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SOAP BUBBLES IN $\mathbb{R}^2$ AND IN SURFACES

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We prove existence and regularity for "soap bubbles" in $\mathbb{R}^2$ and in surfaces, i.e., the least-perimeter way to enclose and separate regions of prescribed area. They consist of constant-curvature arcs meeting in threes at 120 degrees. If one prescribes the combinatorial type too, then the arcs may bump up against each other.

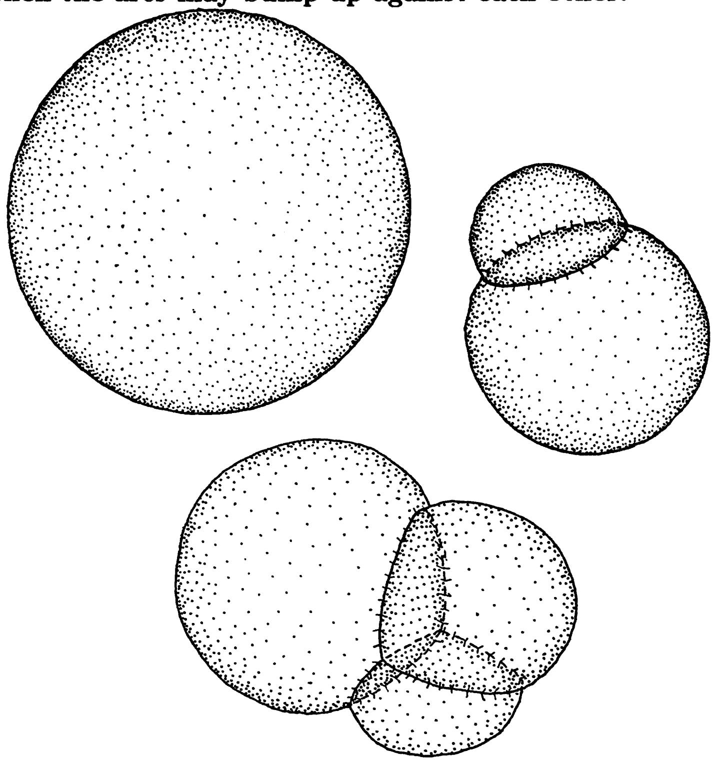

FIGURE 1.1. Single, double, and triple bubbles in $\mathbb{R}^3$ presumably provide the least-area way to enclose and separate the given volumes of air.
Drawings by J. Bredt [M5]

1. Introduction. The general soap bubble problem seeks the least-area way to enclose and separate m given volumes in $\mathbb{R}^n$ or in a smooth compact Riemannian manifold. Examples in $\mathbb{R}^3$ presumably include the bubble clusters of Figure 1.1. Existence and regularity almost everywhere for $n \geq 3$ was established by F. Almgren [**Alm**, VI.2, IV.3(1)] and improved for the case $n = 3$ by J. Taylor ([**T1**], [**T2**]). A simplified discussion appears in the new edition of [**M2**, 13.3]. No such regularity results in the literature seem to apply to the case $n = 2$.

Theorem 2.3 gives a relatively simple, direct treatment of the planar case $n = 2$. The proof provides a simplified illustration of the rather technical methods of Almgren [**Alm**]. The main difficulty is the possibility of regions of infinitely many connected components. To eliminate this possibility, the proof reduces perimeter by eliminating tiny components and delicately readjusting the areas elsewhere (Lemma 2.2).

In a compact manifold, the regions need not be connected. For example, the least-perimeter way to enclose a suitable given area in a sphere with two thin tentacles would be two circles enclosing the two tentacles.

In $\mathbb{R}^n$, it is an open question whether the regions must be connected. Of course for a single prescribed volume, the round sphere is the unique solution (see [**M4**, 10.5]). Recent work by the Williams College SMALL Undergraduate Research Geometry Group ([**Fo**], see [**M1**], [**M3**]), featured in the 1994 AMS *What's Happening in the Mathematical Sciences*, has shown that for two given areas in $\mathbb{R}^2$, the standard double bubble, consisting of two connected regions, is the unique solution (see Figure 1.2). In $\mathbb{R}^3$, it is an open question whether the standard double bubble of Figure 1.1 is a solution.

Indeed, for all $n \geq 2$, it has been an open question whether there exists a least-area way to enclose and separate *connected* regions of prescribed volumes, although it is deceptively easy to think it obvious for $n = 2$ (cf. [**B**]). There are already substantial technical hazards in obtaining *discs* of prescribed area of least perimeter in compact surfaces (where perimeter counts twice when the curves bump up against themselves). Exactly such difficulties delayed for 77 years the completion of Poincare's argument for the existence of a simple closed geodesic on a convex 2-sphere. See Section 3.4 and [**HM**] for more information and for a new, simple approach, which also provides discs of prescribed area of least perimeter in compact surfaces. In his honors thesis [**Ho**], Hugh Howards (Williams '92) gives a number of specific examples of perimeter-minimizing regions in spheres, tori, projective planes, and Klein bottles.

Our Corollary 3.3 provides a least-perimeter way to enclose and separate connected regions of prescribed areas, although the regions may be connected only by infinitesimal strips. In addition to meeting in threes at 120° angles, the edges may in theory bump up against each other differentiably, joining and separating at isolated points, as in Figure 3.2.

Corollary 3.3 follows from Theorem 3.2 on the existence and regularity of soap bubble clusters of prescribed combinatorial type. It is easy to prove existence, using standard compactness for Lipschitz functions and curves. It is difficult to control how wildly the solution curves may bump up against themselves. Such control follows from comparisons and weak curvature bounds, based on a standard variational argument (Lemma 2.2).

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FIGURE 1.2. The standard double bubble is the unique least-perimeter way to enclose and separate two given areas in the plane [**Fo**].

2. Planar soap bubbles of unrestricted combinatorial type. Theorem 2.3 gives the existence and regularity of the least-perimeter way to enclose and separate planar regions of prescribed areas. This formulation does not restrict the combinatorial type of the solution or even require each region to be connected.

The variational arguments will require the ability to make small readjustments in the areas with controlled increase in perimeter. Lemma 2.2 extracts the required arguments from F. Almgren [**Alm**, VI.2(3)], which hold in general dimension and ambient manifolds. First Lemma 2.1 states a standard fact from geometric measure theory. (Nonorientability of M poses no real problem, since the arguments are essentially local.)

LEMMA 2.1. [**Fe**, 4.5.12, 2.10.6]. *Let $R_0, R_1, \ldots, R_m$ be disjoint subsets of $M = \mathbb{R}^n$ or any n-dimensional smooth, compact Riemannian manifold M. Suppose that the topological boundary of each R_i has finite $(n-1)$-dimensional measure:*

$$\mathcal{H}^{n-1}(\mathit{Bdry}\, R_i) < \infty .$$

Then each R_i is measurable and its current boundary $T_i = \partial R_i$ is a rectifiable set.

LEMMA 2.2. (Almgren, see [**Alm**, VI.2(3)]). *Let $R_0, R_1, \ldots, R_m$ be disjoint measurable subsets of positive n-dimensional measure of $M = R^n$ or any n-dimensional smooth, compact Riemannian manifold M, so that $M - \cup R_i$ has measure 0. Suppose that each current boundary $T_i = \partial R_i$ is a rectifiable set.*

(1) For $\mathcal{H}^{n-1}$ almost every point $x \in T_i$, x belongs to precisely one other T_j; T_i and T_j have multiplicity one and a common tangent plane at x, and R_i and R_j have the two associated halfspaces as tangent cones at x.

(2) There are $c, \varepsilon > 0$ and disjoint balls $D_1, D_1', D_2, D_2', \ldots, D_m, D_m'$ such that balls of twice the radius remain disjoint and such that for arbitrary choices of D_i or D_i' and real numbers $t_0, \ldots, t_m$ satisfying $\sum t_i = 0$ and $\max |t_i| < \varepsilon$, there are smooth diffeomorphisms supported in

$$(D_1 \text{ or } D_1') \cup (D_2 \text{ or } D_2') \cup \ldots \tag{3}$$

with local changes in measure

$$\Delta\mathcal{H}^n(R_i) = t_i \ ,$$
$$\Delta\mathcal{H}^{n-1}(T_i) \le c \max |t_i| \ .$$

Proof. Since $\sum T_i = 0$, $\mathcal{H}^{n-1}$ almost every point in $\cup T_i$ belongs to at least two of the T_i. Statement (1) follows by the Gauss-Green-DeGiorgi-Federer Theorem ([**M2**, 12.1] or [**Fe**, 4.5.6]).

As in [**Alm**, VI.9], given small $\eta > 0$, there is a $c_1 > 0$, such that for any two R_i, R_j bordering as in (1), there is a small ball D, such that for small $t > 0$, there is a diffeomorphism supported in D such that

$$\begin{array}{ll} |\Delta\mathcal{H}^n(R_i) + t| \le \eta t & \\ |\Delta\mathcal{H}^n(R_j) - t| \le \eta t & \\ |\Delta\mathcal{H}^n(R_k)| \le \eta t & k \ne i, j \\ |\Delta\mathcal{H}^{n-1}(T_k)| \le c_1 t & 0 \le k \le m. \end{array}$$

The diffeomorphism simply locally pushes the halfspace approximating R_j smoothly into the halfspace approximating R_i. The bound on $\Delta\mathcal{H}^{n-1}(T_k)$ follows from general principles [**All**, 4.1].

Indeed there are many such disjoint D; we will need just two, say D, D'. Furthermore, given any proper subset I of the full set of indices $\{0, 1, \dots, m\}$, there are $i \in I$ and $j \in I^C$ associated as in (1), as follows from applying the Gauss-Green-DeGiorgi-Federer Theorem to $\cup_{i \in I} R_i$. Hence by reordering the indices if necessary, we may obtain such small balls D_1, D'_1, ..., D_m, D'_m, with D_1, D'_1 associated with regions R_0, R_1 and generally with D_j, D'_j associated with regions R_i, R_j with $i < j$. Moreover we may assume these balls, even if doubled in size, are all disjoint. Linear combinations of these deformations yield the desired diffeomorphisms of statement (2). □

THEOREM 2.3. (PLANAR SOAP BUBBLES). *Given prescribed areas A_1, ..., $A_m > 0$, there is a set $S \subset \mathbb{R}^2$ of least 1-dimensional Hausdorff measure $\mathcal{H}^1(S)$ such that $\mathbb{R}^2 - S$ is a disjoint union of (not necessarily connected) components R_0, ..., R_m, with only R_0 unbounded, and* $\mathrm{area}(R_i) = A_i$ $(1 \le i \le m)$. *S consists of finitely many arcs of circles or straight line segments meeting in threes at*

$120°$ angles at finitely many points (possibly plus an unnecessary additional set of $\mathcal{H}^1$ measure 0). All curves separating a specific pair of regions have the same curvature.

Proof. Consider any set $S \subset \mathbb{R}^2$ with $\mathcal{H}^1(S) < \infty$ such that $\mathbb{R}^2 - S$ is a disjoint union of components $R_0, \ldots, R_m$, with only R_0 unbounded, and $\text{area}(R_i) = A_i$. By Lemma 2.1, each R_i is measurable and its current boundary $T_i = \partial R_i$ is rectifiable. By [**Fe**, 4.2.25, p. 421], each T_i is a countable collection of simple closed Lipschitz curves. Therefore the infimum for the following modified problem is no greater than the infimum for the original problem.

(4) *Modified problem.* Find disjoint, bounded sets $R_1, \ldots, R_m$, of prescribed positive areas $A_1, \ldots, A_m$, with rectifiable current boundaries $T_1, \ldots, T_m$, to minimize $\mathcal{H}^1(\cup T_i)$. (Let $R_0 = \mathbb{R}^2 - \cup_{i=1}^m R_i - \cup_{i=1}^m T_i$, $T_0 = \partial R_o$.)

The proof has four steps.

Step 1. There exists a solution to the modified problem (4).
Step 2. The T_i's are actually *finite* collections of simple closed Lipschitz curves. It follows that the R_i are components of $\mathbb{R}^2 - \cup T_i$ and hence admissible solutions to the original problem.
Step 3. A solution to the modified problem has the asserted regularity.
Step 4. Any solution S to the original problem differs from a solution $\cup T_i$ to the modified problem by an inconsequential set of measure 0.

Step 1: Existence. Consider a sequence R_i^j, $T_i^j = \partial R_i^j$, with $\mathcal{H}^1(\cup T_i^j)$ approaching its infimum a. We may assume $\mathcal{H}^1(\cup T_i^j) \leq 2a$.

First we modify the sequence to stay inside a fixed large disc. Consider the simple Lipschitz curves composing T_0^j which are not enclosed by others and hence are surrounded by R_0^j on the outside. Since they have total length at most $2a$, we can certainly move them and all they enclose inside the disc about 0 of radius $3a$, which then contains every R_i^j. Now by compactness (e.g., [**M2**, 5.5] or [**Fe**, 4.2.17]), we may assume that the R_i^j converge to R_i, the T_i^j converge to $T_i = \partial R_i$, and area $R_i = A_i$. (In the limit, each R_i is defined *a priori* only up to a set of measure 0, but viewing it as a region bounded by but not including the countable collection of Lipschitz curves T_i removes the ambiguity.) Moreover by Lemma 2.2(1),

$$\mathcal{H}^1(\cup T_i) = \frac{1}{2}\sum \mathcal{H}^1(T_i) \leq \frac{1}{2}\sum \underline{\lim}\mathcal{H}^1(T_i^j) = \underline{\lim}\mathcal{H}^1(\cup T_i^j) = a \ .$$

We have solved the modified problem (4).

Step 2: We show that each T_i is actually a finite *collection of rectifiable cycles.* Otherwise, for some $0 < \varepsilon < 4\pi/cm^2$, T_i contains a cycle C of length ε, and C lies in a ball disjoint from 2.2(3), for some choice of the alternatives D_j or D_j'. By 2.2(1), there is some other T_j which contains a piece of C of length at least ε/m. Making everything inside C part of R_j reduces $\mathcal{H}^1(\cup T_i)$ by at least ε/m. Since the area inside C is at most $\varepsilon^2/4\pi$, deformations supported in 2.2(3) can restore the areas at a cost of increasing $\mathcal{H}^1(\cup T_i)$ by at most $mc\varepsilon^2/4\pi$. By the minimizing property of $\cup T_i$,

$$\varepsilon/m \leq mc\varepsilon^2/4\pi \ ,$$

a contradiction of the choice of ε. Therefore each T_i is a finite collection of rectifiable cycles. It follows that the R_i are components of $\mathbb{R}^2 - \cup T_i$ and hence admissible solutions to the original problem.

Step 3: Regularity. Since R_i and R_j are disjoint, T_i and T_j cannot cross. They may coincide only at finitely many intervals, since each time they separate and come back together there is at least one cycle of some T_k in between. As shown in Step 2 there are only finitely many such cycles. Therefore $\cup T_i$ is a union of embedded Lipschitz curves which come together and separate at finitely many points. By a variational argument, these curves must be circular arcs or straight line segments, and all curves separating a specific pair of regions have the same curvature.

We claim that at any such point, the set of $k \geq 3$ outward unit tangent vectors must comprise a length-minimizing network connecting the m points at their heads. If not, for some $\alpha > 0$, in a small ball of radius r, length $\cup T_i$ could be reduced by at least αr, while the change $-t_i$ in area R_i satisfies $|t_i| \leq \pi r^2$. Restoring the areas by a deformation supported elsewhere as in 2.2(2) would leave a net decrease in length T of at least $\alpha r - mc\pi r^2$, which is positive for small enough r. This contradiction of the minimality of $\cup T_i$ establishes the claim that the k unit tangent vectors must be length minimizing.

Consider variations moving the tails of two of the unit tangent vectors v, w a distance t in the direction of any unit vector u and adding a segment of length t connecting them to the origin. Then the initial rate of change of length must be nonnegative.

$$0 \leq \frac{dL}{dt} = 1 - (v + w) \cdot u \ .$$

Taking u in the direction of $v + w$ yields

$$1 \geq |v + w| = \sqrt{2 + 2v \cdot w} \ ,$$

so that $v \cdot w \leq -1/2$ and the angle between v and w is at least $120°$. Therefore there must be exactly three tangent vectors at angles of precisely 120 degrees.

Step 4: Original problem. Finally consider any solution S, R_i to the original problem which must, by Step 2, attain the same minimum a as a solution to the modified problem (4). Therefore if $T_i = \partial R_i$, $S' = \cup T_i \subset S$ is a solution to the modified problem and hence to the original problem. It follows that S differs from S' by an inconsequential set of $\mathcal{H}^1$ measure 0. □

2.4. Soap bubbles in surfaces. Theorem 2.3 and its proof extend immediately to any smooth compact Riemannian surface. The curves have constant geodesic curvature.

2.5. Segmentation problem. Our approach similarly provides a simple existence and regularity proof for the segmentation problem of Mumford and Shah [**MS**, Thm. 5.1], in which they approximate a continuous function g by a function f constant on pieces of relatively minimal perimeter. The analog of our Lemma 2.2 is trivial, since the cost of incorporating one piece into another is at most its area times twice the supremum of $|g|$.

3. Soap bubbles of prescribed combinatorial type. Theorem 3.2 provides 2-dimensional soap bubble clusters of prescribed combinatorial type, with weaker regularity properties. Corollary 3.3 minimizes perimeter over all combinatorial types with connected regions, although in the solution a region may be connected only by an infinitesimal strip. Section 3.4 gives an application to Poincare's argument for the existence of a closed geodesic on a 2-sphere of positive Gauss curvature.

The regularity arguments will require the following lemma, which says roughly that two curves curving away from each other cannot rejoin again too soon.

LEMMA 3.1. *Let M be a smooth (C^∞) compact Riemannian surface. Given $c >> 1 >> \varepsilon > 0$, exists $1/c >> \delta > 0$, such that*
(1) δ-balls are topological discs;
(2) any curve C with geodesic curvature $|\kappa| \leq c$, contained in a disc of radius $r \leq \delta$, from some point p_1 to some point p_2, has length at most $(1+\varepsilon)\ell_0$, where $\ell_0 = \mathrm{dist}(p_1, p_2)$;
(3) any two such curves C, C' which do not cross enclose a region R of area at most $\varepsilon \ell_0$; and
(4) if $\partial R = C' - C$, with C above C', then the geodesic curvatures satisfy

$$\sup \kappa' > \inf \kappa - \varepsilon \ ,$$

(with upward curvature counted positive).

REMARK. It is not necessary to assume the curves C^2, only that the curvature bound holds weakly (so they are $C^{1,1}$).

Proof. Of course in establishing each successive conclusion, we may assume that the previous ones hold for larger c and smaller ε, and we may further decrease δ. Conclusion (1) is standard, (2) and (3) are obvious.

To prove (4), take $c > -G$, where G denotes the Gauss curvature of M. Let $\varepsilon_1 < \varepsilon/3c$. We may pick δ such that (1)-(3) hold for ε_1. Let α_1, α_2 denote the interior angles of R at p_1 and p_2. By the Gauss-Bonnet formula,

$$\int_{C'} \kappa' - \int_C \kappa + (\pi - \alpha_1) + (\pi - \alpha_2) = 2\pi - \int_R G \ .$$

Hence the lengths ℓ, ℓ' satisfy

$$\ell' \sup \kappa' - \ell \inf \kappa \geq -\int_R G \geq -c\varepsilon_1 \ell_0 \ ,$$

so

$$\begin{aligned} &\ell_0 \sup \kappa' + \varepsilon_1 \ell_0 c - \ell_0 \inf \kappa + \varepsilon_1 \ell_0 c \geq -c\varepsilon_1 \ell_0 \ , \\ &\sup \kappa' - \inf \kappa \geq -3\varepsilon_1 c > -\varepsilon \ , \end{aligned}$$

as desired. $\square$

The following theorem provides 2-dimensional soap bubbles of combinatorial type prescribed by an embedded graph G_0. In the limit solution, edges of G_0 may shrink to points or bump up against each other (perhaps coinciding along arcs), but they may not cross. Where two edges coincide, the length counts twice. See Figure 3.2.

THEOREM 3.2. *Let M be $\mathbb{R}^2$ or a smooth, compact, connected Riemannian surface. Let G_0 be a graph with faces $F_0, \ldots, F_m$ smoothly embedded in M. (If M is $\mathbb{R}^2$, count the unbounded region as F_0.) Let $A_1, A_2, \ldots, A_m > 0$, with $\sum A_i <$ area M.*

There is a continuous deformation f_t of M such that $f_0 = \mathrm{id}$, f_t is injective $(0 \le t < 1)$, area $f_1(F_i) = A_i$, and $G = f_1(G_0)$ minimizes length among such.

G consists of disjoint or coincident constant-curvature arcs meeting

(1) at vertices of G with the unit tangent vectors summing to 0,

(2) at other isolated points where the edges remain C^1 (actually $C^{1,1}$).

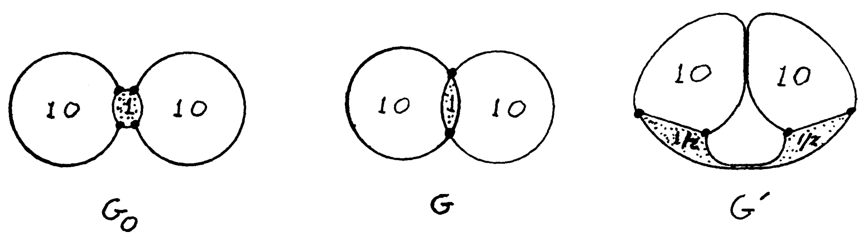

FIGURE 3.2. Given an embedded graph G_0 and prescribed areas $A_1 = 10$, $A_2 = 1$, $A_3 = 10$, Theorem 3.2 provides a least-perimeter deformation G of G_0. In this figure, two edges of G_0 have degenerated to points in the limit G. In the limit, edges are allowed to bump up against each other, as in the poorer admissible competitor G'. Where two edges coincide, the length counts twice.

REMARKS. It two or more vertices of G coincide in the limit, then (1) means that all of the remaining unit tangents there sum to 0.

If the length of overlapping edges counted just once, the prescription of combinatorial type would be nullified and the problem would revert to that of arbitrary regions of Theorem 2.3. Moreover, since there would be no *a priori* bound on multiplicity, there would be no bound on mapping length (which counts multiplicity) and no easy compactness argument for existence.

Proof. Take a sequence of smooth embeddings of G_0 with area $F_i \to A_i$ and length converging to the infimum. By an argument based on the compactness of Lipschitz maps, we may assume convergence to a Lipschitz limit G, which may bump up against itself.

Since the total length is finite, about any point p of G, there is a small circle which intersects G in finitely many points of finite multiplicity. Assume p is not a vertex of G. Some small open disc D about p intersects G in finitely many Lipschitz curves $C_1, \ldots, C_n$ beginning and ending on the boundary of D. These curves may overlap, but they do not cross each other.

Let $\gamma\colon [0, a] \to \mathbb{R}^2$ be an arclength parameterization of one of these curves, and let $0 \le x < y \le a$. Consider replacing $\gamma[x, y]$ by a geodesic Γ, saving $|x - y| - \operatorname{dist}(\gamma(x), (\gamma(y))$. If other curves cross Γ, reroute them along Γ, with further savings. Any distortion of area is bounded by the area between $\gamma[x, y]$ and Γ, which by an isoperimetric inequality is bounded by $c_1|x - y|^2$. Since our small disc D is disjoint from (some choice in) 2.2(3), the diffeomorphisms of 2.2(2) may be used to restore each area to its original value at cost at most $cc_1|x - y|^2$. The resulting modification of G may be approximated by modifications of the original embedded sequence. Since G is length-minimizing among limits of such sequences,

$$|x - y| - \operatorname{dist}(\gamma(x), \gamma(y)) \le c_2|x - y|^2$$

and hence

$$|x - y| \le \operatorname{dist}(\gamma(x), \gamma(y)) + c_3 \operatorname{dist}(\gamma(x), \gamma(y))^2 .$$

It follows that γ is C^1 (in fact, $C^{1,1/2}$; see **[M6]** for a proof for $\mathbb{R}^n$ in a more general context). By shrinking our disc D further if necessary we may assume each C_i is a nearly horizontal C^1 curve through p, labeled $C_1, \ldots, C_n$ from top to bottom.

Count upward curvature positive. Since the top curve C_1 is free to move upward, and any loss of area ΔA may be recovered at cost $c\Delta A$ by 2.2(2), any savings in length $\Delta L \le c\Delta A$, which means that its geodesic curvature $\kappa_1 \le c$ (weakly).

Next $\kappa_2 \leq c$, by the previous argument at points not in C_1 and by the previous result at points in C_1. Then successively $\kappa_2 \leq c$, $\ldots$, $\kappa_n \leq c$. Similarly from the bottom $\kappa_n \leq -c$, $\ldots$, $\kappa_1 \leq -c$. In particular, the curves are $C^{1,1}$.

If $C_1 = C_2 = \ldots$, lump them all together as C_1. Now if $C_2 = C_3 = \ldots$, lump them all together as C_2, and so on. Thus we may assume each C_i is disjoint from the other C_j on some interval, where by a standard variational argument κ_i must be a constant c_i. On an interval where C_i, C_{i+1}, $\ldots$, C_{i+j} overlap, each has curvature $\kappa = (c_i + c_{i+1} + \ldots + c_{i+j})/(j+1)$.

Choose $\varepsilon > 0$ so that all such distinct values of κ differ by at least ε. If necessary, shrink the disc D further so that Lemma 3.1 applies.

We claim that once C_1 and C_2 separate after leaving p, they never meet again inside D. Otherwise focus on an interval of C_1 and an interval of C_2 which coincide only at their endpoints. We may assume that along some subinterval C_2 is disjoint from the other C_j (if along this interval $C_2 = C_3 = \ldots$, lump them together and call them C_2). Upward variations imply that the curvature of C_2 satisfies $\kappa_2 \leq c_2$. By Lemma 3.1, $c_2 > c_1 - \varepsilon$. By choice of ε, $c_2 \geq c_1$. Therefore at both endpoints, for C_1 and C_2 to separate, C_2 must coincide with C_3. But by induction, once C_2 and C_3 separate, they cannot come together again. This contradiction proves the claim that once C_1 and C_2 separate after leaving p, they never meet again inside D.

It follows that the points where edges of G meet and separate are isolated. Standard variational arguments show that elsewhere G consists of constant-curvature arcs and that the unit tangent vectors sum to 0 at vertices.As a nice limit of continuous deformations of G_0, G is a continuous deformation of G_0. $\square$

REMARK. Singularities of type (2) do occur. For example, on a long, skinny torus, the shortest circle enclosing a sufficiently large area will bump up against itself. Michael Hutchings points out that a long string of bubbles in the Euclidean plane $\mathbb{R}^2$ can curve around and touch itself.

The following corollary provides a least-perimeter way to enclose and separate *connected* regions of prescribed area. Each region, however, might be connected only by an infinitesimal strip. See

Figure 3.3 and the remark after the proof.

COROLLARY 3.3. *Let M be $\mathbb{R}^2$ or a smooth, compact, connected Riemannian surface. Given $A_1, \ldots, A_m > 0$ satisfying $\sum A_i <$ area M, there is a shortest graph G with faces of areas $A_1, \ldots, A_m$. (The edges but not the faces of G are allowed to overlap. For overlapping edges, count length with multiplicity.)*

G consists of disjoint or coincident circular arcs meeting
(1) *in threes at $120°$ angles at vertices of G,*
(2) *at other isolated points where the edges remain $C^{1,1}$.*

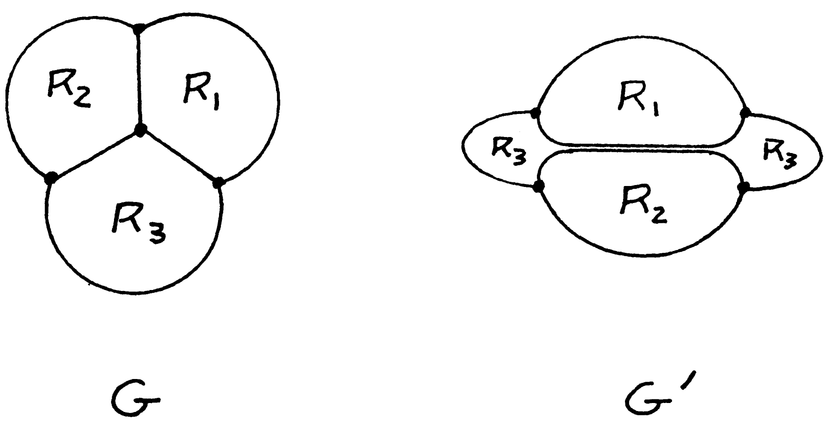

FIGURE 3.3. Corollary 3.3 provides least-perimeter enclosures of connected regions of prescribed areas. The theory allows regions connected only by an infinitesimal strip as in G' as well as the conjectured solution G. Where two edges coincide, the length counts twice.

Proof. Apply Theorem 3.2 to the finitely many combinatorial possibilities for G. The arcs meet in threes at $120°$ by a standard variational argument, as in the end of the proof of 2.2, using the freedom to alter combinatorial type. □

REMARK. We conjecture that the edges of G are disjoint and hence that the faces are connected open sets and there are no singularities of type (2). Even in $\mathbb{R}^2$, the only known solutions are for one area (the circle) and two areas (the standard double bubble, proved by Foisy, Alfaro, Brock, Hodges, and Zimba [**Fo**]), with forthcoming results on three areas by Cox, Harrison, Hutchings, Kim, Light, Mauer, and Tilton [**CHK**].

3.4. Poincare's geodesic. Theorem 3.2 and its proof generalize from area to the integral of any continuous positive function g on a smoothly compact, connected Riemannian surface M. Along the arcs not the curvature κ but instead κg is constant.

This extension provides another simple way to complete Poincare's argument for the existence of a simple closed geodesic on a smooth compact Riemannian surface of positive Gauss curvature g (see **[P]**, **[HM]**). Indeed, by the theorem, there is a nice shortest circle C enclosing half the total Gauss curvature of 4π. A standard variational argument yields a constant c such that along any arc of C with multiplicity m, κg is c/m or 0, according to whether m is odd or even. By the Gauss-Bonnet Theorem, $\int \kappa = 0$. It follows that C is a simple closed geodesic.

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