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THE NONHOMOGENEOUS MINIMAL SURFACE EQUATION INVOLVING A MEASURE

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We find existence of a minimum in BV for the variational problem associated with $\operatorname{div} A(Du) + \mu = 0$, where A is a mean curvature type operator and μ a nonnegative measure satisfying a suitable growth condition. We then show a local L^∞ estimate for the minimum. A similar local L^∞ estimate is shown for sub-solutions that are Sobolev rather than BV.

1. Introduction. In this paper we initiate an investigation of weak solutions of the

$$\operatorname{div} A(Du) + \mu = 0 \tag{1.1}$$

in a bounded Lipschitz domain $\Omega \subset R^n$. Here A is a function for which the mean curvature operator is a prototype and μ is a nonnegative Radon measure supported in Ω that satisfies

$$\mu(B(r)) \leq Mr^{q(n-1)} \text{ for all } B(r) \subset \Omega, \tag{1.2}$$

where $M > 0$ and $1 < q \leq \frac{n}{n-1}$.

This paper has its origins in the work of [**LS**] where it was shown that if u is a weak solution of

$$\Delta u = \mu,$$

where μ is a measure that satisfies the growth condition

$$\mu(B(r)) \leq Mr^{n-2+\varepsilon}$$

for some $\varepsilon > 0$ and for all balls $B(r)$ of radius r, then u is Hölder continuous. In [**RZ**] this result was generalized to equations of the form

$$\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) + \mu = 0 \tag{1.3}$$

where μ is a nonnegative Radon measure and A and B are Borel measurable functions satisfying structural conditions that allow, for example, the p-Laplacian. It is shown that if u is a Hölder continuous solution of 1.3, then μ satisfies

$$\mu(B(r)) \leq Mr^{n-p+\varepsilon}$$

for some $\varepsilon > 0$. Under further restrictions on the structural conditions, it was shown this growth condition on μ was sufficient for Hölder continuity of u.

Recently, Lieberman [**L**] improved the results in [**RZ**] by proving supremum inequalities for solutions of 1.3 without the restrictive structural conditions, thereby establishing necessary and sufficient conditions on the growth of μ for the Hölder continuity of solutions.

All of this analysis takes place in the framework surrounding the p-Laplacian, $p > 1$. It is our purpose to address the situation of $p = 1$. We first consider the question of existence of solutions of 1.5 in the case A is the mean curvature operator. We establish a variational solution by minimizing

$$\int_\Omega \sqrt{1 + |\nabla u|^2}\, dx + \int_\Omega u\, d\mu \tag{1.4}$$

in the class $u \in BV(\Omega)$ where u satisfies the Dirichlet condition $u^* = f$ on $\partial\Omega$, with f an integrable function on $\partial\Omega$. In order to ensure the existence of a minimum, it is necessary to assume that the constant M in 1.2 is chosen sufficiently small. This is analogous to the assumption made in [**M**], in which μ is taken as a bounded measurable function. We then show that the minimizer u is bounded. In this context, it is not possible to utilize the argument given in [**L**] to obtain an L^∞ bound since there is no variational equation associated with 1.4 . Rather, we employ a technique used in [**RZ**] modeled on the method of DeGiorgi.

Next, we investigate an equation which contains the formal Euler-Lagrange equation of 1.4. Thus, we consider a weak solution $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ of the equation

$$\operatorname{div} A(Du) + \mu = 0 \tag{1.5}$$

where we assume there exist non-negative constants a_1, a_2 such that

$$p \cdot A(p) \geq |p| - a_1 \tag{1.6}$$

and

$$|A(p)| \le a_2. \tag{1.7}$$

It is assumed that μ is a nonnegative Radon measure supported in the bounded domain Ω and satisfies 1.2. We show that if $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ is a weak solution of 1.5, then $|u|$ is bounded by the L^1-norm of u with respect to the measure $d\nu = dx + d\mu$. Specifically, we show that u satisfies a supremum inequality, 6.4. The proof of this follows the proof in the corresponding result of [**L**]. The method of DeGiorgi will still work in this case, however the Moser iteration method used in [**L**] gives a slightly different result and is included for this reason. It is well known that weak solutions of 1.5 are not necessarily continuous, even under the assumption that μ is an absolutely continuous measure with bounded density (c.f. [**M**]). Therefore, it is not possible to obtain the weak Harnack inequality involving a lower bound for the solution.

The results of this paper are valid for equations with a more general structure. For the sake of simplicity, we employ this simple structure which fully illustrates the method. In a forthcoming paper, we will address the question of regularity of solutions of 1.4 in which almost everywhere continuity is established. The existence of an *a priori* L^∞ bound will be essential in this future investigation.

2. Preliminaries. Throughout, we assume that Ω is a bounded Lipschitz domain in R^n. The space $W^{1,1}(\Omega)$ is the space of $L^1(\Omega)$ functions whose distributional derivatives also lie in $L^1(\Omega)$.

The class of all functions in $L^1(\Omega)$ whose distributional partial derivatives are measures with finite total variation in Ω comprise the space $BV(\Omega)$. The notation

$$\int_\Omega |Du|\, dx$$

will be used to represent the total variation of the vector-valued measure, Du, the gradient of u. Specifically, the total variation of Du is

$$\sup\left\{\int_\Omega u \operatorname{div} v\, dx \;:\; v = (v_1, \dots, v_n) \in C_0^\infty(\Omega; R^n),\ |v| \le 1\right\}.$$

We also make the notational definition

$$\int_\Omega \sqrt{1+|Du|^2}\,dx$$
$$= \sup\left\{ \int_\Omega (f \operatorname{div} v + v_0)\, dx \;:\; v=(v_1,\dots,v_n)\in C_0^\infty(\Omega), \right.$$
$$\left. v_0 \in C_0^\infty(\Omega),\ |v|^2+|v_0|^2 \le 1 \right\}.$$

The space $BV(\Omega)$ is equipped with the norm

$$\|u\|_{BV} = \int_\Omega |u|\, dx + \int_\Omega |Du|\, dx.$$

The trace of u on $\partial\Omega$ is denoted by u^* (c.f. [**Z**, Section 5.10]). We will make use of the following lemma on the convergence of the traces of BV functions.

LEMMA 2.1. *Let $\Omega \subset R^n$ a bounded Lipschitz domain, and let $\{u_k\}$, u in $BV(\Omega)$ with*

$$\lim_{k\to\infty} \int_\Omega |u_k - u|\, dx = 0$$
$$\lim_{k\to\infty} \int_\Omega \sqrt{1+|Du_k|^2}\,dx = \int_\Omega \sqrt{1+|Du|^2}\,dx.$$

Then

$$\lim_{k\to\infty} \int_{\partial\Omega} |u_k^* - u^*|\, dH^{n-1} = 0,$$

with H^{n-1} the $n-1$ dimensional Hausdorff measure.

The proof follows directly from the proof in [**G**, Proposition 2.6; p.34].

We will also have need for the following compactness result for BV functions [**Z**, Corollary 5.3.4; p. 227].

THEOREM 2.2. *Let $\Omega \in R^n$ be a bounded Lipschitz domain. Then $BV(\Omega) \cap \{u : \|u\|_{BV(\Omega)} \le 1\}$ is compact in $L^1(\Omega)$.*

It was shown in [**MZ**] that if μ satisfies the growth condition $\mu(B(r)) \le Mr^{n-1}$ on all balls $B(r)$ (and therefore condition 1.2 in particular), then μ can be identified with an element of the dual of $BV(\Omega)$. Furthermore, its norm

$$\tilde{M} = \|\mu\| = \sup\left\{\int_\Omega u\,d\mu \;:\; \|u\|_{BV(\Omega)} \le 1\right\}$$

is comparable to M. Thus,

$$\begin{aligned}(2.1)\qquad \left|\int_\Omega u\,d\mu\right| &\le \int_\Omega |u|\; d\mu \\ &\le \|\mu\|\;\|u\|_{BV(\Omega)} \\ &\le \tilde{M}\;\|u\|_{BV(\Omega)}\end{aligned}$$

The following well known result, [**M**], will be used in the existence proof below.

$$(2.2)\qquad \int_\Omega |u|\; dx \le C\left(\int_\Omega |Du|\; dx + \int_{\partial\Omega} u^*\, dH^{n-1}\right)$$

with the constant $C = C(\Omega)$. This yields

$$(2.3)\qquad \|u\|_{BV(\Omega)} \le C\left(\int_\Omega |Du|\; dx + \int_{\partial\Omega} u^*\, dH^{n-1}\right)$$

Finally, we state the following Sobolev inequalities which are of critical importance in our development.

THEOREM 2.3. *Let Ω be a bounded Lipschitz domain and suppose μ is a measure supported in Ω satisfying condition 1.2. Then there exists a constant $C = C(\Omega, q, n)$ such that*

$$(2.4)\qquad \left(\int_\Omega u^q d\mu\right)^{1/q} \le CM^{1/q}\int_\Omega |Du|\; dx$$

whenever $u \in BV(\Omega)$ with compact support in Ω.

The proof may be found in [**Z**, Lemma 4.9.1; p. 209]. Also needed is the standard Sobolev inequality for $W^{1,1}$.

If $u \in W_0^{1,1}(\Omega)$ then there exists a constant $C = C(\Omega, q, n)$ such that

$$(2.5)\qquad \left(\int_\Omega u^q dx\right)^{1/q} \le C\,\|Du\|_1\,.$$

This is simply the above lemma in the special case that μ is Lebesgue measure.

3. Existence of a Minimum. With Ω a bounded Lipschitz domain and $f \in L^1(\partial\Omega)$, we define $I(u;\Omega)$ as follows,

$$I(u;\Omega) = \int_\Omega \sqrt{1+|Du|^2}\,dx + \int_\Omega u\,d\mu + \int_{\partial\Omega} |u^* - f|\;dH^{n-1}.$$

We wish to minimize I over all $u \in BV(\Omega)$. That is, we wish to find a function $u \in BV(\Omega)$ such that

$$I(u;\text{supp }\varphi) \le I(u+\varphi;\text{supp }\varphi),\ \forall\, \varphi \in C_0^\infty(\Omega).$$

THEOREM 3.1. *Let Ω be a bounded Lipschitz domain. With I defined as above, there exists $u \in BV(\Omega)$ such that*

$$I(u;\Omega) = \min_{v\in BV(\Omega)} I(v;\Omega).$$

Proof. Following [**G**, Section 14.4], the first step is to consider a slightly different Dirichlet problem in the complement of Ω. For this purpose, let B be a ball that contains $\overline{\Omega}$, the closure of Ω. Use Theorem 2.16 of [**G**] to extend f to a $W^{1,1}$ function in $B-\overline{\Omega}$ that will still be denoted by f. Let

$$J(u;B) = \int_B \sqrt{1+|Du|^2} + \int_B u\,d\mu.$$

Note that since $\text{supp}\,\mu \subset \Omega$, the second integral could have been taken over Ω. We wish to show that there exists $u \in BV(B)$, coinciding with f in $B-\overline{\Omega}$, that minimizes $J(u;B)$. We proceed by showing that J is bounded below if the constant M in 1.2 is sufficiently small.

$$\begin{aligned}
J(u;B) &\ge \int_B |Du|\;dx + \int_\Omega u\,d\mu \\
\text{(by 2.1)} \qquad &\ge \int_B |Du|\;dx - \tilde{M}\,\|u\|_{BV(\Omega)} \\
&\ge \int_B |Du|\;dx - \tilde{M}\Big(C\int_{\partial\Omega} u_\Omega^*\,dH^{n-1} \\
\text{(by 2.3)} \qquad &\qquad + (C+1)\int_\Omega |Du|\;dx\Big) \\
&\ge \frac{1}{2}\int_B |Du|\;dx - \tilde{M}C\int_{\partial\Omega} f\,dH^{n-1}.
\end{aligned}$$

The last inequality is obtained when $\tilde{M}$ is small enough to insure $1 - \tilde{M}(C+1) \geq \frac{1}{2}$.

Let $J(u_k) \to \lambda$ a minimum of J. We wish to find $u \in BV(B)$ such that $J(u; B) = \lambda$. For sufficiently large k we obtain from the above inequality that

$$\lambda + 1 \geq \frac{1}{2} \int_B |Du_k| \, dx - MC \int_\Omega f \, dH^{n-1}.$$

Thus the terms $\int_B |Du_k| \, dx$ are uniformly bounded, which implies by 2.3 and Theorem 2.2 that there exists $u \in BV(B)$ with $u_k \to u$ in $L^1(B)$. The gradient is lower semi-continuous with respect to $L^1(B)$ convergence so that

$$\liminf_{k\to\infty} \int_B \sqrt{1 + |Du_k|^2} \, dx \geq \int_B \sqrt{1 + |Du|^2} \, dx.$$

From Theorem 2.3, the uniform bound on $\int_B |Du_k| \, dx$ also implies that the terms

$$\left(\int_\Omega {u_k}^q \, d\mu \right)^{1/q}$$

are uniformly bounded. Thus there exists a subsequence, denote it by $\{u_k\}$, that converges weakly in $L^q(\Omega; \mu)$ to some $w \in L^q(\Omega; \mu)$. The Banach–Saks Theorem implies that there exists a subsequence of $\{u_k\}$, again denote it by $\{u_k\}$, such that the sequence of Césaro sums, $\{v_k\}$, defined by

$$v_k = \frac{u_1 + \cdots + u_k}{k}$$

converges strongly to w in $L^q(\Omega; \mu)$. Moreover, the sequence v_k also converges strongly to u in $L^1(\Omega)$. This can be seen as follows: choose $\varepsilon > 0$ and let N denote an integer for which $\|u_j - u\|_{L^1(\Omega)} < \varepsilon$ for $j, k \geq N$. Then for $j \leq k$,

$$\begin{aligned}
\|v_k - u\| &= \left\| \frac{(u_1 - u) + \cdots + (u_k - u)}{k} \right\| \\
&\leq \frac{\|u_1 - u\| + \cdots + \|u_{j-1} - u\|}{k} + \frac{\|u_j - u\| + \cdots + \|u_k - u\|}{k} \\
&\leq \frac{\|u_1 - u\| + \cdots + \|u_{j-1} - u\|}{k} + \frac{(k - j + 1)\varepsilon}{k}.
\end{aligned}$$

Thus,

$$\limsup_{k\to\infty} \|v_k - u\| \le \varepsilon,$$

which yields the desired result since ε is arbitrary. To show that $w = u$ almost everywhere in Ω note that the strong convergence of $\{v_k\}$ to w in $L^q(\Omega;\mu)$ implies the existence of a subsequence that converges pointwise to w μ-almost everywhere and therefore (Lebesgue) almost everywhere, since Lebesgue measure is absolutely continuous with respect to μ in Ω. But the strong convergence of $\{v_k\}$ to u in $L^1(\Omega)$ implies the almost everywhere pointwise convergence of a further subsequence to u in Ω. Hence, $u = w$ almost everywhere in Ω.

Since u_k converges weakly to u in $L^q(\Omega;\mu)$, the lower semicontinuity of the gradient with respect to $L^1(\Omega)$ convergence implies

$$\lambda = \liminf_{k\to\infty} J(u_k;B) \ge J(u;B). \tag{3.1}$$

Since u_k agrees almost everywhere with f in $B - \overline{\Omega}$, it follows that $u = f$ a.e. in $B - \overline{\Omega}$, thus showing that $J(u;B) \ge \lambda$. This completes the first step.

We now proceed with the second and final step of the proof. For each function $v \in BV(\Omega)$, define

$$v_f(x) = \begin{cases} v(x) & x \in \Omega \\ f(x) & x \in B - \Omega \end{cases}$$

Then $v_f \in BV(B)$ and by (2.15) of [**G**],

$$\begin{aligned} \int_B \sqrt{1+|Dv_f|^2}\,dx &+ \int_B v_f\,d\mu \\ &= \int_B \sqrt{1+|Dv|^2}\,dx + \int_{B-\overline{\Omega}} \sqrt{1+|Df|^2}\,dx \\ &\quad + \int_B v_f\,d\mu + \int_{\partial\Omega} |v_\Omega^* - f|\;dH^{n-1} \\ &= I(v;\Omega) + \int_{B-\overline{\Omega}} \sqrt{1+|Df|^2}\,dx \end{aligned}$$

That is,

$$J(v_f;B) = I(v;\Omega) + \int_{B-\overline{\Omega}} \sqrt{1+|Df|^2}\,dx.$$

Thus, a minimizer of $J(v;B)$ with $v = f$ on $B - \overline{\Omega}$ produces a minimizer of $I(v;\Omega)$. □

4. An energy inequality. Now that we have obtained existence of a solution $u \in BV(\Omega)$ to 1.4, we will show that u is bounded. Before doing this we will obtain an energy estimate to be used in the DeGiorgi type argument of section 5.

Let B_R denote the ball of radius R in R^n. Let η be a cutoff function, $\eta = 1$ on B_r, $0 < r < r^* \leq R$, $\eta = 0$ on ∂B_{r^*} with $0 \leq \eta \leq 1$ on B_{r^*} and $|D\eta| \leq \frac{2}{r^*-r}$. Let $\varphi = -\eta(u-k)^+$, then supp $\varphi = A_k = \{u > k\} \cap B_{r^*}$ and

$$I(u; A_k) \leq I(u+\varphi; A_k) \tag{4.1}$$

Using

$$\int_{A_k} |Du|\, dx \leq \int_{A_k} \sqrt{1+|Du|^2}\, dx \leq \int_{A_k} |Du| + 1\, dx \tag{4.2}$$

and that on A_k

$$D(u+\varphi) = (1-\eta)D(u-k)^+ - D\eta(u-k)^+,$$

we obtain from 4.1

$$\begin{aligned}\int_{A_k} \left|D(u-k)^+\right| dx \leq \int_{A_k} (1-\eta)\left|D(u-k)^+\right| dx \\ + \frac{2}{r^*-r}\int_{A_k} \left|(u-k)^+\right| dx \\ + \int_{A_k} \eta \left|(u-k)^+\right| d\mu + |A_k|\end{aligned}$$

where $|A_k|$ is the Lebesgue measure of A_k. This immediately implies

(4.3)
$$\begin{aligned}\int_{B_r} \left|D(u-k)^+\right| dx &\leq \int_{B_{r^*}} \eta\left|D(u-k)^+\right| dx \\ &\leq \frac{2}{r^*-r}\int_{B_{r^*}} \left|(u-k)^+\right| dx \\ &\quad + \int_{B_{r^*}} \left|(u-k)^+\right| d\mu + |A_k|.\end{aligned}$$

5. Supremum estimate for variational solutions.

THEOREM 5.1. *Let $\sigma \in (0,1)$, Ω a bounded Lipschitz domain, and $B_R \subset \Omega$ with $R < 1$. Then for $u \in BV(\Omega)$ a minimum of I there exists a constant $C = C(\sigma, M)$ such that*

$$\sup_{B_{\sigma R}} u \le C\left(R^{-n}\int_{B_R} u^+\,dx + R^{-q(n-1)}\int_{B_R} u^+\,d\mu\right)$$

where q is the constant from 1.2 and u^+ is the positive part of u.

Proof. Let k be a positive constant to be specified later. Set

$$k_i = k(1-2^{-i}),\ r_i = \sigma R + 2^{-i}R(1-\sigma),$$
$$\text{and } \tilde{r}_i = \frac{1}{2}(r_i + r_{i+1}).$$

For notational convenience, denote by B_i the ball of radius r_i, $\tilde{B}_i$ the ball of radius $\tilde{r}_i$, and let

$$A_i = B_i \cap \left\{(u-k_{i+1})^+ > 0\right\}.$$

Note that $B_{i+1} \subset \tilde{B}_i \subset B_i$. Also, for all j we will use the notation

$$\fint_{B_j} dx = R^{-n}\int_{B_j} dx \qquad \text{and} \qquad \fint_{B_j} d\mu = R^{-q(n-1)}\int_{B_j} d\mu.$$

Let φ_i be the cutoff functions on $\tilde{B}_i$ so that $\varphi_i \equiv 1$ on B_{i+1} and

$$|D\varphi_i| \le \frac{2}{\tilde{r}_i - r_{i+1}} = \frac{2^{i+3}}{R(1-\sigma)}. \tag{5.1}$$

Then 4.3 implies

$$\begin{aligned}\fint_{B_{i+1}} \left|D(u-k_{i+1})^+\right| dx &\\ \le \frac{2^{i+3}}{R(1-\sigma)}&\fint_{\tilde{B}_i}(u-k_{i+1})^+\,dx \\ &+ R^{-n+q(n-1)}\fint_{\tilde{B}_i}(u-k_{i+1})^+\,d\mu + R^{-n}\left|A_i\right|.\end{aligned} \tag{5.2}$$

Now, by 2.4 and 5.1,

$$
\begin{aligned}
&\fint_{B_{i+1}} (u-k_{i+1})^+\,d\mu \\
&\quad\le \fint_{\tilde B_i} \varphi_i (u-k_{i+1})^+\,d\mu \\
&\quad\le \left(\fint_{\tilde B_i} \left(\varphi_i(u-k_{i+1})^+\right)^q\,d\mu\right)^{1/q} (R^{-q(n-1)}\mu(A_i))^{1-1/q} \\
&\quad\le CM^{1/q}R\fint_{\tilde B_i} \left|D\left(\varphi_i(u-k_{i+1})^+\right)\right|\,dx\,(R^{-q(n-1)}\mu(A_i))^{1-1/q} \\
&\quad\le CRM^{1/q}\left(\fint_{\tilde B_i} \left|D(u-k_{i+1})^+\right|\varphi_i\,dx\right. \\
&\qquad\left. + \fint_{\tilde B_i} (u-k_{i+1})^+\,|D\varphi_i|\,dx\right)(R^{-q(n-1)}\mu(A_i))^{1-1/q} \\
&\quad\le CRM^{1/q}\left(\fint_{\tilde B_i} \left|D\,(u-k_{i+1})^+\right|\,dx\right. \\
&\qquad\left. + \frac{2^{i+3}}{R(1-\sigma)}\fint_{\tilde B_i} (u-k_{i+1})^+\,dx\right)(R^{-q(n-1)}\mu(A_i))^{1-1/q}.
\end{aligned}
$$

Applying 5.2 we have

$$
\begin{aligned}
&\fint_{B_{i+1}} (u-k_{i+1})^+\,d\mu \\
&\quad\le CRM^{1/q}\left(\frac{2^{i+4}}{R(1-\sigma)}\fint_{B_i} (u-k_{i+1})^+\,dx\right. \\
&\qquad + R^{-n+q(n-1)}\fint_{B_i} (u-k_{i+1})^+\,d\mu \\
&\qquad\left. + R^{-n}\,|A_i|\right)(R^{-q(n-1)}\mu(A_i))^{1-1/q}.
\end{aligned}
$$

Thus we have the following iteration inequality,

(5.3)

$$
\begin{aligned}
&\fint_{B_{i+1}} (u-k_{i+1})^+\,d\mu \\
&\quad\le CM^{1/q}\frac{2^{i+4}}{(1-\sigma)}\left(\fint_{B_i} (u-k_i)^+\,dx\right. \\
&\qquad\left. + \fint_{B_i} (u-k_i)^+\,d\mu + R^{-n}\,|A_i|\right)(R^{-q(n-1)}\mu(A_i))^{1-1/q}.
\end{aligned}
$$

To estimate the quantity $\mu(A_i)$ recall that $A_i = \{u > k_{i+1}\} \cap B_i$, and note that

$$\begin{aligned} k_{i+1} - k_i &= k\left(1 - 2^{-(i+1)}\right) - k\left(1 - 2^{-i}\right) \\ &= 2^{-i}k\left(1 - 2^{-1}\right) \\ &= 2^{-(i+1)}k. \end{aligned}$$

which implies

$$2^{-(i+1)}k < u - k_i \text{ on } A_i.$$

Thus

$$\begin{aligned} R^{-q(n-1)}\mu(A_i) &\leq 2^{i+1}k^{-1}\fint_{B_i} (u - k_i)^+ \, d\mu \\ &\leq 2^{i+1}Y_i. \end{aligned} \tag{5.4}$$

where

$$Y_i = k^{-1}\fint_{B_i} (u - k_i)^+ \, dx + k^{-1}\fint_{B_i} (u - k_i)^+ \, d\mu.$$

We estimate $|A_i|$ in the same manner, obtaining

$$R^{-n}|A_i| \leq 2^{i+1}Y_i. \tag{5.5}$$

Using 5.4 and 5.5 in 5.3 we obtain

(5.6)

$$\begin{aligned} &k^{-1}\fint_{B_{i+1}} (u - k_{i+1})^+ \, d\mu \\ &\leq CM^{1/q}\frac{2^{i+4}}{(1-\sigma)}\left(k^{-1}\fint_{B_i} (u - k_i)^+ \, dx \right. \\ &\qquad \left. + k^{-1}\fint_{B_i} (u - k_i)^+ \, d\mu + k^{-1}2^{i+1}Y_i\right)\left(2^{i+1}Y_i\right)^{1-1/q} \\ &\leq CM^{1/q}\frac{2^{i+4}}{(1-\sigma)}\left(\left(1 + k^{-1}2^{i+1}\right)Y_i\right)\left(2^{i+1}Y_i\right)^{1-1/q} \\ &\leq CM^{1/q}\frac{2^{i+4}}{(1-\sigma)}(k^{-1} + 2^{-i-1})\left(2^{i+1}Y_i\right)^{1+\alpha}. \end{aligned}$$

where $\alpha = 1 - 1/q > 0$. Following the same analysis for dx instead of $d\mu$ we obtain

$$(5.7)\quad k^{-1}\fint_{B_{i+1}} (u-k_{i+1})^+\,dx \le CM^{1/q}\frac{2^{i+4}}{(1-\sigma)}(k^{-1}+2^{-i-1})\left(2^{i+1}Y_i\right)^{1+\alpha}.$$

Combining 5.6 and 5.7, we have

$$(5.8)\quad \begin{aligned} Y_{i+1} &\le CM^{1/q}\frac{2^{i+4}}{(1-\sigma)}(k^{-1}+2^{-i-1})\left(2^{i+1}Y_i\right)^{1+\alpha} \\ &\le CM^{1/q}\frac{2^{i+4}}{\kappa(1-\sigma)}\left(2^{i+1}Y_i\right)^{1+\alpha} \end{aligned}$$

where $\kappa = \min(1, 1/(k^{-1}+2^{-1}))$. The recursion lemma of [**LU**, lemma 4.7; p. 66] then implies that $Y_i \to 0$, and thus

$$\sup_{B_{\sigma R}} u \le k,$$

provided that

$$\begin{aligned} Y_0 &= k^{-1}\fint_{B_R} u^+\,dx + k^{-1}\fint_{B_R} u^+\,d\mu \\ &\le \left(CM^{1/q}\frac{2^{5+\alpha}}{\kappa(1-\sigma)}\right)^{-1/\alpha}\left(2^{2+\alpha}\right)^{-1/\alpha^2}. \end{aligned}$$

This is true if

$$\kappa^{1/\alpha}k \ge \left(\frac{CM^{1/q}2^{\alpha+6+2/\alpha}}{(1-\sigma)}\right)^{1/\alpha}\left(\fint_{B_R} u^+\,dx + \fint_{B_R} u^+\,d\mu\right).$$

Since $\kappa^{1/\alpha} \le 1$, the result follows. □

6. A supremum estimate for weak solutions. We will use a different version of the Sobolev inequalities 2.4 and 2.5.

COROLLARY 6.1. *Let B_R a ball of radius R in R^n. Suppose $u \in W_0^{1,1}(B_R)$ and μ is a measure satisfying* 1.2, *then there exists a constant $C = C(q,n)$ such that*

$$(6.1)\quad \left(R^{-q(n-1)}\int_{B_R} u^q\,d\mu\right)^{1/q} \le M^{1/q}CR^{1-n}\int_{B_R}|Du|\,dx$$

and

$$\left(R^{-n}\int_{B_R} u^q\,dx\right)^{1/q} \le CR^{1-n}\int_{B_R}|Du|\,dx. \tag{6.2}$$

Let u^+ denote the positive part of u.

THEOREM 6.2. *Let $B_R \subset R^n$ a ball of radius $R < 1$. Suppose that $u \in W^{1,1}(B_R)\cap L^\infty(B_R)$ satisfies the inequality*

$$\operatorname{div}\ A(Du) + \mu \ge 0 \quad \text{in } B_R \tag{6.3}$$

with A satisfying 1.6 *and* 1.7, *and μ a Radon measure satisfying* 1.2. *Then for any $\varepsilon > 0$ there exists a constant $C = C(q, n, (a_1 + a_2)/\varepsilon)$ such that*

$$\sup_{B_{R/2}} |u| \le C\left(R^{-n}\int_{B_R} u^+\,dx + R^{-q(n-1)}\int_{B_R} u^+\,d\mu\right) + \varepsilon \tag{6.4}$$

Proof. Let $\varepsilon > 0$ and $R < 1$. Fix a cutoff function $\eta \in C_0^\infty(B_R)$ such that $\eta = 1$ in $B_{R/2}$, $\eta = 0$ on ∂B_R, and $0 \le \eta \le 1$ in B_R with $|D\eta| \le 4/R$. Set $\zeta = \eta(1-\frac{\varepsilon}{u})^+$ and $A_\varepsilon = \{\zeta > 0\} = \{u > \varepsilon\} \subset B_R$. Consider the weak formulation of 6.3 with test function $\zeta^{ks-t}u^s$, for constants k, s and t to be chosen later.

$$\begin{aligned}(ks-t)&\int_{A_\varepsilon} \zeta^{ks-t-1}u^s D\zeta\cdot A(Du)\,dx\\ &+s\int_{A_\varepsilon}\zeta^{ks-t}u^{s-1}Du\cdot A(Du)\,dx \le \int_{A_\varepsilon}\zeta^{ks-t}u^s\,d\mu.\end{aligned}$$

Use that $D\zeta = D\eta(1-\frac{\varepsilon}{u}) + \eta\varepsilon u^{-2}Du$ and 1.6 to obtain

$$\begin{aligned}(ks-t)&\int_{A_\varepsilon}\zeta^{ks-t-1}u^s(1-\frac{\varepsilon}{u})D\eta\cdot A(Du)\,dx\\ &+(ks-t)\int_{A_\varepsilon}\zeta^{ks-t-1}u^s\eta\varepsilon u^{-2}(|Du|-a_1)\,dx\\ &+s\int_{A_\varepsilon}\zeta^{ks-t}u^{s-1}(|Du|-a_1)\,dx\\ &\qquad\qquad\le \int_{A_\varepsilon}\zeta^{ks-t}u^s\,d\mu.\end{aligned}$$

Which implies that

$$\begin{aligned} s\int_{A_\varepsilon} \zeta^{ks-t}u^{s-1}\,|Du|\,dx &\le \int_{A_\varepsilon} \zeta^{ks-t}u^s\,d\mu \\ &+ (ks-t)\int_{A_\varepsilon} \zeta^{ks-t-1}u^s(1-\frac{\varepsilon}{u})D\eta\cdot A(Du)\,dx \\ &+ (ks-t)\int_{A_\varepsilon} \zeta^{ks-t-1}u^s\eta\varepsilon u^{-2}(a_1)\,dx \\ &+ s\int_{A_\varepsilon} \zeta^{ks-t}u^{s-1}(a_1)\,dx. \end{aligned}$$

Use 1.7 and that $\varepsilon/u < 1$ in A_ε to obtain

(6.5)

$$\begin{aligned} s&\int_{A_\varepsilon} \zeta^{ks-t}u^{s-1}\,|Du|\,dx \\ &\le \int_{A_\varepsilon} \zeta^{ks-t}u^s\,d\mu + \frac{a_2 4(ks-t)}{R}\int_{A_\varepsilon} \zeta^{ks-t-1}u^s\,dx \\ &\quad + (ks-t)\int_{A_\varepsilon} \zeta^{ks-t-1}u^s(a_1u^{-1})\,dx \\ &\quad + s\int_{A_\varepsilon} \zeta^{ks-t}u^{s-1}(a_1)\,dx \\ &\le \int_{A_\varepsilon} \zeta^{ks-t-1}u^s\,d\mu + \frac{a_2 4(ks-t)}{R}\int_{A_\varepsilon} \zeta^{ks-t-1}u^s\,dx \\ &\quad + \int_{A_\varepsilon} \zeta^{ks-t-1}u^s(a_1\varepsilon^{-1}(ks-t+s)\,dx \\ &\le \int_{A_\varepsilon} \zeta^{ks-t-1}u^s\,d\mu \\ &\quad + \frac{a_2 4(ks-t)+a_1(ks-t+s)}{\varepsilon R}\int_{A_\varepsilon} \zeta^{ks-t-1}u^s\,dx. \end{aligned}$$

Set $w = \zeta^{ks-t}u^s$ and consider

$$\begin{aligned} \int_{A_\varepsilon} |Dw|\,dx &\le s\int_{A_\varepsilon} \zeta^{ks-t}u^{s-1}\,|Du|\,dx \\ &\quad + (ks-t)\int_{A_\varepsilon} \zeta^{ks-t-1}u^s\,|D\zeta|\,dx \end{aligned}$$

$$\leq s \int_{A_\varepsilon} \zeta^{ks-t} u^{s-1} |Du| \, dx$$
$$+ (ks-t) \int_{A_\varepsilon} \zeta^{ks-t-1} u^s (\frac{1}{R} + u^{-1} |Du|) \, dx$$
$$\leq (s+ks-t) \int_{A_\varepsilon} \zeta^{ks-t-1} u^{s-1} |Du| \, dx$$
$$+ \frac{(ks-t)}{R} \int_{A_\varepsilon} \zeta^{ks-t-1} u^s \, dx.$$

Then use 6.5 to obtain the energy type estimate

(6.6)
$$\int_{A_\varepsilon} |Dw| \, dx$$
$$\leq \frac{s+ks-t}{s} \biggl(\int_{A_\varepsilon} \zeta^{ks-t-2} u^s \, d\mu + \frac{a_2 4(ks-t-1) + a_1(ks-t-1+s)}{\varepsilon R} \int_{A_\varepsilon} \zeta^{ks-t-2} u^s \, dx \biggr)$$
$$+ \frac{(ks-t)}{R} \int_{A_\varepsilon} \zeta^{ks-t-2} u^s \, dx$$
$$\leq s(1+k) \biggl(\int_{A_\varepsilon} \zeta^{ks-t-2} u^s \, d\mu + \Bigl(4k \frac{a_1 + a_2}{\varepsilon} + 1 \Bigr) \cdot \frac{1}{R} \int_{A_\varepsilon} \zeta^{ks-t-2} u^s \, dx \biggr), \quad \textit{for } s \geq 1, \ t \geq 0, \ \textit{and } k \geq 1/5.$$

Sobolev inequalities 6.1 and 6.2 imply

$$\Bigl(R^{-n} \int_{A_\varepsilon} w^q \, dx \Bigr)^{1/q} + \Bigl(M^{-1} R^{-q(n-1)} \int_{A_\varepsilon} w^q \, d\mu \Bigr)^{1/q} \leq C R^{-(n-1)} \int_{A_\varepsilon} |Dw| \, dx \tag{6.7}$$

with $C = C(n,q)$. Define $v = \zeta^k u$ and set $t = \frac{2}{q-1}$, so that $tq = t+2$. Also, define a measure ν by

$$d\nu = \frac{dx}{R^n \zeta^{t+2}} + \frac{d\mu}{R^{q(n-1)} \zeta^{t+2}},$$

which is supported on $A_\varepsilon = \{u > \varepsilon\} \cap B_R$. We combine inequalities 6.6 and 6.7 to yield

$$\left(\int_{A_\varepsilon} v^{sq}\, d\nu\right)^{1/q} \le Cs \int_{A_\varepsilon} v^s\, d\nu. \tag{6.8}$$

where $C = C(q, n, (a_1 + a_2)/\varepsilon)$, since k will be chosen late r to be $\frac{2}{q-1} + 2$ and $s \ge 1$ will be used.

We now iterate on the inequality 6.8. Take $s = 1$ in the first iteration,

$$\frac{1}{C}\left(\int_{A_\varepsilon} v^q\, d\nu\right)^{1/q} \le \int_{A_\varepsilon} v\, d\nu.$$

Take $s = q$ in the second iteration,

$$\frac{1}{C}\left(\frac{1}{Cq}\left(\int_{A_\varepsilon} v^{q^2}\, d\nu\right)^{1/q}\right)^{1/q} \le \int_{A_\varepsilon} v\, d\nu.$$

Proceeding with $s = q^{m-1}$ in the m^{th} iteration will yield

$$K_m\left(\frac{1}{C}\right)^{S_m}\left(\int_{A_\varepsilon} v^m\, d\nu\right)^{1/m} \le \int_{A_\varepsilon} v\, d\nu. \tag{6.9}$$

with the constants K_m and S_m given by

$$K_m = \prod_{j=0}^{m-1}\left(\frac{1}{q^j}\right)^{\frac{1}{q^j}}, \quad S_m = \sum_{j=0}^{m-1} 1/q^j.$$

As $m \to \infty$ the constants $S_m \to \frac{q}{q-1}$ and $K_m \to K$, $0 < K < \infty$. Since $K_1 > K_2 > \ldots > K$ we have, for all m, from 6.9

$$\begin{aligned}\left(\int_{A_\varepsilon} v^m\, d\nu\right)^{1/m} &\le C^{S_m}\frac{1}{K}\int_{A_\varepsilon} v\, d\nu \\ &\le \frac{C^{\frac{q}{q-1}}}{K}\int_{A_\varepsilon} v\, d\nu.\end{aligned}$$

This then implies (with C replacing $\frac{C^{\frac{q}{q-1}}}{K}$)

$$\sup_{A_\varepsilon} v \le C\int_{A_\varepsilon} v\, d\nu. \tag{6.10}$$

On $B_{R/2}$ we have that $\zeta = (1 - \frac{\varepsilon}{u})^+$. Thus when $u \geq 2\varepsilon$, we have $\zeta \geq \frac{1}{2}$. Set $k = t + 2$, and 6.10 implies

$$\begin{aligned}\sup_{B_{R/2}} u &\leq 2^k \sup_{A_\varepsilon} u + 2\varepsilon \\ &\leq C\left(R^{-n}\int_{A_\varepsilon} u\,dx + R^{-q(n-1)}\int_{A_\varepsilon} u\,d\mu\right) + 2\varepsilon\end{aligned}$$

and the result follows, noting that $\int_{A_\varepsilon} u\,dx \leq \int_{B_R} u^+\,dx$. □

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