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NON-COMPACT TOTALLY PERIPHERAL 3-MANIFOLDS

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A 3-manifold is totally peripheral if every loop is freely homotopic into the boundary. It is shown that an orientable 3-manifold M is totally peripheral if and only if there is a boundary component F of M such that the inclusion of F in M induces a surjective map of fundamental groups. If M is non-orientable, there are essentially two counterexamples.

A 3-manifold is *totally peripheral* if every loop is freely homotopic into the boundary. Brin, Johannson and Scott studied compact totally peripheral 3-manifolds. They showed that when M is orientable, compact and totally peripheral, then there is a boundary component F of M such that the natural map $\pi_1(F) \to \pi_1(M)$ is surjective. When M is non-orientable, they showed that this result is almost true but that there are essentially two counterexamples. In this paper, we show that the same results hold if the compactness hypothesis on M is omitted. The results remain true even if the fundamental group of M is not finitely generated.

Brin, Johannson and Scott [**1**] also proved a relative version of their results. We say that a 3-manifold is totally peripheral relative to a subsurface B of ∂M (possibly B is disconnected), or TP rel B, if every loop in M is freely homotopic into B. They showed that if M is orientable, compact and totally peripheral relative to a compact subsurface B of ∂M, then there is a component C of B such that the natural map $\pi_1(C) \to \pi_1(M)$ is surjective. This relative result is also a consequence of our result for the non-compact case as, given a compact manifold M and a compact subsurface B in ∂M such that M is TP rel B, one can remove the closure of ∂M - B from M to obtain a non-compact totally peripheral 3-manifold M' with boundary equal to the interior of B. However, we use the relative case of [**1**] in the proof of our results.

We organise the paper into three sections. In §1, we consider the case when M has finitely generated fundamental group. This is an easy consequence of our work in [**3**] on the finiteness of the boundary of M, and of McCullough's result in [**5**] which says that given a 3-manifold M with finitely generated fundamental group and a compact subsurface C of the boundary of M, then there exists a compact core X for M with $X \cap \partial M = C$. See also [**7**]. We then extend to the case when M has infinitely generated fundamental group, by considering covers of M with finitely generated fundamental group and applying our results for that case. We handle the orientable case in §2, and the non-orientable case in §3. There are no exceptional cases when M has fundamental group which is not finitely generated.

The work in this paper is part of the Liverpool Ph.D. thesis of Luke Harris completed under the supervision of Peter Scott in 1988. Since then Harris obtained a job not in the academic world and has never had time to prepare this work for publication. Finally, Scott agreed to prepare this for publication, to avoid the complete disappearance of his work.

1. The case when M has finitely generated fundamental group. In this section we prove the following results.

THEOREM 1.1. *Let M be an orientable totally peripheral 3-manifold with finitely generated fundamental group. Then there is a component F of ∂M with the natural map $\pi_1(F) \to \pi_1(M)$ surjective.*

THEOREM 1.2. *Let M be a non-orientable totally peripheral 3-manifold with finitely generated fundamental group. Then either there is a component F of ∂M with the natural map $\pi_1(F) \to \pi_1(M)$ surjective or it has a compact core which is one of the counterexamples in the compact case given in* [**1**].

Before embarking on the proofs of these results, we will discuss briefly our work in [**3**]. In that paper, we consider the question of the uniqueness of a compact core for a 3-manifold M with finitely generated fundamental group. In [**6**], McCullough, Miller and Swarup showed that if N_1 and N_2 are irreducible compact cores of a $\mathbb{P}^2$-

irreducible 3-manifold M, then N_1 and N_2 are homeomorphic. In [**3**], we give an example to show that there is no exactly analogous result when M and its cores are not irreducible. However, in Theorem 2.1 of [**3**], we show that there are only finitely many different cores for M up to a natural equivalence relation which we call almost homeomorphism. Two compact 3-manifolds are *almost homeomorphic* if they are homeomorphic up to connected sum with compact simply connected 3-manifolds (3-balls and fake 3-spheres) and up to replacing $\mathbb{P}^2 \times I$'s with fake $\mathbb{P}^2 \times I$'s. As almost homeomorphic 3-manifolds have homeomorphic boundaries apart from the number of sphere components, it follows that there are only finitely many possibilities for the topological type of the boundary of a compact core for M apart from the number of sphere components.

Now we consider the boundary of a 3-manifold M with finitely generated fundamental group G. Let F be a component of ∂M, and let $H = \mathrm{Im}\,(\pi_1(F) \to \pi_1(M))$ be the image of the fundamental group of F under the natural induced map into G. Then H is finitely generated, by the result of Jaco in [**4**], though $\pi_1(F)$ need not be finitely generated. Now we can take a regular neighbourhood of based loops in F representing the generators of H, and add compressing discs in F to get a compact subsurface C of F with $\mathrm{Im}\,(\pi_1(C) \to \pi_1(M)) = H$, and C incompressible in F. We define such a surface C to be an *essential core* for F. Note that C need not be incompressible in M. Also, if H is infinite cyclic, we can choose a simple closed curve on F to represent a generator of H, and thus we may choose the essential core to be an annulus or a Möbius band in this case.

We can now state the relevant result from [**3**].

THEOREM 3.2 of [**3**]. *Let M be a 3-manifold with finitely generated group. Then:*

(i) *There are only finitely many boundary components F of M with* $\mathrm{Im}\,(\pi_1(F) \to \pi_1(M))$ *not trivial or infinite cyclic,*

(ii) *There are only finitely many boundary components F of M with* $\mathrm{Im}\,(\pi_1(F) \to \pi_1(M))$ *infinite cyclic and with essential core a Möbius band,*

(iii) *Of those components of the boundary F_i with*

$$\mathrm{Im}(\pi_1(F_i) \to \pi_1(M))$$

infinite cyclic and with essential core an annulus, there are only finitely many conjugacy classes in $\pi_1(M)$ *of*

$$\operatorname{Im}(\pi_1(F_i) \to \pi_1(M)).$$

McCullough gives a result equivalent to parts (i) and (ii) of this theorem in the case when ∂M is incompressible as a corollary to his main theorem in [**5**]. See also [**7**]. Our result follows by a similar argument using the finiteness result mentioned in the preceding paragraphs.

Proof of Theorem 1.1. For each component F of ∂M, we may find an essential core C, so that C is a compact subsurface of F with $\operatorname{Im}(\pi_1(C) \to \pi_1(M))$ equal to $\operatorname{Im}(\pi_1(F) \to \pi_1(M))$. Thus we can find an essential core for each of the boundary components F_i of M with $\operatorname{Im}(\pi_1(F_i) \to \pi_1(M))$ not 1 or $\mathbb{Z}$, and also an essential core for one representative from each conjugacy class of components of ∂M with image in $\pi_1(M)$ infinite cyclic. Call the union of these essential cores B.

Theorem 3.2 of [**3**] tells us that B is compact, so by the theorem of McCullough in [**5**], we may find a compact core X for M with $X \cap \partial M = B$. But every element of $\pi_1(M)$ is conjugate into the fundamental group of some boundary component, and hence conjugate into the fundamental group of one of the components of B. Thus this is also true for elements of $\pi_1(X)$, and so X must be totally peripheral relative to B. Now we can apply the theorem of Brin, Johannson and Scott in [**1**] to deduce that there is a component B_1 of B with the natural map $\pi_1(B_1) \to \pi_1(X)$ surjective. But B_1 was an essential core of some component F_1 of ∂M, and so we must have that the natural map $\pi_1(F_1) \to \pi_1(M)$ is also surjective. This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 2.1. As in the orientable case, we can find a core X for M with $X \cap \partial M = B$ with B a union of essential cores of boundary components of M and with X totally peripheral relative to B.

B need not be π_1-injective in ∂X, but we can easily make it so by adding discs lying in ∂X to B to form B'. So B' is π_1-injective in ∂X, (but not necessarily π_1-injective in X) and also X

is totally peripheral relative to B'. Now we can apply the theorem of Brin, Johannson and Scott. We find that either a component C' of B' has the natural map $\pi_1(C') \to \pi_1(X)$ surjective, in which case the corresponding component C of B is the essential core of a component F of ∂M with the map $\pi_1(F) \to \pi_1(M)$surjective, or else that X and B' are one of the counterexamples to the theorem in the compact case. $\square$

2. M is orientable and has infinitely generated fundamental group.

THEOREM 2.1. *Let M be an orientable totally peripheral* 3- *manifold with $\pi_1(M)$ not finitely generated. Then there is a component F of ∂M with the natural map $\pi_1(F) \to \pi_1(M)$surjective.*

Proof of Theorem 2.1. Let $G = \pi_1(M)$. Then we may take an exhausting sequence G_i for G, with each G_i a finitely generated subgroup of G such that $G_i \subset G_{i+1}$ $\forall i$ and $\bigcup_{i=1}^{\infty} G_i = G$. We may assume G_1 is not trivial.

First we will rule out the case of all the G_i being infinite cyclic. If $G_i \cong \mathbb{Z}$ for all i, then G must be an infinitely generated subgroup of $\mathbb{Q}$, the additive group of rationals.

In this case, let α be any simple closed curve in ∂M representing a non-trivial element a in $\pi_1(M)$. A regular neighbourhood of α is an annulus A which is incompressible in M.

Now $\pi_1(M)$ is an infinitely generated subgroup of $\mathbb{Q}$, so we can find an element b in $\pi_1(M)$ such that $b^m = a$ where m is not equal to ± 1 or 0. M is totally peripheral, so there is a loop β in ∂M representing b. Take a regular neighbourhood B of β. As β may be singular, B is not necessarily incompressible in M, so we need to compress it. Consider a compressing disc D for B, with boundary ∂D embedded in B. D must separate M, and one component of the result must be simply connected, since $\pi_1(M)$ is not a non-trivial free product. So ∂D separates B into B_1 and B_2, and one component, B_2 say, has fundamental group with trivial image in $\pi_1(M)$.

As β represents something non-trivial in $\pi_1(M)$, it cannot be completely contained in B_2. Suppose that λ is a sub-arc of β lying in B_2. The endpoints of λ bound a sub-arc μ of ∂D. We cut out λ from β and replace it with μ. The resulting curve, which we shall continue to call β, still represents b in $\pi_1(M)$ since μ and λ lie in B_2 and hence are homotopic (preserving endpoints) in M. We may repeat the process until β lies entirely in B_1.

We now find a compressing disc for B_1, and repeat the whole process until we end up with B' incompressible, containing a loop β representing b in $\pi_1(M)$. B' must be an annulus, since it is compact and thus has image in $\pi_1(M)$ trivial or infinite cyclic. Clearly a closed curve on B' representing $a = b^m$ cannot be simple. Going back to A, the only non-trivial essential closed curve is α. However, we can apply the Annulus Theorem (see [**8**] for a brief history and list of references) to get an embedded essential annulus running from A to B'. This annulus must carry the element a of $\pi_1(M)$, since the intersection of the annulus with A must be simple. But this is a contradiction, since it would imply that a can be represented by a simple closed curve on B'. So we deduce that an orientable 3-manifold with fundamental group an infinitely generated subgroup of $\mathbb{Q}$ cannot be totally peripheral.

So, after deleting the first few G_i in the sequence, we may suppose that no $G_i \cong Z$.

For any i, M_{G_i}, the cover of M with fundamental group G_i, is totally peripheral and has finitely generated fundamental group. Then we may apply the theorem of the previous section to deduce that some component F_i of ∂M_{G_i} has the map $\pi_1(F_i) \to \pi_1(M_{G_i})$ surjective.

Now lift to the universal cover $\tilde{M}$. We see that a boundary component $\tilde{F}_i$ of $\tilde{M}$, the lift of F_i, is stabilised by the action of G_i on $\tilde{M}$.

We wish to show that G_i can stabilise at most two boundary components of $\tilde{M}$. Suppose to the contrary that G_i stabilises three distinct boundary components $\tilde{F}_i$, $\tilde{F}_i'$ and $\tilde{F}_i''$. Then these project down to three distinct boundary components F_i, F_i' and F_i'' of M_{G_i}.

Now take essential cores C_i, C_i' and C_i'' for these boundary components. We can use the result of McCullough in [**5**] to get a compact core X for M_{G_i} with $X \cap \partial M_{G_i} = B = C_i \cup C_i' \cup C_i''$, and X is TP rel B.

We know from the results in [**1**] that if X is TP rel B then either the component C of B with the natural map $\pi_1(C) \to \pi_1(X)$ surjective is unique or $(X, C) \cong (C \times I\#M', C \times 0)$ where M' is simply connected.

In either case we have a contradiction since C_i, C_i' and C_i'' may be assumed to be injective in ∂X and are not discs, spheres or annuli, since the image of their maps into $\pi_1(X)$ is G_i, which itself is not trivial or infinite cyclic.

Thus G_i stabilises at most two boundary components of $\tilde{M}$ for each i. If G_j, $j \geq i$, stabilises a boundary component $\tilde{F}_j$ of $\tilde{M}$, then so does G_i. Hence some boundary component $\tilde{F}$ of $\tilde{M}$ is stabilised by every G_i. But then $\bigcup_{i=1}^{\infty} G_i = G$ stabilises $\tilde{F}$, and so the projection F of $\tilde{F}$ to M has the natural map $\pi_1(F) \to \pi_1(M)$surjective. This completes the proof of Theorem 2.1. □

3. M is not orientable and has infinitely generated fundamental group.

THEOREM 3.1. *Let M be a totally peripheral non-orientable 3-manifold with infinitely generated fundamental group. Then there is a component F of ∂M such that the natural map $\pi_1(F) \to \pi_1(M)$is surjective.*

Proof of Theorem 3.1. Consider the orientable double cover $\tilde{M}$ of M. It is totally peripheral, so we apply Theorem 2.1 to see that there is a boundary component $\tilde{F}$ of $\tilde{M}$ with the natural map $\pi_1(\tilde{F}) \to \pi_1(\tilde{M})$ surjective. Also we have an involution τ acting on $\tilde{M}$. If $\tau\tilde{F} = \tilde{F}$ then $\tilde{F}$ is stabilised by the action of τ and so projects down to a boundary component $F = \tilde{F}/\tau$ of M with the natural map $\pi_1(F) \to \pi_1(M)$surjective, as required.

So assume instead that we have two boundary components of $\tilde{M}$, $\tilde{F}$ and $\tau\tilde{F}$, which project down to a single surface F in ∂M. To complete the proof, we will show that this is impossible. Notice that in this case, $\mathrm{Im}\,(\pi_1(F) \to \pi_1(M)) = \pi_1(\tilde{M})$ is normal of index two in $\pi_1(M)$. Note also that F is orientable as it is homeomorphic to $\tilde{F}$.

Now $\pi_1(\tilde{M})$ cannot be abelian since an infinitely generated 3-manifold group cannot be abelian of rank two or greater by a result of Evans and Moser [**2**], and if $\pi_1(\tilde{M})$ is abelian of rank one then the argument in the previous section gives a contradiction. Thus we can find a finitely generated subgroup H of $\mathrm{Im}\,(\pi_1(F) \to \pi_1(M))$ which is not $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$.

Note that τ induces an automorphism τ_* on $\pi_1(\tilde{M})$. From H we can construct a τ_* -invariant subgroup H_1 of $\pi_1(F)$ generated by H and $\tau_* H$, $H_1 = \langle H, \tau_* H\rangle$, with H_1 finitely generated, and contained with index two in some finitely generated subgroup G_1 of $\pi_1(M)$.

We take the cover M_1 of M with fundamental group G_1. It is non-orientable by our choice of G_1, and has orientable double cover $\tilde{M}_1$ with fundamental group H_1. Let σ denote the covering involution on $\tilde{M}_1$. Note that $\tilde{M}_1$ covers $\tilde{M}$ and that M_1 and $\tilde{M}_1$ are totally peripheral.

In $\tilde{M}_1$, $\tilde{F}$ and $\tau\tilde{F}$ lift to components $\tilde{F}_1$ and $\tilde{F}_2$ of $\partial\tilde{M}_1$, with the maps $\pi_1\left(\tilde{F}_i\right) \to \pi_1\left(\tilde{M}_1\right)$ $i = 1, 2$ surjective. Consider the action of σ on $\tilde{M}_1$. If $\sigma\tilde{F}_1 = \tilde{F}_1$, then $\tilde{F}_1$ projects down to a component F_1 of ∂M_1 with the map $\pi_1(F_1) \to \pi_1(M_1)$ surjective, which is a contradiction since M_1 is non-orientable, but F_1 is orientable since it covers F. We deduce that the action of σ on $\tilde{M}_1$ interchanges $\tilde{F}_1$ and $\tilde{F}_2$.

So $\sigma\tilde{F}_1 = \tilde{F}_2$, and they project down to a component F_1 of ∂M_1. As in section one, we can find a compact core X of M_1 with X totally peripheral relative to $B = X \cap \partial M$. We may suppose that X contains an essential core C_1 of F_1. The pre-image $\tilde{X}$ of X in $\tilde{M}_1$ is a compact core for $\tilde{M}_1$ and $\tilde{X}$ is totally peripheral relative to $\tilde{B}$. $\tilde{B}$ contains $\tilde{C}_1$ and $\tilde{C}_2$, essential cores for $\tilde{F}_1$ and $\tilde{F}_2$, and these are transposed by the involution σ which acts on $\tilde{X}$. Now we can apply the results of [**1**] to deduce that X has infinite cyclic fundamental group or that X is one of the exceptional cases to the theorem in the compact case. In any case, $\tilde{X}$ has fundamental group $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. But we chose $\pi_1(\tilde{X}) = \pi_1\left(\tilde{M}_1\right) = H_1$ to contain H, a non-abelian group, which gives the desired contradiction and completes the proof of Theorem 3.1. $\square$

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