-empty i-component iA of A is an admissible subset of X_i. Denote by $\mathscr{P}(X_i, I)$ the set of all admissible subsets of $\bigcup_{i \in I} X_i$.

DEFINITION 5. Let $A \subseteq \bigcup_{i \in I} X_i$ be a finite subset and apply the process which constructs the admissible set to each non-empty i-component iA of A. The result, to be denoted by $\overline{A}$, will be termed the *admissible set generated by* A. Defining the $\wedge$-operation in $\mathscr{P}(X_i, I)$ by $A \wedge B = \overline{A \cup B}$ then $\mathscr{P}(X_i, I)$ becomes a semilattice.

The so obtained semilattice $(\mathscr{P}(X_i, I), \wedge)$ is the free product of the semilattices $(\mathscr{P}(X_i), \wedge_i)$, $i \in I$.

DEFINITION 6. Let $A \in \mathscr{P}(X_i, I)$ be an admissible set. An equivalence relation α on I_A is *admissible* if

(1) $\alpha|iI_A$ is the identical relation on each nonempty i-component iI_A of I_A,

(2) for any $x, y \in I_A$ there exist $x_1, y_1, x_2, y_2, \dots, x_n, y_n \in I_A$ where $x_k, y_k \in I_{\alpha_k}$ for certain $\alpha_k \in A$ such that

$$x = x_1, y_1 \alpha x_2, \dots, y_{n-1} \alpha x_n, y_n = y,$$

(3) at most finitely many α-classes contain more than one element.

For $A \in \mathscr{P}(X_i, I)$ denote by X_A the set of all admissible equivalence relations on I_A.

Notice that the set X_A may be empty for an admissible set A. Let A be an admissible subset of some X_i. If α is an admissible relation on I_A then α is the identical relation on I_A by condition (1) and $iI_A = I_A$. By condition (2) this is only possible if A consists of only one element. Hence $X_A = \varnothing$ if $A \subseteq X_i$ and $|A| > 1$. Condition (1) reflects the second example in §3 whereas conditions (2) and (3) reflect that the free product is constituted by all finite products $(x_1, y_1)(x_2, y_2)\cdots(x_n, y_n)$ where $(x_j, y_j) \in S_{i_j}$. By condition (1) it follows that each α class contains at most finitely many elements. In the following, we shall call an equivalence class *trivial* if it is a singleton.

DEFINITION 7. Put $X = \bigcup_{A \in \mathscr{P}(X_i, I)} X_A$. For $\alpha \in X_A$, $\beta \in X_B$ $(A, B \in \mathscr{P}(X_i, I))$ let $\alpha \geq \beta$ if and only if

(1) for each $\alpha^i \in A$ there is some $\beta^i \in B$ such that $\alpha^i \geq_i \beta^i$ $(i \in I)$,

(2) if $x \in I_{\alpha^i}$, $y \in I_{\alpha^j}$ for $\alpha^i, \alpha^j \in A$ such that $x \alpha y$ then also $x f_{\alpha^i, \beta^i} \beta y f_{\alpha^j, \beta^j}$. Here β^i and β^j denote the (uniquely determined) lower bounds of α^i and α^j in B.

Notice that (1) in fact states that $A \geq B$ in $\mathscr{P}(X_i, I)$.

LEMMA 4.1. *$(X, \leq)$ is a partially ordered set.*

Proof. Obviously, $\alpha \leq \alpha$ for each $\alpha \in X$. Let $\alpha \in X_A$, $\beta \in X_B$ for some $A, B \in \mathscr{P}(X_i, I)$ such that $\alpha \geq \beta$ and $\beta \geq \alpha$. Then $A \geq B$ and $B \geq A$. Hence $A = B$. Condition (2) of Definition 7 now implies that $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$ (considered as sets of ordered pairs). Consequently, $\alpha = \beta$. Finally, let $\alpha \in X_A$, $\beta \in X_B$, $\gamma \in X_C$

such that $\alpha \geq \beta$ and $\beta \geq \gamma$. Then $A \geq B$ and $B \geq C$ so that $A \geq B \geq C$. Let $\alpha^i, \alpha^j \in A$. Then there are (unique) $\beta^i, \beta^j \in B$, $\gamma^i, \gamma^j \in C$ such that $\alpha^i \geq_i \beta^i \geq_i \gamma^i$ and $\alpha^j \geq_j \beta^j \geq_j \gamma^j$. If $x \alpha y$ for some $x \in I_{\alpha^i}$, $y \in I_{\alpha^j}$ then $x f_{\alpha^i, \beta^i} \beta y f_{\alpha^j, \beta^j}$ and thus also $x f_{\alpha^i, \gamma^i} = x f_{\alpha^i, \beta^i} f_{\beta^i, \gamma^i} \gamma y f_{\alpha^j, \beta^j} f_{\beta^j, \gamma^j} = y f_{\alpha^j, \gamma^j}$ which implies that $\alpha \geq \gamma$.

LEMMA 4.2. *For each $i \in I$ the mapping $\phi_i \colon X_i \to X$, defined by $\alpha^i \mapsto \varepsilon_{I_{\{\alpha^i\}}}$ provides an isomorphic embedding of the partially ordered set X_i into X.*

We will not use this result so that the straightforward proof is omitted. The partially ordered set X of Definition 7 will be the structure set of the free product of the semigroups S_i. We proceed to define the respective sets I_α and the structure mappings $\overline{f}_{\alpha,\beta}$ (here $^-$ indicates the difference to the mappings f_{α^i,β^i}).

DEFINITION 8. For each $\alpha \in X_A$, $A \in \mathscr{P}(X_i, I)$, put $I_\alpha = I_A/\alpha$. Further, let $A, B \in \mathscr{P}(X_i, I)$, $\alpha \in X_A$, $\beta \in X_B$ such that $\alpha \geq \beta$. Let $x\alpha \in I_\alpha$ and $x' \in I_{\alpha^i}$ for some $\alpha^i \in A$ such that $x' \in x\alpha$. By definition of $\geq$ there is a unique $\beta^i \in B$ such that $\alpha^i \geq_i \beta^i$. Put

$$x\alpha\overline{f}_{\alpha,\beta} = (x' f_{\alpha^i,\beta^i})\beta .$$

By condition (2) of Definition 7, the value of $x\alpha\overline{f}_{\alpha,\beta}$ does not depend on the special choice of $x' \in x\alpha$. Therefore, $\overline{f}_{\alpha,\beta} \colon I_\alpha \to I_\beta$ is a well defined mapping.

LEMMA 4.3. *The mappings $\overline{f}_{\alpha,\beta}$ satisfy the following.*
(1) $\overline{f}_{\alpha,\alpha} = \mathrm{id}_{I_\alpha}$ *for all* $\alpha \in X$.
(2) $\overline{f}_{\alpha,\beta}\overline{f}_{\beta,\gamma} = \overline{f}_{\alpha,\gamma}$ *whenever* $\alpha \geq \beta \geq \gamma$.

Proof. (1) Let $x\alpha \in I_\alpha$ and $x' \in x\alpha$ for some $x' \in I_{\alpha^i}$. Then $x\alpha\overline{f}_{\alpha,\alpha} = (x' f_{\alpha^i,\alpha^i})\alpha = x'\alpha = x\alpha$.

(2) Let $\alpha \geq \beta \geq \gamma$, $\alpha \in X_A$, $\beta \in X_B$, $\gamma \in X_C$. Let $x\alpha \in I_\alpha$ and $x' \in x\alpha$, $x' \in I_{\alpha^i}$. There are (unique) $\beta^i \in B$, $\gamma^i \in C$ such that $\alpha^i \geq_i \beta^i \geq_i \gamma^i$. By definition, $x\alpha\overline{f}_{\alpha,\gamma} = (x' f_{\alpha^i,\gamma^i})\gamma$, $x\alpha\overline{f}_{\alpha,\beta} = (x' f_{\alpha^i,\beta^i})\beta$ and $(x\alpha\overline{f}_{\alpha,\beta})\overline{f}_{\beta,\gamma} = (x' f_{\alpha^i,\beta^i}) f_{\beta^i,\gamma^i}\gamma$ since $x' f_{\alpha^i,\beta^i} \in x\alpha\overline{f}_{\alpha,\beta}$. Now $f_{\alpha^i,\beta^i} f_{\beta^i,\gamma^i} = f_{\alpha^i,\gamma^i}$ implies that $x\alpha\overline{f}_{\alpha,\beta}\overline{f}_{\beta,\gamma} = x\alpha\overline{f}_{\alpha,\gamma}$. $\square$

In the following we shall prove that the system $(X; I_\alpha, \overline{f}_{\alpha,\beta})$ defines a combinatorial strict inverse semigroup. We therefore have to find a δ-function $\delta \colon \bigcup I_\alpha \times \bigcup I_\alpha \to X$.

Construction of the δ-function for $(X; I_\alpha, \overline{f}_{\alpha,\beta})$. Let $\alpha, \beta \in X$, $x\alpha \in I_\alpha$, $y\beta \in I_\beta$. Suppose that $\alpha \in X_A$, $\beta \in X_B$. Let $C_1 = \overline{A \cup B}$ be the admissible set generated by $A \cup B$ (as described in Definition 5). For each $\alpha^i \in A$, $\beta^j \in B$ there are (unique) α_1^i, $\beta_1^j \in C_1$ such that $\alpha^i \geq_i \alpha_1^i$ and $\beta^j \geq_j \beta_1^j$. Define binary relations U_1, V_1, W_1 on I_{C_1} as follows:

$$U_1 = \{(uf_{\alpha^i,\alpha_1^i}, vf_{\alpha^j j,\alpha_1^j}) | u \in I_{\alpha^i},\ v \in I_{\alpha^j},\ u\alpha v,\ \alpha^i, \alpha^j \in A\},$$
$$V_1 = \{(uf_{\beta^i,\beta_1^i}, vf_{\beta^j,\beta_1^j}) | u \in I_{\beta^i},\ v \in I_{\beta^j},\ u\beta v,\ \beta^i, \beta^j \in B\},$$
$$W_1 = \{(x'f_{\alpha^i,\alpha_1^i}, y'f_{\beta^j,\beta_1^j}) | x' \in x\alpha \cap I_{\alpha^i},\ y' \in y\beta \cap I_{\beta^j},\ \alpha^i \in A,\ \beta^j \in B\}.$$

Now let η_1 be the equivalence relation on I_{C_1} which is generated by $U_1 \cup V_1 \cup W_1$. Admissibility of α and β ensures that this latter relation contains only a finite number of pairs with distinct entries. Next suppose that for $k = 1, \dots, n-1$, admissible sets C_k and equivalence relations η_k on I_{C_k} have already been defined and η_k is generated by a relation $U_k \cup V_k \cup W_k$ which has only finitely many pairs with distinct entries. For each $i \in I$ such that the i-component $iI_{C_{n-1}}$ of $I_{C_{n-1}}$ is not empty put $\eta_{n-1}^i = \eta_{n-1}|iI_{C_{n-1}}$. Then $iI_{C_{n-1}}/\eta_{n-1}^i$ is a collection of finite (pairwise disjoint) subsets of $I_i = \bigcup_{\alpha^i \in X_i} I_{\alpha^i}$ and only a finite number of them contains more than one element. Let $\delta_i(iI_{C_{n-1}}/\eta_{n-1}^i)$ be as it is described by Definition 1 in §2 where δ_i denotes the δ-function of S_i. The so obtained set is finite. Let $C_n^i = \overline{\delta_i(iI_{C_{n-1}}/\eta_{n-1}^i)}$ be the admissible set generated by $\delta_i(iI_{C_{n-1}}/\eta_{n-1}^i)$. Put $C_n = \bigcup C_n^i$ where the union is taken over all $i \in I$ for which the i-component $iI_{C_{n-1}}$ is not empty. Each $\gamma_{n-1}^i \in C_{n-1}$ has a unique lower bound γ_n^i in C_n. That is, $C_{n-1} \geq C_n$ and by induction, $A, B \geq C_1 \geq \cdots \geq C_{n-1} \geq C_n$. For each $\alpha^i \in A$, $\beta^j \in B$ let α_n^i, $\beta_n^j \in C_n$ be the uniquely determined elements such that $\alpha^i \geq_i \alpha_n^i$, $\beta^j \geq_j \beta_n^j$. Define binary relations U_n, V_n, W_n, respectively, on I_{C_n} as follows:

$$U_n = \{(uf_{\alpha^i,\alpha_n^i}, vf_{\alpha^j,\alpha_n^j}) | u \in I_{\alpha^i},\ v \in I_{\alpha^j},\ u\alpha v,\ \alpha^i, \alpha^j \in A\},$$
$$V_n = \{(uf_{\beta^i,\beta_n^i}, vf_{\beta^j,\beta_n^j}) | u \in I_{\beta^i},\ v \in I_{\beta^j},\ u\beta v,\ \beta^i, \beta^j \in B\},$$
$$W_n = \{(x'f_{\alpha^i,\alpha_n^i}, y'f_{\beta^j,\beta_n^j}) | x' \in x\alpha \cap I_{\alpha^i},\ y' \in y\beta \cap I_{\beta^j},\ \alpha^i \in A,\ \beta^j \in B\},$$

and let η_n be the equivalence relation on I_{C_n} which is generated by $U_n \cup V_n \cup W_n$.

LEMMA 4.4. *For each* $n \in \mathbb{N}$, *the equivalence relation* η_n *on* I_{C_n} *satisfies the conditions* (2) *and* (3) *of Definition* 6.

Proof. By construction of C_n, each $\gamma_n^i \in C_n$ has an upper bound $\gamma_{n-1}^i \in C_{n-1}$ and, by induction, has an upper bound $\gamma_1^i \in C_1$. Each $\gamma_1^i \in C_1$ has an upper bound either $\alpha^i \in A$ or $\beta^i \in B$. Consequently, each $\gamma_n^i \in C_n$ has an upper bound either in A or in B. For the moment, the element of C_n will be denoted by τ^i rather than by γ_n^i. Let $s, t \in I_{C_n}$, $s \in I_{\tau^i}$, $t \in I_{\tau^j}$. We assume that τ^i has an upper bound α^i in A and τ^j has an upper bound β^j in B. The cases of both upper bounds α^i, β^j being contained either in A or in B are proved analogously but more easily. Let $v \in x\alpha$; then $v \in I_{\alpha^{i'}}$ for some $\alpha^{i'} \in A$. By Definition 6 (2), there exist $v_1, u_2, v_2, \ldots, u_{n-1}, v_{n-1}, u_n$ where $u_k, v_k \in I_{\alpha_k^{i_k}}$ for some $\alpha_k^{i_k} \in A$ (and we omit the upper indices in the following) such that

$$v_1 \in I_{\alpha^i} = I_{\alpha_1}, \quad v_1 \, \alpha \, u_2, \ldots, v_{n-1} \, \alpha \, u_n, \quad v \in I_{\alpha_n} = I_{\alpha^{i'}}.$$

Further, let $w \in y\beta$; then $w \in I_{\beta^{j'}}$ for some $\beta^{j'} \in B$. Again there exist $z_1, w_2, \ldots, z_{m-1}, w_m$ where $z_k, w_k \in I_{\beta_k}$ for some $\beta_k \in B$ such that

$$w \in I_{\beta^{j'}} = I_{\beta_1}, \quad z_1 \, \beta \, w_2, \ldots, z_{m-1} \, \beta \, w_m, \quad w_m \in I_{\beta_m} = I_{\beta^j}.$$

Each of the elements α_k, β_l has a (unique) lower bound in C_n, say $\alpha_k \geq \tau_k$ and $\beta_l \geq \tau'_l$ (and $\geq$ denotes the partial order in the respective set X_i). We now may apply the mappings f_{α_k, τ_k} respectively f_{β_l, τ'_l} and obtain a finite sequence

$$v'_1 \, U_n \, u'_2, \ldots, v'_{n-1} \, U_n \, u'_n, \quad v' \in I_{\tau_n}, \; w' \in I_{\tau'_1},$$
$$z'_1 \, V_n \, w'_2, \ldots, z'_{m-1} \, V_n \, w'_m$$

such that $u'_k, v'_k \in I_{\tau_k}$, $z'_l, w'_l \in I_{\tau'_l}$. By construction we also have $s \in I_{\tau_1} = I_{\tau^i}$, $t \in I_{\tau'_m} = I_{\tau^j}$ and $(v', w') \in W_n$. Consequently, η_n satisfies condition (2) of Definition 6. It has been already mentioned that the generating relation $U_n \cup V_n \cup W_n$ contains only a finite number of pairs with distinct entries and this implies (3).

LEMMA 4.5. *If η_n is admissible then $C_n = C_{n+1} = \cdots$ and $\eta_n = \eta_{n+1} = \cdots$. Conversely, if η_n is not admissible then $C_n > C_{n+1}$ and $\eta_n \neq \eta_{n+1}$.*

Proof. If η_n is admissible then all η_n^i-classes are trivial. Therefore, $\delta_i(iI_{C_n}/\eta_n^i) = \{\delta_i\{x\} | x \in iI_{C_n}\} = iC_n$, the i-component of C_n. Consequently, $C_{n+1} = \bigcup iC_n = C_n$. By definition, $U_n = U_{n+1}$, $V_n = V_{n+1}$, $W_n = W_{n+1}$ and thus $\eta_n = \eta_{n+1}$. On the other hand, if η_n is not admissible then there is $i \in I$ such that η_n^i has a non-trivial equivalence class. The construction of C_{n+1} now implies $C_n > C_{n+1}$.

LEMMA 4.6. *Let e_n denote the number of non-trivial η_n-classes. Then $e_n \geq e_{n+1}$ for all $n \in \mathbb{N}$. If η_{n+1} is not admissible then $e_n > e_{n+1}$.*

Proof. Each $\gamma_n^i \in C_n$ has a unique lower bound γ_{n+1}^i in C_{n+1}. Let $f_n = \bigcup f_{\gamma_n^i, \gamma_{n+1}^i}$ where the union is taken over all $\gamma_n^i \in C_n$. That is, for $x \in I_{C_n}$ let $xf_n = xf_{\gamma_n^i, \gamma_{n+1}^i} \in I_{C_{n+1}}$ provided $x \in I_{\gamma_n^i}$. Then f_n is a well defined mapping from I_{C_n} to $I_{C_{n+1}}$. Take $a \in I_{C_{n+1}}$ such that $a\eta_{n+1}$ is not trivial. By definition of U_{n+1}, V_{n+1}, W_{n+1} and U_n, V_n, W_n, respectively, it follows that there are $u, v \in I_{C_n}$ such that $u \neq v$, $u \,\eta_n\, v$ and $uf_n, vf_n \in a\eta_{n+1}$. On the other hand, if $u \,\eta_n\, v$ for $u, v \in I_{C_n}$ then also $uf_n \,\eta_{n+1}\, vf_n$. From this it follows that each non-trivial η_{n+1}-class contains the f_n-image of a non-trivial η_n-class. Thus $e_n \geq e_{n+1}$. Now suppose that η_{n+1} is not admissible. There are two distinct elements a, b belonging to the same i-component $iI_{C_{n+1}}$ satisfying $a \,\eta_{n+1}\, b$. Take any $u, v \in I_{C_n}$ such that $a = uf_n$, $b = vf_n$. (By definition of η_{n+1} and η_n such elements exist.) Then u and v belong to the same i-component iI_{C_n}. It follows that $(u, v) \notin \eta_n$ since otherwise, by construction of C_{n+1}, $uf_n = vf_n$. By definition of η_{n+1} there are $a_1, \ldots, a_k, b_1, \ldots, b_k \in I_{C_{n+1}}$ such that $a = a_1$, $b_k = b$, $b_j = a_{j+1}$, $a_j \neq b_j$ and $(a_j, b_j) \in (U_{n+1} \cup V_{n+1} \cup W_{n+1})^{\pm 1}$ for all j. For each j there are $u_j, v_j \in I_{C_n}$ such that $a_j = u_jf_n$, $b_j = v_jf_n$ and $(u_j, v_j) \in (U_n \cup V_n \cup W_n)^{\pm 1}$. Put $u = u_1$ and $v = v_k$. Then, since $u_1 \neq v_1$ and $u_k \neq v_k$, $u\eta_n$ and $v\eta_n$ are non-trivial η_n-classes. Since $(u, v) \notin \eta_n$ these η_n-classes are distinct. Since $uf_n = a \,\eta_{n+1}\, b = vf_n$, $u\eta_nf_n$ and $v\eta_nf_n$ are contained in the same η_{n+1}-class $a\eta_{n+1}$. As mentioned above, each non-trivial η_{n+1}-class contains the f_n-image of a non-trivial η_n-class. Therefore $e_n > e_{n+1}$.

Combining Lemmas 4.4, 4.5 and 4.6 we have the following

COROLLARY 4.7. *There is a unique least* $n \in \mathbb{N}$ *such that* η_n *is admissible. In this case,* $\eta_n = \eta_{n+1} = \cdots$ *and* $C_n = C_{n+1} = \cdots$.

DEFINITION 9. Let $\tau = \tau(x, \alpha, \beta, y) = \eta_n$ be the (uniquely determined) admissible relation for x, y and α, β in the sequence $\eta_1, \eta_2, \ldots$.

In the following, the admissible set C_n where $\tau(x, \alpha, \beta, y) = \eta_n$ will be denoted by D, its elements will be denoted by τ^i rather than by γ_n^i.

LEMMA 4.8. *The admissible equivalence relation* $\tau = \tau(x, \alpha, \beta, y)$ *as constructed above is the greatest element of*

$$D(x, \alpha, \beta, y) = \{\gamma \leq \alpha, \beta | x\alpha\overline{f}_{\alpha,\gamma} = y\beta\overline{f}_{\beta,\gamma}\}.$$

Proof. First, $D \leq A, B$. If $u\alpha v$ then $uf_{\alpha^i,\tau^i} U_n vf_{\alpha^j,\tau^j}$ and the analogous assertion is true for β. That is, $\tau \leq \alpha, \beta$. Let $x' \in x\alpha \cap I_{\alpha^i}$, $y' \in y\beta \in I_{\beta^j}$ and denote by τ^i and τ_1^j the respective lower bounds of α^i and β^j in D. Then $x\alpha\overline{f}_{\alpha,\tau} = (x'f_{\alpha^i,\tau^i})\tau = (y'f_{\beta^j,\tau_1^j})\tau = y\beta\overline{f}_{\beta,\tau}$ since τ contains W_n. In particular, $\tau \in D(x, \alpha, \beta, y)$. On the other hand, let $\gamma \leq \alpha, \beta$ be such that $x\alpha\overline{f}_{\alpha,\gamma} = y\beta\overline{f}_{\beta,\gamma}$. Let G be the admissible set such that $\gamma \in X_G$. Since $\gamma \leq \alpha, \beta$, so by Definition 7 it follows that each element $\gamma^i \in C_1 = \overline{A \cup B}$ has a unique lower bound ν^i in G. Next apply the appropriate functions f_{γ^i,ν^i} to the relations U_1, V_1, W_1, respectively, which have been defined for the construction of τ. We obtain the relations U', V', W', to be defined as follows:

$$\begin{aligned}
U' &= \{(uf_{\alpha^i,\nu^i}, vf_{\alpha^j,\nu^j}) | u \in I_{\alpha^i},\ v \in I_{\alpha^j},\ u\alpha v,\ \alpha^i, \alpha^j \in A\},\\
V' &= \{(uf_{\beta^i,\nu_1^i}, vf_{\beta^j,\nu_1^j}) | u \in I_{\beta^i},\ v \in I_{\beta^j},\ u\beta v,\ \beta^i, \beta^j \in B\},\\
W' &= \{x'f_{\alpha^i,\nu^i}, f_{\beta^j,\nu_1^j}) | x' \in x\alpha \cap I_{\alpha^i},\\
&\qquad\qquad y' \in y\beta \cap I_{\beta^j},\ \alpha^i \in A,\ \beta^j \in B\}.
\end{aligned}$$

Since $\gamma \leq \alpha, \beta$ and $x\alpha\overline{f}_{\alpha,\gamma} = y\beta\overline{f}_{\beta,\gamma}$ it follows that $U' \cup V' \cup W' \subseteq \gamma$. Let the equivalence relation η_1 on I_{C_1} be defined as in the construction of τ and let a, b be distinct η_1-equivalent elements contained in the same i-component iI_{C_1}, that is, $a\,\eta_1^i\, b$. Notice that (a, b) is contained in the symmetric-transitive closure of $U_1 \cup V_1 \cup W_1$. Let $u \in I_{\alpha^i}$, $v \in I_{\beta^i}$ be such that $a = uf_{\alpha^i,\gamma^i}$ and $b = vf_{\beta^i,\gamma_1^i}$ for appropriate α^i, $\beta^i \in i(A \cup B)$ and γ^i, $\gamma_1^i \in iC_1$.

Then $af_{\gamma^i,\nu^i} = uf_{\alpha^i,\nu^i}\,\gamma\, vf_{\beta^i,\nu_1^i} = bf_{\gamma_1^i,\nu_1^i}$ (for appropriate $\nu^i \leq_i \gamma^i$ and $\nu_1^i \leq_i \gamma_1^i$) since the pair $(uf_{\alpha^i,\nu^i}, vf_{\beta^i,\nu_1^i})$ is contained in the symmetric-transitive closure of $U' \cup V' \cup W'$. These elements belong to the same i-component iI_G of I_G. Since γ is admissible, $uf_{\alpha^i,\nu^i} = vf_{\beta^i,\nu_1^i}$ and in particular $\nu^i = \nu_1^i$. Consequently, $\delta_i(a, b) = \delta(uf_{\alpha^i,\gamma^i}, vf_{\beta^i,\gamma_1^i}) \geq_i \nu^i = \nu_1^i$. Also, if $b\,\eta_1^i\,c$ then for the same reason $\delta_i(b, c) \geq_i \nu^i$. In particular, $\delta_i(a, b) \wedge \delta_i(b, c) \geq_i \nu^i$. Using Lemma 2.4 it can be seen that the elements of $\delta_i(iI_{C_1}/\eta_1^i)$ are of the form

$$(\cdots(\delta_i(x_1, x_2) \wedge \delta_i(x_2, x_3)) \wedge \cdots) \wedge \delta_i(x_{l-1}, x_l)$$

if $\{x_1, \ldots, x_l\}$ is a non-trivial η_1^i-class and $\delta_i\{x\} = \gamma^i \in iC_1$ if $\{x\}$ is a trivial η_1^i-class. By this description it follows that each element of $\delta_i(iI_{C_1}/\eta_1^i)$ has a lower bound in G. By Lemma 2.7, each element of $C_2^i = \overline{\delta_i(iI_{C_1}/\eta_1^i)}$ has a lower bound in G. This is true for each non-empty i-component C_2^i of C_2 so that each element of C_2 has a lower bound in G, that is, $C_2 \geq G$. Repeating this procedure $n-1$ times it can be seen that $C_n = D \geq G$. Now consider distinct elements $u, v \in I_D$ such that $u\,\tau\,v$. Then (u, v) is contained in the equivalence relation which is generated by $U_n \cup V_n \cup W_n$. Applying the appropriate mappings f_{τ^i,ν^i} yields the relation $U' \cup V' \cup W'$ which is contained in γ. In particular $uf_{\tau^i,\nu^i}\,\gamma\, uf_{\tau^j,\nu^j}$, that is, $\gamma \leq \tau$. □

We have thus shown that the system $(X; I_\alpha, \overline{f}_{\alpha,\beta})$ satisfies the conditions (1)–(3) of Theorem 2.1 and thus defines a combinatorial strict inverse semigroup. However, the sets I_α are not necessarily pairwise disjoint for distinct $\alpha, \beta \in X$. In order to ensure disjointness for distinct $\mathscr{D}$-classes they actually will be realized by the sets $I_\alpha \times \{\alpha\} \times I_\alpha$ rather than by $I_\alpha \times I_\alpha$. In fact we have shown the following.

THEOREM 4.9. *Let I be an index set and $\{S_i = (X_i; I_{\alpha^i}, f_{\alpha^i,\beta^i}) | i \in I\}$ be a collection of pairwise disjoint combinatorial strict inverse semigroups S_i (having pairwise disjoint structure sets X_i). For the admissible set $A \in \mathscr{P}(X_i, I)$ (Definition 4) let X_A denote the set of all admissible relations on I_A (Definition 6) and let $X = \bigcup_{A\in\mathscr{P}(X_i,I)} X_A$. For each $\alpha \in X_A$ put $I_\alpha = I_A/\alpha$ and for $\alpha \geq \beta$ (Definition 7) let $\overline{f}_{\alpha,\beta}\colon I_\alpha \to I_\beta$ be as in Definition 8. Let $S = S_{\{S_i|i\in I\}} = \bigcup_{\alpha\in X} I_\alpha \times \{\alpha\} \times I_\alpha$, endowed with the multiplication*

$$(x\alpha, \alpha, y\alpha)(u\beta, \beta, v\beta) = (x\alpha\overline{f}_{\alpha,\tau}, \tau, v\beta\overline{f}_{\beta,\tau})$$

where the δ-function $\tau = \tau(y, \alpha, \beta, u)$ (Definition 9) is constructed as above. Then S is a combinatorial strict inverse semigroup. Its inverse operation is given by

$$(x\alpha, \alpha, y\alpha)^{-1} = (y\alpha, \alpha, x\alpha).$$

Notice that the structure of S is uniquely determined by the structure of the semigroups S_i, $i \in I$.

5. The main theorem. Now we show that the semigroup $(X; I_\alpha, \overline{f}_{\alpha,\beta})$, as it is presented in Theorem 4.9, is the free product of the combinatorial strict inverse semigroups $(X_i; I_{\alpha^i}, f_{\alpha^i,\beta^i})$ within the class of all combinatorial strict inverse semigroups. First we need the following result.

PROPOSITION 5.1. *Let $\{S_i = (X_i; I_{\alpha^i}, f_{\alpha^i,\beta^i}) | i \in I\}$ be a collection of pairwise disjoint combinatorial strict inverse semigroups S_i having also pairwise disjoint structure sets X_i. For $\alpha^i \in X_i$ let $\varepsilon_{\alpha^i} = \varepsilon_{I_{\{\alpha^i\}}}$. Then the mapping $\psi_i: S_i \to S$, defined by*

$$\psi_i : (k, l) \mapsto (k\varepsilon_{\alpha^i}, \varepsilon_{\alpha^i}, l\varepsilon_{\alpha^i}) \qquad (k, l \in I_{\alpha^i}, \alpha^i \in X_i)$$

embeds S_i isomorphically into S. The semigroups $S_i\psi_i$ are pairwise disjoint.

Proof. It is clear that $\psi_i: S_i \to S$ is injective and that the sets $S_i\psi_i$ are pairwise disjoint. It suffices to show that ψ_i is a homomorphism. Let $k, l \in I_{\alpha^i}$, $s, t \in I_{\beta^i}$. Then $(k, l)(s, t) = (kf_{\alpha^i,\delta^i}, tf_{\beta^i,\delta^i})$ for $\delta^i = \delta_i(l, s)$. In the following we shall omit the upper index in α^i etc. On the other hand,

$$(k, l)\psi_i(s, t)\psi_i = (k\delta_\alpha, \varepsilon_\alpha, l\varepsilon_\alpha)(s\varepsilon_\beta, \varepsilon_\beta, t\varepsilon_\beta).$$

To find $\tau = \tau(l, \varepsilon_\alpha, \varepsilon_\beta, s)$ as it is described in §4, we first have to find the admissible set generated by $\{\alpha\} \cup \{\beta\} = \{\alpha, \beta\}$. Two cases are possible.

Case (1). The elements α and β do not have a common upper bound in X_i. Then $C = \{\alpha, \beta\}$ itself is admissible. Now consider the relation η on I_C which is generated by the binary relation $U_1 \cup V_1 \cup W_1$ as it is defined in §4. Since ε_α and ε_β are identical relations it follows that η is the equivalence relation on I_C generated by the pair (l, s). Consequently, $\eta = \eta^i$ is the equivalence relation on $I_C = I_\alpha \cup I_\beta$ which identifies l and s and all other equivalence classes are singletons. Let $\delta_i(iI_C/\eta^i)$ (Definition 1) be shortly denoted by $\delta_i\eta$. Then

$$\delta_i\eta = \{\delta_i(l, s), \delta_i(x, x) | x \in I_C, \ x \notin \{l, s\}\}.$$

That is, $\delta_i\eta$ consists of the element $\delta_i(l, s)$ to which perhaps α and/or β are/is adjoined (depending on whether I_α respectively I_β contain elements distinct from l respectively s). In any case, the admissible set generated by $\delta_i\eta$ consists entirely of the element $\delta_i(l, s)$. The procedure for obtaining τ therefore has to be applied only once. The domain of $\tau = \tau(l, \varepsilon_\alpha, \varepsilon_\beta, s)$ then is the set $I_{\delta_i(l,s)}$, shortly denoted by I_δ. From this, $\tau = \varepsilon_{I_\delta}$, shortly denoted by ε_δ, and therefore

$$\begin{aligned}(k, l)\psi_i(s, t)\psi_i &= (k\delta_\alpha, \varepsilon_\alpha, l\varepsilon_\alpha)(s\varepsilon_\beta, \varepsilon_\beta, t\varepsilon_\beta)\\ &= (k\varepsilon_\alpha\overline{f}_{\varepsilon_\alpha, \tau}, \tau, t\varepsilon_\beta\overline{f}_{\varepsilon_\beta, \tau})\\ &= (k\varepsilon_\alpha\overline{f}_{\varepsilon_\alpha, \varepsilon_\delta}, \varepsilon_\delta, t\varepsilon_\beta\overline{f}_{\varepsilon_\beta, \varepsilon_\delta})\\ &= (kf_{\alpha,\delta}\varepsilon_\delta, tf_{\beta,\delta}\varepsilon_\delta)\\ &= (kf_{\alpha,\delta}, tf_{\beta,\delta})\psi_i = [(k, l)(s, t)]\psi_i.\end{aligned}$$

Case (2). If α and β have a common upper bound then the meet $\alpha \wedge \beta$ exists. The admissible set C generated by $\{\alpha, \beta\}$ consists entirely of the element $\nu = \alpha\wedge\beta$. Now consider again the equivalence relation $\eta = \eta^i$ on $I_C = I_\nu$ which is generated by $U_1 \cup V_1 \cup W_1$. Since ε_α and ε_β are identical relations we only have to consider the set W_1. Since $W_1 = \{(lf_{\alpha,\nu}, sf_{\beta,\nu})\}$, similarly as in case (1),

$$\delta_i\eta = \{\delta_i(lf_{\alpha,\nu}, sf_{\beta,\nu}), \delta_i(x, x) | x \in I_C, \ x \notin \{lf_{\alpha,\nu}, sf_{\beta,\nu}\}\}.$$

That is, $\delta_i\eta$ consists of the element $\delta_i(lf_{\alpha,\nu}, sf_{\beta,\nu})$ to which perhaps ν is adjoined. Since $\nu = \alpha \wedge \beta$ we have $\nu \geq \delta_i(lf_{\alpha,\nu}, sf_{\beta,\nu}) = \delta_i(l, s) = \delta$. Consequently, the admissible set generated by $\delta_i\eta$ is given by $\overline{\delta_i\eta} = \{\delta_i(l, s)\} = \{\delta\}$. Again, the domain of $\tau = \tau(l, \varepsilon_\alpha, \varepsilon_\beta, s)$ is I_δ. By construction and similarly as in case (1) it follows that $\tau(l, \varepsilon_\alpha, \varepsilon_\beta, s)$ is the identical relation on I_δ, shortly denoted by ε_δ. As in case (1) we now observe that $(k, l)\psi_i(s, t)\psi_i = [(k, l)(s, t)]\psi_i$.

Next we show that $S = (X; I_\alpha, \overline{f}_{\alpha,\beta})$ is generated by its subsemigroups $S_i\psi_i \cong (X_i; I_{\alpha^i}, f_{\alpha^i,\beta^i})$.

THEOREM 5.2. *The semigroup $S = (X; I_\alpha, \overline{f}_{\alpha,\beta})$ as it is constructed in Theorem* 4.9 *is generated by the set $\bigcup_{i\in I} S_i\psi_i$.*

Proof. Let $A \in \mathscr{P}(X_i, I)$, $\alpha \in X_A$, $x, y \in I_A$. We have to show that $(x\alpha, \alpha, y\alpha) \in \langle \bigcup_{i\in I} S_i\psi_i \rangle$. Denote the latter semigroup by T. Notice that T is closed under inversion. If $|A| = 1$ then $A = \{\alpha^i\}$ for some $\alpha^i \in X_i$ and $i \in I$. Then $\alpha = \varepsilon_{I_{\{\alpha^i\}}}$ and $(x\alpha, \alpha, y\alpha) \in S_i\psi_i$. Let $A \in X$, $|A| = n > 1$ and suppose that the assertion be true for all $\beta \in X_B$ where $B \in \mathscr{P}(X_i, I)$ and $|B| < n$. Assume first that α is maximal in X_A. That is, if $\beta \in X_A$ such that $\beta \geq \alpha$ then $\beta = \alpha$. For $\alpha, \beta \in X_A$ we have $\alpha \geq \beta$ if and only if $\alpha \subseteq \beta$ (regarded as sets of ordered pairs). Now there exists $\alpha^i \in A$ such that for $B = A \setminus \{\alpha^i\}$ the relation $\beta = \alpha|I_B$ is admissible on I_B, that is, $\beta \in X_B$. This can be seen as follows. First notice that B is an admissible set since each non-empty subset of an admissible set is admissible. Further, it is clear that the restriction of β to any non-empty i-component is the identical relation since $\beta \subseteq \alpha$ and α is admissible. For the same reason also condition (3) of Definition 6 holds for β. Now consider the graph $g(\alpha)$ defined as follows. Let

$$V(g(\alpha)) = A$$

be the set of vertices of $g(\alpha)$ and

$$E(g(\alpha)) = \{\{\alpha^i, \alpha^j\} | \alpha^i \neq \alpha^j, u\,\alpha\,v \text{ for some } u \in I_{\alpha^i}, v \in I_{\alpha^j}\}$$

be the set of edges of $g(\alpha)$. Condition (2) of Definition 6 holds for the relation α if and only if the graph $g(\alpha)$ is connected. By a well-known graph theoretic result (see, for instance, Behzad and Chartrand [3]) there exists a vertex $\alpha^i \in V(g(\alpha))$ which is not a cut-vertex. That is, removing the vertex α^i from $g(\alpha)$ (and also the edges containing α^i) yields a connected graph g. Since $g \cong g(\beta)$ (defined in the same way as $g(\alpha)$ for $\beta = \alpha|I_B$) is connected, also the second condition for admissibility of β holds. In particular, $\beta \in X_B$. By admissibility of α there exist $u \in I_{\alpha^i}$, $v \in I_B$ such that $u\,\alpha\,v$. Next define an equivalence relation α' on I_A as follows:

$$\begin{aligned}
w\alpha' &= w\beta && \text{if } w \in I_B,\ w\beta \neq v\beta, \\
v\alpha' &= v\beta \cup \{u\} = u\alpha', && \\
w\alpha' &= \{w\} && \text{if } w \in I_{\alpha^i},\ w \neq u.
\end{aligned}$$

That is, α' arises from β (and $\varepsilon_{I_{\alpha^i}}$) by adjoining u to $v\beta$ and leaving the remaining equivalence classes of β and $\varepsilon_{I_{\alpha^i}}$ unchanged. Since $\alpha' \geq \alpha$ and α' is admissible on I_A it follows that $\alpha = \alpha'$. By hypothesis of induction, $(x\beta, \beta, v\beta) \in T$ and $(u\varepsilon, \varepsilon, w\varepsilon) \in T$

where $\varepsilon = \varepsilon I_{\alpha^i}$ for any $x \in I_B$ and $w \in I_{\alpha^i}$. Then also

$$(x\beta, \beta, v\beta)(u\varepsilon, \varepsilon, w\varepsilon) = (x\beta\overline{f}_{\beta,\xi}, \xi, w\varepsilon\overline{f}_{\varepsilon,\xi}) \in T$$

for a certain $\xi \in X_A$. By definition of multiplication in S, ξ is the equivalence relation on I_A which is generated by $\beta \cup v\beta \times \{u\}$, that is, $\xi = \alpha$. Further, it can be seen easily that $x\beta\overline{f}_{\beta,\alpha} = x\alpha$ and $w\varepsilon\overline{f}_{\varepsilon,\alpha} = w\alpha$. For any $x \in I_B$ and $w \in I_{\alpha^i}$ therefore $(x\alpha, \alpha, w\alpha) \in T$. Since T is closed under inversion, also $(w\alpha, \alpha, x\alpha) \in T$. Taking both alternatives for either different $x, y \in I_B$ or different $w, z \in I_{\alpha^i}$ and multiplying the so obtained to elements appropriately yields $(x\alpha, \alpha, y\alpha) \in T$ for any $x, y \in I_B$ and $(w\alpha, \alpha, z\alpha) \in T$ for any $w, z \in I_{\alpha^i}$. Summarizing these four cases, $(s\alpha, \alpha, t\alpha) \in T$ for any $s, t \in I_A$. Now let $\alpha \in X_A$ and assume that the assertion be true for all $\beta \in X_A$ for which $\beta > \alpha$. Let $x, y \in I_A$. If α is not maximal in X_A then by conditions (1) and (3) of Definition 6 there is $\beta \in X_A$ which covers α. That is, $\beta > \alpha$ and $\beta > \gamma \geq \alpha$ for $\gamma \in X_A$ imply $\gamma = \alpha$. Now there are $u, v \in I_A$ such that $u\alpha v$ but $(u, v) \notin \beta$. By our assumption, $(x\beta, \beta, u\beta), (v\beta, \beta, y\beta) \in T$. Now

$$(x\beta, \beta, u\beta)(v\beta, \beta, y\beta) = (x\beta\overline{f}_{\beta,\xi}, \xi, y\beta\overline{f}_{\beta,\xi}) \in T.$$

Similarly as above it can be seen that $\xi = \alpha$, $x\beta\overline{f}_{\beta,\alpha} = x\alpha$, $y\beta\overline{f}_{\beta,\alpha} = y\alpha$. Consequently $(x\alpha, \alpha, y\alpha) \in T$. Again, by conditions (1) and (3) of Definition 6, for each $\alpha \in X_A$ there is $\beta \in X_A$ such that $\beta \geq \alpha$, β is maximal in X_A and the interval $[\alpha, \beta] = \{\gamma \in X | \alpha \leq \gamma \leq \beta\}$ is finite. In fact we have obtained that $(x\alpha, \alpha, y\alpha) \in T$ for all $\alpha \in X_A$, $x, y \in I_A$. Therefore, $S = T$, that is, S is generated by $\bigcup_{i\in I} S_i\psi_i$. □

LEMMA 5.3. *Let $S = (X; I_\alpha, \overline{f}_{\alpha,\beta})$ be as in Theorem 4.9 and let $(l_1, \varepsilon_1, r_1), \dots, (l_n, \varepsilon_n, r_n) \in \bigcup_{i\in I} S_i\psi_i$; each ε_t being the identical relation on some $I_{\alpha_t^{i_t}}$ and l_t, r_t standing for their own respective ε_t-classes. (We shall omit the upper index in $\alpha_t^{i_t}$ in the sequel.) Let $C_1 = \{\gamma_{11}, \dots, \gamma_{n1}\}$ be the admissible set generated by $\{\alpha_1, \dots, \alpha_n\}$ where $\alpha_t \geq \gamma_{t1}$ (and several of the elements γ_{t1} may coincide). Let $l_t^1 = l_t f_{\alpha_t,\gamma_{t1}}$ and $r_t^1 = r_t f_{\alpha_t,\gamma_{t1}}$. Now let $k \geq 1$ and suppose that the elements $\{l_1^k, r_1^k, \dots, l_n^k, r_n^k\}$ have already been defined. Let η_k be the equivalence relation on $\{l_1^k, r_1^k, \dots, l_n^k, r_n^k\}$ which is generated by the pairs $\{(r_1^k, l_2^k), \dots, (r_{n-1}^k, l_n^k)\}$. For each $i \in I$ let η_k^i be the restriction of η_k to the i-component $\{l_1^k, r_1^k, \dots, l_n^k, r_n^k\} \cap \bigcup_{\alpha^i\in X_i} I_{\alpha^i}$. Let*

$\delta_i\eta_k^i$ be the set described in Definition 1 by applying δ_i to each η_k^i-class. Let $\delta\eta_k = \bigcup \delta_i\eta_k^i$ and let C_{k+1} be the admissible set generated by $\delta\eta_k$. Let $\gamma_{1,k+1}, \dots, \gamma_{n,k+1}$ be the (unique) lower bounds of $\alpha_1, \dots, \alpha_n$ in C_{k+1}. Finally, let $l_t^{k+1} = l_t f_{\alpha_t, \gamma_{t,k+1}}$ and $r_t^{k+1} = r_t f_{\alpha_t, \gamma_{t,k+1}}$. Then there is a least $q \in \mathbb{N}$ such that for $D = C_q$ the equivalence relation $\gamma = \eta_q \cup \varepsilon_{I_D}$ is admissible on I_D and

$$(*) \qquad (l_1, \varepsilon_1, r_1)\cdots(l_n, \varepsilon_n, r_n) = (l_1^q\gamma, \gamma, r_n^q\gamma).$$

Proof. Let $j \in \mathbb{N}$. If for some $i \in I$, η_j^i is not the identical relation then

$$|\{l_1^j, r_1^j, \dots, l_n^j, r_n^j\}| > |\{l_1^{j+1}, \dots, l_n^{j+1}, r_n^{j+1}\}|$$

since at least two distinct η_j^i-related elements of $\{l_1^j, r_1^j, \dots, l_n^j, r_n^j\}$ are mapped onto the same element of $\{l_1^{j+1}, r_1^{j+1}, \dots, l_n^{j+1}, r_n^{j+1}\}$ when the δ_i-function is applied. Consequently, there is a (unique) least $q \in \mathbb{N}$ such that η_q^i is the identical relation for all involved $i \in I$. It follows that $C_q = C_{q+1} = \cdots$ and $\eta_q = \eta_{q+1} = \cdots$. Denoting $D = C_q$ then $\gamma = \eta_q \cup \varepsilon_{I_D}$ is an admissible equivalence relation on I_D. By definition of the mappings $\overline{f}_{\alpha,\beta}$ (Definition 8) and the multiplication in S (as defined in the statement of Theorem 4.9), the product on the left-hand side of $(*)$ is given by

$$(l_1, \varepsilon_1, r_1)\cdots(l_n, \varepsilon_n, r_n) = (l_1 f_{\alpha_1,\nu_1}\tau, \tau, r_n f_{\alpha_n,\nu_n}\tau).$$

Here τ is the uniquely determined greatest element of the set

$$A = \{\xi \in X | \xi \le \varepsilon_1, \dots, \varepsilon_n, r_t\varepsilon_t\overline{f}_{\varepsilon_t,\xi} = l_{t+1}\varepsilon_{t+1}\overline{f}_{\varepsilon_{t+1},\xi} \quad \text{for } t = 1, \dots, n-1\},$$

ν_t is the unique lower bound of α_t in D' and D' is the uniquely determined admissible set such that $I_{D'}$ is the domain of τ. Since each element α_t has the lower bound $\gamma_{tq} = \gamma_t$ in D and

$$r_t f_{\alpha_t,\gamma_t} = r_t^q \gamma l_{t+1}^q = l_{t+1} f_{\alpha_{t+1},\gamma_{t+1}},$$

by definition of the mappings $\overline{f}_{\varepsilon_t,\gamma}$ it follows that γ is contained in A. Conversely, let $\xi \in A$ with $\xi \in X_G$. Denote by ξ_t the (unique) lower bound of α_t in G. By Lemma 2.7 it follows that $\gamma_{t1} \ge \xi_t$ for all t. Since $r_t\varepsilon_t\overline{f}_{\varepsilon_t,\xi} = r_t f_{\alpha_t,\xi_t}\xi$ and $l_t\varepsilon_t\overline{f}_{\varepsilon_t,\xi} = l_t f_{\alpha_t,\xi_t}\xi$ we have $x f_{\gamma_{t1},\xi_t}\xi\, y f_{\gamma_{t'1},\xi_{t'}}$, for all $x, y \in \{l_1^1, r_1^1, \dots, l_n^1, r_n^1\}$ for which $x\,\eta_1^i\, y$ for some $i \in I$. Since ξ is admissible, it follows that $x f_{\gamma_{t1},\xi_t} = y f_{\gamma_{t'1},\xi_{t'}}$ (and in particular $\xi_t = \xi_{t'}$) and thus $\delta_i(x, y) \ge_i \xi_t$ whenever

$x\,\eta_1^i\,y$. If $z\,\eta_1^i\,x$ then for the same reason $\delta_i(x, z) \geq_i \xi_t$ and thus also $\delta_i\{x, y, z\} = \delta_i(x, y) \wedge \delta_i(x, z) \geq_i \xi_t$. By induction it follows that for each η_1^i-class, $\delta_i(x\eta_1^i) \geq \xi_t$ (similarly as in the proof of Lemma 4.8). This can be done for each i-component so that each element of $\delta\eta_1 = \bigcup \delta_i\eta_1^i$ has a lower bound in G. Since C_2 is the admissible set generated by $\delta\eta_1$, by Lemma 2.7 each $\gamma_{t2} \in C_2$ has the (unique) lower bound ξ_t in G. That is, $G \leq C_2$. By induction it can be seen that $C_j \geq G$ for all j. In particular, $C_q = D \leq G$. By definition of γ and since $\xi \in A$ it follows that $uf_{\gamma_t, \xi_t}\xi\, vf_{\gamma_{t'}, \xi_{t'}}$, whenever $u\gamma v$. Consequently, $\xi \leq \gamma$ and thus $\gamma = \tau$, as required.

We now are ready to prove the main result.

THEOREM 5.4. *Let* $\{S_i = (X_i, I_{\alpha^i}, f_{\alpha^i, \beta^i}) | i \in I\}$ *be a collection of pairwise disjoint combinatorial strict inverse semigroups* S_i *(with pairwise disjoint structure sets* X_i*). Define admissible sets* $A \in \mathscr{P}(X_i, I)$, *admissible relations* $\alpha \in X_A$, $I_\alpha = I_A/\alpha$, $X = \bigcup_{A \in \mathscr{P}(X_i, I)} X_A$, *the partial order* $\geq$ *on* X *and mappings* $\overline{f}_{\alpha, \beta}\colon I_\alpha \to I_\beta$ *as in* §4. *For any* $x\alpha \in I_\alpha$, $y\beta \in I_\beta$, $\alpha, \beta \in X$ *let* $\tau = \tau(x, \alpha, \beta, y)$ *be the* δ*-function as it is defined by the process described in* §4. *Let* $S = \bigcup_{\alpha \in X} I_\alpha \times \{\alpha\} \times I_\alpha$ *and define a multiplication on* S *by*

$$(x\alpha, \alpha, y\alpha)(u\beta, \beta, v\beta) = (x\alpha\overline{f}_{\alpha, \tau}, \tau, v\beta\overline{f}_{\beta, \tau})$$

where $\tau = \tau(y, \alpha, \beta, u)$. *Then* S *is the free product of the combinatorial strict inverse semigroups* S_i *within the variety of all combinatorial strict inverse semigroups. The embeddings* $\psi_i\colon S_i \to S$ *are given by*

$$\psi_i\colon (k, l) \mapsto (k\varepsilon_{\alpha^i}, \varepsilon_{\alpha^i}, l\varepsilon_{\alpha^i}) \quad (k, l \in I_{\alpha^i},\ \alpha^i \in X_i)$$

where $\varepsilon_{\alpha^i} = \varepsilon_{I_{\alpha^i}}$. *The inverse operation in* S *is given by*

$$(x\alpha, \alpha, y\alpha)^{-1} = (y\alpha, \alpha, x\alpha).$$

Proof. Let $T = (Y; J_\alpha, g_{\alpha, \beta})$ be the free product of the (pairwise disjoint) combinatorial strict inverse subsemigroups $S_i = (X_i; I_{\alpha^i}, f_{\alpha^i, \beta^i})$ and assume that $X_i \subseteq Y$ for all i in the appropriate way. For each $i \in I$ let $\psi_i\colon S_i \to S$ be the embedding $(j, k) \mapsto (j\varepsilon, \varepsilon, k\varepsilon)$ as described by Proposition 5.1 where $j, k \in I_{\alpha^i}$ and $\varepsilon = \varepsilon_{I_{\alpha^i}}$ for $\alpha^i \in X_i$. Let $\psi\colon T \to S$ denote the unique extension of the mappings ψ_i. By Proposition 5.2, S is generated by $\bigcup_{i \in I} S_i\psi_i$. Hence ψ is surjective. For $j = 1, \ldots, n$, $k = 1, \ldots, m$ let $l_j, r_j \in I_{\alpha_j}$ where

$\alpha_j \in X_{i_j}$ and $s_k, t_k \in I_{\beta_k}$ where $\beta_k \in X_{i_k}$ (again we omit the upper indices of the elements α_j, β_k). Assume that

$$(**) \qquad [(l_1, r_1)\cdots(l_n, r_n)]\psi = [(s_1, t_1)\cdots(s_m, t_m)]\psi .$$

Let $(l_j, r_j)\psi = (l_j, \varepsilon_j, r_j)$ for $\varepsilon_j = \varepsilon_{I_{\alpha_j}}$ and $(s_k, t_k)\psi = (s_k, \iota_k, t_k)$ for $\iota_k = \varepsilon_{I_{\beta_k}}$. The condition $(**)$ states that

$$(l_1, \varepsilon_1, r_1)\cdots(l_n, \varepsilon_n, r_n) = (s_1, \iota_1, t_1)\cdots(s_m, \iota_m, t_m)$$

holds in S. Let $L_1 = \{\lambda_{11}, \dots, \lambda_{n1}\}$ be the admissible set generated by $\{\alpha_1, \dots, \alpha_n\}$ where $\lambda_{j1} \le \alpha_j$. Let $l_j^1 = l_j g_{\alpha_j, \lambda_{j1}}$ and $r_j^1 = r_j g_{\alpha_j \lambda_{j1}}$ and define s_k^1, t_k^1 by analogy. Let λ_1 be the equivalence relation on the set $\{l_1^1, r_1^1, \dots, l_n^1, r_n^1\}$ generated by the pairs $\{(r_1^1, l_2^1), \dots, (r_{n-1}^1, l_n^1)\}$. By analogy, let ρ_1 denote the equivalence relation on $\{s_1^1, t_1^1, \dots, s_m^1, t_m^1\}$ which is generated by $\{(t_1^1, s_2^1), \dots, (t_{m-1}^1, s_m^1)\}$. For each i-component λ_1^i and ρ_1^i consider the sets $\delta_i\lambda_1^i$ and $\delta_i\rho_1^i$ (defined as in Lemma 5.3). Let $\delta\lambda_1 = \bigcup \delta_i\lambda_1^i$ and $\delta\rho_1 = \bigcup \delta_i\rho_1^i$, the union being taken over the non-empty i-components. Let $L_2 = \overline{\delta\lambda_1}$ and $R_2 = \overline{\delta\rho_1}$ be the admissible sets generated by $\delta\lambda_1$ and $\delta\rho_1$, respectively. For $j = 1, \dots, n$ and $k = 1, \dots, m$ let λ_{j2} denote the lower bound of α_j in L_2 and ρ_{k2} the lower bound of β_k in R_2. Let $l_j^2 = l_j g_{\alpha_j, \lambda_{j2}}$, $r_j^2 = r_j g_{\alpha_j, \lambda_{j2}}$, $s_k^2 = s_k g_{\beta_k, \rho_{k2}}$, $t_k^2 = t_k g_{\beta_k, \rho_{k2}}$. Notice that the involved structure mappings in fact are structure mappings of the semigroups S_i. Now repeat the same procedure sufficiently often, say $u-1$ (respectively $v-1$) times until each i-component of λ_u (respectively ρ_v) has only trivial equivalence classes. Let $L = L_u = \overline{\delta\lambda_{u-1}}$ and $R = R_v = \overline{\delta\rho_{v-1}}$ be the admissible sets generated by $\delta\lambda_{u-1}$, respectively $\delta\rho_{v-1}$ (defined in the same way as for the first step). For each α_j let $\tau_j = \lambda_{ju}$ denote the (unique) lower bound of α_j in L. Dually let $\sigma_k = \rho_{kv}$ be the lower bound of β_k in R. For each j, k let $l_j' = l_j^u = l_j g_{\alpha_j, \tau_j}$, $r_j' = r_j^u = r_j g_{\alpha_j, \tau_j}$, $s_k' = s_k^v = s_k g_{\beta_k, \sigma_k}$, $t_k' = t_k^v = t_k g_{\beta_k, \sigma_k}$. By Lemma 2.3,

$$(l_1, r_1)\cdots(l_n, r_n) = (l_1', r_1')\cdots(l_n', r_n')$$

and

$$(s_1, t_1)\cdots(s_m, t_m) = (s_1', t_1')\cdots(s_m', t_m').$$

Next let ξ denote the equivalence relation on I_L which is generated by the pairs

$$A = \{(r_1', l_2'), \dots, (r_{n-1}', l_n')\}.$$

That is, $\xi = \lambda_u \cup \varepsilon_{I_L}$. By Lemma 5.3 and $(**)$ and taking into account that the mappings g_{α_j, τ_j} and g_{β_k, σ_k} which have been used for the definition of the elements l'_j, r'_j and s'_k, t'_k in fact are structure mappings f_{α_j, τ_j} and f_{β_k, σ_k} of the involved semigroups S_i it follows that $\lambda_u \cup \varepsilon_{I_L} = \rho_v \cup \varepsilon_{I_R}$. In particular, $L = R$. Further, ξ is also generated (as an equivalence relation on $I_L = I_R$) by the pairs

$$B = \{(t'_1, s'_2), \dots, (t'_{m-1}, s'_m)\}.$$

That is,

$$\xi = (A \cup \varepsilon_{I_L} \cup A^{-1})^t = (B \cup \varepsilon_{I_L} \cup B^{-1})^t$$

(^{t} denoting the transitive closure). Let δ denote the δ-function of T and let

$$\mu = \inf\{\delta(r'_1, l'_2), \dots, \delta(r'_{n-1}, l'_n)\}$$

and

$$\nu = \inf\{\delta(t'_1, s'_2), \dots, \delta(t'_{m-1}, s'_m)\}.$$

Taking into account that $\delta(y, x) = \delta(x, y) \le \delta(x, x)$ it follows that $\delta(t'_k, s'_{k+1}) \ge \mu$ and $\delta(r'_j, l'_{j+1}) \ge \nu$ for all $k = 1, \dots, m-1$ and $j = 1, \dots, n-1$. We obtain that $\mu = \nu$. Further, $(**)$ and Lemma 5.3 imply that $l'_1 \,\xi\, s'_1$ and $r'_n \,\xi\, t'_m$. Again, since ξ is generated by A it follows that $l'_1 g_{\tau_1, \mu} = s'_1 g_{\sigma_1, \mu}$ and $r'_n g_{\tau_n, \mu} = t'_m g_{\sigma_m, \mu}$. We obtain that

$$\begin{aligned}(l_1, r_1) \cdots (l_n, r_n) &= (l'_1, r'_1) \cdots (l'_n, r'_n) \\ &= (l'_1 g_{\tau_1, \mu}, r'_n g_{\tau_n, \mu}) = (s'_1 g_{\sigma_1, \mu}, t'_m g_{\sigma_m, \mu}) \\ &= (s'_1, t'_1) \cdots (s'_m, t'_m) = (s_1, t_1) \cdots (s_m, t_m).\end{aligned}$$

Consequently, ψ is injective and thus is an isomorphism between T and S.

Lemma 5.3 and Theorem 5.4 provide the following criterion for equality of two words in the free product of the combinatorial strict inverse semigroups $S_i = (X_i, I_{\alpha^i}, f_{\alpha^i, \beta^i})$ (this generalizes Theorem 4.1 in [**1**]).

THEOREM 5.5. *Let* $\{S_i = (X_i; I_{\alpha^i}, f_{\alpha^i, \beta^i}) | i \in I\}$ *be a collection of pairwise disjoint combinatorial strict inverse semigroups* S_i *having pairwise disjoint structure sets* X_i. *For* $k = 1, \dots, n$ *and* $q = 1, \dots, m$ *let* $\alpha_k, \beta_q \in \bigcup_{i \in I} X_i$ *(omitting the upper indices) and let* $l_k, r_k \in I_{\alpha_k}$, $s_q, t_q \in I_{\beta_q}$. *Construct* γ *and* D *for* $(l_1, r_1), \dots, (l_n, r_n)$ *as in Lemma* 5.3 *and analogously,* γ' *and* D' *for* $(s_1, t_1), \dots, (s_m, t_m)$.

For each k and q let γ_k and γ'_q be the respective (unique) lower bounds of α_k in D and β_q in D'. Let $a = (l_1, r_1) \cdots (l_n, r_n)$ and $b = (s_1, t_1) \cdots (s_m, t_m)$. Then $a = b$ holds in the (combinatorial strict inverse) free product of the semigroups S_i if and only if

(1) *$\gamma = \gamma'$ (which implies that $D = D'$).*

(2) *$l_1 f_{\alpha_1, \gamma_1} \gamma s_1 f_{\beta_1, \gamma'_1}$,*

(3) *$r_n f_{\alpha_n, \gamma_n} \gamma t_m f_{\beta_m, \gamma'_m}$.*

Furthermore, $a \mathscr{D} b$ if and only if (1) holds, $a \mathscr{L} b$ if and only if (1) and (3) hold and $a \mathscr{R} b$ if and only if (1) and (2) hold.

The assertions about the Green's relations hold since for $S = (X; I_\alpha, f_{\alpha,\beta})$, $(i, j) \in I_\alpha \times I_\alpha$, $(k, l) \in I_\beta \times I_\beta$ we have $(i, j) \mathscr{D} (k, l)$ if and only if $\alpha = \beta$, $(i, j) \mathscr{L} (k, l)$ if and only if $j = l$ (which includes $\alpha = \beta$) and $(i, j) \mathscr{R} (k, l)$ if and only if $i = k$.

The triple $(l_1 f_{\alpha_1, \gamma_1} \gamma, \gamma, r_n f_{\alpha_n, \gamma_n} \gamma)$ can be interpreted as a "canonical form" of the product $(l_1, r_1) \cdots (l_n, r_n)$. However, the process described in Lemma 5.3 for obtaining such a "canonical form" in general is not "effective" or "computable". It is not expressed purely in terms of the algebraic operation of the inverse semigroups S_i. If all involved semigroups S_i are finite then the procedure can be effectively computed. This follows from the fact that for a given (that is, the elements and the multiplication are completely known) finite combinatorial strict inverse semigroup S, the partial order $X = S/\mathscr{J}$, the sets I_α and all mappings $f_{\alpha,\beta}$ can be effectively computed. Also, for two given elements it is decidable whether or not the respective $\mathscr{J}$-classes have a common upper bound. In particular, Theorem 5.5 provides a solution to the word problem for the free product of *finite* combinatorial strict inverse semigroups. The next section shows that this is not true for the general case.

6. A counterexample. The following example is obviously influenced by the example of Jones-Olin in **[9]**. A function $f: \mathbb{N} \to \mathbb{N}$ is *computable* if there is an algorithm (a computer program) which for any given $n \in \mathbb{N}$ as input computes nf. For a precise definition and a characterization of computable functions see, for instance, Cohen **[4**, Chapters 2–4]. An important theorem on computable functions says that there is such a function $f: \mathbb{N} \to \mathbb{N}$ whose range is not recursive. That is, there is no algorithm which for given $n \in \mathbb{N}$ decides whether or not $n \in \mathbb{N}f$. Now take such a function f and define a semilattice S as follows. For $i = 1, 2$ let $\mathbb{N}_i = \{n_i | n \in \mathbb{N}\}$ and $S = \{0\} \cup \mathbb{N}_1 \cup \mathbb{N}_2$, endowed with the following multiplication:

$$\begin{aligned} n_2 m_2 &= \min\{n, m\}_2, \\ n_2 m_1 &= m_1 \quad \text{iff } m \in \{1f, 2f, \dots, nf\}, \\ n_1 m_1 &= n_1 \quad \text{iff } n = m \end{aligned}$$

and let all other products be defined to be 0. For $n, m \in \mathbb{N}$ it is decidable whether or not $m \in \{1f, \dots, nf\}$. So the multiplication in S can be effectively computed. On the other hand, for given $n \in \mathbb{N}$ the elements n_1 and $(1f)_1$, $n \neq 1f$, have a common upper bound in S if and only if n_1 has an upper bound among the elements $\mathbb{N}_2$. The latter holds if and only if $n \in \{1f, \dots, mf\}$ for some $m \in \mathbb{N}$, that is, if and only if $n \in \mathbb{N}f$. This is undecidable by construction. If a semilattice S is considered as a combinatorial strict inverse semigroup $S = (X; I_\alpha, f_{\alpha,\beta})$ then $S = X$ and $I_\alpha = \{\alpha\}$ for all $\alpha \in X$. According to Theorem 2.1, S is realized by the pairs $\{(\alpha, \alpha) | \alpha \in S\}$. Now let T be any combinatorial strict inverse semigroup not being a semilattice, for instance, $T = B_2$, the combinatorial Brandt semigroup with two non zero idempotents. Let $a \in T$, $a \neq a^2$. In a representation of T according to Theorem 2.1, a can be identified with the pair $(aa^{-1}, a^{-1}a)$ where $aa^{-1} \neq a^{-1}a$. Consider the two words $p, q \in S * T$, defined as follows:

$$p = (0, 0)(aa^{-1}, a^{-1}a)(0, 0)$$

and

$$q = (n_1, n_1)(aa^{-1}, a^{-1}a)((1f)_1, (1f)_1)$$

where $(0, 0)$ is the zero of S and $n_1 \neq (1f)_1$. Applying Theorem 5.5 to the products p and q one obtains that $q = p$ if and only if n_1 and $(1f)_1$ have a common upper bound in X $(= S)$. However, this is undecidable by construction.

Acknowledgment. I wish to express my gratitude to the organizers of the Conference on Semigroups in Oberwolfach in July 1991, Professor Howie, Professor Munn and Professor Weinert for inviting me to this meeting. Several discussions there, especially with Peter Jones, have contributed essentially to the construction of the above example.

References

[1] K. Auinger, *Free strict inverse semigroups*, J. Algebra, **157** (1993), 26–42.

[2] ——, *Free objects in joins of strict inverse and completely simple semigroup varieties*, J. London Math. Soc., **45** (1992), 491–507.

[3] M. Behzad and G. Chartrand, *Introduction to the Theory of Graphs*, Allyn and Bacon, Boston, 1971.
[4] D. E. Cohen, *Computability and Logic*, Ellis Horwood Ltd., Chichester, 1987.
[5] G. Grätzer, *Universal Algebra*, Van Nostrand, Princeton, 1968.
[6] P. R. Jones, *Completely simple semigroups: free products, free semigroups and varieties*, Proc. Royal Soc. Edinberg, **88A** (1981), 293–313.
[7] ——, *A graphical representation for the free product of E-unitary inverse semigroups*, Semigroup Forum, **24** (1982), 195–221.
[8] ——, *Free products of inverse semigroups*, Trans. Amer. Math. Soc., **282** (1), (1984), 293–317.
[9] ——, *On free products of bands*, Semigroup Forum, **42** (1991), 53–62.
[10] ——, *Free products of inverse semigroups: A survey, with examples*, Proc. of the 1990 Monash Conference on Semigroups.
[11] ——, *On free products of completely regular semigroups*, J. Austral. Math. Soc., to appear.
[12] P. R. Jones, S. W. Margolis, J. C. Meakin, and J. B. Stephen, *Free products of inverse semigroups* II, Glasgow Math. J., **33** (1991), 373–387.
[13] S. W. Margolis, J. C. Meakin and J. B. Stephen, *Free objects in certain varieties of inverse semigroups*, Canad. J. Math., **XLII** 6 (1990), 1084–1097.
[14] M. Petrich, *Inverse Semigroups*, Wiley, New York, 1984.
[15] N. R. Reilly, *Free combinatorial strict inverse semigroups*, J. London Math. Soc., **39** (1989), 102–120.

Received November 19, 1991 and in revised form April 15, 1992.

INSTITUT FÜR MATHEMATIK
STRUDLHOFGASSE 4
A-1090 WIEN, AUSTRIA