PACIFIC JOURNAL OF MATHEMATICS
Vol. 165, No. 1, 1994

COMPLETE OPEN MANIFOLDS OF NON-NEGATIVE RADIAL CURVATURE

YOSHIROH MACHIGASHIRA

We generalize the Toponogov hinge theorem and the Alexandrov convexity to the context of radial curvature, and study complete open Riemannian manifolds of non-negative radial curvature.

0. Introduction. It is well-known that a non-negative curved manifold has some interesting characters as exemplified in the Soul theorem ([CG]) or the Toponogov splitting theorem ([T]). In such theorems, Toponogov's comparison theorem plays an essential role.

Throughout this paper let M be a connected complete Riemannian manifold of dimension $n \geq 2$. For a point $o \in M$, the sectional curvature K_M of M restricted to those planes that are tangent to some minimal geodesic starting from o is called minimal radial curvature from o and is denoted by $K_o^{\min}$. The notion of radial curvature was initiated by Klingenberg in [K] to prove a homotopy sphere theorem for compact simply-connected manifolds with $\frac{1}{4}$-pinched radial curvature. Also in the case where M is noncompact and o is a pole of M, Greene and Wu have shown some results related to the radial curvature from o (see [GW]).

In [M], it is shown that Toponogov's comparison theorem holds for the edge angles at x_1 and x_2 of a minimal geodesic triangle with vertices at o, x_1, and x_2 under suitable condition on $K_o^{\min}$. Moreover by using this fact, some results related to the radial curvature from o were obtained in [M] or [MS]. For example,

THEOREM 0.1 (*Theorem* A *in* [MS]). *A complete noncompact Riemannian manifold M which contains a point o such that $K_o^{\min} > 0$ has exactly one end.*

THEOREM 0.2 (*Theorem* C *in* [MS]). *Let M be noncompact with a point o such that $K_o^{\min} \geq 0$. If*

$$\lim_{r\to\infty} \frac{\mathrm{vol} B(o, r)}{b_0(r)} > \frac{1}{2},$$

then M is diffeomorphic to $\mathbf{R}^n$, where $\mathrm{vol}B(o, r)$ is the volume of the r-ball $B(o, r)$ in M around o and $b_0(r)$ is the volume of the r-ball of $\mathbf{R}^n$.

In this paper we prove that Toponogov's comparison theorem also holds for edge angles at o (see Theorem 1.3) and investigate the topology of complete open manifolds of non-negative radial curvature. By using Theorem 1.3 we will obtain the

MAIN THEOREM. *Let M be noncompact. Assume that $K_o^{\min} \geq 0$ for some point $o \in M$. Then:*

(A) *The set of critical points of the distance function from o is bounded and consequently M is finitely-connected.*

(B) *M has at most two ends.*

(C) *If M has a line, then M is diffeomorphic to $N \times \mathbf{R}$, where N is a hypersurface in M. Moreover the projection $M \to \mathbf{R}$ is a Riemannian submersion.*

1. The Toponogov hinge theorem for radial curvature. For any $\delta \in \mathbf{R}$, let M^δ denote the simply-connected surface of constant Gauss curvature δ. First of all we recall the

THEOREM 1.1 (*Proposition* 1.1 *in* [**M**] *or Theorem* 1.1 *in* [**MS**]). *Assume that $K_o^{\min} \geq \delta$ for $o \in M$ and $\delta \in \mathbf{R}$. Let γ_1 and γ_2 be length-minimizing segments in M with $\gamma_1(0) = \gamma_2(1) = o$ and let γ_0 be a length-minimizing segment such that $\gamma_0(0) = \gamma_1(1)$ and $\gamma_0(1) = \gamma_2(0)$. Then, there exist length-minimizing segments $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, and $\tilde{\gamma}_0$ in M^δ with $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(1)$, $\tilde{\gamma}_0(0) = \tilde{\gamma}_1(1)$, and $\tilde{\gamma}_0(1) = \tilde{\gamma}_2(0)$ which are such that*

$$L(\gamma_i) = L(\tilde{\gamma}_i) \qquad \textit{for } i = 0, 1, 2$$

and

$$\theta_1 := \angle(-\dot{\gamma}_1(1), \dot{\gamma}_0(0)) \geq \angle(-\dot{\tilde{\gamma}}_1(1), \dot{\tilde{\gamma}}_0(0)) =: \tilde{\theta}_1,$$
$$\theta_2 := \angle(-\dot{\gamma}_0(1), \dot{\gamma}_2(0)) \geq \angle(-\dot{\tilde{\gamma}}_0(1), \dot{\tilde{\gamma}}_2(0)) =: \tilde{\theta}_2.$$

Moreover if $\theta_1 = \tilde{\theta}_1 \neq \pi$, then there exists a piece of totally geodesic surface of constant curvature δ bounded by γ_1, γ_0, and a minimizing geodesic joining o to $\gamma_0(1)$ (which is not necessarily γ_2) in M.

The above theorem is shown by dividing γ_0 into sufficiently small sub-arcs $\{\gamma_0|_{[t_{i-1}, t_i]}\}_{i=1,\dots,N}$, where $t_0 = 0$ and $t_N = 1$, and applying Berger's comparison theorem to obtain the angle estimates at

$\gamma_0(t_{i-1})$ and $\gamma_0(t_i)$ of a geodesic triangle $\Delta(o, \gamma_0(t_{i-1}), \gamma_0(t_i))$ (cf. Theorem 2.2 in **[CE]**). If $\theta_1 = \tilde{\theta}_1 \neq \pi$, then the angles at $\gamma_0(t_{i-1})$ and $\gamma_0(t_i)$ of any geodesic triangle $\Delta(o, \gamma_0(t_{i-1}), \gamma_0(t_i))$ must be equal to the angles of the corresponding triangle in M^δ respectively for $i = 1, \cdots, N-1$. Also the angle at $\gamma_0(t_{N-1})$ of $\Delta(o, \gamma_0(t_{N-1}), \gamma_2(0))$ equals the angle of the corresponding triangle. Moreover for every $i = 1, \cdots, N$, there is a minimal geodesic γ joining o to $\gamma_0(t_i)$ such that $\gamma \subset \exp_o(X)$, where $X \subset T_oM$ is the plane spanned by $\dot{\gamma}_1(0)$ and the vector parallel to $\dot{\gamma}_0(0)$ along γ_1. Hence we obtain the minimal geodesic γ_2' joining o to $\gamma_0(1) = \gamma_2(0)$ such that $\dot{\gamma}_2'(0) \subset \exp_o(X)$ and a totally geodesic surface of constant curvature δ bounded by γ_1, γ_2' and γ_0. (For detail see **[M]**.)

We can check the following corollary by dividing a geodesic triangle into two triangles.

COROLLARY 1.2. *Under the assumption of Theorem* 1.1,

$$d_M(o, \gamma_0(s)) \geq d_\delta(\tilde{o}, \tilde{\gamma}_0(s)) \qquad \text{for } s \in [0, 1],$$

where d_M and d_δ denote the distance functions on M and M^δ respectively.

The following theorem implies that edge angles at o can be compared.

THEOREM 1.3. *Assume that $K_o^{min} \geq \delta$ for $o \in M$ and $\delta \in \mathbf{R}$. For any minimizing geodesics $\sigma_1 : [0, 1] \to M$ and $\sigma_2 : [0, 1] \to M$ starting from o, we have the following results*

(1) *Let $\tilde{\sigma}_i : [0, 1] \to M^\delta$ for $i = 1, 2$ be minimizing geodesics starting from same point such that*

$$L(\sigma_i) = L(\tilde{\sigma}_i) \qquad \text{for } i = 0, 1, 2$$

and

$$\angle(\dot{\sigma}_1(0), \dot{\sigma}_2(0)) = \angle(\dot{\tilde{\sigma}}_1(0), \dot{\tilde{\sigma}}_2(0)).$$

Then

$$d_M(\sigma_1(1), \sigma_2(1)) \leq d_\delta(\tilde{\sigma}_1(1), \tilde{\sigma}_2(1)).$$

(2) (The Alexandrov convexity). *Let $\tilde{\theta}_{s,t}$ be the angle at $\tilde{o}$ of the triangle $\tilde{\Delta}(\tilde{o}, \tilde{x}_s, \tilde{y}_t)$ in M^δ corresponding to $\Delta(o, \sigma_1(s), \sigma_2(t))$ in M. Then $\tilde{\theta}_{s,t}$ is monotone non-increasing in s, t.*

(3) *In* (1), *if equality holds, then there is a piece of totally geodesic surface of constant curvature δ bounded by σ_1, σ_2 and a minimal geodesic joining $\sigma_1(1)$ to $\sigma_2(1)$.*

Proof of Theorem 1.3. (1) We work with $M^{\delta-\epsilon}$ instead of M^δ, where ϵ is any small positive number.

Put

$$t_0 := \sup_{t\in[0,1]} \{t | \text{for } s \leq t,\ d_M(\sigma_1(s), \sigma_2(s)) \leq d_{\delta-\epsilon}(\tilde\sigma_1(s), \tilde\sigma_2(s))\}.$$

Then Rauch's comparison theorem implies that $t_0 > 0$. Suppose that $t_0 < 1$. Then we see that

$$d_M(\sigma_1(t_0), \sigma_2(t_0)) = d_{\delta-\epsilon}(\tilde\sigma_1(t_0), \tilde\sigma_2(t_0)).$$

Thus we can apply Theorem 1.1 to the geodesic triangles

$$\Delta(o, \sigma_1(t_0), \sigma_2(t_0)) \quad \text{in } M$$

and

$$\tilde\Delta(\tilde\sigma_1(0), \tilde\sigma_1(t_0), \tilde\sigma_2(t_0)) \quad \text{in } M^{\delta-\epsilon},$$

that is, if we let θ_i and $\tilde\theta_i$ for $i = 1, 2$ be the angles at $\sigma_i(t_0)$ and $\tilde\sigma_i(t_o)$, then

$$\theta_1 \geq \tilde\theta_1 \quad \text{and} \quad \theta_2 \geq \tilde\theta_2.$$

In the case where $\theta_1 > \tilde\theta_1$, the first variation formula implies that

$$d_M(\sigma_1(t_0+h), \sigma_2(t_0+h)) < d_{\delta-\epsilon}(\tilde\sigma_1(t_0+h), \tilde\sigma_2(t_0+h))$$

for sufficiently small $h > 0$. This contradicts the definition of t_0. Next we consider the case where $\theta_1 = \tilde\theta_1$. Since the assumption that $t_0 < 1$ says $\theta_1 \neq \pi$, the later half of Theorem 1.1 implies that there exists a piece of totally geodesic surface of constant curvature $\delta - \epsilon$ bounded by $\sigma_1|_{[0,t_0]}$, $\sigma_2|_{[0,t_0]}$ and a minimal geodesic joining $\sigma_1(t_0)$ to $\sigma_2(t_0)$. This contradicts $K_o^{\min} \geq \delta$. This completes the proof of (1).

(2) It suffice to show that for arbitrary fixed $s \in (0, 1]$ and $t \in (0, 1)$,

$$\tilde\theta_{s,t} \geq \tilde\theta_{s,t+h} \quad \text{for small } h > 0. \tag{1.1}$$

By continuity of $\tilde\theta_{s,t}$, we may assume $s < 1$. Put

$$\theta := \angle(\dot\sigma_1(0), \dot\sigma_2(0)).$$

Restating (1), we see that $\tilde\theta_{s,t} \leq \theta$ for all $s, t \in [0, 1]$. Thus in the case where $\tilde\theta_{s,t} = \theta$, clearly (1.1) holds. Hence we consider only the case $\tilde\theta_{s,t} < \theta$.

Let $\tilde\sigma$ be minimal geodesic in M^δ starting from $\tilde o$ and passing y_t parameterized as $\tilde\sigma(t) = \tilde y_t$. From Theorem 1.1, the angle at $\tilde y_t$ of

$\tilde{\Delta}(\tilde{o}, \tilde{x}_s, \tilde{y}_t)$ does not exceed the angle at $\sigma_2(t)$ of $\Delta(o, \sigma_1(s), \sigma_2(t))$. If the angles are equal to each other, by the latter half of Theorem 1.1, it must be that $\tilde{\theta}_{s,t} \geq \theta$, because the minimal geodesic joining o to $\sigma_1(s)$ is unique. This contradicts $\tilde{\theta}_{s,t} < \theta$. If the angle at y_t is smaller than the angle at $\sigma_2(t)$, then the first variation formula implies that

$$d_\delta(\tilde{o}, \tilde{\sigma}(t+h)) \geq d_M(o, \sigma_2(t+h))$$

for small $h > 0$, that is,

$$\tilde{\theta}_{s,t+h} \leq \tilde{\theta}_{s,t}.$$

This completes the proof of (2).

(3) We apply (2) to obtain that if

$$d_M(\sigma_1(1), \sigma_2(1)) = d_\delta(\tilde{\sigma}_1(1), \tilde{\sigma}_2(1)),$$

then $\theta = \tilde{\theta}_{s,t}$ for all $s, t \in [0, 1]$. Hence the angles at $\tilde{x}_s$ and $\tilde{y}_t$ of $\tilde{\Delta}(\tilde{o}, \tilde{x}_s, \tilde{y}_t)$ equal the angles at $\sigma_1(s)$ and $\sigma_2(t)$ of $\Delta(o, \sigma_1(s), \sigma_2(t))$ respectively for all $s, t \in (0, 1)$. Thus there is a piece of totally geodesic surface of constant curvature δ bounded by $\sigma_1|_{[0,s]}$, $\sigma_2|_{[0,t]}$ and a minimal geodesic $\sigma_{s,t}$ joining $\sigma_1(s)$ to $\sigma_2(t)$. Hence $\dot{\sigma}_{s,t}(0)$ is contained the plane spanned by $\dot{\sigma}_1(s)$ and the vector parallel to $\dot{\sigma}_2(0)$ along $\sigma_1|_{[0,s]}$. Taking $s, t \to 1$, we obtain a minimal geodesic joining $\sigma_1(1)$ to $\sigma_2(1)$.

The proof of Theorem 1.3 is completed.

REMARK 1.4. By using Theorem 1.3 (2), we can construct the ideal boundary $M(\infty)$ of a complete open manifold M of nonnegative radial curvature and the Titz metric on it. However it is not needed in this article.

2. Proof of the main theorem. Now part (A) of the main theorem is shown directly from Theorem 1.3 in the same way as the proof of the corollary to Theorem 1.5.A in [G] or Corollary 2.9 in [C].

In the remainder of this paper, we agree that geodesics will be parameterized by the arc-length.

Proof of part (B) *of the main theorem.* Suppose that M has three or more ends. Then there are three rays γ_1, γ_2, and γ_3 starting from o going to different ends. Let l_t be a minimal geodesic joining $\gamma_1(t)$ to $\gamma_2(t)$ and $\tilde{\theta}_t$ the angle at $\tilde{o} \in \mathbf{R}^2$ of $\tilde{\Delta}(\tilde{o}, \tilde{x}_t, \tilde{y}_t)$ in $\mathbf{R}^2$ such that $d_0(\tilde{o}, \tilde{x}_t) = d_0(\tilde{o}, \tilde{y}_t) = t$ and $d_0(\tilde{x}_t, \tilde{y}_t) = L(l_t)$. Since the distance between o and l_t is bounded from above by some constant

C independent of t, Corollary 1.2 implies that the distance between $\tilde{o}$ and the segment joining $\tilde{x}_t$ to $\tilde{y}_t$ is also bounded by C. Thus we see that

$$\tilde{\theta}_t \to \pi \quad \text{as } t \to \infty$$

and consequently

$$\angle(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) = \pi.$$

Similarly, we have that

$$\angle(\dot{\gamma}_2(0), \dot{\gamma}_3(0)) = \angle(\dot{\gamma}_3(0), \dot{\gamma}_1(0)) = \pi.$$

This contradicts and hence completes the proof of (B).

In Lemma 1.3 in **[MS]** it is shown that if a non-negative minimal radial curved M with base point o has a line, then there is a line passing through o. We will show that a similar thing is realized for any $x \in M$. We define a Busemann function F_γ on noncompact M for a ray γ by

$$F_\gamma(x) := \lim_{t\to\infty}[t - d_M(x, \gamma(t))] \qquad \text{for } x \in M.$$

LEMMA 2.1. *Under the assumption of the main theorem, if there is a line σ through o, then for any $x \in M$ there is a unique line l_x through x which is biasymptotic to σ. Moreover there exists a flat totally geodesic strip bounded by $\sigma(\mathbf{R})$ and $l_x(\mathbf{R})$.*

Proof. Choose the parameter of σ such that $\sigma(0) = o$ and set $\sigma_\pm(t) := \sigma(\pm t)$ for $t \geq 0$. Let β be a minimal geodesic joining o to x and put $\theta_\pm := \angle(\dot{\beta}(0), \dot{\sigma}_\pm(0))$. Let $\tilde{\theta}_{\pm t}$ be the angles at $\tilde{o} \in \mathbf{R}^2$ of $\tilde{\Delta}(\tilde{o}, \tilde{x}, \tilde{y}_{\pm t})$ in $\mathbf{R}^2$ corresponing to $\Delta(o, x, \sigma_\pm(t))$ in M, and put $\tilde{\theta}_{\pm\infty} := \lim_{t\to\infty}\tilde{\theta}_{\pm t}$. (Theorem 1.3 (2) guarantees the existence of $\tilde{\theta}_{\pm\infty}$.) Then

$$F_{\sigma_\pm}(x) = d(o, x)\cos\tilde{\theta}_{\pm\infty}.$$

Thus we obtain that

$$\cos\tilde{\theta}_{+\infty} + cos\tilde{\theta}_{-\infty} \leq 0 \tag{2.1}$$

because it follows from the triangle inequality that

$$F_{\sigma_+}(x) + F_{\sigma_-}(x) \leq 0.$$

On the other hand, by Theorem 1.3 we see that

$$\tilde{\theta}_{+\infty} + \tilde{\theta}_{-\infty} \leq \theta_+ + \theta_- = \pi. \tag{2.2}$$

The formulas (2.1) and (2.2) imply that $\tilde{\theta}_{+\infty} + \tilde{\theta}_{-\infty} = \pi$, and consequently $\tilde{\theta}_{+t} + \tilde{\theta}_{-t} = \pi$ for all $t > 0$ from Theorem 1.3. Thus we obtain that $\tilde{\theta}_{\pm t} = \theta_{\pm}$ for all $t > 0$ and there are two pieces of totally geodesic surface of constant curvature 0 bounded by $\sigma_{\pm}|_{[0,t]}$, β and $l_{\pm t}$, where $l_{\pm t}$ are geodesics joining x to $\sigma_{\pm}(t)$. Hence there exist two rays $l_{x\pm}$ starting from x and asymptotic to $\sigma_{\pm}$ such that $\dot{l}_{x\pm}(0)$ are parallel to $\dot{\sigma}_{\pm}(0)$ along β. Moreover there exists a flat totally geodesic strip bounded by σ and $l_x := l_{x+} \cup l_{x-}$. To prove that l_x is a line, we consider $l_{x+}(-s) =: x_s$ for arbitrary $s > 0$ instead of x. Then $l_{x-}|_{[s,\infty]}$ is the unique ray starting from x_s and asymptotic to σ_- because x_s is an interior point of a ray l_{x-} (see Theorem 1.1 in [S]). Hence we see that $l_{x+}|_{[-s,\infty]}$ is a ray and l_x must be a line. This completes the proof of Lemma 2.1.

Proof of part (C) *of the main theorem.* Let σ be a line passing through o parameterized as $\sigma(0) = o$, constructed in Lemma 1.3 in [**MS**]. Let σ_+ be $\sigma|_{[0,\infty)}$ and put $N_s := (F_{\sigma_+})^{-1}(s)$ for $s \in \mathbf{R}$. Then N_s is a smooth hypersuface of M because the gradient vector at x of F_{σ_+} is unique and its length equals 1 for any $x \in M$ by Lemma 2.1. For $x \in M$ let $x_0 \in l_x(\mathbf{R})$ be the point such that $d_M(o, x_0) = d_M(o, l_x(\mathbf{R}))$, where l_x is as in Lemma 3.1. Then Lemma 3.1 implies that x_0 is unique and contained in N_0. Thus we can define a map $g_s : N_s \to N_0$ for all $s \in \mathbf{R}$ by $g_s(x) := x_0$. This map is clearly bijective and a local diffeomorphism, that is, a global diffeomorphism. Hence at last we obtain the desired map $G : M \to N_0 \times \mathbf{R}$ by $G(x) := (x_0, F_{\sigma_+}(x))$.

REMARK 2.2. Each hypersurface N_s is a star-shaped subset of M, that is, a minimal geodesic joining $\sigma(s)$ to a point in N_s is contained in N_s. Moreover if the Busemann functions $F_{\sigma|_{[a,\infty]}}$ and $F_{\sigma|_{[-\infty,a]}}$ for $a \in \mathbf{R}$ are convex, then N_s is totally convex for any $s \in \mathbf{R}$ and the map G is an isometry. But we do not know whether it is true or not that the Busemann function for a ray passing through o is convex under the condition $K_o^{\min} \geq 0$.

Acknowledgment. The author would like to express thanks to Professor Katsuhiro Shiohama for his advice and encouragement. The author also thanks Takashi Shioya and Takao Yamaguchi. Remark 1.4 is Shioya's idea. He is preparing a paper related to the ideal boundary of a nonnegative curved manifold and the Titz metric on it.

References

[C] J. Cheeger, *Critical points of distance functions and applications to geometry*, Proc. Sympos. Pure Math., (to appear).

[CG] J. Cheeger and D. Gromoll, *On the structure of complete manifolds and non-negative curvature*, Ann. of Math., **96** (1972), 413–443.

[G] M. Gromov, *Curvature, diameter and Betti numbers*, Comment. Math. Helv., **56** (1981), 179–195.

[GW] R. E. Greene and H. Wu, *Function Theory on Manifolds Which Possess a Pole,* Lecture Notes in Math., vol. 699, Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[K] W. Klingenberg, *Manifolds with restricted conjugate locus*, Ann. of Math., **78** (1963), 527–547.

[M] Y. Machigashira, *Manifolds with pinched radial curvature*, Proc. Amer. Math. Soc., **118** (1993), 979–985.

[MS] Y. Machigashira and K. Shiohama, *Riemannian manifolds with positive radial curvature*, Japan. J. Math., (to appear).

[S] K. Shiohama, *Topology of complete noncompact manifolds*, Geometry of Geodesics and Related Topics, Advanced Studies in Pure Mathematics, vol. 3, 1984, pp. 423–450.

[T] V. Toponogov, *Riemannian spaces with straight lines*, Amer. Math. Soc. Transl. Ser. 2, **37** (1964), 287–290.

Received May 21, 1992.

Department of Information Science
Saga University
Saga 840, Japan