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ENVELOPING ALGEBRAS AND REPRESENTATIONS OF TOROIDAL LIE ALGEBRAS

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This paper is about Toroidal Lie algebras which generalize the notion of an Affine Lie algebra. We study Verma type modules for these Toroidal algebras and prove an irreducibility criterion when the number of variables is two. We use the fact that the universal enveloping algebra is an Ore domain to obtain facts about the Verma type modules. Moreover, we are able to characterize the Affine Kac-Moody Lie algebras as those whose universal enveloping algebras are non-Noetherian Ore domains.

Introduction. A toroidal Lie algebra is a perfect central extension of the Lie algebra $\mathfrak{T}_{[m]}(\mathfrak{g}) = R_{[m]} \otimes \mathfrak{g}$ where $\mathfrak{g}$ is one of the finite dimensional simple Lie algebras over $\mathbb{C}$ and $R_{[m]}$ is the ring of Laurent polynomials in m variables $t_1, \ldots, t_m$ over $\mathbb{C}$. Here, the multiplication in $\mathfrak{T}_{[m]}(\mathfrak{g})$ is the obvious one defined componentwise. It turns out that these algebras are homomorphic images of some of the G.I.M. and I.M.Lie algebras defined by P. Slodowy (see [**2**], [**16**] and [**17**]) but it is not clear, at the outset, if there is a nontrivial kernel. In [**3**] realizations of certain of these I.M. Lie algebras are given (when $\mathfrak{g}$ is simply laced) and there it is shown, in a computational way using roots, that the kernel is non-trivial. A more conceptual way was sought by the present authors and we thought that, roughly speaking, the fact that the root spaces of $\mathfrak{T}_{[m]}(\mathfrak{g})$ have bounded dimension should be enough to allow one to see that the kernel is non-zero. This turns out to be true but much more is true as well.

Recall that in the paper [**15**] that fundamental use is made of the fact that for the affine algebras or the Virasoro algebras one has a root space decomposition with root spaces of bounded dimension, and this is used to prove that the universal enveloping algebras of the negative part of the algebra, namely $U(\mathfrak{n}^-)$, is a left or right Ore domain. They then go on to exploit the Ore condition in investigating Verma type modules, and derive some facts about these modules via this method. It turned out that we could mimic this approach, making minor changes when necessary to obtain similar results about Verma type modules for toroidal Lie algebras. Moreover, this approach could be used as well, in seeing that there is a non-trivial kernel for the homomorphism of the G.I.M. algebra (or in some cases from the I.M. algebras) to $\mathfrak{T}_{[m]}(\mathfrak{g})$. Along the way we noticed that we could give the following characterization for the Kac-Moody Lie algebras $\mathfrak{g}(A)$, where A is an indecomposable symmetrizable generalized Cartan matrix and $U(\mathfrak{g}(A))$ the universal enveloping algebra of $\mathfrak{g}(A)$:

(i) $U(\mathfrak{g}(A))$ is both left and right Noetherian if and only if A is of finite type,

(ii) $U(\mathfrak{g}(A))$ is a left and right Ore domain if and only if A is of finite type or affine type.

In some sense this result would be obvious to ring theorists if they knew enough about the Kac-Moody algebras $\mathfrak{g}(A)$ while on the other hand, it would be obvious to Lie theorists if they knew the relevant ring theory. Thus we can hardly claim any originality here (the major results used for this are due to Kac and Rocha-Caridi, Wallach) but we have included it because we thought it should be recorded somewhere and besides, the methods used are needed in our investigation of the toroidal algebras $\mathfrak{T}_{[m]}(\mathfrak{g})$.

After recovering some of the results of [**15**] for the Verma type modules it became clear that closer investigation of these modules was called for. These Verma type modules are very complicated due to the fact that they have some of their weight spaces being infinite dimensional, and so this seems to render many of the usual techniques fruitless. Moreover, the roots of the toroidal algebras, $\mathfrak{T}_{[m]}(\mathfrak{g})$, has coefficients with mixed signs when $m \geq 2$, so the usual techniques of Kac-Moody theory don't seem to work. For us, the paper [**5**] served as inspiration where, in an investigation of certain modules for affine algebras, the author uses a close analysis of a particular Poincaré-Birkhoff-Witt basis. We found that an analogous technique worked to allow us to establish an irreducibility criterion for our Verma type modules for certain toroidal algebras. We carry this out when the number of variables is $m = 2$. Roughly speaking our theorem says the Verma type module $M(\lambda)$ is irreducible if and only if the corresponding Verma type module $M_{\hat{\mathfrak{g}}}(\hat{\lambda})$ for an affine subalgebra $\hat{\mathfrak{g}}$ of $\mathfrak{T}_{[m]}(\mathfrak{g})$ is also irreducible. (An exact statement, and proof, is given in Section three.) We then use this, and some known facts about Verma modules of affine algebras to strengthen our previous results which were implied by the Ore condition.

There have been several other investigations of modules for toroidal algebras but from different points of view. In [**12**] certain representations which arise from the vertex operator construction are defined and studied, and along these lines see also [**13**] and the thesis [**4**]. In another direction one may consult the paper [**7**] for some results about toroidal algebras and [**18**] for a study of unitary representations.

The present paper is organized as follows. In Section one we set up the necessary notation to be used and then prepare our foundation by showing the Ore condition holds for $U(\mathfrak{T}_{[m]}(\mathfrak{g}))$ and for $U(\hat{\mathfrak{T}}_{[m]}(\mathfrak{g}))$ where $\hat{\mathfrak{T}}_{[m]}(\mathfrak{g})$ is the universal central extension of $\mathfrak{T}_{[m]}(\mathfrak{g})$. This section closes with our above mentioned application to the G.I.M. algebras of P. Slodowy. In Section two we define and prove some initial remarks on Verma type modules and then go on to specialize to the case when there are just two variables. We then define a total ordering on a particular Poincaré-Birkhoff-Witt basis of the universal enveloping algebra which is needed in Section three. Section two closes with three lemamas which are crucial for our irreducibility criterion. Section three is concerned with the proof of this irreducibility criterion and this is accomplished by proving three more lemmas. The first two are quite straightforward while the third, Lemma 3.3, is the heart of the matter and rather long and technically complex. Actually, this Lemma was first understood by us in the case when $\mathfrak{g}$ is the Lie algebra $sl_2(\mathbb{C})$ and with this restriction Lemma 3.3 is easier to understand due to the fact that the affine algebra $\widehat{sl_2(\mathbb{C})}$ has all of it's non-zero root spaces being one dimensional. We then found it was natural to extend the argument to the general case where then one must account for root spaces with dimensions greater than one. Section three closes with a sharpening of some of our results in light of the irreducibility criterion.

1. The Ore Condition. In this section we are going to investigate the universal enveloping algebra, $U(\mathfrak{l})$, of some Lie algebra $\mathfrak{l}$ defined over the field $\mathbb{C}$ of complex numbers. Recall that an integral domain U is a *left Ore domain* if and only if for all non-zero elements $a, b \in U$ we have $Ua \cap Ub \neq (0)$. Right Ore domains are similarly defined and we will drop any left, right distinction and just say U is an Ore domain by which we shall mean it is both a left and right Ore domain. We will prove that if A is an indecomposable generalized Cartan matrix and $\mathcal{L}(A)$ is the Kac-Moody Lie algebra attached to A then $U(\mathcal{L}(A))$ is an Ore domain if and only if A is not of indefinite type. That is $U(\mathcal{L}(A))$ is an Ore domain if and only if A is of finite or affine type. We will also show that if $\mathcal{L}$ is a toroidal Lie algebra then $U(\mathcal{L})$ is an Ore domain and then go on to use this in investigating ideals of some of the G.I.M. algebras of P. Slodowy which are covers of the toroidal algebras.

It is well-known that if $\mathcal{L}$ is any finite dimensional Lie algebra then $U(\mathcal{L})$ is Noetherian (either left or right) so that one has that $U(\mathcal{L})$ is an Ore domain because this is implied by the Noetherian condition. Also, we have the following useful result.

PROPOSITION 1.1. ([**1**]) *If $\mathcal{L}$ is a Lie algebra then $\mathcal{L}$ satisfies the ascending chain condition for subalgebras if $U(\mathcal{L})$ satisfies the ascending chain condition for right (or left) ideals.*

To use this one only needs to note that if a Lie algebra contains a subalgebra which is a free Lie algebra on two generators then it contains a subalgebra which is a free Lie algebra on countably infinitely many generators and hence an infinite strictly ascending chain of subalgebras so that then $U(\mathcal{L})$ can not be Noetherian. Similarly if a Lie algebra $\mathcal{L}$ contains an infinite dimensional abelian subalgebra then $U(\mathcal{L})$ cannot be Noetherian and so in particular this is true for Lie algebras which contain an infinite dimensional Heisenberg Lie algebra. Now by Corollary 9.12 in [**9**] one knows that if A is a symmetrizable generalized Cartan matrix of indefinite type then the Kac-Moody algebra $\mathcal{L}(A)$ has a subalgebra isomorphic to a free algebra on two or more generators while if A is one of the sixteen types of indecomposable affine generalized Cartan matrices then the Kac-Moody Lie algebra $\mathcal{L}(A)$ has an infinite dimensional Heisenberg algebra as subalgebra so this gives the following result:

PROPOSITION 1.2. *If A is a symmetrizable generalized Cartan matrix and $\mathcal{L}(A)$ is the Kac-Moody Lie algebra attached to A then $U(\mathcal{L}(A))$ is Noetherian if and only if A is of finite type.*

Next recall that A. Rocha-Caridi and N. Wallach have shown that if A is one of the sixteen types of indecomposable affine Cartan matrices then $U(\mathcal{L}(A)^-)$ is an Ore domain where here $\mathcal{L}(A)^-$ is the usual negative subalgebra of the Kac-Moody algebra $\mathcal{L}(A)$. More precisely one has:

PROPOSITION 1.3. ([**15**]) *Let $\mathfrak{m} = \cup_{n\geq 1}\mathfrak{m}_n$ be a filtered Lie algebra where $\mathfrak{m}_1 \subseteq \mathfrak{m}_2 \subseteq \cdots$. Assume that $d_i = \dim(\mathfrak{m}_i/\mathfrak{m}_{i-1}) < \infty$ for all $i \geq 1$ where we take $\mathfrak{m}_0 = (0)$ and that $\limsup_i (d_i)^{1/i} \leq 1$. Then $U(m)$ is a left and right Ore domain.*

We want to apply this result to the toroidal Lie algebra so we now recall their definition and some of the properties which we will use. Let $\mathfrak{g}$ be any finite dimensional simple Lie algebra over $\mathbb{C}$ and let $R_{[m]}$ be the Laurent polynomial ring in $m \geq 1$ commuting variables $t_1, \ldots, t_m$ and their inverses so that $R_{[m]} = \mathbb{C}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$. We form the Lie algebra $\mathfrak{T}_{[m]}(\mathfrak{g}) = R_{[m]} \otimes \mathfrak{g}$ with the obvious multiplication and denote this by $\mathfrak{T}$ when $\mathfrak{g}$ and m are understood. $\mathfrak{T}$ is a perfect Lie algebra. One knows that (see [**10**] and [**11**]) if $A = R_{[m]}$ then the so called space of Kähler differentials modulo exact forms, Ω_A/dA, gives the universal central extension of $\mathfrak{T}$. To be more specific recall there is a linear map $d : A \to \Omega_A$ such that if M is an A-module and $D : A \to M$ any derivation then there is a unique A-module map $f : \Omega_A \to M$ such that $f \circ d = D$. Here we have $d(ab) = (da)b + a(db)$ for any $a, b \in A$. The image of A under d, dA, is just a subspace of Ω_A and we form the vector space, Ω_A/dA where we denote the image of an element z by $\overline{z}$. Thus, in Ω_A/dA we have $\overline{a\,db} = -\overline{b\,da}$, for $a, b \in A$. Now form the space $\hat{\mathfrak{T}} = \mathfrak{T} \oplus (\Omega_A/dA)$ and define the bracket by $[a \otimes x, b \otimes y] = ab \otimes [x, y] + (x, y)\overline{(da)b}$ where $x, y \in \mathfrak{g}$, $a, b \in A$, $(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is the Killing form and Ω_A/dA is central. Then we have the short exact sequence

$$0 \to \Omega_A/dA \to \hat{\mathfrak{T}} \to \mathfrak{T} \to 0$$

and one knows that $\hat{\mathfrak{T}}$ is the universal central extension of $\mathfrak{T}$.

We next describe a grading on $\hat{\mathfrak{T}}$. As usual we have that $\mathfrak{g}$ is graded by it's root lattice $Q \cong \mathbb{Z}^l$, where l is the rank of $\mathfrak{g}$ with the non-zero root spaces being one dimensional and the zero root space having dimension l. Also, we have that $A = R_{[m]}$ is graded by $\mathbb{Z}^m$ where the degree of t_i is denoted by $(0, \ldots, 1, \ldots, 0)$, the m-tuple with a one in the i-th place and 0's elsewhere. Because the t_i's commute, each homogeneous space in A is of dimension one. It now follows that $\mathfrak{T} = A \otimes \mathfrak{g}$ has a $\mathbb{Z}^{m+l}$ grading with each homogeneous space of dimension less than or equal to l. Next we note that the grading on A gives rise to one on Ω_A and hence on Ω_A/dA where if a and b are homogeneous in A then $\overline{a\,db}$ is homogeneous of degree equal to the sum of the degrees of a and b. It is easy to see that Ω_A is a free A-module with basis $dt_1, \ldots, dt_m$ and from this it follows that each non-zero homogeneous space in Ω_A/dA is of dimension $m-1$ while the dimension of the space of elements of degree 0 is m. It follows that in $\hat{\mathfrak{T}}$ with it's $\mathbb{Z}^{m+l}$ grading we have that the space of homogeneous elements of an fixed degree is of dimension less than or equal to $m+l$ and so the dimension of these spaces is bounded above. The following lemma allows us to make use of these facts.

LEMMA 1.4.

(a) *Let $m = \oplus_{i\in\mathbb{Z}} m_i$ be a graded Lie algebra with $d_i = \dim m_i < \infty$ for all $i \in \mathbb{Z}$. If $\lim_{i\to\infty, i\neq 0}(d_i + d_{-i})^{1/i} \leq 1$ then $U(m)$ is an Ore domain.*

(b) *Let $m = \oplus_{a\in G} m_a$ be a graded Lie algebra with G a finitely generated abelian group such that there exits a fixed number M with $\dim m_a \leq M$ for all $a \in G$. Then $U(m)$ is an Ore domain.*

Proof. (a) Let $n_k = \sum_{j=-k}^{k} m_j$ for $k \geq 1$. Then $[n_k, n_s] \subset n_{k+s}$ and $\cup_{k=1}^{\infty} n_k = m$ so m is filtered by the spaces n_k. Moreover, if $k \geq 2$ then $\dim(n_k/n_{k-1}) = d_k + d_{-k} < \infty$ and $\limsup_k \dim(n_k/n_{k-1})^{1/k} \leq 1$. Applying Proposition 1.3 we get the result.

(b) Because G is a finitely generated abelian group we can write $G \cong F \times \mathbb{Z}^s$ where F is a finite abelian group and $s \in \mathbb{Z}_{\geq 0}$. If $s = 0$, then $\dim m$ is finite so $U(m)$ is Noetherian and by a result of Goldie $U(m)$ is an Ore domain, see [**8**]. Assume $s > 0$ and let $\pi_i : G \to \mathbb{Z}$ be the canonical projection of G onto the i-th copy of

$\mathbb{Z}$ for $i = 1, 2 \dots, s$. Set $n_k = \oplus m_a$ where the sum is over all a such that $|\pi_i(a)| \leq k$ for all i. Then $n_1 \subset n_2 \subset \cdots \subset m$, $[n_k, n_s] \subset n_{k+s}$ and $\cup_{i=1}^{\infty} n_i = m$ so that $n_0 \subset n_1 \subset \cdots \subset m$ is a filtration of m. Moreover for $k \geq 2$

$$\dim(n_k/n_{k-1}) \leq \dim n_k \leq |F||2k+1|^s M$$

where $|F|$ denotes the cardinality of F. Hence

$$\begin{aligned}\limsup_k \dim(n_k/n_{k-1})^{1/k} &\leq \limsup_k (|F|(2k+1)^s M)^{1/k} \\ &\leq \limsup_k \frac{|F|(2k+3)^s M}{|F|(2k+1)^s M} = 1.\end{aligned}$$

where the last inequality is derived from Theorem 3.37 of [**14**]. By Proposition 1.3 $U(m)$ is an Ore domain. □

Applying this we have the following result.

COROLLARY 1.5. *If $\mathfrak{g}$ is any finite dimensional simple Lie algebra over $\mathbb{C}$ and $m \geq 1$ is any integer then the universal enveloping algebra of either $\mathfrak{T}_{[m]}(\mathfrak{g})$ or $\hat{\mathfrak{T}}_{[m]}(\mathfrak{g})$ is an Ore domain.*

Next we need another Lemma.

LEMMA 1.6. *Let $\mathcal{L}$ be a Lie algebra with a subalgebra $\mathcal{S}$ isomorphic to a free algebra on two generators. Then $U(\mathcal{L})$ is not an Ore domain.*

Proof. Let the two generators of $\mathcal{S}$ be denoted by x and y so that $U(\mathcal{S})$ is isomorphic to the free associative algebra with identity on these two generators and so has a basis consisting of the standard monomials

$$\mathcal{M} = \{1, x, y, x^2, xy, y^2, \dots\}.$$

Let $\{z_i\}_{i \in I}$ be any basis of a vector space complement of $\mathcal{S}$ in $\mathcal{L}$ with I a totally ordered index set. Then $U(\mathcal{L})$ is a free right $U(\mathcal{S})$-module with basis B consisting of elements of the form $z_{i_1}^{n_1} \cdots z_{i_r}^{n_r}$ where $r \geq 0$, $i_1 < i_2 < \cdots < i_r$ belong to I and $n_{i_j} \geq 1$ for $1 \leq j \leq r$. Thus a vector space basis of $U(\mathcal{L})$ is given by the elements bm for $b \in B$, $m \in \mathcal{M}$. Now let $\mathcal{M}_x$ (resp $\mathcal{M}_y$) be the monomials in $\mathcal{M}$

ending in x (resp. y) (i.e. x on the right of the monomial) so that we have the disjoint union $\mathcal{M} = \{1\} \cup \mathcal{M}_x \cup \mathcal{M}_y$ and $\mathcal{M}_x = \mathcal{M}x$, $\mathcal{M}_y = \mathcal{M}y$. If $u = \sum a(b,m)bm \in U$ then $ux = \sum a(b,m)bmx$ so that the elements $\{bmx\}$ (resp. $\{bmy\}$) for $b \in B$, $m \in \mathcal{M}$ form a basis of Ux (resp. Uy) and as $\{bmx\} \cup \{bmy\}$ are linearly independent we get $Ux \cap Uy = (0)$. □

This now gives the following characterization of affine Kac-Moody Lie algebras. The reader should consult [**9**] for the necessary background material.

PROPOSITION 1.7. *Let $\mathcal{L}$ be the Kac-Moody Lie algebra based on the indecomposable symmetrizable generalized Cartan matrix A of rank $l < \infty$. Then $U(\mathcal{L})$ is an Ore domain which is not Noetherian if and only if A is one of the sixteen types of affine Cartan matrices.*

REMARK. The diagram of these sixteen types of affine Cartan matrices are listed in the tables on pages 55-56 of V. Kac's book [**9**].

In order to make another application of some of these ideas the following lemma will be useful.

LEMMA 1.8. *Let $\mathcal{S} = \cup_{n\geq 1}\mathcal{S}_n$ be a filtered Lie algebra so that $\mathcal{S}_1 \subseteq \mathcal{S}_2 \ldots$, and $[\mathcal{S}_i, \mathcal{S}_j] \subseteq \mathcal{S}_{i+j}$ for all $i,j \geq 1$. Let S be the associated graded Lie algebra so that $S = \oplus_{n\geq 1} S_n$ where $S_n = \mathcal{S}_n/\mathcal{S}_{n-1}$ for $n \geq 1$ and $\mathcal{S}_0 = (0)$. Assume that $z_1 \in \mathcal{S}_{n_1}\backslash\mathcal{S}_{n_1-1}$; $z_2 \in \mathcal{S}_{n_2}\backslash\mathcal{S}_{n_2-1}$ where $\overline{z_1} = z_1 + \mathcal{S}_{n_1-1} \in S_{n_1}$, $\overline{z_2} = z_2 + \mathcal{S}_{n_2-1} \in S_{n_2}$, generate a free Lie algebra F so that $F = \mathfrak{FrL}(\overline{z_1}, \overline{z_2})$ is a subalgebra of S. Let $\mathfrak{T}$ be the subalgebra of $\mathcal{S}$ generated by z_1 and z_2. Then $\mathfrak{T}$ is also isomorphic to $\mathfrak{FrL}(z_1, z_2)$ the free Lie algebra on z_1 and z_2.*

Proof. Let B be any basis of F consisting of monomials of the form $[\overline{z_{i_1}}, \overline{z_{i_2}}, \ldots, \overline{z_{i_k}}] := ad\,\overline{z_{i_1}}\, ad\overline{z_{i_2}} \cdots (\overline{z_{i_k}})$ where each $i_j \in \{1,2\}$ so if $k \geq 2$ then $i_{k-1} \neq i_k$. If $b = [\overline{z_{i_1}}, \ldots, \overline{z_{i_k}}] \in B$ we pull it back to $\mathcal{S}$ and let $p(b) = [z_{i_1}, \cdots, z_{i_k}] \in \mathfrak{T} \subseteq \mathcal{S}$. We call the number $s = n_{i_1} + \cdots + n_{i_k}$ the degree of b or $p(b)$ so $p(b) \in \mathcal{S}_s$ and we have

$$\begin{aligned} p(b) + \mathcal{S}_{s-1} &= [z_{i_1}, \cdots, z_{i_k}] + \mathcal{S}_{s-1} \\ &= [z_{i_1} + \mathcal{S}_{s-1}, \cdots, z_{i_k} + \mathcal{S}_{s-1}] \\ &= [\overline{z_{i_1}}, \ldots, \overline{z_{i_k}}] = b \in S_{s-1}. \end{aligned}$$

Next, say that we have a finite linear relation $0 = \sum_{b\in B} a_b p(b)$, where not all the coefficients are zero. Let s denote the largest degree of an element $b \in B$ occurring with a non-zero coefficient. Then we have

$$0 = \sum_{b\in B, \deg b<s} a_b p(b) + \sum_{b\in B, \deg b=s} a_b p(b)$$

so that

$$\sum_{b\in B, \deg b<s} a_b p(b) \in \mathcal{S}_{s-1}$$

and hence we obtain

$$\begin{aligned} 0 &= \sum_{b\in B, \deg b=s} a_b p(b) + \mathcal{S}_{s-1} \\ &= \sum_{b\in B, \deg b=s} a_b (p(b) + \mathcal{S}_{s-1}) = \sum_{b\in B, \deg b=s} a_b b \end{aligned}$$

because $p(b) + \mathcal{S}_{s-1} = b$ as above. This contradicts the fact that B is a basis of F so that the elements $\{p(b) | \, b \in B\}$ are linearly independent in $\mathfrak{T}$.

Next, since F is free with generators $\overline{z_1}$ and $\overline{z_2}$ there is a unique homomorphism $\phi : F \to \mathfrak{T}$ satisfying $\phi(\overline{z_i}) = z_i$ for $i = 1, 2$. It is clear that $\phi(b) = p(b)$ for all $b \in B$ and hence it follows that ϕ is one to one so that F and $\mathfrak{T}$ are isomorphic. $\square$

We want to apply Lemma 1.8 to some of the G.I.M. algebras of P. Slodowy and for the sake of brevity we refer the reader to the papers [**16**], [**17**], [**2**] and [**3**] for the relevant facts about these algebras. We recall only a few of the ones we need here. Now if A is any $l \times l$ G.I.M. matrix then there is a Lie algebra, $\mathcal{L}(A)$, attached to A called the G.I.M. algebra of A. Also, if A is given, there is a $2l \times 2l$ generalized Cartan matrix, $C(A)$, obtained by a process of doubling A and one says that A is *unoriented* if and only if $C(A)$ is indecomposable. Letting $\mathcal{L}(C(A))$ denote the Kac-Moody Lie algebra of $C(A)$ we have (see [**2**]) that there is an involutory automorphism σ of $\mathcal{L}(C(A))$ such that $\mathcal{L}(A)$ is isomorphic to the subalgebra $\mathcal{A}$ of fixed points of σ in $\mathcal{L}(C(A))$. Moreover, $\mathcal{S}$ is filtered and it's associated graded algebra contains a subalgebra isomorphic to the positive subalgebra $\mathcal{L}(C(A))^+$ of our Kac-Moody Lie algebra based on $C(A)$. Now if $C(A)$ is indecomposable and not of finite or affine type (i.e. is of indefinite type) then by Kac's result (see Corollary 9.12 in [**9**]) $\mathcal{L}(C(A))^+$ contains a free Lie algebra on two generators. It then follows, by Lemma 1.8, that our G.I.M. algebra, $\mathcal{L}(A)$, which is isomorphic to $\mathcal{S}$, contains a free Lie algebra on two generators and hence $U(\mathcal{L}(A))$ can not be an Ore domain. This proves the following:

PROPOSITION 1.9. *Let A be any G.I.M. matrix whose associated Cartan matrix, $C(A)$, is indecomposable and of indefinite type. Then the G.I.M. algebra of A, $\mathcal{L}(A)$, contains a free Lie algebra on two generators. In particular, $U(\mathcal{L}(A))$ is not an Ore domain.*

REMARK 1.10. If A is an indecomposable G.I.M. matrix and $C(A)$ is not indecomposable then it splits into two equal components, say B, where B is an indecomposable Cartan matrix, and in this situation $\mathcal{L}(A)$ and $\mathcal{L}(B)$ are isomorphic. Thus, in this situation we also have that if B is of indefinite type then $\mathcal{L}(A)$ contains a free subalgebra and $U(\mathcal{L}(A))$ is not an Ore domain.

Next we let $m \geq 1$ and let A be any simply laced finite Cartan matrix of rank $l \geq 2$ and then let $A^{[m]}$ be any m-fold affinization of A. Thus, $A = (\langle \alpha_i, \alpha_j \rangle)_{1 \leq i,j \leq l}$ where $\Pi = \{\alpha_i, \dots, \alpha_j\}$ is a system of simple roots of the root system Δ of our matrix A so that $A^{[m]} = (\langle \alpha_i, \alpha_j \rangle)_{1 \leq i,j \leq l+m}$ where $\alpha_{l+1}, \dots, \alpha_{l+m}$ are any roots in Δ. We say $A^{[m]}$ is the *standard affinization* of A if $m = 1$ and α_{l+1} is $\pm\xi$ where ξ is the highest root of Δ. Otherwise we say $A^{[m]}$ is *non-standard.* If $A^{[m]}$ is non-standard then one knows (see [**3**]) that $A^{[m]}$ is unoriented and $C(A^{[m]})$ is of indefinite type. Thus $U(\mathcal{L}(A^{[m]}))$ is not an Ore domain. However, we always have a surjective homomorphism ϕ of $\mathcal{L}(A^{[m]})$ onto $\hat{\mathfrak{T}}_{[m]}(\mathfrak{g})$ where $\mathfrak{g}$ is the finite dimensional simple Lie algebra whose Cartan matrix is A and we know that $U(\hat{\mathfrak{T}}_{[m]}(\mathfrak{g}))$ is an Ore domain. This establishes the next result.

PROPOSITION 1.11. *Let $A^{[m]}$ be any non-standard affinization of the finite simply laced Cartan matrix A of rank $l \geq 2$. Then the natural surjective homomorphism of $\mathcal{L}(A^{[m]})$ onto $\hat{\mathfrak{T}}_{[m]}(\mathfrak{g})$ has a nontrivial kernel.*

REMARK 1.12. When A is of type D_l for $l \geq 4$ or of type E_6, E_7, E_8 then we know by the realization theorem [**3**] that the

I.M. algebra of $A^{[m]}$ is isomorphic to $\hat{\mathfrak{T}}_{[m]}(\mathfrak{g})$ so that if $A^{[m]}$ is non-standard the natural map of $\mathcal{L}(A^{[m]})$ to the I.M. algebra of $A^{[m]}$ has a non-trivial kernel. This gives another way of proving Proposition 4.14 of [**3**] when A is of type D or E.

2. Initial remarks on Verma type modules for toroidal algebras. In this section we define the modules $M(\lambda)$ in a way analogous to the usual Verma module construction via induction and then exploit the Ore condition to obtain some results about these modules as in [**15**]. We have chosen to work with an algebra we call $\mathfrak{t}$ which just has central elements of degree 0 (so we factor by most of Ω_A/dA) but does have added to it the degree derivations. These derivations allow us to use weight space decompositions rather than just gradations and so submodules then also have weight space decompositions. Killing the homogeneous elements of Ω_A/dA of degree different from zero makes then many computations we encounter later more tractable. These algebras $\mathfrak{t}$ also occur in the paper [**7**]. After giving basic definitions and recovering results of [**15**] for our toroidal algebra $\mathfrak{t}$, it is natural to investigate the irreducibility of our Verma module. We do this in the two variable case in the next section and are able to prove that the module $M(\lambda)$ (for definitions see below) is irreducible if and only if the corresponding module for the loop algebra $\hat{\mathfrak{g}}$, namely $M_{\hat{\mathfrak{g}}}(\hat{\lambda})$, is irreducible. Our proof is computational and makes heavy use of a total ordering for a particular Poincaré-Birkhoff-Witt basis of $U(\mathfrak{t})$ which we define in this section.

For any finite dimensional simple Lie algebra $\mathfrak{g}$ and any $m \geq 1$ we have already defined the algebras $\mathfrak{T}_{[m]}(\mathfrak{g}) = \mathfrak{g} \otimes R_{[m]}$ where $A = R_{[m]} = \mathbb{C}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ and $\hat{\mathfrak{T}} = \hat{\mathfrak{T}}_{[m]}(\mathfrak{g}) = \mathfrak{T}_{[m]}(\mathfrak{g}) \oplus (\Omega_A/dA)$. Letting z be the central ideal of $\hat{\mathfrak{T}}$ consisting of the span of the homogeneous element of non-zero degree in Ω_A/dA we obtain the algebra

$$\overline{\mathfrak{T}_{[m]}(\mathfrak{g})} = \overline{\mathfrak{T}} = \mathfrak{T} \oplus (\oplus_{i=1}^{m} \mathbb{C}c_i)$$

where c_i denotes the class of the element $\overline{t_i^{-1}\, dt_i} \in \Omega/dA$ in $\overline{\mathfrak{T}}$. $\overline{\mathfrak{T}}$ is still graded by $\mathbb{Z}^{l+m}$ with $\deg c_i = 0$, for $1 \leq i \leq m$, and we let d_i be the i-th degree derivation on $\overline{\mathfrak{T}}$ so that $d_i(x \otimes t_1^{n_1} \ldots t_m^{n_m}) = n_i$ and $d_i(c_j) = 0$ for $1 \leq i, j \leq m$. Let $\mathcal{D}$ be the abelian Lie algebra spanned by $d_1, \ldots, d_m$. We form the semi-direct product of $\overline{\mathfrak{T}}$ and $\mathcal{D}$ and denote this by $\mathfrak{t} = \mathfrak{t}_{[m]}(\mathfrak{g})$. Thus

$$\mathfrak{t} = \mathfrak{T} \oplus (\oplus_{i=1}^{m} \mathbb{C}c_i) \oplus (\oplus_{i=1}^{m} \mathbb{C}d_i).$$

We have that $[d_i, x \otimes t_1^{n_1} \ldots t_m^{n_m}] = n_i(x \otimes t_1^{n_1} \ldots t_m^{n_m})$ and $[d_i, c_j] = 0$ for $1 \leq i, j \leq m$. $\mathfrak{t}$ is graded by $\mathbb{Z}^{l+m}$ and the space of degree 0, $\mathfrak{t}_0$, is of dimension $2m + l$ where l is the rank of $\mathfrak{g}$. For $1 \leq k \leq m$ we let $R_{[k]} = \mathbb{C}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$ and let $R_{[0]} = \mathbb{C}$. Also, define

$$R_{[k]}^{+} = \mathbb{C}[t_1^{\pm 1}, \ldots, t_{k-1}^{\pm 1}, t_k]t_k,$$

and

$$R_{[k]}^{-} = \mathbb{C}[t_1^{\pm 1}, \ldots, t_{k-1}^{\pm 1}, t_k^{-1}]t_k^{-1}$$

so that $R_{[k]}^{\pm} = R_{k-1}[t_k^{\pm 1}]t_k^{\pm 1}$ for $1 \leq k \leq m$ and we view all of these in $R_{[m]}$. Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be a simple system of roots of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{H}$ and let $\Delta = \Delta^+ \cup \Delta^-$ be the root system so that we have the usual triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{H} \oplus \mathfrak{n}^-$ where $\mathfrak{n}^{\pm} = \oplus_{\alpha \in \Delta^{\pm}} \mathfrak{g}_{\alpha}$. We define

$$\mathfrak{t}^+ = \mathfrak{n}^+ \oplus (\oplus_{k=1}^{m} (\mathfrak{g} \otimes R_{[k]}^{+}))$$

and

$$\mathfrak{t}^- = \mathfrak{n}^- \oplus (\oplus_{k=1}^{m} (\mathfrak{g} \otimes R_{[k]}^{-})).$$

Thus, we have the decomposition $\mathfrak{t} = \mathfrak{t}^+ \oplus \mathfrak{t}_0 \oplus \mathfrak{t}^-$. For example, all roots of $\mathfrak{t}$ are of the form $\gamma = (\beta, (n_1, \ldots, n_m))$ where$\beta \in \Delta \cup \{0\}$ and $n_i \in \mathbb{Z}$, $1 \leq i \leq m$ and we have that the corresponding root space is in $\mathfrak{t}^+$ if and only if either there is some j satisfying $1 \leq j \leq m$ and $n_j > 0$ but $n_{j+k} = 0$ for $1 \leq k \leq m - j$ or $n_1 = \cdots = n_m = 0$ but $\beta \in \Delta^+$. In this case we write $\gamma > 0$.

Next, let $\mathfrak{b}^+ = \mathfrak{t}^+ \oplus \mathfrak{t}_0$ and define for any $\lambda \in \mathfrak{t}_0^*$ a one dimensional $\mathfrak{b}^+$-module $\mathbb{C}_\lambda = \mathbb{C}v^+$ by requiring that $\mathfrak{t}^+ v^+ = 0$ and $h.v^+ = \lambda(h)v^+$ for all $h \in \mathfrak{t}_0$. Then our generalized Verma modules are the induced modules

$$M(\lambda) = U(\mathfrak{t}) \otimes_{U(\mathfrak{b}^+)} \mathbb{C}_\lambda$$

which are vector spaces isomorphic to $U(\mathfrak{t}^-)$. Also it is clear that $M(\lambda)$ has a weight space decomposition with weights in $\mathfrak{t}_0^*$ and so does any submodule. We let $w(N)$ be the set of weights of any

$\mathfrak{t}$-module N having a weight space decomposition and so $M(\lambda) = \oplus_{\beta \in w(M(\lambda))} M(\lambda)_\beta$ where $M(\lambda)_\beta$ denotes the weight space corresponding to $\beta \in \mathfrak{t}_0^*$. It is clear that $\dim M(\lambda)_\lambda = 1$ and the usual argument shows that $M(\lambda)$ has a unique maximal proper submodulewhich we denote by Rad $(M(\lambda))$.

Next, note that the root spaces of $\mathfrak{t}^-$ are of bounded dimension so that Lemma 1.4 applies to give us that $U(\mathfrak{t}^-)$ is an Ore domain. Thus, the following result from [**15**] applies in our case.

PROPOSITION 2.1. *Let the notation be as above and let* $\lambda, \mu \in \mathfrak{t}_0^*$ *and assume* $M(\mu)$ *is a subquotient of* $M(\lambda)$. *Then*

(1) $M(\mu)$ *is a submodule of* $M(\lambda)$.

(2) *If* $N \subset M(\lambda)$ *is a non-zero submodule then* $N \cap M(\mu) \neq 0$.

(3) *If* $M(\mu)$ *is irreducible then* $M(\mu)$ *is the unique irreducible submodule of* $M(\lambda)$.

(4) *If* $M(\mu)$ *is irreducible then* $\dim Hom_{\mathfrak{t}}(M(\lambda'), M(\lambda)) \leq 1$ *for any* $\lambda' \in \mathfrak{t}_0^*$.

We now go on to develop an irreducibility criterion for the modules $M(\lambda)$ in the case when $m = 2$. Here we let $s = t_1$ and $t = t_2$ denote the variables and c_s, c_t, d_s and d_t the corresponding central elements and degree derivations. Thus, we have

$$\mathfrak{t} = \mathfrak{t}_2(\mathfrak{g}) = (\mathfrak{g} \otimes R_{[2]}) \oplus \mathbb{C}c_s \oplus \mathbb{C}c_t \oplus \mathbb{C}d_s \oplus \mathbb{C}d_t$$

and

$$[d_s, x \otimes s^{n_1} t^{n_2}] = n_1(x \otimes s^{n_1} t^{n_2}), \quad [d_t, x \otimes s^{n_1} t^{n_2}] = n_2(x \otimes s^{n_1} t^{n_2})$$

for $x \in \mathfrak{g}$, $n_1, n_2 \in \mathbb{Z}$. The usual affine Lie algebra, $\hat{\mathfrak{g}}$, is the subalgebra of $\mathfrak{t}$ given by $\hat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}[s, s^{-1}]) \oplus \mathbb{C}c_s \oplus \mathbb{C}d_s$ so that $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}^+ \oplus \hat{\mathfrak{H}} \oplus \hat{\mathfrak{n}}^-$ where

$$\hat{\mathfrak{n}}^+ = \mathfrak{n}^+ \oplus (\mathfrak{g} \otimes \mathbb{C}[s]s), \quad \hat{\mathfrak{n}}^- = \mathfrak{n}^- \oplus (\mathfrak{g} \otimes \mathbb{C}[s^{-1}]s^{-1}),$$

and

$$\hat{\mathfrak{H}} = \mathfrak{H} \oplus \mathbb{C}c_s \oplus \mathbb{C}d_s.$$

We also let

$$\mathfrak{t}_{\pm 1} = \hat{\mathfrak{n}}^{\pm}, \quad \mathfrak{t}_2 = \mathfrak{g} \otimes \mathbb{C}[s, s^{-1}, t]t, \quad \mathfrak{t}_{-2} = \mathfrak{g} \otimes \mathbb{C}[s, s^{-1}, t^{-1}]t^{-1}$$

so that $\mathfrak{t}_{\pm} = \mathfrak{t}_{\pm 1} \oplus \mathfrak{t}_{\pm 2}$ and of course $\mathfrak{t} = \mathfrak{t}_{+} \oplus \mathfrak{t}_0 \oplus \mathfrak{t}_{-}$. Let $\delta_s, \delta_t, \Lambda_s$, and Λ_t in $\mathfrak{t}_0^*$ be the linear functionals in $\mathfrak{t}_0^*$ dual to d_s, d_t, c_s, and c_t respectively. We thus have

$$\mathfrak{t}_0^* = \mathfrak{H}^* \oplus \mathbb{C}\Lambda_s \oplus \mathbb{C}\Lambda_t \oplus \mathbb{C}\delta_s \oplus \mathbb{C}\delta_t$$

and

$$\hat{\mathfrak{H}}^* = \mathfrak{H}^* \oplus \Lambda_s \oplus \mathbb{C}\delta_s.$$

For $\lambda \in \mathfrak{t}_0^*$ we let $\hat{\lambda}$ denote the restriction of λ to $\hat{\mathfrak{H}}$. Now the usual Verma module corresponding to $\hat{\lambda}$ for $\hat{\mathfrak{g}}$ is $M_{\hat{\mathfrak{g}}}(\hat{\lambda}) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{n}}^+ \oplus \hat{\mathfrak{H}})} \mathbb{C}_\lambda$ and it is easy to see this may be regarded as a submodule of $M(\lambda)$ when we treat $M(\lambda)$ as $\hat{\mathfrak{g}}$-module. The irreducibility criterion we will prove then states that $M(\lambda)$ is irreducible if and only if $M_{\hat{\mathfrak{g}}}(\hat{\lambda})$ is irreducible.

The Poincaré-Birkhoff-Witt theorem implies that if we let

$$P_1 = Q_- = \{\sum_{i=1}^{l} n_i\alpha_i \mid n_i \in \mathbb{Z}, n_i \leq 0\}, \quad Q = \{\sum_{i=1}^{l} n_i\alpha_i \mid n_i \in \mathbb{Z}\},$$

$$P_2 = Q + \{n\delta_s \mid n \in \mathbb{Z}, n < 0\}, \quad \text{and}$$

$$P_3 = Q + \mathbb{Z}\delta_s + \{n\delta_t \mid n \in \mathbb{Z}, n < 0\}$$

then the weights of $M(\lambda)$ are just the elements of the form $\lambda + \mu$ for $\mu \in P_1 \cup P_2 \cup P_3$ so if $P = P_1 \cup P_2 \cup P_3$ then $w(M(\lambda)) = \lambda + P \subset \mathfrak{t}_0^*$. Moreover, one has that if $\mu \in P$, then $\dim M(\lambda)_{\lambda+\mu} < \infty$ if and only if $\mu \in P_1 \cup P_2$ so that the weight space $M(\lambda)_{\lambda+\mu}$ is infinite dimensional if $\mu \in P_3$.

We next construct a Poincaré-Birkhoff-Witt basis of $U(\mathfrak{t}_-)$ and also a total ordering of it. We will make use of this in our investigation of irreducibility. Thus, let $\{x_i\}_{i\in\mathbb{Z}}$ be a basis of root vectors for the subspace $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[s, s^{-1}]$ of $\hat{\mathfrak{g}} \subset \mathfrak{t}$ where x_i is in the root space $\hat{\mathfrak{g}}_{\beta_i}$ and we demand that if $\beta_i < \beta_j$ in the usual ordering, then $i < j$. Thus, our indexing of the x_i's is consistent with the usual ordering of roots on $\hat{\mathfrak{g}}$. We also stipulate that $\{c_s, d_s, x_i\}_{i\geq 0}$ is a basis of $\mathfrak{H} \oplus \mathfrak{n}^+$ while $\{x_i\}_{i\leq -1}$ is a basis of $\mathfrak{n}^-$. We are choosing $x_i = x \otimes s^n$ for a root vector of $\mathfrak{g}$ and some $n \in \mathbb{Z}$. Notice that we may have that $\beta_i = \beta_j$ even if $i \neq j$ and that there are exactly $\dim \hat{\mathfrak{g}}_{\beta_i}$ indices j for which $\beta_j = \beta_i$ as long as $\beta_j \neq 0$. When $\beta_i = 0$

there are $\dim \hat{\mathfrak{H}} - 2$ such indices. In any case there are only finitely many such indices.

Next, for $m < 0$, $m \in \mathbb{Z}$ we let $x_i(m) = x \otimes s^n t^m$ where $x_i = x \otimes s^n$. When $m = 0$ we let $x_i(0) = x_i$ for $i \leq -1$. Thus, with this notation we find that $\mathfrak{t}_-$ has as a basis the set of elements $B(\mathfrak{t}_-) = \{x_i(0)\}_{i \leq -1} \cup \{x_i(m) | \, i, m \in \mathbb{Z}, m < 0\}$ and $\{x_i(0)\}_{i \leq -1}$ is a basis of $\hat{\mathfrak{n}}_-$. If $x = \sum a_i x_i \in L(\mathfrak{g})$ we let $x(m) = \sum a_i x_i(m)$. We totally order $B(\mathfrak{t}_-)$ by saying that for two elements $x_i(m)$, $x_j(r)$ that we have $x_i(m) < x_j(r)$ if and only if either $m < r$ or $m = r$ and $i < j$.

Note that a basis of $U(\mathfrak{t}_-)$ consists of monomials $x_{i_r}(m_r) \cdots x_{i_1}(m_1)$ where $r \geq 0$ and $x_{i_1}(m_1) \geq x_{i_2}(m_2) \geq \cdots \geq x_{i_r}(m_r)$; where by the usual convention we have if $r = 0$ this element represents 1.

Allowing for powers we use multi-index notation and write $\mathbf{a} = (a_1, \ldots, a_r)$ so that we let

$$x_{\mathbf{i},\mathbf{m},\mathbf{p}} = x_{i_r}(m_r)^{p_r} \cdots x_{i_1}(m_1)^{p_1}$$

where $\mathbf{i} = (i_r, \ldots, i_1)$, $\mathbf{m} = (m_r, \ldots, m_1)$, $\mathbf{p} = (p_r, \ldots, p_1)$ and we also have that $i_j \leq -1$ if $m_j = 0$ but if $m_j \leq -1$ then $i_j \in \mathbb{Z}$; and also $p_j \geq 1$ but that $x_{i_{j+1}}(m_{j+1}) < x_{i_j}(m_j)$. If $\bar{x} = x_{\mathbf{i},\mathbf{m},\mathbf{p}}$ then we define

$$\bar{x}^{\hat{j}} = x_{i_r}(m_r)^{p_r} \cdots x_{i_j}(m_j)^{p_j - 1} \cdots x_{i_1}(m_1)^{p_1}$$

for $1 \leq j \leq r$. For $1 \leq j \leq \xi \leq r$ define

$$\bar{x}^{\hat{\xi}\hat{j}} = \bar{x}^{\hat{j}\hat{\xi}}$$

$$= \begin{cases} x_{i_r}(m_r)^{p_r} \cdots x_{i_j}(m_j)^{p_j - 1} \cdots \\ \qquad \cdots x_{i_\xi}(m_\xi)^{p_\xi - 1} \cdots x_{i_1}(m_1)^{p_1} & \text{if } j \neq \xi, \\ x_{i_r}(m_r)^{p_r} \cdots x_{i_j}(m_j)^{p_j - 2} \cdots x_{i_1}(m_1)^{p_1} & \text{if } j = \xi,\ p_j \geq 2, \\ 0 & \text{if } j = \xi \text{ and } p_j = 1. \end{cases}$$

In other words $\bar{x}^{\hat{\xi}\hat{j}}$ is just $\bar{x}$ where we've decreased the exponents p_ξ and p_j by one.

By the Poincaré-Birkhoff-Witt theorem we have that

$$B = B(U(\mathfrak{t}_-)) := \{x_{\mathbf{i},\mathbf{m},\mathbf{p}}\} \cup \{1\}$$

is a basis of $U(\mathfrak{t}_-)$ so that an arbitrary element of $M(\lambda)$ can be written as

$$\sum u_{\mathbf{i},\mathbf{m},\mathbf{p}} x_{\mathbf{i},\mathbf{m},\mathbf{p}} v^+ + uv^+ \tag{1}$$

where u, $u_{\mathbf{i},\mathbf{m},\mathbf{p}} \in \mathbb{C}$ and the sum is over a finite number of the above allowable $(\mathbf{i}, \mathbf{m}, \mathbf{p})$ with $r \geq 1$.

Let B_2 be the set of those $x_{\mathbf{i},\mathbf{m},\mathbf{p}}$ with $m_j \neq 0$ for all j together with 1 and let B_1 be the set of $x_{\mathbf{i},\mathbf{m},\mathbf{p}}$ with $m_j = 0$ for all j together with 1. Then the above basis is just

$$B = B_2 \cdot B_1 = \{b_2 b_1 \,|\, b_i \in B_i,\ i = 1, 2\}$$

and B_2 (resp. B_1) is just a basis of $U(\mathfrak{t}_{-2})$ (resp. $U(\mathfrak{t}_{-1})$).

We now totally order B_2 (respectively B_1) as follows. We write

$$x_{\mathbf{i},\mathbf{m},\mathbf{p}} = x_{i_r}(m_r)^{p_r} \cdots x_{i_1}(m_1)^{p_1} < x_{j_\zeta}(n_\zeta)^{q_\zeta} \cdots x_{j_1}(n_1)^{q_1} = x_{\mathbf{j},\mathbf{n},\mathbf{q}}$$

if and only if there is some $k \geq 0$ with

$$x_{i_k}(m_k)^{p_k} \cdots x_{i_1}(m_1)^{p_1} = x_{j_k}(n_k)^{q_k} \cdots x_{j_1}(n_1)^{q_1}$$

(so no condition here if $k = 0$) and either $x_{i_{k+1}}(m_{k+1}) < x_{j_{k+1}}(n_{k+1})$ or $x_{i_{k+1}}(m_{k+1}) = x_{j_{k+1}}(n_{k+1})$ and $p_{k+1} > q_{k+1}$ (i.e. $-p_{k+1} < -q_{k+1}$ and this conforms with the power of t involved or if no t power is involved with the power of s). We make the convention that if $r > \zeta$ and if

$$x_{i_\zeta}(m_\zeta)^{p_\zeta} \cdots x_{i_1}(m_1)^{p_1} = x_{j_\zeta}(n_\zeta)^{q_\zeta} \cdots x_{j_1}(n_1)^{q_1}$$

then $x_{\mathbf{i},\mathbf{m},\mathbf{p}} < x_{\mathbf{j},\mathbf{n},\mathbf{q}}$ so that in this case this amounts to taking $x_{j_{\zeta+1}}(n_{\zeta+1})^{q_{\zeta+1}} = 1$ and also taking $1 \geq x_{\mathbf{i},\mathbf{m},\mathbf{p}}$. Note that this total ordering on B_2 (resp. B_1) extends the ordering we already have on the basis elements $\{x_i(m) | \, i \in \mathbb{Z}, m \in \mathbb{Z}, m \leq -1\}$ and $\{x_i(0) | \, i \leq -1\}$). Also we have for $p, q \geq 1$ that $x_i(m)^p < x_i(m)^q$ if and only if $q < p$.

Next we let $B_2^* = \{b \in B_2 | \, b \neq 1\}$ and notice that $B = B_2 B_1 = B_2^* B_1 \cup B_1$ where the union is disjoint. We now totally order B by declaring

$$b_2 b_1 < b_2' b_1'$$

for $b_i, b_i' \in B_i$, $i = 1, 2$ to mean that either $b_2 < b_2'$ or $b_2 = b_2'$ but $b_1 < b_1'$. Thus 1 is the greatest element of B and $B_2^* B_1 < B_1$ as seen by taking $b_2' = 1$ in the above. This total ordering on B extends the previous ordering on B_1 and B_2.

We now close this section with three lemmas which we shall use in the next section to prove our irreducibility criterion. It should not confuse the reader if we write elements as $x_{i_1}(m_1)^{p_1} \cdots x_{i_r}(m_r)^{p_r}$ rather than $x_{i_r}(m_r)^{p_r} \cdots x_{i_1}(m_1)^{p_1}$ as we have been doing. This was convenient in defining the ordering but is not so necessary now. Of course, if we write $x_{\mathbf{i},\mathbf{m},\mathbf{p}} = x_{i_1}(m_1)^{p_1} \cdots x_{i_r}(m_r)^{p_r}$ then $(\mathbf{i}, \mathbf{m}; \mathbf{p})$ must be allowable so that $x_{i_1}(m_1) < x_{i_2}(m_2) < \cdots < x_{i_r}(m_r)$.

LEMMA 2.2. *Let $\bar{x}(\bar{m}) = x_{i_1}(m_1)^{p_1} \cdots x_{i_k}(m_k)^{p_k}$ with $m_j \leq -1$ for $1 \leq j \leq k$. If $z \in L(\mathfrak{g})$ and $m \in \mathbb{Z}$ is such that $m_j < m$ for all j then*

$$[z(-m), \bar{x}(\bar{m})] \in U(\mathfrak{t}_{-2})\mathfrak{t}_{-2}.$$

Proof. As $ad\, z(-m)$ is a derivation on $U(\mathfrak{t}_{-2})$ one has

$$[z(-m), \bar{x}(\bar{m})] = \sum_{j=1}^{k} x_{i_1}(m_1)^{p_1} \cdots [z(-m), x_{i_j}(m_j)^{p_j}] \cdots x_{i_k}(m_k)^{p_k}.$$

Then the lemma follows as

$$\begin{aligned}[z(-m), x_{i_j}(m_j)^{p_j}] &= \sum_{a=0}^{p_j-1} x_{i_j}(m_j)^a [z(-m), x_{i_j}(m_j)] x_{i_j}^{p_j-1-a}(m_j) \\ &= \sum_{a=0}^{p_j-1} x_{i_j}(m_j)^a [z, x_{i_j}](m_j - m) x_{i_j}^{p_j-1-a}(m_j)\end{aligned}$$

is an element of $U(\mathfrak{t}_{-2})\mathfrak{t}_{-2}$ since $m_j - m < 0$ for all j. $\square$

Next we recall that the roots of our finite dimensional simple Lie algebra are in $\Delta \cup \{0\}$ so the roots of the affine algebra, $\hat{\mathfrak{g}}$, are of the form $\alpha + q\delta_s$ for $\alpha \in \Delta \cup \{0\}$, $q \in \mathbb{Z}$. We use this notation in our next lemma.

LEMMA 2.3. *Let $m \in \mathbb{Z}_{<0}$ and $z \in \hat{\mathfrak{g}}_{\alpha+q\delta_s}$ with $q \neq 0$ then*

$$[z(-m), x_{i_1}(m)^{p_1} \cdots x_{i_\zeta}(m)^{p_\zeta}] = y + \sum_{j=1}^{\zeta} p_j x_{\mathbf{i},\mathbf{m},\mathbf{p}}^{\hat{j}} [z(-m), x_{i_j}(m)]$$

where $y \in U(\mathfrak{t}_{-2})\mathfrak{t}_{-2}$ *and* $x_{\mathbf{i},\mathbf{m},\mathbf{p}} = x_{i_1}(m)^{p_1} \cdots x_{i_\zeta}(m)^{p_\zeta}$.

Proof. Since $[z(-m), x_{i_j}(m)] = [z, x_{i_j}] + c_j$ for some central element c_j we have

$$[z(-m), x_{i_1}(m)^{p_1} \cdots x_{i_\zeta}(m)^{p_\zeta}]$$

$$= \sum_{j=1}^{\zeta} x_{i_1}(m)^{p_1} \cdots [z(-m), x_{i_j}(m)^{p_j}] \cdots x_{i_\zeta}(m)^{p_\zeta}$$

$$= \sum_{j=1}^{\zeta} \sum_{a=0}^{p_j-1} x_{i_1}(m)^{p_1} \cdots x_{i_j}(m)^{a} \cdot [z(-m), x_{i_j}(m)] x_{i_j}(m)^{p_j-1-a} \cdots x_{i_\zeta}(m)^{p_\zeta}.$$

Now since c_j is central we get

$$[z(-m), x_{i_1}(m)^{p_1} \cdots x_{i_\zeta}(m)^{p_\zeta}]$$

$$= \sum_{j=1}^{\zeta} \sum_{a=0}^{p_j-1} x_{i_1}(m)^{p_1} \cdots x_{i_j}(m)^{a}[[z, x_{i_j}], x_{i_j}(m)^{p_j-1-a} \cdots x_{i_\zeta}(m)^{p_\zeta}]$$
$$+ \sum_{j=1}^{\zeta} p_j x_{i_1}(m)^{p_1} \cdots x_{i_j}(m)^{p_j-1} \cdots x_{i_\zeta}(m)^{p_\zeta} [z(-m), x_{i_j}(m)].$$

Let y be the first summation in the right side of the above equality. By Lemma 2.2 y is in $U(\mathfrak{t}_{-2})\mathfrak{t}_{-2}$ which gives us the desired result. □

REMARK 2.4. We will later use Lemma 2.3 by choosing z in such a way that we have

$$[z(-m), x_{i_1}(m)^{p_1} \cdots x_{i_\zeta}(m)^{p_\zeta}] = y + \sum_{j=1}^{\zeta} p_j x^{\hat{j}}_{\mathbf{i},\mathbf{m},\mathbf{p}}[z, x_{i_j}]$$

so that no central term appears.

Basic to our proof of our irreducibility criterion is the idea that we can raise the t value (towards 0 as these values are negative) and hence end up in $\hat{\mathfrak{g}}$. We do this later by lowering the s value so as to avoid any trouble. Thus, we are really exploiting the fact that in $\hat{\mathfrak{g}}$ there is no bottom. For example, if we are given a finite number of elements $x_{i_1}, \dots x_{i_n}$ in $L(\mathfrak{g})$ where $x_{i_1} \neq 0$ and where $x_{i_j} \in \hat{\mathfrak{g}}_{\beta_{i_j}}$ then say $x_{i_j} = y_j \otimes s^{e_j}$ where either y_j is in a root space of $\mathfrak{g}$ or in the Cartan subalgebra $\mathfrak{H}$ and $e_j \in \mathbb{Z}$. Then $x_{i_1} \neq 0$ implies $y_1 \neq 0$ and since $\mathfrak{g}$ is simple there is an element y in a root space of $\mathfrak{g}$ satisfying $[y, y_1] \neq 0$. Thus by choosing m large enough we can insure that $z = y \otimes s^{-m}$ satisfies $z \in \hat{\mathfrak{g}}_\beta$ for some root β of $\hat{\mathfrak{g}}$ and $0 \neq [z, x_{i_1}] \in \hat{\mathfrak{g}}_{\beta+\beta_{i_1}}$ where $\beta + \beta_{i_1} < \beta_{i_j}$ for $1 \leq j \leq n$. More strongly, we can insure that m is large enough to give us that if $\beta + \beta_{i_j}$ is a root of $\hat{\mathfrak{g}}$ for some $j \in \{1, \dots, n\}$ then $\beta + \beta_{i_j} < \beta_{i_k}$ for all $k \in \{1, \dots, n\}$ and moreover that $-m + e_j \neq 0$ so that $[z(r), x_{i_j}(-r)] = [z, x_{i_j}]$ for all $r \in \mathbb{Z}$, and all $j \in \{1, \dots, n\}$. For easy reference we record this as follows:

LEMMA 2.5. *Let $I = \{i_1, \dots, i_n\}$ be a finite subset of $\mathbb{Z}$. Then there is an element $z \in L(\mathfrak{g}) \cap \hat{\mathfrak{g}}_\beta$ for some root β of $\hat{\mathfrak{g}}$ satisfying*

(i) $0 \neq [z, x_{i_1}]$,

(ii) *if $\beta + \beta_{i_j}$ is a root of $\hat{\mathfrak{g}}$ then $\beta + \beta_{i_j} < \beta_{i_k}$ for all $1 \leq k \leq n$ and*

(iii) $[z(r), x_{i_j}(-r)] = [z, x_{i_j}]$ *for $r \in \mathbb{Z}$, and $1 \leq j \leq n$.*

REMARK 2.6. In the above it is clear that we do not even need to require that the elements $x_{i_1}, \dots, x_{i_n}$ are our chosen basis elements but only that $x_{i_1} \neq 0$ and each x_{i_j} is an element of the root space $\hat{\mathfrak{g}}_{\beta_j}$ for $1 \leq j \leq n$.

3. Irreducibility Criterion. In this section we establish our irreducibility criterion for the modules $M(\lambda)$ defined in Section two. Recall $\hat{\mathfrak{g}}$ is a subalgebra of $\mathfrak{t}$ so that $M(\lambda)$ is also a $\hat{\mathfrak{g}}$ module and we have let $\hat{\lambda}$ denote the restriction of our $\lambda \in \mathfrak{t}_0^*$ to $\hat{\mathfrak{H}}$ and $M_{\hat{\mathfrak{g}}}(\hat{\lambda})$ is the $\hat{\mathfrak{g}}$-submodule of $M(\lambda)$ generated by our generating element v^+ of $M(\lambda)$ so that $M_{\hat{\mathfrak{g}}}(\hat{\lambda})$ is the Verma module with highest weight $\hat{\lambda}$ for the affine algebra $\hat{\mathfrak{g}}$. We denote this by $\widehat{M}$.

If N is any $\mathfrak{t}$-module with a weight space decomposition we define $N^{(m)} = \{n \in N | \, d_t \cdot n = mn\}$ for any $m \in \mathbb{C}$ so that $N = \oplus_{m \in \mathbb{C}} N^{(m)}$ and, in particular, we have $M(\lambda) = \oplus_{m \in \mathbb{Z}, m \leq 0} M(\lambda)^{(m+\lambda(d_t))}$ because $M(\lambda) = U(\mathfrak{t}_-)v^+$ and $U(\mathfrak{t}_-) = \oplus_{m \in \mathbb{Z}, m \leq 0} U(\mathfrak{t}_-)^{(m)}$. Thus, we have that $M(\lambda)^{(m+\lambda(d_t))} = (0)$ for $m > 0, m \in \mathbb{Z}$. Recall also that $M(\lambda)$

has a maximal proper submodule which we denote by Rad $M(\lambda)$.

LEMMA 3.1. *If $M(\lambda)$ is an irreducible $\mathfrak{t}$-module then $\widehat{M}$ is an irreducible $\hat{\mathfrak{g}}$-module.*

Proof. Assume $N \subset \widehat{M}$ is a proper $\hat{\mathfrak{g}}$-submodule. Then we have $N_{\hat{\lambda}} = 0$. Also, since d_t and $U(\hat{\mathfrak{g}})$ commute we find that if $m > 0$, $m \in \mathbb{Z}$ and $u \in U(\mathfrak{t}_2) \cap U(\mathfrak{t})^{(m)}$ then

$$uN \subset M(\lambda)^{(m+\lambda(d_t))} = 0.$$

Hence $U(\mathfrak{t}_2)N = \oplus_{n\geq 0} U(\mathfrak{t}_2)^{(n)}N = N$. By the Poincaré-Birkhoff-Witt theorem we have $U(\mathfrak{t})N = U(\mathfrak{t}_{-2})U(\mathfrak{t}_{-1})U(\mathfrak{t}_0)U(\mathfrak{t}_1)U(\mathfrak{t}_2)N = U(\mathfrak{t}_{-2})N$. This implies that

$$w(U(\mathfrak{t})N) = \{\mu + \psi | \mu \in w(U(\mathfrak{t}_{-2}))\backslash\{0\},\, \psi \in w(N)\} \cup w(N).$$

But then $\lambda \notin w(U(\mathfrak{t})N)$ so $U(\mathfrak{t})N$ is a proper $\mathfrak{t}$-submodule of $M(\lambda)$. □

The more difficult direction in proving the irreducibility criterion is to show the irreducibility of $\widehat{M}$ implies that $M(\lambda)$ is irreducible as a $\mathfrak{t}$-module. The first step of this is the following lemma.

LEMMA 3.2. *Suppose $\widehat{M}$ is an irreducible $\hat{\mathfrak{g}}$-module. Then $Rad\, M(\lambda)^{(\lambda(d_t))} = 0$.*

Proof. Since $M(\lambda) = U(\mathfrak{t}^-)v^+$ then we find that $M(\lambda)^{(\lambda(d_t))} = \widehat{M}$ so that

$$(\mathrm{Rad}\, M(\lambda))^{(\lambda(d_t))} \subset M(\lambda)^{(\lambda(d_t))} \cap \mathrm{Rad}\, M(\lambda) = \widehat{M} \cap \mathrm{Rad}\, M(\lambda)$$

and this latter space, being a $\hat{\mathfrak{g}}$-submodule, is either (0) or all of $\widehat{M}$. If the intersection is $\widehat{M}$ then $v^+ \in \mathrm{Rad}\, M(\lambda)$ and this is impossible for then $M(\lambda) = \mathrm{Rad}\, M(\lambda)$. □

The next result is the crucial one in establishing the irreducibility criterion.

LEMMA 3.3. *Suppose that $\widehat{M}$ is irreducible. If* $Rad\,M(\lambda)^{(-n+\lambda(d_t))} \neq 0$ *for some* $n \in \mathbb{Z}$, $n > 0$, *then* $Rad\,M(\lambda)^{(-n+m+\lambda(d_t))} \neq 0$ *for some* $m \in \mathbb{Z}$ *satisfying* $0 < m \leq n$.

Proof. Let $u \in \operatorname{Rad} M(\lambda)^{(-n+\lambda(d_t))}$ be nonzero and write $u = \sum u_{\mathbf{i},\mathbf{m},\mathbf{p}} x_{\mathbf{i},\mathbf{m},\mathbf{p}} v^+$ where the coefficients $u_{\mathbf{i},\mathbf{m},\mathbf{p}}$ belong to $\mathbb{C}$. We let X denote the finite set of indices $(\mathbf{i},\mathbf{m},\mathbf{p})$ with $u_{\mathbf{i},\mathbf{m},\mathbf{p}} \neq 0$. Using our totally ordered basis of $U(\mathfrak{t}_-)$ we obtain a unique index $(\mathbf{a},\mathbf{b},\mathbf{c})$ in X with $x_{\mathbf{a},\mathbf{b},\mathbf{c}}$ maximal among all monomials $x_{\mathbf{i},\mathbf{m},\mathbf{p}}$ with $(\mathbf{i},\mathbf{m},\mathbf{p}) \in X$. Write $\mathbf{a} = (a_1, \dots, a_\zeta)$, $\mathbf{b} = (b_1, \dots, b_\zeta)$, $\mathbf{c} = (c_1, \dots, c_\zeta)$ and because $n > 0$ we find that not all b_i's are zero so we fix r satisfying $1 \leq r \leq \zeta$ with $\mathbf{b} = (b_1, \dots, b_r, 0, \dots, 0)$ and $b_i < 0$ for $1 \leq i \leq r$. If $b_j \neq b_r$ for some $j < r$ let k be the such that $b_{k-1} < b_k = b_{k+1} = \cdots = b_r$ and otherwise let $k = 1$. Recalling that β_j denotes the root associated to the root vector x_j of $L(\mathfrak{g})$ we take account of some of the root spaces possibly having multiplicity bigger than one as follows. Let l be such that $a_r \geq l$ and $\beta_{l-1} \neq \beta_l = \beta_{l+1} \cdots \beta_{a_r-1} = \beta_{a_r}$.

In order to simplify notation in the following argument we break up $x_{\mathbf{a},\mathbf{b},\mathbf{c}}$ into an initial segment (with t values less than b_r), a middle segment (with t-values equal to b_r), and a final segment (in $U(\mathfrak{t}_{-1})$). More precisely we let

$$\bar{x} = x_{a_1}(b_1)^{c_1} \cdots x_{a_{k-1}}(b_{k-1})^{c_{k-1}}, \quad \tilde{x} = x_{a_k}(b_r)^{c_k} \cdots x_{a_r}(b_r)^{c_r},$$
$$\bar{x}' = x_{a_{r+1}}^{c_{r+1}} \cdots x_{a_\zeta}^{c_\zeta}$$

so we are using the fact that $b_k = \cdots = b_r$.

Next we define a special set of indices in X which we denote by $I_{\mathbf{a},\mathbf{b},\mathbf{c}}$. If $c_r \geq 2$ we let $I_{\mathbf{a},\mathbf{b},\mathbf{c}} = \{(\mathbf{a},\mathbf{b},\mathbf{c})\}$. If $c_r = 1$ then for any index j satisfying $l \leq j \leq a_r$ we let $x_{\mathbf{a},\mathbf{b},\mathbf{c}^{(j)}}$ be the same as the monomial $x_{\mathbf{a},\mathbf{b},\mathbf{c}}$ except the term $x_{a_r}(b_r) = x_{a_r}(b_r)^{c_r}$ is deleted and in it's place we put the term $x_j(b_r)$, so that $(\mathbf{a},\mathbf{b},\mathbf{c})^{(j)} = (\mathbf{i},\mathbf{m},\mathbf{p})$ where

$$\mathbf{i} = (a_1, \dots, a_r, j, a_{r+1}, \dots, a_\zeta), \quad \mathbf{m} = (b_1, \dots, b_{r-1}, b_r, b_r, 0, \dots, 0)$$
$$\text{and} \quad \mathbf{p} = (c_1, \dots, c_{r-1}, c_r - 1, 1, c_{r+1}, \dots, c_\zeta).$$

Thus $x_{\mathbf{a},\mathbf{b},\mathbf{c}^{(j)}} = \bar{x}\hat{\tilde{x}^j} x_j(b_r)\bar{x}'$. Consequently, if $j = a_r$ then $(\mathbf{a},\mathbf{b},\mathbf{c})^{(j)} = (\mathbf{a},\mathbf{b},\mathbf{c})$. Note that it may be that j is less than the index a_{r-1} but if this is the case then $\beta_{a_{r-1}} = \beta_j = \beta_{a_r}$ because $\beta_{a_r} = \beta_j \leq \beta_{a_{r-1}} \leq \beta_{a_r}$. Also, if γ is any root of $L(\mathfrak{g})$ then we have $[L(\mathfrak{g})_\gamma, L(\mathfrak{g})_\gamma] = (0)$ so that in our case we have $x_{a_{r-1}}(b_r)^{c_{r-1}} x_j(b_r) = x_j(b_r) x_{a_{r-1}}(b_r)^{c_{r-1}}$. Thus, we can commute $x_j(b_r)$ around enough terms $x_\mu(b_r)^\mu$ so that the monomial is written in proper order. That is, $x_{\mathbf{a},\mathbf{b},\mathbf{c}^{(j)}}$ is one of our basis elements of $U(\mathfrak{t}_-)$. We let

$$I_{\mathbf{a},\mathbf{b},\mathbf{c}} = \{(\mathbf{a},\mathbf{b},\mathbf{c})^{(j)} \,|\, l \leq j \leq a_r\} \cap X.$$

Thus, in either case we have $(\mathbf{a},\mathbf{b},\mathbf{c}) \in I_{\mathbf{a},\mathbf{b},\mathbf{c}}$.

If we let $I = \{i_j \,|\, \mathbf{i} = (i_1, \ldots, i_j, \ldots) \quad \text{for some } (\mathbf{i},\mathbf{m},\mathbf{p}) \in X\}$ so I is the finite set of all indices i_j that appear in any $\mathbf{i}$ for various $(\mathbf{i},\mathbf{m},\mathbf{p})$ belonging to X. Also using our definition of r above, we define the element y as follows. When $c_r \geq 2$ let $y = c_r u_{\mathbf{a},\mathbf{b},\mathbf{c}} x_r$ and when $c_r = 1$ let

$$y = \sum_{j=l}^{a_r} u_{(\mathbf{a},\mathbf{b},\mathbf{c})^j} x_j = \sum_{j=l}^{a_r} c_r u_{(\mathbf{a},\mathbf{b},\mathbf{c})^j} x_j.$$

Note that y is a non-zero element of the root space $L(\mathfrak{g})_{\beta_{a_r}}$ for $L(\mathfrak{g})$. Thus, using Lemma 2.5 and the remark following it we find there is some root vector $z \in L(\mathfrak{g})_\beta$ satisfying

1. $0 \neq [z, y]$,
2. if $j \in I$ and $\beta + \beta_{i_j}$ is a root of $\hat{\mathfrak{g}}$ then $\beta + \beta_{i_j} < \beta_{i_k}$ for all $i \in I$ and
3. $[z(r), x_j(-r)] = [z, x_j]$ and $[z(r), y(-r)] = [z, y]$ for $r \in \mathbb{Z}$, and $j \in I$.

In the rest of the proof we consider $z(-b_r)$ acting on our element u. We will show that $0 \neq z(-b_r)u$ and then the Lemma follows immediately from this since $z(-b_r)u$ is in $(\operatorname{Rad} M(\lambda))^{(-n-b_r+\lambda(d_t))}$ and $b_r < 0$ so clearly $0 < -b_r \leq n$.

We now split the argument into two cases. The first case is when we assume the power $c_r \geq 2$. Here $I_{\mathbf{a},\mathbf{b},\mathbf{c}} = \{(\mathbf{a},\mathbf{b},\mathbf{c})\}$. We have

$$z(-b_r) u_{\mathbf{a},\mathbf{b},\mathbf{c}} x_{\mathbf{a},\mathbf{b},\mathbf{c}} v^+ = u_{\mathbf{a},\mathbf{b},\mathbf{c}} [z(-b_r), \bar{x}] \tilde{x} \bar{x}' v^+ + u_{\mathbf{a},\mathbf{b},\mathbf{c}} \bar{x} [z(-b_r), \tilde{x}] \bar{x}' v^+,$$

because $z(-b_r)$ applied to $\bar{x}' v^+$ is zero as $-b_r > 0$. Applying Lemmas 2.2 and 2.3 we obtain then for some $y \in U(\mathfrak{t}_-) U(\mathfrak{t}_{-2}) \mathfrak{t}_{-2}$ (where we can ignore any central elements arising by our choice of z and so write $[z, x_{a_\mu}]$ instead of $[z(-b_r), x_{a_\mu}(b_r)]$)

$$z(-b_r)u_{\mathbf{a},\mathbf{b},\mathbf{c}}x_{\mathbf{a},\mathbf{b},\mathbf{c}}v^+ = y\bar{x}'v^+ + \sum_{\mu=k}^{r-1} c_\mu u_{\mathbf{a},\mathbf{b},\mathbf{c}}\bar{x}\tilde{x}^{\widehat{c_\mu}}[z, x_{a_\mu}]\bar{x}'v^+$$
$$+ c_r u_{\mathbf{a},\mathbf{b},\mathbf{c}}\bar{x}\tilde{x}^{\widehat{c_r}}[z, x_{a_r}]\bar{x}'v^+.$$

Let

$$S_1 = y\bar{x}'v^+, \quad S_2 = \sum_{\mu=k}^{r-1} c_\mu u_{\mathbf{a},\mathbf{b},\mathbf{c}}\bar{x}\tilde{x}^{\widehat{c_\mu}}[z, x_{a_\mu}]\bar{x}'v^+$$

and

$$S_3 = c_r u_{\mathbf{a},\mathbf{b},\mathbf{c}}\bar{x}\tilde{x}^{\widehat{c_r}} \cdot [z, x_{a_r}]\bar{x}'v^+.$$

Then $z(-b_r)u_{\mathbf{a},\mathbf{b},\mathbf{c}}x_{\mathbf{a},\mathbf{b},\mathbf{c}}v^+ = S_1 + S_2 + S_3$. Notice that condition (1) in our choice of z guarantees that $S_3 \neq 0$. Also as $y \in U(\mathfrak{t}_{-2})\mathfrak{t}_{-2}$, condition (2) in our choice of z gives that S_1 cannot contribute to cancel S_3. (That is, the part of the monomials involving no t terms in S_1 are higher, by our choice of z, than at least the one term $[z, x_{a_r}]$ appearing in S_3). Finally, in all monomials involved in S_2 there is some $\mu \in \{k, \ldots, r-1\}$ so that the monomial is of the form $\bar{x}\tilde{x}^{\hat{c}_\mu}[z, x_{a_\mu}]\bar{x}$ and these cannot contribute to cancel S_3 because the powers of $x_{a_r}(b_r)$ don't match up. Thus, $z(-b_r)u_{\mathbf{a},\mathbf{b},\mathbf{c}}x_{\mathbf{a},\mathbf{b},\mathbf{c}} \neq 0$.

Next we consider an arbitrary non-zero summand $u_{\mathbf{i},\mathbf{m},\mathbf{p}}x_{\mathbf{i},\mathbf{m},\mathbf{p}}v^+$ of uv^+ where

$$(\mathbf{i},\mathbf{m},\mathbf{p}) \in X\text{but}(\mathbf{i},\mathbf{m},\mathbf{p}) \notin I_{\mathbf{a},\mathbf{b},\mathbf{c}}\text{so}(\mathbf{i},\mathbf{m},\mathbf{p}) \neq (\mathbf{a},\mathbf{b},\mathbf{c})$$

We expand $z(-b_r)u_{\mathbf{i},\mathbf{m},\mathbf{p}}x_{\mathbf{i},\mathbf{m},\mathbf{p}}v^+$ and show it cannot contribute to cancel our term S_3 above. Write

$$x_{\mathbf{i},\mathbf{m},\mathbf{p}} = x_{i_1}(m_1)^{p_1} \cdots x_{i_q}(m_q)^{p_q} x_{i_{q+1}}^{p_{q+1}} \cdots x_{i_\epsilon}^{p_\epsilon}.$$

Now if $m_q < b_r$ then using Lemma 2.2 and the fact that $z(-b_r)U(\mathfrak{t}_{-1})v^+ = 0$ we find that

$$z(-b_r)x_{\mathbf{i},\mathbf{m},\mathbf{p}}v^+ = y' x_{i_{q+1}}^{p_{q+1}} \cdots x_{i_\epsilon}^{p_\epsilon} v^+$$

for some $y' \in U(\mathfrak{t}_{-2})\mathfrak{t}_{-2}$. By condition (2) in our choice of z such a term cannot contribute to cancel S_3.

Next note that since $(\mathbf{a},\mathbf{b},\mathbf{c})$ was chosen maximal in X by the definition of the total ordering on our basis of $U(\mathfrak{t})$ we must have that $m_1 \leq m_2 \leq \cdots \leq m_q = b_r$ so that we may suppose that $m_\gamma < m_{\gamma+1} = \cdots = m_q = b_r$ and set

$$\bar{w} = x_{i_1}(m_1)^{p_1} \cdots x_{i_\gamma}(m_\gamma)^{p_\gamma}\ , \quad \tilde{w} = x_{i_{\gamma+1}}(m_{\gamma+1})^{p_{\gamma+1}} \cdots x_{i_q}(m_q)^{p_q}\ ,$$
$$\bar{w}' = x_{i_{q+1}}^{p_{q+1}} \cdots x_{i_\epsilon}^{p_\epsilon}\ .$$

Thus, by Lemmas 2.2 and 2.3 we have that for some $y \in U(\mathfrak{t}_{-2})\mathfrak{t}_{-2}$

$$z(-b_r)x_{\mathbf{i},\mathbf{m},\mathbf{p}}v^+ = y\bar{w}'v^+ + \sum_{j=\gamma+1}^{q} p_j \bar{w}\tilde{w}^{\hat{p_j}}[z, x_{i_j}]\bar{w}'v^+$$

so letting T_1 be the first of these terms and T_2 the second summation we have $z(-b_r)x_{\mathbf{i},\mathbf{m},\mathbf{p}}v^+ = T_1 + T_2$. As before, T_1 cannot contribute to cancel the summand S_3. If T_2 contributes to cancel S_3 then we would have

$$\bar{x}x_{a_k}(b_r)^{c_k} \cdots x_{a_{r-1}}(b_r)^{c_{r-1}}x_{a_r}(b_r)^{c_r-1} =$$
$$\bar{w}x_{i_{\gamma+1}}(b_r)^{p_{\gamma+1}} \cdots x_{i_j}(b_r)^{p_j-1} \cdots x_{i_q}(b_r)^{p_q}$$

for some j. Moreover, as $\bar{x}$ and $\bar{w}$ are monomials in $x_j(b)$'s with $b < b_r$ we must have $\bar{x} = \bar{w}$ and

$$x_{a_k}(b_r)^{c_k} \cdots x_{a_{r-1}}(b_r)^{c_{r-1}}x_{a_r}(b_r)^{c_r-1} =$$
$$x_{i_{\gamma+1}}(b_r)^{p_{\gamma+1}} \cdots x_{i_j}(b_r)^{p_j-1} \cdots x_{i_q}(b_r)^{p_q}.$$

Then by the maximality of $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ we must have $j = q$ for otherwise

$$x_{i_{\gamma+1}}(b_r)^{p_{\gamma+1}} \cdots x_{i_q}(b_r)^{p_q}$$

is a monomial with $p_q = c_r - 1 < c_r$ ($c_r - 1 \geq 1$ by hypothesis) and this gives a monomial which is larger than $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. Thus we get $j = q$ so that

$$x_{a_k}(b_r)^{c_k} \cdots x_{a_{r-1}}(b_r)^{c_{r-1}}x_{a_r}(b_r)^{c_r-1} =$$
$$x_{i_{\gamma+1}}(b_r)^{p_{\gamma+1}} \cdots x_{i_{q-1}}(b_r)^{p_{q-1}} \cdots x_{i_q}(b_r)^{p_q-1}$$

and we obtain $(\mathbf{i}, \mathbf{m}, \mathbf{p}) = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ which is a contradiction. This establishes the Lemma in this case.

We now do the more difficult case when $c_r = 1$. Here we recall that

$$I_{\mathbf{a},\mathbf{b},\mathbf{c}} = \{(\mathbf{a}, \mathbf{b}, \mathbf{c})^{(j)} \,|\, l \leq j \leq a_r\} \cap X$$

may contain more than the single element $(\mathbf{a},\mathbf{b},\mathbf{c}) = (\mathbf{a},\mathbf{b},\mathbf{c})^{(a_r)}$ Let

$$u_1 = \sum_{I_{\mathbf{a},\mathbf{b},\mathbf{c}}} u_{\mathbf{i},\mathbf{m},\mathbf{p}} x_{\mathbf{i},\mathbf{m},\mathbf{p}} v^+$$

so that

$$\begin{aligned} z(-b_r)u_1 &= \sum_{j=l}^{a_r} u_{(\mathbf{a},\mathbf{b},\mathbf{c})^{(j)}}[z(-b_r),\bar{x}]\tilde{x}^{\widehat{c_r}} x_j(b_r)\bar{x}'v^+ \\ &\quad + \sum_{j=l}^{a_r} u_{(\mathbf{a},\mathbf{b},\mathbf{c})^{(j)}}\bar{x}[z(-b_r),\tilde{x}^{\widehat{c_r}} x_j(b_r)]\bar{x}'v^+ \end{aligned}$$

because $z(-b_r)$ applied to $\bar{x}'v^+$ is zero. Applying Lemmas 2.2 and 2.3 we obtain that for some $y \in U(\mathfrak{t}_{-2})\mathfrak{t}_{-2}$ we have $z(-b_r)u_1 = S_1 + S_2 + S_3$ where

$$\begin{aligned} S_1 &= y\bar{x}'v^+, \\ S_2 &= \sum_{j=l}^{a_r}\sum_{\mu=k}^{r-1} c_\mu u_{(\mathbf{a},\mathbf{b},\mathbf{c})^{(j)}}\bar{x}\tilde{x}^{\widehat{c_\mu}\widehat{c_r}} x_j(b_r)[z,x_{a_\mu}]\bar{x}'v^+, \\ S_3 &= u_{\mathbf{a},\mathbf{b},\mathbf{c}}\bar{x}\tilde{x}^{\widehat{c_r}}[z,x_{a_r}]\bar{x}'v^+ + \sum_{j=l}^{a_r-1} u_{(\mathbf{a},\mathbf{b},\mathbf{c})^{(j)}}\bar{x}\tilde{x}^{\widehat{c_r}}[z,x_j]\bar{x}'v^+ \\ &= \bar{x}\tilde{x}^{\widehat{c_r}}[z, u_{\mathbf{a},\mathbf{b},\mathbf{c}}x_{a_r} + \sum_{j=l}^{a_r-1} u_{(\mathbf{a},\mathbf{b},\mathbf{c})^{(j)}}x_j]\bar{x}'v^+. \end{aligned}$$

Notice that condition (1) in our choice of z guarantees S_3 is non-zero. Also, as $y \in U(\mathfrak{t}_{-2})\mathfrak{t}_{-2}$ then condition (2) in our choice of z gives S_1 cannot contribute to cancel S_3. Finally, all monomials involved in S_2 have the term $x_j(b_r)$ for some $a_r \le j \le l$ in them while no term in S_3 does so that S_2 cannot contribute to cancel S_3. Thus $z(-b_r)u_1 \neq 0$.

Next we consider an arbitrary non-zero summand $u_{\mathbf{i},\mathbf{m},\mathbf{p}}x_{\mathbf{i},\mathbf{m},\mathbf{p}}$ of uv^+ where $(\mathbf{i},\mathbf{m},\mathbf{p}) \in X$ but $(\mathbf{i},\mathbf{m},\mathbf{p}) \notin I_{\mathbf{a},\mathbf{b},\mathbf{c}}$. We will expand $z(-b_r)x_{\mathbf{i},\mathbf{m},\mathbf{p}}$ and see it cannot contribute to cancel our term S_3 above. Write

$$x_{\mathbf{i},\mathbf{m},\mathbf{p}} = x_{i_1}(m_1)^{p_1}\cdots x_{i_q}(m_q)^{p_q}x_{i_{q+1}}^{p_{q+1}}\cdots x_{i_\epsilon}^{p_\epsilon}.$$

Now if $m_q < b_r$ then using Lemma 2.2 and the fact that $z(-b_r)U(\mathfrak{t}_{-1})v^+ = 0$ (as $-b_r > 0$) we find that

$$z(-b_r)x_{\mathbf{i},\mathbf{m},\mathbf{p}}v^+ = y'x_{i_{q+1}}^{p_{q+1}}\cdots x_{i_\epsilon}^{p_\epsilon}v^+$$

for some $y' \in U(\mathfrak{t}_{-2})\mathfrak{t}_{-2}$ and such a term cannot contribute to cancel S_3 by (2) in our choice of z.

Next note that since $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ was chosen maximal in X that by the definition of the total ordering on our basis of $U(\mathfrak{t})$ we must have $m_1 \leq m_2 \leq \cdots \leq m_q \leq b_r$ so that we may suppose that $m_\gamma < m_{\gamma+1} = \cdots = m_q = b_r$, and set

$$\bar{w} = x_{i_1}(m_1)^{p_1} \cdots x_{i_\gamma}(m_\gamma)^{p_\gamma} , \quad \tilde{w} = x_{i_{\gamma+1}}(m_{\gamma+1})^{p_{\gamma+1}} \cdots x_{i_q}(m_q)^{p_q} ,$$
$$\bar{w}' = x_{i_{q+1}}^{p_{q+1}} \cdots x_{i_\epsilon}^{p_\epsilon} .$$

Thus, by Lemmas 2.2 and 2.3 we have that for some $y \in U(\mathfrak{t}_{-2})\mathfrak{t}_{-2}$

$$z(-b_r)x_{\mathbf{i},\mathbf{m},\mathbf{p}}v^+ = y\bar{w}'v^+ + \sum_{j=\gamma+1}^{q} p_j \bar{w}\tilde{w}^{\hat{p_j}}[z, x_{i_j}]\bar{w}'v^+$$

(again we are writing $[x(-r), x_{i_j}(r)] = [z, x_{i_j}]$ since by (3) in our choice of z no central elements will be involved). Let $T_1 = y\bar{w}'v^+$ be the first term above and T_2 the second summation. As before, T_1 cannot contribute to cancel our term S_3 because of (2) in our choice of z.

If T_2 could contribute to cancel S_3 then we would have

$$\bar{x}x_{a_k}(b_r)^{c_k} \cdots x_{a_{r-1}}(b_r)^{c_{r-1}} = \bar{w}x_{i_{\gamma+1}}(b_r)^{p_{\gamma+1}} \cdots x_{i_j}(b_r)^{p_j-1} \cdots x_{i_q}(b_r)^{p_q}$$

for some j. Moreover, as $\bar{x}$ and $\bar{w}$ are monomials in $x_j(b)$'s with $b < b_r$ we must have $\bar{x} = \bar{w}$ and

$$x_{a_k}(b_r)^{c_k} \cdots x_{a_{r-1}}(b_r)^{c_{r-1}} = x_{i_{\gamma+1}}(b_r)^{p_{\gamma+1}} \cdots x_{i_j}(b_r)^{p_j-1} \cdots x_{i_q}(b_r)^{p_q}.$$

so these two monomials are equal termwise. We are going to show that the root attached to x_{i_j} above, namely β_{i_j}, equals β_{a_r}. Clearly $\beta_{i_j} \leq \beta_{a_r}$. The above implies $[z, x_{i_q}]\bar{w}'$ lies in the space

$$\mathbb{C}[z, c_r u_{\mathbf{a},\mathbf{b},\mathbf{c}} x_{a_r} + \sum_{j=l}^{a_r-1} u_{(\mathbf{a},\mathbf{b},\mathbf{c})^{(j)}} x_j]\bar{x}'$$

where $\bar{w}'$ and $\bar{x}'$ are monomials in the x_i's for $i \in I$ and also i_j and a_r belong to I. Recalling again (2) in our choice of z we see that if $\beta_{i_j} < \beta_{a_r}$ then $[z, x_{i_j}]\bar{w}'$ would have it's lowest term in the root space $L(\mathfrak{g})_{\beta_{i_j}+\beta}$ while that of the other element is in $L(\mathfrak{g})_{\beta_{a_r}+\beta}$ and this is impossible if they cancel. Thus, we conclude that $\beta_{i_j} = \beta_{a_r}$ so that

$$\bar{w}x_{i_{\gamma+1}}(b_r)^{p_{\gamma+1}} \cdots x_{i_q}(b_r)^{p_q}\bar{w}' = \bar{w}x_{a_k}(b_r)^{c_k} \cdots x_{a_{r-1}}(b_r)^{c_{r-1}}x_{i_j}(b_r)\bar{w}'$$
$$= \bar{x}x_{a_k}(b_r)^{c_k} \cdots x_{a_{r-1}}(b_r)^{c_{r-1}}x_{i_\mu}(b_r)\bar{x}'$$

for some $\mu \in \{l, \ldots, a_r\}$. Thus, $(\mathbf{i}, \mathbf{m}, \mathbf{p}) \in I_{\mathbf{a},\mathbf{b},\mathbf{c}}$ contrary to our assumption and so we conclude that T_2 cannot contribute to cancel S_3. □

Putting together Lemmas 3.1, 3.2, and 3.3 we now have proven our irreducibility criterion.

THEOREM 3.4. *$M(\lambda)$ is an irreducible $\mathfrak{t}$-module if and only if $\widehat{M}$ is an irreducible $\hat{\mathfrak{g}}$-module.*

Putting this together with Proposition 2.1 we obtain a sharpening of that result.

COROLLARY 3.5. *If $\lambda, \mu \in \mathfrak{t}_0^*$ and let $M(\mu)$ be a subquotient of $M(\lambda)$, then*

(i) *$M(\mu)$ is a submodule of $M(\lambda)$,*

(ii) *if $N \subset M(\lambda)$ is a nonzero submodule, then $N \cap M(\mu) \neq 0$,*

(iii) *if $M_{\hat{\mathfrak{g}}}(\hat{\mu})$ is irreducible then $M(\mu)$ is the unique irreducible submodule of $M(\lambda)$,*

(iv) *if $M_{\hat{\mathfrak{g}}}(\hat{\mu})$ is irreducible then*

$$\dim Hom_{\mathfrak{t}}(M(\lambda'), M(\lambda)) \leq 1 \quad \text{for any} \quad \lambda' \in \mathfrak{H}^*.$$

Next as in [**15**] we let ψ denote the principal null root of $(\hat{\mathfrak{g}}, \hat{\mathfrak{H}})$ and let $\rho \in \hat{\mathfrak{H}}$ satisfy $\rho(h_i) = 1$ for all i where h_i is the coroot dual to α_i. Then we have the following:

COROLLARY 3.6. *Let $\lambda \in \mathfrak{H}^*$ be such that $(\hat{\lambda} + \hat{\rho}, \psi) < 0$. Then*

(i) *$M(\lambda)$ has a unique irreducible submodule $M(\mu)$, for some $\mu \in \mathfrak{t}_0^*$;*

(ii) $\dim Hom_{\mathfrak{t}}(M(\lambda'), M(\lambda)) \leq 1$ *for any* $\lambda' \in \mathfrak{t}_0^*$.

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