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IMMERSIONS UP TO JOINT-BORDISM

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A necessary and sufficient condition for a map to be joint-bordant to an immersion is given in terms of Stiefel-Whitney numbers.

1. Introduction. This note is devoted to a study of immersions of manifolds into manifolds up to joint-bordism. We will work throughout in the category of smooth manifolds and smooth maps.

A map of dimension (n, k) is a map of a closed n-manifold into a closed $(n+k)$-manifold. Two maps $f_0 : M_0 \to N_0$ and $f_1 : M_1 \to N_1$ of dimension (n, k) are said to be *joint-bordant* if there is a map $F : V \to W$ extending $f_0 \cup f_1$ where V and W are compact manifolds with $\partial V = M_0 \cup M_1$ and $\partial W = N_0 \cup N_1$. Joint-bordism classes of maps of dimension (n, k) form an abelian group under the disjoint union which we denote by $M(n, k)$. It is well known that Stiefel-Whitney numbers form a complete system of invariants for the joint-bordism theory [**9**]. So one may hope to characterize maps joint-bordant to immersions or embeddings in terms of these numbers whenever $k > 0$. For the case of embeddings this has already been settled by Brown [**3**]; his proof is based on a construction suggested by Stong. In this note, using the *model construction* of Koschorke [**6**], we shall give such a criterion for maps joint-bordant to immersions in the "metastable" range $n \leq 2k$.

Our method of proof can also be applied to study immersions up to various oriented joint-bordism relations. These are naturally defined for the following restricted classes of maps (see Stong [**9**]):

C_1: maps with oriented source manifolds;
C_2: maps with oriented target manifolds;
C_3: maps with oriented stable normal bundles;
C_4: maps with oriented source and target manifolds.

In fact, let $M_r(n,k)$ be the resulting oriented joint-bordism group of maps of dimension (n,k) belonging to C_r and let $\rho_r : M_r^{(n,k)} \to M(n,k)$ be the natural forgetful homomorphism. Then $x \in M_r(n,k)$ contains an immersion if $\rho_r(x)$ does provided $n \leq 2k$.

We now summarize our main results by the following

MAIN THEOREM. *Let $n \leq 2k$. Then under all five orientedness assumptions above a map $f : M \to N$ of dimension (n,k) is joint-bordant to an immersion if and only if*

$$w_m(f)f^*w_{\mu_1}(N)w_{\mu_2}(M) \cap [M] = 0$$

for all $m > k$ and all partions μ_1, μ_2 with $|\mu_1| + |\mu_2| = n - m$. Here $w(f) = f^*w(N)\overline{w}(M)$ is the total Stiefel-Whitney class of the stable normal bundle of f and $[M]$ is the $\mathbf{Z}_2$-fundamental class of M.

REMARK. Stong [9] showed that the unoriented joint-bordism class of f is completely determined by Stiefel-Whitney numbers of the form

$$w_{\mu_1}(N)f_!w_{\mu_2}(M)\cdots f_!w_{\mu_l}(M) \cap [N]$$

which, in the case of $l > 1$, are equal to

$$(*) \qquad f^*w_{\mu_1}(N)w_{\mu_2}(M)f^*f_!w_{\mu_3}(M)\cdots f^*f_!w_{\mu_l}(M) \cap [M].$$

Here μ_i are partitions with $\sum |\mu_i| = n-(l-2)k$ and $f_! : H^*(M, \mathbf{Z}_2) \to H^{*+k}(N, \mathbf{Z}_2)$ is the *Umkehr* homomorphism defined by taking a cohomology class x into the Poincaré dual of $f_*(x \cap [M])$. In the case of $n \leq 2k$ the $f^*f_!w_{\mu_i}(M)$ factors above disappear if $|\mu_1| + |\mu_2| > k$. So our main theorem is equivalent to saying that: "if $n \leq 2k$ then under all five orientedness assumptions above f is joint-bordant to an immersion if and only if all Stiefel-Whitney numbers of the form (*) involving $w_m(f)$ where $m > k$ are zero." I do not know whether this statement remains true if $n > 2k$.

REMARK. It was conjectured by Olk [8] that in a certain "metastable" range (probably $n \leq 2k - 1$) a closed n-manifold M can be immersed into $\mathbb{R}^{n+k}$ up to bordism, or equivalently, a map $f : M \to \mathbb{S}^{n+k}$ is bordant in the sense of Atiyah [1] to an immersion if and only if

$$\overline{w}_m(M)w_\mu(M) \cap [M] = 0$$

for all $m > k$ and all partions μ of $n - m$. By our results, these numbers are zero if and only if f is joint-bordant in $M_2(n,k)$ (even in $M_4(n,k)$ if M is oriented) to an immersion provided $n \leq 2k$. Olk [**8**] showed that the above statement is always true if $k \geq n - 7$. So in these cases "joint-bordant to an immersion" and "bordant to an immersion" are the same for a map into a sphere.

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2. The unoriented case. We start by describing the behavior of a generic map around its singularity subset. Recall that a map is *generic* if its self-intersections are transversal and its jet sections are transversal to the Boardman manifolds [**2**]. Given a generic map $f : M \to N$ of dimension (n,k), we denote by Σ_f the set of singular points of f and by $\overline{\Sigma}_f = f(\Sigma_f)$ its image. Hereafter we always assume $k > 0$. In the case of $n \leq 2k+2$, Σ_f is a closed submanifold of dimension $n-k-1$ containing only points over which df is of rank $n-1$, f is an embedding restricted to Σ_f and the bordism class of Σ_f depends only on the joint-bordism class of f.

Let us denote by Ker and Coker the 1- and $(k+1)$-dimensional vector bundles over Σ_f and $\overline{\Sigma}_f$ which are determined by kernels and cokernels of $df|\Sigma_f$ respectively. By Koschorke [**6**] the normal bundle ν of Σ_f in M is canonically isomorphic to $\mathrm{Ker} \otimes \mathrm{Coker}$. Here and henceforth we shall always omit the pull-backs of vector bundles by $f|\Sigma_f$. Now let $\overline{\nu}$ be the normal bundle of $\overline{\Sigma}_f$ in N and let $\overline{f} : E\nu \to E\overline{\nu}$ be the map between total spaces of normal bundles determined by the k-morphism $\nu \to \overline{\nu}$ induced by df. Then by applying the model construction given in Section 1 of Koschorke [**6**] to the bundle homomorphism $df|\Sigma_f$ of constant rank $n-1$, we obtain a nondegenerate $(n-1)$-morphism $T(E\nu) \to T(E\overline{\nu})$ covering $\overline{f}$ with Σ_f its singularity subset, and hence a generic map

$$\alpha_f : E\nu \to E\overline{\nu}$$

with Σ_f its singularity subset by Feit [**5**]. Moreover, suitably identifying $E\nu$ and $E\overline{\nu}$ with certain tubular neighborhoods of Σ_f in M

and of $\overline{\Sigma}_f$ in N respectively, we can identify α_f with f restricted to a small tubular neighborhood of Σ_f in M (see [**6**]).

Now let $x \in M(n,k)$ be represented by a generic map $f : M \to N$ and let $n \leq 2k+2$. We denote by

$$\beta_f : \Sigma_f \to BO_{k+1} \times \mathbb{P}^\infty$$

the map defined by the classifying maps of Coker and Ker. The bordism obstruction $\Phi(x)$ is defined to be the unoriented bordism class of β_f. It is readily seen that

$$\Phi : M(n,k) \to \mathfrak{R}_{n-k-1}(BO_{k+1} \times \mathbb{P}^\infty)$$

is a well defined homomorphism.

THEOREM . *If* $n \leq 2k$ *then* $x \in M(n,k)$ *contains an immersion if and only if* $\Phi(x) = 0$.

Proof. If x contains an immersion then clearly $\Phi(x) = 0$. To prove the contrary let $f : M \to N$ be a generic map of dimension (n,k) so that β_f is null-bordant. We have to show that f is joint-bordant to an immersion.

In the dimensional range $n \leq 2k+1$ there is a k-dimensional vector bundle ν_0 over Σ_f so that $\nu \cong \nu_0 \oplus \text{Ker}$. In this case it is easy to see that $\overline{\nu} \cong \nu_0 \oplus \text{Coker}$ and that the k-morphism $\nu \to \overline{\nu}$ induced by df then corresponds to the k-morphism

$$\nu_0 \oplus \text{Ker} \to \nu_0 \oplus \text{Coker}$$

defined by $a \oplus b \to a$. Let us denote by λ and γ the canonical vector bundles over $\mathbb{P}^\infty$ and BO_{k+1} respectively, and by $G : \Sigma \to BO_{k+1} \times \mathbb{P}^\infty$ a null-bordism of β_f. For dimensional reasons ν_0, which is stably isomorphic to $G^*\lambda \otimes G^*\gamma - G^*\lambda$ restricted over Σ_f, can be extended to a k-dimensional vector bundle η_0 over Σ so that $G^*\lambda \otimes G^*\gamma \cong \eta_0 \oplus G^*\lambda$. Now let $\eta = G^*\lambda \otimes G^*\gamma$ and let $\overline{\eta} = \eta_0 \oplus G^*\gamma$. Then η and $\overline{\eta}$ restrict to ν and $\overline{\nu}$ over Σ_f respectively, and the k-morphism

$$\eta \cong \eta_0 \oplus G^*\lambda \to \eta_0 \oplus G^*\gamma \cong \overline{\eta}$$

defined by $a \oplus b \to a$ restricts to the k-morphism $\nu \to \overline{\nu}$ induced by df. Applying the model construction of Koschorke to the obvious bundle homomorphism $\eta \oplus T\Sigma_f \to \overline{\eta} \oplus T\Sigma_f$ of constant rank $n-1$, we obtain a generic map

$$F : E\eta \to E\overline{\eta}$$

with Σ_f its singularity subset, which, after identifying $\Sigma_f \subset E\overline{\eta}$ with $\overline{\Sigma}_f$ via f, is readily seen to be an extension of the map α_f.

Now let M_1 be defined by glueing $M - \text{int}T$ and a sphere bundle $S\eta$ of η along the boundaries via a diffeomorphism between ∂T and $S\nu$ where T is a small tubular neighborhood of Σ_f in M, and let N_1 be defined similarly. Then M_1 and N_1 are closed smooth manifolds after straightening possible angles. Moreover, if the above constructions are suitably done, then f and F can be fitted to yield an immersion $f_1 : M_1 \to N_1$ which is readily seen to be joint-bordant to f. This completes the proof of the theorem. □

We now calculate the Stiefel-Whitney numbers of the map β_f : $\Sigma_f \to BO_{k+1} \times \mathbb{P}^\infty$ associated to a given generic map $f : M \to N$ of dimension (n, k). By definition the Stiefel-Whitney numbers of β_f take the form

$$w_{\mu_1}(\text{Ker})w_{\mu_2}(\text{Coker})w_{\mu_3}(\Sigma_f) \cap [\Sigma_f]$$

where μ_i are partitions with $\sum |\mu_i| = n - k - 1$. We have the following simple relations among total Stiefel-Whitney classes:

$$w(\text{Ker})w(\text{Coker}) = w(f)|_{\Sigma_f}$$

$$w(\text{Ker} \otimes \text{Coker})w(\Sigma_f) = w(M)|_{\Sigma_f}.$$

It follows that every Stiefel-Whitney number of β_f is the sum of numbers of the form

$$w_1(\text{Ker})^m(f^*w_{\mu_1'}(N)w_{\mu_2'}(M))|_{\Sigma_f} \cap [\Sigma_f]$$

which by 9.11 of Koschorke [**6**] are equal to

$$w_{m+k+1}(f)f^*w_{\mu_1'}(N)w_{\mu_2'}(M) \cap [M].$$

This proves our main theorem for the unoriented case.

3. Oriented cases. We now extend the argument above to study various oriented cases. For our purpose we shall need the notion of *oriented bordism group with coefficients*. Let ϕ be a vector bundle over X, we denote by $\overline{\Omega}_n(X, \phi)$ the bordism group of triples (M, f, or) where M is a closed n-manifold, $f : M \to X$ is a map and or is an isomorphism between orientation line bundles of TM and $f^*\phi$. This kind of bordism groups was first studied by Atiyah [**1**] and then by Koschorke [**6**]. Our notational convention follows from that of Koschorke. It should be noticed that $\overline{\Omega}_*(X, \phi) \cong \Omega_*(X)$ if ϕ is orientable and that

$$\overline{\Omega}_*(X \times \mathbb{P}^\infty, \lambda + \phi) \cong \mathfrak{R}_*(X).$$

The bordism group $\overline{\Omega}_*(X, \phi)$ has several properties analogous to the usual oriented bordism group, including a generalization of Rochlin's theorem given in [**4**].

Similar to the unoriented case, if $n \leq 2k+2$ then for each $1 \leq r \leq 4$ we can define a homomorphism $\phi_r : M_r(n, k) \to B_r$ where

$$\begin{aligned} B_1 &= \overline{\Omega}_{n-k-1}(BO_{k+1} \times \mathbb{P}^\infty, \lambda \otimes \gamma_{k+1}), \\ B_2 &= \overline{\Omega}_{n-k-1}(BO_{k+1} \times \mathbb{P}^\infty, \lambda \otimes \gamma_{k+1} + \gamma_{k+1} - \lambda), \\ B_3 &= \mathfrak{R}_{n-k-1}(BSO_l \times \mathbb{P}^\infty), \quad l \gg 0, \\ B_4 &= \overline{\Omega}_{n-k-1}(BSO_l \times \mathbb{P}^\infty, \lambda \otimes (\tilde{\gamma}_l - \mathbb{R}^{l-k-1})), \quad l \gg 0, \end{aligned}$$

γ_{k+1} and $\tilde{\gamma}_l$ are canonical vector bundles over BO_{k+1} and BSO_l respectively. To see this notice that if f is a generic map belonging to the class C_r then the map β_f defined in Section 1 also represents a bordism class of B_r. For example, if $r = 3$ then in the definition of β_f the bundle $\text{Coker} - \text{Ker}$ can be canonically oriented. Let $\tau : BO_{k+1} \times \mathbb{P}^\infty \to BO_l \times \mathbb{P}^\infty$ be defined by the classifying map of $\gamma_{k+1} - \lambda$ and λ. Then $\tau \circ \beta_f$ can be lifted to a map into $BSO_l \times \mathbb{P}^\infty$ which represents a bordism class of B_3.

Now, a similar argument as in the unoriented case leads to the following

THEOREM . *If $n \leq 2k$ then for each $1 \leq r \leq 4$ a joint-bordism class $x \in M_r(n, k)$ contains an immersion if and only if $\Phi_r(x) = 0$.*

REMARK. In [**7**] we defined a normal bordism obstruction for a map to be bordant in the sense of Atiyah to an immersion as well as its various oriented versions. Since the invariants Φ_r here are images under certain forgetful homomorphisms and projections of our invariants in [**7**], the images of Φ_r contain only 2-primary torsion elements. Later we will see that all these elements are in fact of order 2.

We now proceed to prove our main theorem for the remaining four oriented cases. In what follows, when l is large enough, $B(S)O_l$ and γ_l will be simply denoted by $B(S)O$ and γ respectively.

If $r = 1$ and k is even, or if $r = 2$ and k is odd, or if $r = 3$, or if $r = 4$ and k is even, then B_r is canonically isomorphic to $\mathfrak{R}_{n-k-1}(B(S)O)$ or to $\mathfrak{R}_{n-k-1}(BSO \times \mathbb{P}^\infty)$. In these cases, our result follows by calculating characteristic numbers as in the unoriented case.

If $r = 2$ and k is even, or if $r = 4$ and k is odd, then B_r is canonically isomorphic to $\Omega_{n-k-1}(B(S)O \times \mathbb{P}^\infty)$ which contains only $\mathbb{Z}$ and $\mathbb{Z}_2$ factors. It follows that $\Phi_r(x)$ is an element of order 2 and hence is determined by its Stiefel-Whitney numbers (see [**4**]). In these cases, our result can again be proved by calculating characteristic numbers.

It remains to consider the case when $r = 1$ and k is odd. Since $B_1 \cong \overline{\Omega}_{n-k-1}(BO \times \mathbb{P}^\infty, \gamma)$ if k is odd and since $\Phi_1(x)$ is a 2-primary torsion by our last remark, it suffices to show that the natural forgetful homomorphism

$$\rho : \overline{\Omega}_*(BO \times \mathbb{P}^\infty, \gamma) \to \mathfrak{R}_*(BO \times \mathbb{P}^\infty)$$

is injective restricted to the 2-primary torsion subgroup of $\overline{\Omega}_*(BO \times \mathbb{P}^\infty, \gamma)$. For this purpose we fit ρ into the following commutative diagram

$$\begin{array}{ccccc}
\Omega_*(BSO \times \mathbb{P}^\infty) & \xrightarrow{\pi_*} & \overline{\Omega}_*(BO \times \mathbb{P}^\infty, \gamma) & \longrightarrow & \Omega_{*-1}(BO \times \mathbb{P}^\infty) \\
 & \searrow^{\overline{e}} & \downarrow e & & \downarrow i_* \\
 & & \mathfrak{R}_*(BO \times \mathbb{P}^\infty) & \longrightarrow & \Omega_{*-1}(BO \times \mathbb{P}^\infty \times \mathbb{P}^\infty)
\end{array}$$

where $\overline{\rho}$ is the natural forgetful homomorphism, i_* is the injection induced by the natural inclusion and π_* is induced by the projection. The top line is the exact Gysin sequence corresponding to the orientation line bundle of γ and the lower line is defined similarly by identifying $\mathfrak{R}_*(BO \times \mathbb{P}^\infty)$ with $\overline{\Omega}_*(BO \times \mathbb{P}^\infty \times \mathbb{P}^\infty, \lambda + \gamma)$ (see 9.21(ii) of [**6**]).

Now let $x \in \overline{\Omega}_*(BO \times \mathbb{P}^\infty, \gamma)$ be a 2-primary torsion so that $\rho(x) = 0$. To prove $x = 0$ we first conclude from the exactness of the top line that there exists $y \in \Omega_*(BSO \times \mathbb{P}^\infty)$ with $\pi_*(y) = x$. Since $\overline{\rho}$ is injective restricted to the 2-primary torsion part of $\Omega_*(BSO \times \mathbb{P}^\infty)$, it suffices to show that y is a 2-primary torsion. For this, let us consider the homomorphism

$$d : \overline{\Omega}_*(BO \times \mathbb{P}^\infty, \gamma) \longrightarrow \Omega_*(BSO \times \mathbb{P}^\infty)$$

defined by taking double covers. By definition $d \circ \pi_* = id + T$ where T is the involution on $\Omega_*(BSO \times \mathbb{P}^\infty)$ induced by t and t is the involution on BSO defined by interchanging the two sheets. Since the involution on $H_*(BSO, \mathbb{Z})$ induced by t is the identity, so is the involution T. It follows that $2y = d \circ \pi_*(y) = d(x)$, and hence y itself, is a 2-primary torsion as desired.

This completes the proof of our main theorem for the oriented cases.

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