PACIFIC JOURNAL OF MATHEMATICS
Vol. 167, No. 1, 1995

LOCAL REPRODUCING KERNELS ON WEDGE-LIKE DOMAINS WITH TYPE 2 EDGES

AL BOGGESS AND ALEXANDER NAGEL

We represent holomorphic functions on a wedge-like domain by positive integral kernels which are defined on the edge of the wedge. Type 2 edges are considered. As an application, we show that an H^p function on a wedge has pointwise almost everywhere limits on the edge within admissable approach regions in the wedge.

A striking fact about function theory in several variables is that, under suitable convexity hypotheses on a domain $\Omega \subset \mathbb{C}^n$ with $n > 1$, if $z_0 \in \Omega$ is "close" to the boundary $\partial\Omega$, then there are representing measures for z_0 whose support on $\partial\Omega$ is compactly supported and "close" to z_0^*, the projection of z_0 onto the boundary. This is false for domains in $\mathbb{C}$, for general non-convex domains in $\mathbb{C}^n$, and for harmonic functions on domains in $\mathbb{R}^{2n}$.

We illustrate this phenomenon with a simple example. Let $\Omega = \{(z,w) \in \mathbb{C}^2; \mathrm{Re}(z) > |w|^2\}$. The boundary of Ω will be denoted by $\Sigma = \{(z,w) \in \mathbb{C}^2; \mathrm{Re}(z) = |w|^2\}$ and can be identified with the Heisenberg group. We wish to represent the value of a holomorphic function F on Ω at the point $(0,r) \in \Omega$ for $r > 0$, by integrating F against a suitable measure on Σ. To do this, we let $\phi \in C_0^\infty(\mathbb{C})$ be a radial function with support in the unit disc and whose integral over $\mathbb{C}$ is one. Let

$$\phi_r(w) = \frac{4}{r^2}\phi\left(\frac{2(w-r)}{r}\right).$$

The function ϕ_r has support in the disc centered at r with radius $r/2$ and the integral of ϕ_r over $\mathbb{C}$ is one. The mean value property for holomorphic functions shows that

$$F(0,r) = \iint_{w\in\mathbb{C}} F(0,w)\phi_r(w)\,dx\,dy$$

where we have written $w = x + \imath y$. The support of ϕ_r is contained in the square $\{r/2 \le x \le 3r/2,\ |y| \le r/2\}$. Therefore, $\sqrt{x}$ is well defined on the support of ϕ_r. Using the mean value property of F in the first variable, we obtain

$$F(0,r) = \frac{1}{2\pi} \iint_{w \in \mathbb{C}} \int_0^{2\pi} F(\sqrt{x}e^{i\theta}, x + \imath y)\phi_r(x + \imath y)\ d\theta dx dy.$$

Now, we change variables by letting $x = t^2$ and then $z = te^{\imath\theta}$. We obtain

$$F(0,r) = \frac{1}{\pi} \iint_{z \in \mathbb{C}} \int_{y \in \mathbb{R}} F(z, |z|^2 + \imath y)\phi_r(|z|^2 + \imath y) d\lambda(z) dy.$$

where $d\lambda(z)$ denotes Lebesgue measure on the complex plane. The map $(z, y) \mapsto (z, |z|^2 + \imath y)$ for $z \in C$ and $y \in R$ is a parameterization for Σ and $d\lambda(z)dy$ is comparable to surface measure on Σ which we denoted by $d\sigma$. Therefore, we obtain

$$F(0,r) = \int_{\Sigma} F(\zeta) K_r(\zeta) d\sigma(\zeta) \tag{1}$$

where $K_r(\zeta)$ is a smooth function of ζ. Notice that the support of K_r is contained in a ball centered at the origin of radius $C\sqrt{r}$ (where C is a uniform constant). Thus, K_r is our desired local representing measure for holomorphic functions on the Heisenberg group (near the origin).

The existence of local representing measures for a domain $\Omega \subset \mathbb{C}^n$ seems to be closely tied to the existence and structure of analytic discs in $\mathbb{C}^n$ whose boundary lies in $\partial\Omega$ close to a given point. This in turn is closely connected to the nonisotropic nature of the boundary tangential Cauchy-Riemann equations, to the associated nonisotropic metrics on the boundary, and to questions about the local polynomial hull of small regions on the boundary.

For strictly pseudoconvex domains, it is easy to study local analytic discs, since after a local biholomorphic change of variables, one can make the boundary of the domain strictly convex (in the linear sense), and one can obtain analytic discs by slicing by appropriate planes, as was done in the example above. The case of weakly pseudoconvex domains of finite type presents certain additional difficulties, since the boundary cannot always be convexified, but here too one can obtain local representing measures by imbedding suitable analytic discs (see [**BDN**]).

In this paper, we generalize these ideas to the case of "wedge domains" with an "edge" which is a submanifold $M \subset \mathbb{C}^n$ of real codimension greater than 1. Our object is to find local representing measures on the edge for points in the wedge near the edge. We study the so called "type 2" case, which is the analogue of the strictly pseudoconvex case for domains with boundary of real codimension 1. We have been greatly influenced in our work by E.M. Stein's seminal observation that a strictly pseudoconvex boundary can be modeled at each point by a nilpotent Lie group, the Heisenberg group, and that the boundary behavior of holomorphic functions on a strictly pseudoconvex domain is intimately connected with the approximating group structure on the boundary. We shall first study a certain "model case" where the edge is a nilpotent Lie group of step 2, and then show that the general case can be obtained by a three stage process which is again inspired by Stein's work: (i) we pass from the original object of study to a "free" object by adding appropriate variables; (ii) we solve the problem on the free object by approximating it suitably by the model case; (iii) we return to the orginal object by integrating out the extra variables.

The plan of the paper is as follows. In Section 1, we recall some definitions and results about CR submanifolds and of domains with edges. These preliminaries are necessary for a precise statement of our main result on the existence of a local integral representation formula for holomorphic functions on a domain with an edge which is a generic CR submanifold of type 2. In Section 2, we study a model example of a generic CR submanifold of type 2. This model carries the structure of a nilpotent Lie group of step 2, and we use this group structure to obtain a local integral representation formula for holomorphic functions defined in the corresponding model wedge. In Section 3, we introduce the notion of a *free* generic CR manifold of type 2. This is one on which there are no linear relations between certain tangential vector fields, or equivalently, one where the real codimension is as large as possible. In Section 3 we also show how a free manifold can be approximated by the model example, and we show how to use Bishop's equation and analytic discs to 'transfer' the integral representation formula from the model to the free manifold. In Section 4, we show how a general generic CR submanifold of type 2 can be "freed" by the addition of variables. It is then possible to obtain the integral representation formula for the general case from the free case by integrating out these added variables. All these ideas are motivated by the work of Folland and Stein [**FS**], Rothschild and Stein [**RS**], *etc.*

In Section 5, we show how these local integral representation formulas can be used to begin the study of H^p functions on domains with edge which is a generic CR submanifold of type 2. In particular, we prove that H^p functions have appropriate admissible limits almost everywhere along the edge, for $0 < p \leq \infty$, and we obtain a necessary condition for a CR distribution on the edge to be the boundary values of an H^p function. In recent work, Rosay [**R**] has shown that H^p functions on a wedge with an arbitrary generic CR edge has admissible limits almost everywhere, at least for $1 \leq p \leq \infty$. It is not clear whether his arguments also work for $p < 1$.

An announcement of the results of this paper appeared in [**BN**]. The authors would like to thank Jean-Pierre Rosay for many useful conversations about this subject.

1. Preliminaries and statement of the main theorem. In this Section, we recall certain basic definitions and results relating to submanifolds of open subsets of $\mathbb{C}^n$. These concepts are necessary for the precise statement of our main result on the existence of a local integral representation formula for holomorphic functions. This theorem is stated at the end of the Section.

Let $U \subset \mathbb{C}^n$ be an open set and let $\rho_l : U \to \mathbb{R}$, $1 \leq l \leq d$ be functions of class C^∞. Set $M = \left\{ z \in U \,\middle|\, \rho_l(z) = 0, \quad 1 \leq l \leq d \right\}$, and assume $d\rho_1 \wedge \ldots \wedge d\rho_d \neq 0$ on M. Then M is a C^∞ submanifold of U of real codimension d. For $p \in M$, let T_pM denote the real tangent space to M at p. The maximal complex subspace of T_pM is denoted by $T_p^{\mathbb{C}} = T_pM \cap J(T_pM)$, where J is the complex structure map given by multiplication by $\imath = \sqrt{-1}$. M is called a CR *submanifold, with* CR *dimension* m if for all $p \in M$, $\dim_{\mathbb{R}} T_p^{\mathbb{C}} M = 2m$. M is called *generic* if for all $p \in M$, $T_pM + J(T_pM) = \mathbb{C}^n$. If M is a generic CR submanifold of real codimension d and CR dimension m, it follows that $n = m + d$.

We shall frequently use the following result on the local representation of generic CR manifolds. We use the notation

$$(z_1, \ldots, z_m, w_1, \ldots, w_d) = (z, w)$$

for coordinates in $\mathbb{C}^{m+d}$.

PROPOSITION 1.1. *Let $M \subset U \subset \mathbb{C}^{m+d}$ be a generic CR submanifold with CR dimension m and real codimension d. For every point $p \in M$, there exist open neighborhoods U_p of p, V_p of $0 \in \mathbb{C}^{m+d}$, a biholomorphic map $\psi_p : U_p \to V_p$, and a C^∞ function $h_p : \mathbb{C}^m \times \mathbb{R}^d \to \mathbb{R}^d$ such that if we write*

$$M_p = \psi_p(M \cap U_p) \subset V_p$$

then

$$M_p = \left\{(z, w) \in V_p \,\middle|\, \mathrm{Re}(w) = h_p\left(z, \mathrm{Im}(w)\right)\right\}.$$

The function h_p satisfies

$$\begin{aligned} h_p(0,0) &= 0 \\ \nabla h_p(0,0) &= 0 \\ \frac{\partial^2 h_p(0,0)}{\partial z_j \partial z_k} &= 0 \quad 1 \leq j, k \leq m. \end{aligned}$$

Proof. This is standard. The existence of a function h_p satisfying the first two conditions follows from the implicit function theorem by viewing M locally near p as a graph over the tangent space T_pM. The further condition on the vanishing of the pure second derivatives of h_p at the origin is achieved by a standard quadratic biholomorphic mapping (see Section 7.2 in [**B**] for details). □

A complex vector field L on an open subset of $\mathbb{C}^n$ is said to be of type $(1,0)$ if it can be written as $L = \sum_{j=1}^n a_j(z)\, \frac{\partial}{\partial z_j}$. Suppose that M is a generic CR submanifold of CR dimension m and real codimension d in an open set $U \subset \mathbb{C}^n$. Then for each point $p \in M$, there is a neighborhood U_p of p in $\mathbb{C}^n$ and m linearly independent C^∞ vector fields of type $(1,0)$ on U_p, $\{L_1, \ldots, L_m\}$, such that $L_j(\rho_l)(z) = 0$ for $1 \leq j \leq m$ and $1 \leq l \leq d$. If we write $L_j = \frac{1}{2}(X_j - \imath X_{m+j})$

where the $\{X_k\}$ are real vector fields, then the real vector fields $\{X_1, \dots X_{2m}\}$ at q span $T_q^{\mathbb{C}}M$ for all $q \in M \cap U_p$.

DEFINITION 1.2. M is of type 2 at p if the vector fields $\{X_1, \dots , X_{2m}\}$ at p together with all the second order commutators $\{\dots , [X_j, X_k], \dots\}$ $1 \leq j, k \leq 2m$ at p span the entire real tangent space T_pM. Equivalently, the vector fields $\{L_1, \dots , L_m, \overline{L}_1, \dots , \overline{L}_m\}$ together with all mixed second order commutators $\{\dots , [L_j, \overline{L}_k], \dots\}$ $1 \leq j, k \leq m$ at p span the complexified tangent space $T_pM \otimes \mathbb{C}$.

This condition is easily seen to be independent of the choice of vector fields $\{L_1, \dots , L_m\}$. This condition is also open; i.e. if M is of type 2 at p then M is of type 2 at all points in some neighborhood of p. We say that M *is of type* 2 if M is of type 2 at every point $p \in M$.

This analytic definition of type is equivalent to a geometric condition on the Levi form on M, whose definition we now recall. For $p \in M$, let Y_pM denote the orthogonal complement to $T_p^{\mathbb{C}}M$ in T_pM, and let $N_pM = J(Y_pM)$. This space is not necessarily orthogonal to $T_p(M)$, but it is transverse, i.e. $T_p(M) \cap N_p(M) = \{0\}$. We have

$$T_pM = T_p^{\mathbb{C}} \oplus Y_pM$$
$$\mathbb{C}^n = T_pM + N_pM.$$

Let $\pi_p : T_pM \to Y_pM$ be the orthogonal projection map. Let $H_p^{1,0}(M)$ denote the subspace of the complexified tangent space to M at p spanned by tangent vectors of type $(1,0)$, (*i.e.* by $\{L_1, \dots , L_m\}$ at p). If Z belongs to $H^{1,0}(M)$, then the vector field $\frac{1}{2\imath}[Z, \overline{Z}]$ is a real tangent vector field.

DEFINITION 1.3. The *Levi form* is the well defined quadratic mapping $\mathcal{L}_p : H_p^{1,0}(M) \to N_pM$ given by

$$\mathcal{L}_p(Z) = -J\left(\pi_p\left(\frac{1}{2\imath}[Z, \overline{Z}]_p\right)\right).$$

The closure of the convex hull of the image of $\mathcal{L}_p$ is a closed cone in N_pM and is denoted by Γ_p.

We shall need a way to compute the Levi form of M at p. For any fixed $p \in M$, we first biholomorphically map M near p to $M_p = \{(z, w) \in \mathbb{C}^m \times \mathbb{C}^d; \mathrm{Re}(w) = h_p(z, \mathrm{Im}(w))\}$ where $h_p =$

$(h_p^1, \dots, h_p^d) : \mathbb{C}^m \times \mathbb{R}^d \to \mathbb{R}^d$ is smooth as in Proposition 1.1. Under this biholomorphism, p gets mapped to the origin; $H_p^{1,0}(M)$ gets mapped to the copy of $\mathbb{C}^m$ given by $\{(z,0);\ z \in \mathbb{C}^m\}$ and $N_p(M)$ gets mapped to the copy of $\mathbb{R}^d$ given by $\{(0,x);\ x \in \mathbb{R}^d\}$. Define the bilinear form $B_p : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^d$

$$B_p(\xi,\eta) = \left(\sum_{j,k=1}^{m} \frac{\partial^2 h_p^1}{\partial z_j \partial \overline{z}_k}(0,0)\, \xi_j \overline{\eta}_k, \dots, \sum_{j,k=1}^{m} \frac{\partial^2 h_p^d}{\partial z_j \partial \overline{z}_k}(0,0)\, \xi_j \overline{\eta}_k \right).$$

It is a standard result that the Levi form of M at p $(= \mathcal{L}_0(M_p))$ is given by the map

$$(z_1, \dots, z_m) \to B_p(z, \overline{z})$$

(see Corollary 1 in Section 10.2 in [**B**]).

The following proposition exhibits the connection between the commutator properties expressed in the type 2 condition, and geometric properties of the cone Γ_p.

PROPOSITION 1.4. *Let $M \subset U \subset \mathbb{C}^n$ be a generic CR submanifold. The following are equivalent.*

(1) *M is of type 2 at $p \in M$.*

(2) *Γ_p has nonempty interior in N_pM.*

(3) *$B_p : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^d$ is surjective.*

Proof. By expanding $[Z + W, \overline{Z+W}]$ and $[Z + \imath W, \overline{Z + \imath W}]$ for $Z, W \in H^{1,0}(M)$, one easily sees that M is of type 2 at p if and only if the set of vector fields L in $H^{1,0}(M)$ and $\overline{L} \in H^{0,1}(M)$ together with all second order commutators of the form $[L, \overline{L}]$ for $L \in H^{1,0}(M)$ span the complexified tangent bundle $T(M) \otimes \mathbb{C}$ near p. Now, (1) and (2) are easily seen to be equivalent. Using the above formula for the Levi form, clearly (2) and (3) are equivalent. □

For any subset $K \subset N_pM$ and any $\epsilon > 0$, let

$$K_\epsilon = \left\{ z \in N_pM \,\middle|\, z \in K, \text{and } |z| < \epsilon \right\}.$$

If γ_1 and γ_2 are two cones in N_pM, we say that γ_1 *is smaller than* γ_2 and write $\gamma_1 < \gamma_2$ if $\overline{\gamma_1} \cap S$ is a compact subset of the interior of

γ_2 where S is the unit sphere in N_pM. We now describe the type of wedge domains which we will study.

DEFINITION 1.5. Let $M \subset U \subset \mathbb{C}^n$ be a generic, CR submanifold of type 2. An open set $\Omega \subset \mathbb{C}^n$ is a domain with edge M if:

(1) $M \subset \overline{\Omega}$;

(2) For each $p \in M$ and for each cone $\gamma < \Gamma_p$ there exists an open set $\omega \subset M$ containing p and an $\epsilon > 0$ such that

$$\omega + \gamma_\epsilon \subset \Omega.$$

Property (2) roughly states that near a point $p \in M$, Ω locally contains translates of M in directions strictly interior to the cone Γ_p.

The set $\omega + \gamma_\epsilon$ is parameterized in a natural way by $\omega \times \gamma_\epsilon$. After shrinking ϵ if necessary, we can consider the projection Π of $\omega + \gamma_\epsilon$ onto M so that for $z \in \omega + \gamma_\epsilon$,

$$z - \Pi(z) \in N_{\Pi(z)}M.$$

We write

$$r(z) = |z - \Pi(z)|$$

where the absolute value denotes the length of a vector in $N_{\Pi(z)}M$. We then have

PROPOSITION 1.6. *Let $M \subset U \subset \mathbb{C}^n$ be a generic CR submanifold of type 2, and let $\Omega \subset \mathbb{C}^n$ be a domain with edge M. Let $p \in M$ and let $\gamma < \Gamma_p$. Then there exist a neighborhood $\omega \subset M$ of p, a constant $\epsilon > 0$, and constants C_1 and C_2 so that for every $q \in \omega$ and every $w \in \gamma_q$, the following holds: if $z = q + w$, then*

$$C_1 |w| \leq r(z) \leq C_2 |w|.$$

We shall need to use the nonisotropic pseudometric and corresponding nonisotropic balls on M induced by the ambient complex structure on $\mathbb{C}^n$. We only summarize the construction, which can be carried out for any CR submanifold M of finite type (see [**NSW**] for more details). Suppose $\{L_1 = X_1 + \imath X_{m+1}, \dots, L_m = X_m + \imath X_{2m}\}$ is a basis for $H^{1,0}(M)$ on an open set $\omega \subset M$.

DEFINITION 1.7. For $p \in \omega$ and $\delta > 0$, let

$$B(p,\delta) = \left\{ \exp\left[\sum_{j=1}^{2m} \alpha_j X_j + \sum_{j,k=1}^{2m} \beta_{j,k}[X_j, X_k]\right](p); \right.$$

$$\left. \alpha_j, \beta_{j,k} \in \mathbb{R} \text{ with } |\alpha_j| < \delta, |\beta_{j,k}| < \delta^2 \right\}$$

where exp denotes the exponential map.

There also exists a pseudo-metric $D : M \times M \mapsto [0,\infty)$ so that

$$B(p,\delta) = \{q \in M;\ D(p,q) < \delta\}.$$

These balls have Euclidean dimension δ in the m complex tangent space directions of M at q and so D behaves roughly Euclidean in these directions. In the d totally real tangent directions, these balls have Euclidean dimension roughly δ^2 and so D behaves roughly like the square root of the Euclidean distance in the totally real directions.

The next lemma summarizes the important properties of these balls.

LEMMA 1.8. **[NSW]** *Given a compact set $K \subset M$, there are constants $0 < C_K^1 < C_K^2 < \infty$ such that*

(1) $C_K^1\, \delta^{2d+2m} \le |B(q,\delta)| \le C_K^2\, \delta^{2d+2m}$ *for* $q \in K$;

(2) *If* $0 < \delta_1 \le \delta_2$ *and* $q_1,\ q_2 \in K$ *with* $B(q_1,\delta_1) \cap B(q_2,\delta_2) \neq \emptyset$, *then* $B(q_1,\delta_1) \subset B\left(q_2, C_K^2 \delta_2\right)$.

Here, $|B(q,\delta)|$ denotes the Lebesgue surface measure of the set $B(q,\delta)$.

We can now state our main result on the existence of a local integral representation formula for holomorphic functions and the corresponding estimates for plurisubharmonic functions on a domain with an edge given by a generic CR submanifold of type 2.

THEOREM 1. *Let $U \subset \mathbb{C}^n$ be an open set and let $M \subset U$ be a generic CR submanifold of type 2. Let Ω be a domain with edge M. Let $p \in M$, let $\gamma < \Gamma_p$, and let $\{L_1, \dots, L_m\}$ be a basis for the space $H^{1,0}(M)$ near p. There exist the following: a neighborhood $\omega \subset M$ of p; a constant $\epsilon > 0$; a constant $C < \infty$; and a C^∞ function*

$$K : \{\omega + \gamma_\epsilon\} \times M \to [0,\infty)$$

with the following properties:

(1) *For $z \in \omega+\gamma_\epsilon$ fixed, the function $K_z(\zeta) = K(z,\zeta)$ has compact support in $B(\Pi(z), C\sqrt{r(z)})$.*

(2) *For every noncommuting polynomial $P(L,\overline{L})$ of degree $k \geq 0$ in the vector fields $L_1, \ldots, L_m, \overline{L}_1, \ldots, \overline{L}_m$, and every compact set K' in M, there is a constant C so that*

$$\left|P\left(L,\overline{L}\right)(K_z)(\zeta)\right| \leq C\, r(z)^{-\frac{k}{2}} \left|B\left(\Pi(z), \sqrt{r(z)}\right)\right|^{-1}$$

for $(z,\zeta) \in \{\omega+\gamma_\epsilon\} \times K'$.

(3) *If F is a function which is continuous on the closure of the set Ω and holomorphic on Ω, then for $z \in \omega+\gamma$,*

$$F(z) = \int_M K(z,\zeta)\, F(\zeta)\, d\sigma(\zeta)$$

where $d\sigma$ is the surface measure on M.

(4) *If u is a function which is continuous on the closure of the set Ω and plurisubharmonic on Ω, then for $z \in \omega+\gamma$,*

$$|u(z)| \leq \int_M K(z,\zeta)\, |u(\zeta)|\, d\sigma(\zeta)$$

where $d\sigma$ is the surface measure on M.

The proof of Theorem 1 is accomplished in Sections 2,3, and 4.

2. The model case. The object of this Section is to study a very special case of a domain with an edge which is a generic CR submanifold of type 2. This example will serve as a model for the general case. As we shall see, this model domain and edge play the same role as the Siegel upper half space and the boundary Heisenberg group do for the study of strictly pseudoconvex domains.

We let $M_m^{\mathbb{C}} \cong \mathbb{C}^{m^2}$ denote the complex vector space of $m \times m$ complex matricies, and let H_m denote the real vector subspace of $m \times m$ Hermitian matricies. For any $r \times s$ complex matrix W, let W^* denote the conjugate transpose $s \times r$ matrix. In particular, for $W \in M_m^{\mathbb{C}}$, set

$$\Re(W) = \frac{1}{2}(W + W^*);$$
$$\Im(W) = \frac{1}{2\imath}(W - W^*).$$

Then for $W \in M_m$, $\Re(W)$ and $\Im(W)$ are Hermitian, $W = \Re(W) + \imath\Im(W)$, and $M_m^{\mathbb{C}} = H_m \oplus \imath H_m$. We can also decompose $M_m^{\mathbb{C}}$ as $M_m^{\mathbb{R}} \oplus \imath M_m^{\mathbb{R}}$ by $M_m^{\mathbb{C}} \ni Z \to \mathrm{Re}\{Z\} + \imath\,\mathrm{Im}\{Z\}$ where $M_m^{\mathbb{R}}$ is the space of real $m \times m$ matrices. Here, $\mathrm{Re}\{Z\}$ and $\mathrm{Im}\{Z\}$ denote the usual real and imaginary parts of a matrix Z with complex entries. It will be useful to know the relationship between these two decompostions. To this end, we define the map $A : M_m^{\mathbb{R}} \to H_m$ as follows. Let X be an element of $M_m^{\mathbb{R}}$. Decompose X as $D + U + L$ where D is the diagonal part of X; U is the upper triangular part of X (with diagonal entries set to 0); and L is the lower triangular part of X (with diagonal entries set to 0). Define $A(X)$ to be the matrix $D + (U + \imath L^t) + (U^t - \imath L)$. We can extend A to act on all of $M_m^{\mathbb{C}}$ by complex linearity (the range of A then becomes $M_m^{\mathbb{C}}$). It is an easy exercise to show the following:

$$\Re(A(Z)) = A(\mathrm{Re}(Z))$$
$$\Im(A(Z)) = A(\mathrm{Im}(Z)).$$

Since $M_m^{\mathbb{R}}$ is a totally real subspace of $M_m^{\mathbb{C}}$, one immediate consequence of this is that the splitting of $M_m^{\mathbb{C}}$ into $H_m \oplus \imath H_m$ exhibits H_m as a maximal totally real subspace of $M_m^{\mathbb{C}}$.

We shall view elements of $\mathbb{C}^m$ as $m \times 1$ complex matricies, and hence if

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} \quad \text{then} \quad z^* = [\overline{z}_1, \ldots, \overline{z}_m].$$

Define a quadratic form $Q_m : \mathbb{C}^m \to H_m$ given by

$$Q_m(z) = z\,z^* = \begin{bmatrix} z_1\overline{z}_1 & z_1\overline{z}_2 & \ldots & z_1\overline{z}_m \\ z_2\overline{z}_1 & z_2\overline{z}_2 & \ldots & z_2\overline{z}_m \\ \vdots & \vdots & \ddots & \vdots \\ z_m\overline{z}_1 & z_m\overline{z}_2 & \ldots & z_m\overline{z}_m \end{bmatrix}$$

DEFINITION 2.1. Define $\rho : \mathbb{C}^m \times M_m^{\mathbb{C}} \to H_m$ by setting

$$\rho(z, Z) = \Re(Z) - Q_m(z).$$

Then set

$$\begin{aligned}\Sigma_m &= \left\{(z; Z) \in \mathbb{C}^m \times M_m^{\mathbb{C}} \,\middle|\, \Re(Z) = z\,z^*\right\} \\ &= \left\{(z; Z) \in \mathbb{C}^m \times M_m^{\mathbb{C}} \,\middle|\, \rho(z; Z) = 0\right\},\end{aligned}$$

and

$$\begin{aligned}\Omega_m &= \left\{(z; Z) \in \mathbb{C}^m \times M_m^{\mathbb{C}} \,\middle|\, \Re(Z) > z\,z^*\right\} \\ &= \left\{(z; Z) \in \mathbb{C}^m \times M_m^{\mathbb{C}} \,\middle|\, \rho(z; Z) > 0\right\}.\end{aligned}$$

Here and below, we adopt the following notation: for matrices M_1 and M_2, we say that $M_1 > M_2$ if $M_1 - M_2$ is positive definite.

It is easy to check that Σ_m is a generic CR submanifold of $\mathbb{C}^m \times M_m^{\mathbb{C}} \cong \mathbb{C}^{m+m^2}$ of type 2, and that Ω_m is a domain with edge Σ_m. We often identify Σ_m with $\mathbb{C}^m \times H_m$ via the correspondence

$$\mathbb{C}^m \times H_m \ni (z, Y) \leftrightarrow (z; z\,z^* + \imath Y) \in \Sigma_m.$$

If $(z, Z) \in \Omega_m$, then $Z = zz^* + X + \imath Y$ with $X, Y \in H_m$, and $X > 0$. There is a natural projection $\pi : \Omega_m \to \Sigma_m$ given by

$$\pi\Big((z; z\,z^* + X + \imath Y)\Big) = (z; z\,z^* + \imath Y).$$

Each element $(w; W) \in \mathbb{C}^m \times M_m^{\mathbb{C}}$ defines a holomorphic mapping of $\mathbb{C}^m \times M_m^{\mathbb{C}}$ to itself as follows:

DEFINITION 2.2. For $(w; W) \in \mathbb{C}^m \times M_m^{\mathbb{C}}$, let $T_{(w;W)} : \mathbb{C}^m \times M_m^{\mathbb{C}} \to \mathbb{C}^m \times M_m^{\mathbb{C}}$ be the holomorphic map given by

$$T_{(w;W)}\Big((z; Z)\Big) = \Big(z + w; Z + W + 2zw^*\Big).$$

We have the following proposition whose easy proof we leave to the reader.

PROPOSITION 2.3. $T_{(w_1;W_1)} \circ T_{(w_2;W_2)} = T_{(w_3,W_3)}$ *where*

$$(w_3; W_3) = (w_2 + w_1; W_2 + W_1 + 2w_2 w_1^*) = T_{(w_1;W_1)}\Big((w_2; W_2)\Big).$$

Moreover

$$\rho\left(T_{(w;W)}(z;Z)\right) = \rho(z;Z) + \rho(w;W).$$

It now follows that if $(w;W) \in \Sigma_m$, then $T_{(w;W)}$ carries Σ_m to itself and Ω_m to itself. These calculations show that Σ_m is a group under the multiplication

$$(w;W)\cdot(z;Z) = (z+w;Z+W+2zw^*).$$

We can also define an action of $\mathrm{GL}(m,\mathbb{C})$ on $\mathbb{C}^m \times M_m^{\mathbb{C}}$ as follows.

DEFINITION 2.4. For $g \in \mathrm{GL}(m,\mathbb{C})$, let $S_g : \mathbb{C}^m \times M_m^{\mathbb{C}} \to \mathbb{C}^m \times M_m^{\mathbb{C}}$ be the holomorphic map given by

$$S_g\big((z;Z)\big) \equiv (g\,z; g\,Z\,g^*).$$

PROPOSITION 2.5. *This action of* $\mathrm{GL}(m,\mathbb{C})$ *preserves* Σ_m *and* Ω_m*, and is a group of automorphisms of* Σ_m*.*

Proof. We have

$$\Re(g\cdot W\cdot g^*) = g\cdot\Re(W)\cdot g^*.$$

If $\Re(W) \geq z\,z^*$, then $\Re(g\cdot W\cdot g^*) \geq (g\cdot z)(g\cdot z)^*$, with equality in one equation if and only if there is equality in the other. Also

$$\begin{aligned} S_g\big((w;W)\cdot(z;Z)\big) &= S_g\big((z+w;Z+W+2zw^*)\big) \\ &= (g\,z+g\,w; g\,Z\,g^* + g\,W\,g^* + 2(g\,z)(g\,w)^*) \\ &= S_g\big((w;W)\big)\cdot S_g\big((z;Z)\big). \end{aligned}$$

□

We are interested in the existence of analytic discs in Ω_m with boundary in Σ_m. Every analytic disc in $\mathbb{C}^m \times M_m^{\mathbb{C}}$ is a continuous map $A = (Z;W) : \overline{\mathbb{D}} \to \mathbb{C}^m \times M_m^{\mathbb{C}}$ which is holomorphic on $\mathbb{D}$. We write

$$Z(\zeta) = \begin{bmatrix} Z_1(\zeta) \\ \vdots \\ Z_m(\zeta) \end{bmatrix},$$

where each $Z_j(\cdot)$ is a (scalar) holomorphic function. $W(\cdot)$ is an $m \times m$ matrix-valued holomorphic function. Such an analytic disc maps the boundary of the unit disc $\mathbb{D}$ into Σ_m if and only if

$$\Re\left(W\left(e^{i\theta}\right)\right) = Z\left(e^{i\theta}\right) \cdot Z\left(e^{i\theta}\right)^*$$

for $0 \leq \theta \leq 2\pi$.

We shall introduce the notation $A_0(\zeta)$ for the special analytic disc

$$A_0(\zeta) = (Z_0(\zeta); W_0(\zeta))$$

where

$$Z_0(\zeta) = \begin{bmatrix} \zeta \\ \zeta^2 \\ \vdots \\ \zeta^m \end{bmatrix},$$

and

$$W_0(\zeta) = \begin{bmatrix} 1 & 0 & 0 & \ldots 0 \\ 2\zeta & 1 & 0 & \ldots 0 \\ 2\zeta^2 & 2\zeta & 1 & \ldots 0 \\ \vdots & \vdots & \vdots & \ddots \vdots \\ 2\zeta^{m-1} & 2\zeta^{m-2} & 2\zeta^{m-3} & \ldots 1 \end{bmatrix}.$$

Note that

$$\Re(W_0(\zeta)) = \begin{bmatrix} 1 & \bar{\zeta} & \bar{\zeta}^2 & \ldots \bar{\zeta}^{m-1} \\ \zeta & 1 & \bar{\zeta} & \ldots \bar{\zeta}^{m-2} \\ \zeta^2 & \zeta & 1 & \ldots \bar{\zeta}^{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \zeta^{m-1} & \zeta^{m-2} & \zeta^{m-3} & \ldots & 1 \end{bmatrix}.$$

Also,

$$Q_m(Z_0(\zeta)) = \begin{bmatrix} \zeta\bar{\zeta} & \zeta\bar{\zeta}^2 & \zeta\bar{\zeta}^3 & \ldots & \zeta\bar{\zeta}^m \\ \zeta^2\bar{\zeta} & \zeta^2\bar{\zeta}^2 & \zeta^2\bar{\zeta}^3 & \ldots & \zeta^2\bar{\zeta}^m \\ \zeta^3\bar{\zeta} & \zeta^3\bar{\zeta}^2 & \zeta^3\bar{\zeta}^3 & \ldots & \zeta^3\bar{\zeta}^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \zeta^m\bar{\zeta} & \zeta^m\bar{\zeta}^2 & \zeta^m\bar{\zeta}^3 & \ldots & \zeta^m\bar{\zeta}^m \end{bmatrix}.$$

Thus when $|\zeta| = 1$,

$$\Re(W_0(\zeta)) = Q_m(Z_0(\zeta)) = Z_0(\zeta) \cdot Z_0(\zeta)^*$$

and hence A_0 maps $\partial\mathbb{D}$ to Σ_m. If $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{C}^m$, then $\left(\Re(W_0(\zeta))\xi, \xi\right)$ is a harmonic function of ζ, while $\left(Q_m(Z_0(\zeta))\xi, \xi\right) = \left|\sum_{j=1}^m \zeta^j \xi_j\right|^2$ is a subharmonic function of ζ. It is easy to check that $\left((\Re(W_0(0)) - Q_m(Z_0(0)))\xi, \xi\right) = |\xi|^2$ and hence it follows from the minimum principle for superharmonic functions that $\Re(W_0(\zeta)) - Q_m(Z(\zeta)) > 0$ for $|\zeta| < 1$. Hence the analytic disc A_0 maps the open unit disc to Ω_m.

We now construct a local integral representation formula for Ω_m. Let F be holomorphic on Ω_m, and continuous on $\overline{\Omega_m}$. Let I denote the $m \times m$ identity matrix. Note that $(0; I) \in \Omega_m$. Our object is to construct a representing measure on Σ_m for the point $(0; I) \in \Omega_m$ which has compact support and C^∞ density; i.e. to find a C^∞ function K_0 with compact support, such that

$$F((0; I)) = \int_{\Sigma_m} F(\zeta)\, K_0(\zeta)\, d\lambda(\zeta)$$

where $d\lambda$ is Lebesgue measure on $\Sigma_m = \mathbb{C}^m \times H_m$. Later, we shall obtain a representation formula for other points in Ω_m by making use of the group structure on Ω_m.

Since F is holomorphic on the subset of the complex subspace

$$\left\{(0; W) \in \mathbb{C}^m \times M_m^{\mathbb{C}} \,\middle|\, \Re(W) > 0\right\},$$

we can use the mean value property to conclude that

$$F(0; I) = \int_{M_m^{\mathbb{C}}} F(0; I + W)\, \varphi(W)\, dW,$$

where φ is any radial function with compact support in a small neighborhood of $0 \in M_m^{\mathbb{C}}$ and with total integral equal to one. Write $W = H + \imath Y$ where both $H, Y \in H_m$. Thus for a suitable function φ with compact support we have

$$F(0; I) = \int_{H_m} \int_{H_m} F(0; (I + H) + \imath Y)\, \varphi(H, Y)\, dH\, dY.$$

Note that the integrand has support in a small neighborhood of the origins of each copy of H_m. Next, we make the change of variables

$$(I + H) = (I + X)^2,$$

that is,

$$H = 2X + X^2$$

where $X \in H_m$. The mapping $X \to 2X + X^2$ is a diffeomorphism from some small neighborhood of the origin of H_m to a neighborhood of the origin of H_m. Hence there is a smooth function ψ with support near the origin of $H_m \times H_m$ so that

$$F(0; I) = \int_{H_m} \int_{H_m} F(0; (I + X)^2 + iY)\psi(X, Y)\, dX\, dY.$$

Let $U(m)$ denote the group of $m \times m$ unitary matricies, and let dg denote Haar measure on $U(m)$. Since $Z_0(0) = 0$ and $W_0(0) = I$, for every $g \in U(m)$, we have

$$\begin{aligned} &F(0; (I + X)^2 + iY) \\ &\qquad = F((I + X)\, g\, Z_0(0); (I + X)\, g\, W_0(0)\, g^*\, (I + X) + iY) \end{aligned}$$

and so $F(0; I)$ is given by the integral

$$\begin{aligned} \iiint\limits_{H_m \times H_m \times U(m)} & F((I + X)\, g\, Z_0(0); \\ & (I + X)\, g\, W_0(0)\, g^*\, (I + X) + iY)\, \psi(X, Y)\, dg\, dX\, dY. \end{aligned}$$

We perform the translation $\hat{X} = I + X$ and let $\hat{\psi}(\hat{X}, Y) = \psi(\hat{X} - I, Y)$. Then, we drop the $\hat{}$. Since the map

$$\mathbb{D} \ni \zeta \to (X\, g\, Z_0(\zeta); X\, g\, W_0(\zeta)\, g^*\, X + iY)$$

is an analytic disc, the mean value property of holomorphic functions implies that $F(0; I)$ is given by the integral

$$\begin{aligned} \frac{1}{2\pi} \int_{H_m} \int_{H_m} \int_{U(m)} \int_0^{2\pi} & F(X\, g\, Z_0(e^{i\theta}); \\ & X\, g\, W_0(e^{i\theta})\, g^*\, X + iY)\, \psi(X, Y)\, d\theta\, dg\, dX\, dY. \end{aligned}$$

Since $\Re(W_0(e^{i\theta})) = Z_0(e^{i\theta}) \cdot Z_0(e^{i\theta})^*$, the point

$$\Psi(X, Y, g, \theta) = \left(X\, g\, Z_0\left(e^{i\theta}\right); X\, g\, W_0\left(e^{i\theta}\right)\, g^*\, X + iY\right)$$

belongs to Σ_m for any $X, Y \in H_m$, $g \in U(m)$ and $\theta \in [0, 2\pi]$. We now study this mapping $\Psi : H_m \times H_m \times U(m) \times [0, 2\pi] \to \Sigma_m$. We have

LEMMA 2.6. *For each fixed $g_0 \in U(m)$ and $\theta_0 \in [0, 2\pi]$, the mapping Ψ has maximal rank at the point $(I, 0, g_0, \theta_0)$.*

Proof. After we identify Σ_m with $\mathbb{C}^m \times H_m$, the mapping Ψ becomes

$$\Psi(X, Y, g, \theta) = (X\, g\, Z_0(e^{i\theta}), Y) \in \mathbb{C}^m \times H_m.$$

Thus it suffices to show that the mapping

$$(X, g, \theta) \to X\, g \begin{bmatrix} e^{i\theta} \\ e^{2i\theta} \\ \vdots \\ e^{mi\theta} \end{bmatrix} \in \mathbb{C}^m$$

has rank m at the point $(I, g_0, \theta_0))$. This is clear if we restrict X to diagonal matricies and g to diagonal multiples of g_0. This completes the proof. □

Using this lemma, we can integrate out the extra variables in the above integral formula for $F(0, I)$ and obtain the following.

LEMMA 2.7. *There is a non-negative function $K_0 \in C_0^\infty(\Sigma_m)$ such that:*

(1) *For every function F which is continuous on $\overline{\Omega_m}$ and holomorphic on Ω,*

$$\begin{aligned} F(0; I) &= \int_{\Sigma_m} F(\xi)\, K_0(\xi)\, d\lambda(\xi) \\ &= \int_{\mathbb{C}^m} \int_{H_m} F(z, Y) K_0(z, Y)\, dY\, dz; \end{aligned}$$

(2) *For every function u which is continuous on* $\overline{\Omega_m}$ *and plurisubharmonic on* Ω,

$$|u(0;I)| \leq \int_{\Sigma_m} |u(\xi)|\, K_0(\xi)\, d\lambda(\xi)$$
$$= \int_{\mathbb{C}^m} \int_{H_m} |u(z,Y)| K_0(z,Y)\, dY\, dz.$$

In the above formulas, we have identified a point $(z, Z) \in \Sigma_m$ with $(z, Y) \in \mathbb{C}^m \times H_m$ where $Y = \Im(Z)$. In the derivation of (1), we dealt with representing the value of a holomorphic function, but if the function is plurisubharmonic, all equalities are replaced by inequalities and (2) is obtained.

We can obtain representation formulas for other points in Ω_m by making use of the group structure and dilations on Σ_m in the formulas in Lemma 2.7.

Let $(z_0; Z_0) = (z_0; z_0 z_0^* + X_0 + \imath Y_0)$ be a point in Ω_m. Since $X_0 > 0$, X_0 has a unique positive square root $X_0^{\frac{1}{2}} \in H_m$. Then

$$(z_0; Z_0) = (z_0; z_0 z_0^* + X_0 + \imath Y_0) = T_{(z_0; \imath Y_0 + z_0 z_0^*)} S_{X_0^{\frac{1}{2}}}\Big((0; I)\Big).$$

If F is continuous on $\overline{\Omega_m}$ and holomorphic on Ω_m, put

$$F_{(z_0;Z_0)}(z; Z) = F(T_{(z_0; \imath Y_0 + z_0 z_0^*)} S_{X_0^{\frac{1}{2}}}\Big((z; Z)\Big),$$

so that $F_{(z_0;Z_0)}$ is again continuous on $\overline{\Omega_m}$ and holomorphic on Ω_m. By the last lemma, we have

$$F\Big((z_0;Z_0)\Big)$$
$$= F_{(z_0;Z_0)}\Big((0;I)\Big)$$
$$= \iint_{\mathbb{C}^m \times H_m} F_{(z_0;Z_0)}(z,Y) K_0(z,Y)\, dY\, dz$$
$$= \iint_{\mathbb{C}^m \times H_m} F(T_{(z_0; \imath Y_0 + z_0 z_0^*)} S_{X_0^{\frac{1}{2}}}\Big((z; Y)\Big) K_0(z,Y)\, dY\, dz$$

$$= \iint_{\mathbb{C}^m \times H_m} F(T_{(z_0; \imath Y_0 + z_0 z_0^*)}\big((z;Y)\big)$$
$$K_0(S^{-1}_{X_0^{\frac{1}{2}}}\big((z,Y)\big) \det(X_0)^{-m-1}\, dY\, dz$$
$$= \iint_{\mathbb{C}^m \times H_m} F\big((z;Y)\big)$$
$$K_0(S^{-1}_{X_0^{\frac{1}{2}}} T^{-1}_{(z_0; \imath Y_0 + z_0 z_0^*)}\big((z,Y)\big) \det(X_0)^{-m-1}\, dY\, dz.$$

Thus we have proved

THEOREM 2.8. *There is a non-negative, C^∞ function $K : \Omega_m \times \Sigma_m \to [0, \infty)$ such that if*

$$(z_0; Z_0) = (z_0; z_0 z_0^* + X_0 + \imath Y_0) \in \Omega_m$$

then:

(1) *For every function F which is continuous on $\overline{\Omega_m}$ and holomorphic on Ω_m,*

$$F(z_0; Z_0) = \iint_{\mathbb{C}^m \times H_m} F(z, Y)\, K\big((z_0; Z_0), (z, Y)\big)\, dY\, dz.$$

(2) *For every function u which is continuous on $\overline{\Omega_m}$ and plurisubharmonic on Ω_m,*

$$|u(z_0; Z_0)| \le \iint_{\mathbb{C}^m \times H_m} |u(z, Y)|\, K\big((z_0; Z_0), (z, Y)\big)\, dY\, dz.$$

Moreover,

$$K\left((z_0; z_0 z_0^* + X_0 + \imath Y_0), (z, Y)\right) =$$
$$\det(X_0)^{-m-1} K_0\left(X_0^{-\frac{1}{2}}(z - z_0), X_0^{-\frac{1}{2}}\left(Y - Y_0 - 2\Im(z z_0^*)\right) X_0^{-\frac{1}{2}}\right).$$

In later Sections, we shall be particularly concerned with the situation when the point (z_0, Z_0) lies in the normal space to the manifold Σ_m at the point $(0, 0)$; i.e., we consider the case when $(z_0, Z_0) = (0, X_0)$ with $X_0 > 0$. In this case, the reproducing kernel takes the form

$$K\big((0, X_0), (z, Y)\big) = \det(X_0)^{-m-1} K_0\left(X_0^{-\frac{1}{2}} z, X_0^{-\frac{1}{2}} Y X_0^{-\frac{1}{2}}\right).$$

In other words, the reproducing kernel for points in the normal to the origin is obtained from the reproducing kernel for the fixed point $(0, I)$ by dilation by the matrix $X_0^{-\frac{1}{2}}$.

Theorem 2.8 can easily be used to establish Theorem 1 for the model case. Statements (1) and (2) of theorem 2.8 establish statements (3) and (4) of Theorem 1. To show the support property (1) of Theorem 1, we first note that Γ_0 is the cone H_m^+ which by definition is the set of positive definite, Hermitian symmetric $m \times m$ matrices. Suppose $\gamma < H_m^+$ is given. The distance from the point $p = (z, z \cdot z^* + X)$ to Σ_m (denoted $r(p)$ in Theorem 1) is proportional to $||X||$, which in turn is proportional to the m^{th} root of $\det X$. The proportionality constant can be chosen to depend only on γ. Since K_0 has support in a fixed compact set, the desired support property for K follows from the above formula for K. The desired estimate on K also follows from this formula and from the fact that $\left|B(\Pi(p), \sqrt{r(p)})\right| \cong r(p)^{m+m^2} \cong (\det X)^{m+1}$.

Though it is nice to have an explicit formula for K, such as the one given above, an explicit formula is not necessary to establish the desired support property and estimate stated in (1) and (2) of Theorem 1. These properties can be established by first finding a representing kernel K for points of the form $z = (0, X)$ for X of unit norm and then rescaling. For example, suppose we have a smooth function $(z, \zeta) \to K(z, \zeta)$ with compact ζ-support in Σ_m, such that

$$F(0, X) = \int_{\zeta \in \Sigma_m} F(\zeta)\, K((0, X), \zeta) d\sigma(\zeta) \quad \text{for } ||X|| = 1$$

for holomorphic F. Now suppose $||X|| = \epsilon$. Define the scale map $S_\epsilon(z, Z) = (\epsilon^{-\frac{1}{2}} z, \epsilon^{-1} Z)$. The sets Ω_m and Σ_m are invariant under this scale map. S_ϵ takes the point $(0, X)$ to $(0, \epsilon^{-1} X)$ and $\epsilon^{-1} X$ has unit norm. Using the above representation, we obtain

$$\begin{aligned} F(0, X) &= F(S_\epsilon^{-1}(0, \epsilon^{-1} X)) \\ &= \int_{\Sigma_m} F(S_\epsilon^{-1}(\zeta))\, K((0, \epsilon^{-1} X), \zeta)\, d\sigma(\zeta) \\ &= \int_{\Sigma_m} F(\zeta)\, K((0, \epsilon^{-1} X), S_\epsilon(\zeta))\, \epsilon^{-(m+m^2)}\, d\sigma(\zeta) \end{aligned}$$

The kernel $K_\epsilon(z,\zeta) = K(z, S_\epsilon(\zeta))$ has ζ-support in the ball $B(0,\sqrt{\epsilon})$ and ϵ is proportional to the distance from z to Σ. Since ϵ^{m+m^2} is comparable to the measure of $B(0,\sqrt{\epsilon})$, the estimate in (2) also follows from the fact that $K(z,\zeta)$ has compact ζ-support.

This idea of first representing holomorphic functions at points of the form $(0,X)$ with $||X|| = 1$ and then rescaling will be used for more general edges where explicit formulas are too complicated to analyze directly.

3. Free generic CR submanifolds of type 2. For a general type 2 submanifold of CR dimension m, if $\{L_1,\dots,L_m\}$ is a basis for the space of tangential vector fields of type $(1,0)$ near a given point p_0, then the collection of $2m+m^2$ vector fields

$$\{L_1,\dots,L_m,\overline{L}_1,\dots,\overline{L}_m,\dots,[L_j,\overline{L}_k],\dots\}$$

span the complexified tangent space at each point near p_0. In general, there may be linear relations between these vector fields, and these relations may change from point to point. (The subset of vector fields $\{L_1,\dots,L_m,\overline{L}_1,\dots,\overline{L}_m\}$ is of course always linearly independent.) These changing relationships may lead to abrupt changes in the local nature of CR functions on M, and present difficulties when studying local analytic discs with boundaries in M near p_0. In this Section, we study a particularly simple class of generic CR submanifolds of type 2 where these difficulties are not present.

DEFINITION 3.1. Let $M \subset U \subset \mathbb{C}^n$ be a generic CR submanifold of type 2. Let p_0 be a point in M. Then M is *free* at p_0 if for some (and hence for any) choice of basis $\{L_1,\dots,L_m\}$ for the tangential vector fields of type $(1,0)$ near p_0, the $2m+m^2$ vector fields

$$\left\{L_1,\dots,L_m,\overline{L}_1,\dots,\overline{L}_m,\dots,[L_j,\overline{L}_k],\dots\right\}$$

are linearly independent at p_0 (and hence in a neighborhood of p_0). This is equivalent to the condition that the m^2 vector fields $\{\dots,[L_j,\overline{L}_k],\dots\}$ are linearly independent near p_0.

On a general generic submanifold of type 2, of CR dimension m and real codimension d, the "missing" d directions in T_pM which are not in $T_p^{\mathbb{C}}M$ are spanned by the m^2 vector fields $\{\dots,[L_j,\overline{L}_k],\dots\}$.

It follows that in general, the codimension d is at most m^2. However if M is free, we must have $d = m^2$. Thus we have

PROPOSITION 3.2. *A generic CR submanifold $M \subset U \subset \mathbb{C}^n$ of type 2 and CR dimension m is free if and only if the real codimension is m^2.*

The first main objective of this Section is to show that a generic CR submanifold of type 2 which is free can be well approximated in an appropriate sense by the model CR submanifold Σ_m. We need to introduce notations for certain spaces of mappings.

Let $C^2(\mathbb{C}^m \times H_m; H_m)_0$ denote the space of C^2 mappings

$$h : \mathbb{C}^m \times H_m \to H_m$$

with compact support which satisfy

$$\begin{aligned} &\text{a)} \quad h(0,0) = 0; \\ &\text{b)} \quad \nabla h(0,0) = 0; \\ &\text{c)} \quad \nabla_z^2 h(0,0) = 0. \end{aligned}$$

Here ∇ denotes the gradient with respect to all variables, and ∇_z denotes the gradient only with respect to the variables $z \in \mathbb{C}^m$. For $h \in C^2(\mathbb{C}^m \times H_m; H_m)_0$ we set

$$||h||_2 = ||h||_{\text{sup}} + ||\nabla h||_{\text{sup}} + ||\nabla^2 h||_{\text{sup}}.$$

The condition $h \in C^2(\mathbb{C}^m \times H_m; H_m)_0$ means that near the origin, we have $h(z, Y) = O(|z|^3 + |z||Y| + |Y|^2)$. If we give the z-coordinate weight 1 and the Y-coordinate weight 2, then h vanishes to third order at the origin.

LEMMA 3.3. *Let $U \subset \mathbb{C}^n$ be open, and let $M \subset U$ be a free generic CR submanifold of U of type 2. Let p_0 be a point in M. Then there exist an open neighborhood V of the origin in $\mathbb{C}^m \times M_m^{\mathbb{C}}$, a neighborhood ω of p_0 in $\mathbb{C}^n$, a C^∞ mapping $h : \omega \cap M \to C^2(\mathbb{C}^m \times H_m; H_m)_0$, and a C^∞ mapping Ψ from $\omega \cap M$ to the space of biholomorphic mappings of $\mathbb{C}^n$ with the following properties (we write h_p and Ψ_p instead of $h(p)$ and $\Psi(p)$):*

(1) *For each $p \in \omega \cap M$ there is a neighborhood U_p of p in $\mathbb{C}^n$ so that Ψ_p is a biholomorphic mapping of U_p to V.*

(2) $\Psi_p(p) = (0,0)$, *and*

$$d\Psi_p(T_pM) = \left\{(z,W) \in \mathbb{C}^m \times M_m^{\mathbb{C}} \,\middle|\, \Re(W) = 0\right\};$$
$$d\Psi_p(T_p^{\mathbb{C}}M) = \left\{(z,W) \in \mathbb{C}^m \times M_m^{\mathbb{C}} \,\middle|\, W = 0\right\}$$
$$d\Psi_p(N_pM \cap U_p) = \left\{(z,W) \in V \,\middle|\, z = 0, \Im(W) = 0\right\}.$$

(3) *If we set* $M_p = \Psi_p(M \cap U_p)$ *then* M_p *is a free generic CR submanifold of* V, *and*

$$M_p = \left\{(z,W) \in V \subset \mathbb{C}^m \times M_m^{\mathbb{C}} \,\middle|\, \Re(W) = z\,z^* + h_p(z, \Im(W))\right\}.$$

(4) *For* $p \in \omega \cap M$, *the function* h_p *is infinitely differentiable, and has compact support on* $\mathbb{C}^m \times H_m$.

The point of this lemma, expressed explicitly in item (3), is that near each point $p \in M$, we can make a biholomorphic change of variables so that M has the same equation as the model case except for a third order error term.

Proof. The argument is fairly standard. Recall that according to the proof of Proposition 1.1, for any $p \in M$, there is a neighborhood U_p of p, a translation and unitary change of variables Ψ'_p, and a neighborhood V' of the origin in $\mathbb{C}^{m+m^2}$ so that

$$\Psi'_p(M \cap U_p) = \left\{(z,w) \in \mathbb{C}^{m+m^2} \,\middle|\, \mathrm{Re}(w) = g_p(z, \mathrm{Im}(w))\right\}$$

where

$$g_p = (g_{p,1}, \dots, g_{p,m^2}) : \mathbb{C}^{m+m^2} \to \mathbb{R}^{m^2}$$

is a function of class $\mathbb{C}^\infty$ with $g_p(0,0) = 0$, $\nabla g_p(0,0) = 0$, and $\dfrac{\partial^2 g_p}{\partial z_j \partial z_k}(0,0) = 0$ for $1 \leq j,k \leq m$. Near a fixed point p_0, the change of variables Ψ'_p can be made to depend smoothly on p. Moreover, the hypotheses on M imply that the Hermitian bilinear form

$$B_p(\xi,\eta) = \left(\sum_{j,k=1}^{m} \frac{\partial^2 g_{p,1}}{\partial z_j \partial \overline{z}_k}(0,0)\xi_j\overline{\eta}_k, \dots, \sum_{j,k=1}^{m} \frac{\partial^2 g_{p,m^2}}{\partial z_j \partial \overline{z}_k}(0,0)\xi_j\overline{\eta}_k\right)$$

is surjective.

To complete the proof of Lemma 3.3, we need a result on the universality of the Hermitian form $B_m : \mathbb{C}^m \times \mathbb{C}^m \to M_m^{\mathbb{C}}$, where

$$B_m(\xi, \eta) = \begin{bmatrix} \xi_1\overline{\eta}_1 & \xi_1\overline{\eta}_2 & \cdots & \xi_1\overline{\eta}_m \\ \xi_2\overline{\eta}_1 & \xi_2\overline{\eta}_2 & \cdots & \xi_2\overline{\eta}_m \\ \vdots & \vdots & \ddots & \vdots \\ \xi_m\overline{\eta}_1 & \xi_m\overline{\eta}_2 & \cdots & \xi_m\overline{\eta}_m \end{bmatrix}.$$

This result will also be needed in §4. □

LEMMA 3.4. *Let $B : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^d$ be any Hermitian bilinear form. There exist a unique real linear map $T_B : H_m \to \mathbb{R}^d$ and associated complex linear map $\tilde{T}_B : M_m^{\mathbb{C}} \to \mathbb{C}^d$ such that for all $\xi, \eta \in \mathbb{C}^m$,*

$$B(\xi, \eta) = \tilde{T}_B(B_m(\xi, \eta)).$$

Here, $\tilde{T}_B(X + \imath Y) = T_B(X) + \imath T_B(Y)$, where $X, Y \in H_m$.

Proof. Let $\hat{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$ be the standard basis element of $\mathbb{C}^m$ which is the m-tuple consisting of zeros except for a 1 in the j^{th} spot. A basis over $\mathbb{C}$ for $M_m^{\mathbb{C}}$ is given by the collection of matricies $\hat{e}_{j,k} = \{B_m(\hat{e}_j, \hat{e}_k)\}$, $1 \leq j, k \leq m$. We then obtain a basis over $\mathbb{R}$ for H_m by setting

$$\begin{aligned} \hat{f}_{j,j} &= \hat{e}_{j,j}, \quad 1 \leq j \leq m, \\ \hat{f}_{j,k} &= \Re(\hat{e}_{j,k}) \quad 1 \leq j < k \leq m, \\ \hat{f}_{j,k} &= \Im(\hat{e}_{k,j}) \quad 1 \leq k < j \leq m. \end{aligned}$$

We define the real linear map T_B by prescribing its action on this basis. We set:

$$\begin{aligned} T_B(\hat{f}_{j,j}) &= B(\hat{e}_j, \hat{e}_j), \quad 1 \leq j \leq m, \\ T_B(\hat{f}_{j,k}) &= \frac{1}{2}\Big(\overline{B(\hat{e}_j, \hat{e}_k)} + B(\hat{e}_j, \hat{e}_k)\Big), \quad 1 \leq j < k \leq m, \\ T_B(\hat{f}_{j,k}) &= \frac{1}{2\imath}\Big(\overline{B(\hat{e}_j, \hat{e}_k)} - B(\hat{e}_j, \hat{e}_k)\Big), \quad 1 \leq k < j \leq m. \end{aligned}$$

It is easy to check that the operator defined in this way satisfies the required properties, and is unique. □

COROLLARY. *If $B : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{R}^d$ is a Hermitian bilinear form which is surjective, then the real linear mapping $T_B : H_m \to \mathbb{R}^d$ for the associated bilinear mapping is surjective.*

We can now complete the proof of Lemma 3.3. Since the bilinear form B_p is surjective, the corresponding linear mapping $\tilde{T}_{B_p}$: $M_m^{\mathbb{C}} \to \mathbb{C}^{m^2}$ is surjective, and hence bijective. If we let Ψ_p denote the composition of the biholomorphic mapping Ψ'_p with the map

$$(z, w) \to (z, \tilde{T}_{B_p}^{-1}(w))$$

we obtain the the biholomorphic mapping whose existence is asserted by the theorem. The rest of the argument is standard. □

For the model domain, our reproducing formula was constructed from an explicit family of analytic discs whose boundary lies in Σ_m and whose centers sweep out an appropriate cone in the normal space. Since M_p is well approximated by Σ_m, we expect that there will be a similar family of discs in this case, at least if the size of the discs is kept small. In order to see that this is so, we need to briefly recall the analysis of Bishop's equation. (The solution to Bishop's equation is discussed thoroughly in [**B**].)

Let $C^{1,\alpha}(S^1; H_m)$ denote the space of continuously differentiable functions from the unit circle S^1 to H_m whose first derivatives satisfy a Hölder condition of order α. Let $A_0^{1,\alpha}(\mathbb{D}; \mathbb{C}^m)$ denote the space of continuously differentiable functions from the unit circle to $\mathbb{C}^m$ whose first derivatives satisfy a Hölder condition of order α, and which are boundary values of holomorphic functions on the unit disc which vanish at the origin.

For $h \in C^2(\mathbb{C}^m \times H_m; H_m)_0$, put

$$M_h = \left\{(z, X + \imath Y) \in \mathbb{C}^m \times M_m^{\mathbb{C}} \,\middle|\, X = zz^* + h(z, Y)\right\}.$$

Here and below, we let $X \in H_m$ denote the coordinate $\Re(W)$ and $Y \in H_m$ denote $\Im(W)$ for $W \in M_m^{\mathbb{C}}$. Define a map

$$H : C^2(\mathbb{C}^m \times H_m; H_m)_0 \times A_0^{1,\alpha}(\mathbb{D}; \mathbb{C}^m) \times C^{1,\alpha}(S^1; H_m) \times H_m \to C^{1,\alpha}(S^1; H_m)$$

by setting

$$H_h(Z, v, Y)(\zeta) = h(Z(\zeta), v(\zeta) + Y)$$

for $|\zeta| = 1$. Then H is a smooth map, and the hypotheses on h imply

$$\begin{aligned} H_h(0,0,0) &= 0 \\ D_Z H_h(0,0,0) &= 0 \\ D_v H_h(0,0,0) &= 0 \\ D_Y H_h(0,0,0) &= 0 \end{aligned}$$

where D_* stands for the Frechet derivative with respect to the given variable $*$.

For a continuous function $u : S^1 \to M_m^{\mathbb{R}}$, the Hilbert transform of u is the function $Tu : S^1 \to M_m^{\mathbb{R}}$ with the property that $u + iTu : S^1 \to M_m^{\mathbb{C}}$ is the boundary values of an analytic function $G : \mathbb{D} \to M_m^{\mathbb{C}}$ with $\operatorname{Im} G(0) = 0$. We wish to modify this definition of the Hilbert transform as follows. Let u be a continuous function from S^1 to H_m; let $G : \mathbb{D} \to M_m^{\mathbb{C}}$ be the unique analytic function with $\Re(G)|_{S^1} = u$ and $\Im(G)(0) = 0$; then define $\tilde{T}u$ to be the restriction of $\Im(G)$ to S^1. Using the map A defined at the beginning of Section 2 and the discussion there, it is clear that $\tilde{T} = A \circ T \circ A^{-1}$. From here on, we drop the $\tilde{}$ and denote this modified Hilbert transform by T. T is a continuous linear mapping from the space $C^{1,\alpha}(S^1; H_m)$ to itself provided $0 < \alpha < 1$. From now on, we fix such an α, say $\alpha = \frac{1}{2}$. We denote the norm on this space by $|| \cdot ||$. Next define

$$\begin{aligned} \Phi : C^2(\mathbb{C}^m \times H_m; H_m)_0 \times A_0^{1,\alpha}(\mathbb{D}; \mathbb{C}^m) \times C^{1,\alpha}(S^1; H_m) \\ \times H_m \to C^{1,\alpha}(S^1; H_m) \end{aligned}$$

by

$$\Phi_h(Z, v, Y) = v - T(Z \cdot Z^* + H_h(Z, v, Y)).$$

The equation $\Phi_h(Z, v, y) = 0$ is called *Bishop's equation*, and we view it as an equation for an unknown function v, given the parameter function h, the analytic mapping Z, and the point Y. By the definition of T, a solution $v = v_h(Z, Y)$ to Bishop's equation gives rise to an analytic disc $G_h = G_h(Z, Y)$ with values in $\mathbb{C}^d$ such that

$$\begin{aligned} \Re(G_h(Z,Y))|_{S^1} &= Z \cdot Z^* + H_h(Z, v, Y) \\ \Im(G_h(Z,Y))|_{S^1} &= v_h(Z, Y) + Y \\ \Im(G_h(Z,Y))(\zeta = 0) &= Y. \end{aligned}$$

From the first of these equations, we see that the analytic disc

$$A_h(Z,Y)(\zeta) = \big(Z(\zeta), G_h(Z,Y)(\zeta)\big)$$

maps the boundary S^1 of the unit disc to M_h.

Note that if $h \equiv 0$, then the analytic disc $A_0(Z,Y)$ has boundary in the model domain Σ_m and is given for $|\zeta| = 1$ by

$$A_0(Z,Y)(\zeta) = (Z(\zeta), Z(\zeta)\, Z(\zeta)^* + \imath T(ZZ^*)(\zeta) + \imath Y).$$

The next theorem summarizes the information we shall need about the existence of solutions to Bishop's equation. For $\delta, t > 0$ set

$$W_\delta = \{h \in C^2(\mathbb{C}^m \times H_m; H_m)_0 \,\big|\, ||h||_2 < \delta\};$$
$$V_t = \left\{(Z,Y) \in A_0^{1,\alpha}(\mathbb{D};\mathbb{C}^m) \times H_m \,\Big|\, ||Z||^2 + |Y| < t^2\right\}.$$

THEOREM 3.5. *Fix $t > 0$ and $\epsilon > 0$. There exists $\delta > 0$ so that for all $h \in W_\delta$, there exists*

$$v_h : V_t \to C^{1,\alpha}(S^1; H_m)$$

which satisfies Bishop's equation:

$$v_h(Z,Y) = T(Z \cdot Z^* + H_h(Z, v_h(Z,Y), Y)).$$

The mapping v_h depends smoothly on h, and gives rise to a family of analytic discs $A_h(Z,Y) = (Z, G_h(Z,Y))$ with boundary in M_h for $(Z,Y) \in V_t$. We have

$$\Im(G_h(Z,Y))(\zeta = 0) = Y.$$

If we write

$$A_h(Z,Y)(\zeta) = A_0(Z,Y)(\zeta) + \big(0, E_h(Z,Y)(\zeta)\big)$$

then the following estimates hold

(i) $$|E_h(Z,Y)(\zeta)| \le \epsilon\Big[||Z||^2 + ||Z||\,|Y| + |Y|^2\Big];$$

(ii) $$|D_Z(E_h(Z,Y)(\zeta)| \le \epsilon\Big[||Z|| + |Y|\Big];$$

(iii) $$|D_Y E_h(Z,Y)(\zeta)| \le \epsilon\Big[||Z|| + |Y|\Big].$$

Proof. Note that $\Phi_h(0,0,Y) = 0$ since the Hilbert transform of a constant is zero. Also

$$D_Z\Phi_h(0,0,0) = 0$$
$$D_v\Phi_h(0,0,0) = I$$
$$D_Y\Phi_h(0,0,0) = 0.$$

From the implicit function theorem, there is a $\delta > 0$ such that Bishop's equation has a solution $v = v_h(Z,Y)$ for $Z \in A_0^{1,\alpha}(\mathbb{D})$ and $Y \in H_m$ with $(Z,Y) \in V_\delta$. We wish to rescale to allow (Z,Y) to be any element in V_t where t is given in the hypothesis of the theorem. To this end, we let $\hat{z} = \delta z/t$ and $\hat{X} + i\hat{Y} = \delta^2(X+iY)/t^2$. This rescaling map takes the set V_t to V_δ. This rescaling map also takes $M = \{X = h(z,Y)\}$ to $\hat{M} = \{\hat{X} = \hat{h}(\hat{z},\hat{Y})\}$, where $\hat{h}(z,Y) = (\delta t^{-1})^2 h(t\delta^{-1}z, (t\delta^{-1})^2 Y)$. Since h is a function of order 3, if $||h||_2 < \delta$, then $||\hat{h}||_2 < t$. This rescaling establishes the existence part of the theorem.

To prove the estimates, we first note the estimate

$$||v_h(Z,Y)|| \leq C\left[||Z||^2 + |Y|^2\right]$$

which follows because the Frechet derivatives of v_h vanish at the origin. (To see this, differentiate Bishop's equation.) The first estimate on the $\Re$-part of the error term E_h follows from the above estimate, the estimates we have on h, and from the formula

$$E_h(Z,Y)(\zeta) = \int P(\zeta, e^{\imath\theta}) h(Z(e^{\imath\theta}), v_h(Z,Y)(e^{\imath\theta}) + Y)\, d\theta.$$

where $P(\cdot,\cdot)$ is the Poisson kernel. The corresponding estimate for the $\Im$-part follows from the estimate on the $\Re$-part and from the fact that the Hilbert transform is a continuous map from $C^{1,\alpha}(S^1; H_m)$ to itself. The other two estimates follow similarly, by differentiating Bishop's equation. □

Let H_m^+ denote the open cone in H_m of strictly positive Hermitian matricies. Define a mapping

$$Z : H_m^+ \times U(m) \to A_0^{1,\alpha}(\mathbb{D}; \mathbb{C}^m)$$

by

$$Z(X,g) = X^{\frac{1}{2}}\, g\, Z_0.$$

Note that

$$||Z(X,g)|| \leq C\,|X|^{\frac{1}{2}}$$

for some universal constant C. It follows from the theorem that given $\epsilon > 0$ and $t > 0$ there exists $\delta > 0$ so that if $h \in W_\delta$, then for all $X \in H_m^+$ with $|X| \leq t^2$, all $Y \in H_m$ with $|Y| \leq t^2$ and all $g \in U(m)$ there exists the analytic disc $A_h(Z(X,g),Y)$ satisfying the theorem. Consider the mapping

$$\Theta : W_\delta \times H_m^+ \times H_m \times U(m) \to M_m^{\mathbb{C}}$$

given by

$$\Theta_h(X,Y,g) = G_h(Z(X,g),Y)(0)$$

Our goal is to invert Θ_h, i.e. we wish to find a map $\psi_h : H_m^+ \times H_m \times U(m) \to H_m^+$ so that $\Theta_h(\psi_h(X,Y,g),Y,g) = X + \imath Y$. Calculations for the model case show that

$$G_0(Z(X,g),Y)(0) = X + \imath Y$$

and so

$$\Theta_h(X,Y,g) = X + \imath Y + E_h(Z(X,g),Y)(0).$$

Since E_h is a higher order error term, it is reasonable to expect that we can invert Θ_h for X and Y small. This is made precise in the next lemma.

LEMMA 3.6. *Let $K_1 \subset H_m^+$ and $K_2 \subset H_m$ be compact subsets. There exists $\delta > 0$ so that for all $h \in W_\delta$, there exists a smooth mapping*

$$\psi_h : K_1 \times K_2 \times U(m) \to H_m^+$$

such that for any $(X,Y) \in K_1 \times K_2$ and any $g \in U(m)$,

$$\Theta_h(\psi_h(X,Y,g),Y,g) = X + iY$$

Sketch of Proof. Fix compact sets $K_1 \subset H_m^+$ and $K_2 \subset H_m$ (so in particular, K_1 avoids the origin). Our estimates in Theorem 3.5 imply that δ can be chosen small enough so that if $h \in W_\delta$ then the X-derivative of the $\Re$-part of $\Theta_h(X,Y,g)$ has maximal rank for $X \in K_1$ and $Y \in K_2$. Since the $\Im$-part of $E_h(Z(X,g),Y)(\zeta = 0)$

is zero, it is clear that the $\Im$-part of $\Theta_h(X,Y,g)$ is Y. The proof of the lemma now follows easily.

We can now repeat the arguments of §2 pertaining to the model case to find a local integral representation formula for holomorphic functions at the point $(0,I)$. First, suppose F is a holomorphic function on all of $\mathbb{C}^m \times M_m^{\mathbb{C}}$ (instead of on just Ω_m as stated in the hypothesis of Theorem 1). Then as in §2 we have

$$F(0,I) = \iint_{H_m \times H_m} F(0, I + X + \imath Y)\varphi(X,Y)dX\,dY$$

where φ is a radial function with compact support in a small neighborhood of the origin $0 \in M_m^{\mathbb{C}}$. In particular, we can find compact sets $K_1 \subset H_m^+$ and $K_2 \subset H_m$ so that the integration takes place only over $(X,Y) \in K_1 \times K_2$. Choose $\delta > 0$ according to Lemma 3.6. Then, for any $h \in W_\delta$ and any $g \in U(m)$,

$$F(0, I + X + \imath Y) = F(A_h(Z(\psi_h(I+X,Y,g),g),Y)(0))$$

and hence

$$\begin{aligned} F(0,I) &= \iiint F(A_h(Z(\psi_h(I+X,Y,g),g),Y)(0) \\ &\qquad \varphi(X,Y)\,dX\,dY\,dg \\ &= \iiint \int_0^{2\pi} F(A_h(Z(\psi_h(I+X,Y,g),g),Y)(e^{i\theta}) \\ &\qquad \varphi(X,Y)\,d\theta\,dX\,dY\,dg. \end{aligned}$$

Using a comparison with the model case, it is clear that this last integral is a compactly supported integral over M_h with respect to surface measure on M_h. We thus obtain the following analogue of Lemma 2.7:

LEMMA 3.7. *There exists $\delta > 0$ so that for all $h \in W_\delta$, there exists a non-negative function $K_h \in C_0^\infty(M_h)$ such that:*
(1) *If F is holomorphic on $\mathbb{C}^m \times M_m^{\mathbb{C}}$ then*

$$F(0,I) = \int_{M_h} F(\zeta)\,K_h(\zeta)d\sigma(\zeta);$$

(2) *If u is continuous and plurisubharmonic on $\mathbb{C}^m \times M_m^{\mathbb{C}}$ then*

$$|u(0,I)| \leq \int_{M_h} |u(\zeta)|\,K_h(\zeta)\,d\sigma(\zeta).$$

(Here $d\sigma$ denotes the surface measure on M_h.) We also have the following uniformity on the functions K_h:

(1) *There is a constant C so that for all $h \in W_\delta$,*

$$\text{suppt } K_h \subset \left\{ z \in M_h \,\middle|\, |z| \leq C \right\};$$

(2) *For each multiindex β there is a constant C_β so that for all $h \in W_\delta$, if D^β is a derivative of order β, then*

$$\sup_{z \in M_h} |D^\beta K_h(z)| \leq C_\beta.$$

So far, we have seen that if $||h||_2$ is small enough, we can represent the point $(0, I)$ by integration against a compactly supported smooth function on M_h. We can now use homogeneity arguments to deal with arbitrary points $(0, X)$ for X belonging to a smaller cone $\gamma < H_m^+$.

LEMMA 3.8. *Let $\gamma < H_m^+$ be a relatively compact subcone. There exists $\delta > 0$ so that if $||h||_2 < \delta$ and $X \in \gamma \cap \{|X| \leq 1\}$ then there is a function $K_{h,X} \in C_0^\infty(M_h)$ with the following properties:*

(1) *If F is holomorphic on $\mathbb{C}^m \times M_m^{\mathbb{C}}$ then*

$$F(0, X) = \int_{M_h} F(\zeta)\, K_{h,X}(\zeta) d\sigma(\zeta);$$

(2) *If u is continuous and plurisubharmonic on $\mathbb{C}^m \times M_m^{\mathbb{C}}$ then*

$$|u(0, X)| \leq \int_{M_h} |u(\zeta)|\, K_{h,X}(\zeta)\, d\sigma(\zeta).$$

Moreover

(1) *There is a constant C_1 so that the function $K_{h,X}$ has compact support in the nonisotropic ball on M_h centered at the origin having radius $C_2 \, |X|^{\frac{1}{2}}$.*

(2) *Let $L_1, \ldots, L_m$ be the standard basis for the vector fields of type $(1,0)$ on M_h near the origin. For every noncommuting polynomial $P(L, \overline{L})$ of degree $k \geq 0$ in the vectorfields $L_1, \ldots, L_m, \overline{L}_1, \ldots, \overline{L}_m$, there is a constant C_P so that*

$$|P(L, \overline{L})(K_{h,X})(\zeta)| \leq C_P \, ||X||^{-\frac{k}{2}} \left| B\left(\Pi(z), \sqrt{||X||}\right)\right|^{-1}.$$

The constants C_1 and C_P are independent of h and X.

Proof. For any $X \in H_m^+$, consider the linear map $S_X : \mathbb{C}^m \times M_m^{\mathbb{C}} \to \mathbb{C}^m \times M_m^{\mathbb{C}}$ given by

$$S_X(z, Z) = \left(X^{-\frac{1}{2}}z, X^{-\frac{1}{2}}\, Z\, X^{-\frac{1}{2}}\right).$$

Then $S_X(0, X) = (0, I)$, and $S_X(M_h) = M_{h_X}$, where

$$h_X(z, Y) = X^{-\frac{1}{2}}h\left(X^{\frac{1}{2}}z, X^{\frac{1}{2}}YX^{\frac{1}{2}}\right)\, X^{-\frac{1}{2}}.$$

Since $h \in C^2(\mathbb{C}^m \times H_m; H_m)_0$ has order 3, we have

$$\begin{aligned} ||h_X||_2 &\le C||h||_2||X||^{\frac{1}{2}} \\ &\le C||h||_2 \quad \text{for } ||X|| \le 1. \end{aligned}$$

By restricting $||h||_2$, we can make $||h_X||_2$ as small as desired. To prove Lemma 3.8 for a given holomorphic function F, it suffices to apply Lemma 3.7 to the function $F \circ S_X^{-1}$. We have

$$F(0, X) = F \circ S_X^{-1}(0, I) = \int\limits_{\zeta \in M_h} F \circ S_X^{-1}(\zeta) K_h(\zeta) d\sigma_h(\zeta)$$

where $d\sigma_h(\zeta)$ denotes surface measure on M_h. Therefore

$$F(0, X) = \int\limits_{\zeta \in M} F(\zeta) K_{h,X}(\zeta) d\sigma(\zeta)$$

where $K_{h,X}(\zeta) = K_h(S_X(\zeta))(S_X^* d\sigma_h(\zeta))$ The estimates on the derivatives of $K_{h,X}$ follow from the chain rule and the volume estimate on the nonisotropic ball given in Lemma 1.8. □

This lemma can be used to complete the proof of Theorem 1 for the case of *free* generic CR submanifolds of type 2 as follows. Let p_0 be a fixed point in M. For p near p_0, we can biholomorphically map M to M_p so that its defining equations are in the normal form given in Lemma 3.3. This biholomorphism takes p to 0 (the origin). Therefore, it suffices to represent holomorphic functions at points of the form $(0, X)$ for X with sufficiently small norm and which belong to some subcone γ of H_m^+.

Now we wish to apply the last lemma. However, there are 2 differences between the statement of Theorem 1 for free generic CR submanifolds of type 2 and the statement of Lemma 3.8. First, Theorem 1 has no restriction on the norm of h. Second, Theorem 1 assumes that the functions to be represented are holomorphic (or plurisubharmonic) only on the open set Ω rather than all of $\mathbb{C}^{m^2+m}$ as stated in Lemma 3.8. The first difference can be handled by the scaling argument given in the proof of Lemma 3.8 with $X = \epsilon I$. Note that in the proof of that lemma, the norm of the function $h_\epsilon(z, Y) = \epsilon^{-1}h(\sqrt{\epsilon}z, \epsilon Y)$ is bounded by a constant factor of $||h||_2\epsilon^{\frac{1}{2}}$. For a given $h \in C^2(\mathbb{C}^m \times H_m; H_m)_0$, we can restrict $||h_\epsilon||_2$ by suitably restricting ϵ. Then, we can apply Lemma 3.8. This rescaling means that the representation given in (1) and (2) of Lemma 3.8 now only applies to $X \in \gamma$ with sufficiently small norm. For the second difference, we need to show that the analytic discs used in the construction of the integral formulas not only have their boundaries in M, but also lie entirely in Ω, for then the proof of Lemma 3.8 applies to functions which are holomorphic only in Ω. This additional fact follows from the following result.

LEMMA 3.9. *Fix constants $C_1 > 0$ and $0 < \epsilon_0 < \epsilon_1 < 1$. There is a constant C_2 such that for any pair of closed convex cones $\gamma < \gamma' < H_m^+$, there is a $t > 0$ so that for any $h \in C^2(\mathbb{C}^m \times H_m; H_m)_0$ with $||h||_2 < t$ the following holds. Let $0 < \delta \leq 1$. Let $Z \in A_0^{1,\alpha}(\mathbb{D}; \mathbb{C}^m)$ with $||Z|| < C_1\delta$ and let $Y \in H_m$ with $|Y| < C_1\delta$ and suppose*

$$\Re\Big(G_h(z, Y)(\zeta = 0)\Big) \in \gamma \cap \{\epsilon_0\delta^2 < |X| < \epsilon_1\delta^2\}.$$

Then $A_h(Z, Y)(\zeta) \in M_h + \{\gamma' \cap \{|X| \leq C_2\delta^2\}\}$ for all $|\zeta| \leq 1$.

The lemma roughly states that if the center of an analytic disc belongs to a set of the form $M_h + \{\gamma \cap S_\delta\}$ where S_δ is some annulus where the inner and outer radii are proportional to δ^2, then the image of the entire disc must belong to $M_h + \{\gamma' \cap B_\delta\}$ where B_δ is a ball of radius proportional to δ^2. Since Ω is a domain with edge M_h, the above lemma shows that the image of the discs constructed in the proof of Lemma 3.7 are contained in Ω as desired.

Proof of Lemma 3.9. An easy scaling argument shows that it suffices to prove the lemma in the case $\delta = 1$. We can write the analytic disc as

$$\begin{aligned}
A_h(Z,Y)(\zeta) & \\
&= \big(Z(\zeta), G_h(\zeta)\big) \\
&= \big(Z(\zeta), Z(\zeta)\cdot Z(\zeta)^* + h(Z(\zeta), \Im(G_h(\zeta))) + \imath\Im(G_h(\zeta))\big) \\
&\quad + \big(0, \Re(G_h(\zeta)) - Z(\zeta)\cdot Z(\zeta)^* - h(Z(\zeta), \Im(G_h(\zeta)))\big).
\end{aligned}$$

The first term belongs to M_h. Therefore, it suffices to show that the term

$$f_h(\zeta) = \Re(G_h(\zeta)) - Z(\zeta)\cdot Z(\zeta)^* - h(Z(\zeta), \Im(G_h(\zeta)))$$

belongs to a compact subset of γ' given that $\Re(G_h(Z,Y)(\zeta = 0))$ belongs to $\gamma \cap \{\epsilon_0 < |X| < \epsilon_1\}$.

Let S be the set of all real-affine linear maps of unit norm which define the convex set γ' (*i.e.* a point X belongs to γ' if and only if $\ell(X) \geq 0$ for all $\ell \in S$). Our hypothesis that $\Re(G_h(Z,Y))(\zeta = 0)$ belongs to $\gamma \cap \{\epsilon_0 < |X| < \epsilon_1\}$ implies that there is an $\hat{\epsilon} > 0$ (depending only on γ, γ' and ϵ_0) with $\ell(\Re(G_h(Z,Y)(\zeta = 0)) > \hat{\epsilon}$ for all $\ell \in S$. Using Theorem 3.5, it follows that $G_h \to G_0$ as $h \to 0$. Hence there exists a $t > 0$ such that if $||h||_2 < t$, then

$$\ell(f_0(0)) = \ell(\Re(G_0(Z,Y))(\zeta = 0) > \frac{\hat{\epsilon}}{2}.$$

In addition, $-\ell \circ f_0$ is subharmonic on $\mathbb{D}$ and vanishes on $\{|\zeta| = 1\}$. The Hopf lemma and the maximum principle together with the previous inequality imply that there is a number $\eta > 0$ (depending only on $\hat{\epsilon}$) such that

$$\ell(f_0(\zeta)) \geq \eta(1 - |\zeta|) \quad \text{for } |\zeta| \leq 1. \tag{3.10}$$

We wish to show that an analogous inequality holds with f_0 replaced by f_h, for this will show that $f_h(\zeta)$ belongs to γ' as desired. We have

$$\begin{aligned}
f_h(\zeta) - f_0(\zeta) &= \Re(G_h(\zeta)) - \Re(G_0(\zeta)) - h(Z(\zeta), \Im(G_h(\zeta))) \\
&= \Re(E_h(Z,Y)(\zeta)) - h(Z(\zeta), \Im(G_h(\zeta))),
\end{aligned}$$

where E_h is the error term introduced in Theorem 3.5. Any first order ζ (or $\overline{\zeta}$) derivative of $f_h - f_0$ can be written as a sum of terms involving first order Z and Y derivatives of E_h and terms involving first order derivatives of h. These derivatives in turn can be controlled by $||h||_2$ in view of Theorem 3.5. Together with the fact that $f_0(\zeta) = f_h(\zeta) = 0$ for $|\zeta| = 1$, we see that if $||h||_2 < t$, then

$$|f_h(\zeta) - f_0(\zeta)| \leq C\, t\, (1 - |\zeta|)$$

where C is a uniform constant which is independent of $Z \in A_0^{1,\alpha}(\mathbb{D}, \mathbb{C}^m)$, with $||Z|| < C_1$ and $Y \in H_m$ with $|Y| < C_1$. If we choose t small enough, then this inequality together with (3.10) implies that

$$\ell(f_h(\zeta)) \geq \frac{\eta}{2}(1 - |\zeta|) > 0 \quad \text{for } |\zeta| \leq 1$$

as desired. □

4. General generic CR submanifolds of type 2. In this Section, we complete the proof of Theorem 1 by reducing the case of a general generic CR submanifold to the free case discussed in the last Section. This reduction is accomplished by a process of adding variables, which we now describe.

Let $M \subset \mathbb{C}^n$ be a generic CR submanifold of type 2, with CR dimension m and real codimension d, so that $n = m + d$. Let p_0 be a point in M. If we denote the coordinates in $\mathbb{C}^{m+d}$ by $(z_1, \dots, z_m, w_1, \dots, w_d) = (z, w)$ then as in Proposition 1.1, we can assume that p_0 is the origin and that there is a neighborhood U of the origin such that

$$M \cap U = \Big\{(z, w) \in U \,\Big|\, \mathrm{Re}(w) = g(z, \mathrm{Im}(w))\Big\}$$

where $g = (g_1, \dots, g_d) : \mathbb{C}^m \times \mathbb{R}^d \to \mathbb{R}^d$ and for $1 \leq l \leq d$,

$$\begin{aligned} g_l(0,0) &= 0 \\ \nabla g_l(0,0) &= 0 \\ \frac{\partial^2 g_l}{\partial z_j \partial z_k}(0,0) &= 0 \quad 1 \leq j, k \leq m. \end{aligned}$$

It follows from Proposition 1.4 that the bilinear form $B : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^d$ given by

$$B(\xi, \eta) = \left(\sum_{j,k=1}^{m} \frac{\partial^2 g_1}{\partial z_j \partial \overline{z}_k}(0,0)\xi_j \overline{\eta}_k, \dots, \sum_{j,k=1}^{m} \frac{\partial^2 g_d}{\partial z_j \partial \overline{z}_k}(0,0)\xi_j \overline{\eta}_k \right)$$

is surjective. Using Lemma 3.4, it follows that there is a real linear map $T_B : H_m \to \mathbb{R}^d$ such that

$$B(\xi, \eta) = \tilde{T}_B(B_m(\xi, \eta)).$$

Since B is surjective, it follows that the mapping T_B is also surjective. Hence the dimension of the kernel of T_B is $m^2 - d$. Let $A^{d+1}, \ldots, A^{m^2}$ be a basis for the kernel of T_B, where A^l is the Hermitian matrix $\{a^l_{j,k}\}$. For $d+1 \leq l \leq m^2$, define the quadratic function g_l on $\mathbb{C}^m \times \mathbb{R}^{m^2}$ which is independent of the last m^2 variables by setting

$$g_l(z_1, \ldots, z_m, w_1, \ldots, w_{m^2}) = \sum_{j,k=1}^{m} a^l_{j,k} z_j \bar{z}_k.$$

Let $\pi : \mathbb{C}^{m+m^2} \to \mathbb{C}^{m+d}$ be the projection onto the first $m+d$ variables. Let $\tilde{U} = \left\{(z,w) \in \mathbb{C}^{m+m^2} \,\middle|\, \pi(z,w) \in U\right\}$. Set

$$\tilde{M} = \left\{(z,w) \in \tilde{U} \,\middle|\, \operatorname{Re}(w_l) = g_l(z, \operatorname{Im}(w)), \quad 1 \leq l \leq m^2\right\}.$$

Clearly $\tilde{M}$ is a submanifold of $\mathbb{C}^{m+m^2}$ of real codimension m^2. The function π is a smooth CR map from $\tilde{M}$ to M. The submanifold $\tilde{M}$ is foliated by the collection of manifolds $\left\{\tilde{M}^{y'};\ y' \in \mathbb{R}^{m^2-d}\right\}$ where $\tilde{M}^{y'}$ is the slice of $\tilde{M}$ with the coordinates

$$\operatorname{Im}(w') = (\operatorname{Im}(w)_{d+1}, \ldots, \operatorname{Im}(y)_{m^2})$$

set to $y' = (y_{d+1}, \ldots, y_{m^2})$. Note that $\tilde{M}^0$ is the graph over M of the function $g' = (g_{d+1}, \ldots, g_{m^2})$ and that each $\tilde{M}^{y'}$ is a translate (by y') of $\tilde{M}^0$ (because the last $m^2 - d$ defining equations for $\tilde{M}$ are independent of $\operatorname{Im}(w')$).

Let $\mathcal{L}_p$ be the Levi form of M at $p \in M$ and let $\tilde{\mathcal{L}}_{\tilde{p}}$ be the Levi form of $\tilde{M}$ at $\tilde{p} \in \tilde{M}$.

LEMMA 4.1. *$\tilde{M}$ is a generic CR submanifold of $\tilde{U}$ of type 2 of CR dimension m and real codimension m^2, and hence is free. Let $p \in M \cap U$, and let $\tilde{p} \in \tilde{M}$ with $\pi(\tilde{p}) = p$. Then $d\pi$ is an isomorphism from $T^{\mathbb{C}}_{\tilde{p}}(\tilde{M})$ to $T^{\mathbb{C}}_p(M)$, and also induces an isomorphism of* *the space of tangential vectors of type $(1,0)$ at $\tilde{p}$ to those at p. In particular, the Levi forms for M and $\tilde{M}$ at p and $\tilde{p}$, $\mathcal{L}_p$ and $\mathcal{L}_{\tilde{p}}$, are related by*

$$\mathcal{L}_p(d\pi(Z)) = \pi\left(\tilde{\mathcal{L}}_{\tilde{p}}(Z)\right)$$

for every Z which is a tangent vector of type $(1,0)$ at $\tilde{p}$.

Proof. Everything in the statement of the lemma is standard (see Chapter 10 in [**B**]) except the claim that $\tilde{M}$ is of type 2. To see this, first note that the bilinear form $\tilde{B}$ associated to the submanifold $\tilde{M}$ at the origin of $\mathbb{C}^{m+m^2}$ is given by

$$\tilde{B}(\xi,\eta) = \left(B(\xi,\eta), \sum_{j,k=1}^{m} a_{j,k}^{d+1}\xi_j\overline{\eta}_k, \dots, \sum_{j,k+1}^{m} a_{j,k}^{m^2}\xi_j\overline{\eta}_k\right).$$

We must show that $\tilde{B} : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{R}^{m^2}$ is surjective. There exists a real linear mapping $T_{\tilde{B}} : H_m \to \mathbb{R}^{m^2}$ such that

$$\tilde{B}(\xi,\eta) = \tilde{T}_{\tilde{B}}(B_m(\xi,\eta))$$

and so it suffices to show that $T_{\tilde{B}}$ is surjective. The lemma now follows from the fact that H_m is isomorphic to $Range\ T_B \oplus Ker\ T_B$ and because the range of T_B is isomorphic to $\mathbb{R}^d$ (since M is of type 2). □

It follows from this lemma that if Ω is a domain in $\mathbb{C}^{m+d}$ with edge M, then the open set

$$\tilde{\Omega} = \left\{(z,w) \in \mathbb{C}^m \times \mathbb{C}^{m^2} \,\middle|\, \pi(z,w) \in \Omega\right\}$$

is a domain with edge $\tilde{M}$. Moreover, holomorphic functions F and plurisubharmonic functions u on Ω lift to holomorphic functions $\tilde{F} = F \circ \pi$ and plurisubharmonic functions $\tilde{u} = u \circ \pi$ on $\tilde{\Omega}$.

To prove Theorem 1, we must show that if $p_0 \in M$ and $\gamma < \Gamma_{p_0}$ are given, then we can find a positive kernel which represents holomorphic functions on Ω at points of the form $p + x$ where p is a point in some neighborhood ω containing p_0 and x belongs to some ϵ - neighborhood of the origin of γ (denoted γ_ϵ in the statement of the theorem). The basic idea is as follows: (1) lift the problem to $\tilde{M}$; (2) apply the kernel from Lemma 3.8 to $\tilde{M}$; and then (3) integrate out any extra variables introduced by the lift from M to $\tilde{M}$. A slight complication arises in that there is no canonical way to lift points $x \in \gamma$ to the convex hull of the image of the Levi form of $\tilde{M}$. Therefore, a localization argument with a partition of unity is used.

Now we present the details. For each point $p \in M$ near p_0, let $\tilde{p}$ be the *unique* point in $\tilde{M}^0$ which lies over p (recall that $\tilde{M}^0$ is the slice of $\tilde{M}$ with $\mathrm{Im}(w') = 0$ and $\tilde{M}^0$ is a graph over M). For $\tilde{p}$ near $\tilde{p}_0$, we can biholomorphically map $\tilde{M}$ to $\tilde{M}_{\tilde{p}}$ so that its defining equations are in the normal form given in Lemma 3.3. Recall that this biholomorphism takes $\tilde{p}$ to 0. This biholomorphism also induces a biholomorphic map defined near M (by restricting the biholomorphism to the first $m + d$ complex coordinates). We let M_p be the image of M and $\gamma_p \subset \mathbb{R}^d$ be the image of γ under this biholomorphism. The convex hull of the image of the Levi form of $\tilde{M}_{\tilde{p}}$ at the origin is the cone H_m^+ by Lemma 3.3. Likewise, we let Ω_p and $\tilde{\Omega}_{\tilde{p}}$ be the images under this biholomorphism of Ω and $\tilde{\Omega}$, respectively. In the new coordinates, there is a projection $\pi_p : \tilde{M}_{\tilde{p}} \to M_p$ which takes $\tilde{\Omega}_{\tilde{p}}$ to Ω_p. To prove Theorem 1, it suffices to find a kernel for M_p which represents holomorphic functions on Ω_p at points of the form $(0, X) \in \mathbb{C}^m \times \gamma_p$. By rescaling as in the discussion after the proof of Lemma 3.8, it suffices to assume $|X| = 1$ and to assume that the norm of the third order terms in the defining functions of $\tilde{M}$ are suitably small. Since the closure of $\gamma_p \cap \{|X| = 1\}$ is compact, it suffices (by a partition of unity argument) to show the following: each point $X_0 \in \gamma_p$ with $|X_0| = 1$ has a neighborhood $U \subset \mathbb{R}^d$ and a kernel which represents holomorphic functions on Ω_p at points of the form $(0, X)$ for $X \in U$. In view of Lemma 4.1, for each such X_0, there is a point $\tilde{X}_0 = (X_0, u_0) \in \mathbb{R}^d \times \mathbb{R}^{m^2-d}$ which lies in H_m^+, the interior of the convex hull of the image of the Levi form of $\tilde{M}_{\tilde{p}}$ at the origin. Furthermore, $\pi_p(\tilde{X}_0) = X_0$. There is a neighborhood U of X_0 in $\mathbb{R}^d$ such that every point of the form (X, u_0) for $X \in \overline{U}$ belongs to H_m^+. We now apply Lemma 3.8 to $\tilde{M}_{\tilde{p}}$ with γ equal to the cone over $U \times \{u_0\}$ in H_m^+. We obtain a kernel $\tilde{K}_{\tilde{p}}(\tilde{z}, \tilde{\zeta})$ which represents holomorphic functions on $\tilde{\Omega}_{\tilde{p}}$ at points of the form $(0, X, u_0)$ for $X \in U$.

Now suppose F is holomorphic on Ω_p and continuous up to M_p. The function F lifts to a function $\tilde{F}$ which is holomorphic on $\tilde{\Omega}_{\tilde{p}}$

and continuous up to $\tilde{M}_{\tilde{p}}$. Let $z = (0, X)$ and $\tilde{z} = (0, X, u_0)$. Using the kernel $\tilde{K}_{\tilde{p}}$

$$\begin{aligned} F(z) &= \tilde{F}(\tilde{z}) \\ &= \int_{\tilde{M}_{\tilde{p}}} \tilde{F}(\tilde{\zeta}) \tilde{K}_{\tilde{p}}(\tilde{z}, \tilde{\zeta})\, d\tilde{\sigma}_{\tilde{p}}(\tilde{\zeta}). \end{aligned}$$

As mentioned earlier, $\tilde{M}_{\tilde{p}}$ can be thought of as a foliation by copies of the graph of M_p (i.e. $\tilde{M}^0_{\tilde{p}}$) parameterized by $\mathbb{R}^{m^2-d}$ with coordinates $y' = (y_{d+1}, \ldots, y_{m^2})$. In particular, each point $\tilde{\zeta} \in \tilde{M}_{\tilde{p}}$ corresponds to a unique point $\zeta \in M_p$ and a unique point $y' \in \mathbb{R}^{m^2-d}$. Surface measure on $\tilde{M}_{\tilde{p}}$, $d\tilde{\sigma}_{\tilde{p}}$, is therefore comparable to $dy_{d+1} \ldots dy_{m^2} d\sigma_p$ where $d\sigma_p$ denotes surface measure on M_p. Thus we may write

$$\tilde{K}_{\tilde{p}}(\tilde{z}, \tilde{\zeta}) d\tilde{\sigma}_p(\tilde{\zeta}) = \tilde{K}^1_p(\tilde{z}, \zeta, y') dy_{d+1} \ldots dy_{m^2} d\sigma_p(\zeta)$$

where $\tilde{K}^1_p$ is a kernel which is smooth and has compact ζ-support. Therefore

$$F(z) = \int_{M_p} F(\zeta) \left[\int_{y' \in \mathbb{R}^{m^2-d}} \tilde{K}^1_p(\tilde{z}, \zeta, y') dy' \right] d\sigma_p(\zeta).$$

Since the point $\tilde{z} = (0, X, u_0)$ is uniquely determined by the point $z = (0, X)$ (because u_0 is fixed), the term in brackets in the above expression only depends on z and $\zeta \in M_p$. So we define

$$K_p(z, \zeta) = \int_{y' \in \mathbb{R}^{m^2-d}} \tilde{K}^1_p(\tilde{z}, \zeta, y') dy'$$

and we obtain

$$F(z) = \int_{\zeta \in M_p} F(\zeta) K_p(z, \zeta) d\sigma_p(\zeta)$$

for $z = (0, X, u_0)$ with $X \in U$, as required. The desired kernel for the original M is then obtained by composing K_p with the biholomorphism which mapped M (and $\tilde{M}$) to M_p (and $\tilde{M}_p$) for p near p_0. Since we have assumed $||X|| = 1$, this kernel only represents holomorphic functions at points $p + x$ for $x \in \gamma$ with norm (approximately) one. More general $x \in \gamma$ can be handled by a scaling argument given at the end of Section 1. The desired support property and estimate of the kernel follow from this scaling argument. This completes the proof of Theorem 1.

5. Boundary behavior of H^p functions. In this Section, we use Theorem 1 to study the boundary behavior along a generic type 2 edge of H^p functions defined on a wedge. We begin with the definition of this class of functions.

DEFINITION 5.1. Let $\Omega \subset \mathbb{C}^n$ be a domain with edge M. For $0 < p < \infty$, let $H^p_{loc}(\Omega, M)$ denote the space of holomorphic functions on Ω such that for every $q \in M$ and every $\gamma < \Gamma_q$ there exists $\epsilon > 0$ and a neighborhood $\omega \subset M$ of q so that

$$\sup_{z \in \gamma_\epsilon} \int_{\zeta \in \omega} |F(\zeta + z)|^p \, d\sigma(\zeta) < +\infty.$$

If $p = \infty$ we require that

$$\sup_{z \in \gamma_\epsilon} |F(z)| < +\infty.$$

Next we define certain approach regions in domains with edges. We will need to consider a conical, or nontangential approach, as well as an admissible approach to a boundary point.

DEFINITION 5.2. Let $\Omega \subset \mathbb{C}^n$ be a domain with edge M. Assume that M is a generic CR submanifold of type 2. Let $q_0 \in M$, let $\gamma < \Gamma_{q_0}$, and let $\alpha > 0$. By the definition of a domain with an edge, there is a neighborhood ω of q_0 in M and an $\epsilon > 0$ so that $\omega + \gamma_\epsilon \subset \Omega$. Let $q \in \omega$. Define

$$(1) \qquad C(\gamma, \alpha, q) = \left\{ z \in \omega + \gamma_\epsilon \,\middle|\, |\Pi(z) - q| \le \alpha \ \mathrm{dist}(z, M) \right\};$$

$$(2) \qquad \mathcal{A}(\gamma, \alpha, q) = \left\{ z \in \omega + \gamma_\epsilon \,\middle|\, D(\Pi(z), q)^2 \le \alpha \ \mathrm{dist}(z, M) \right\}.$$

Recall that Π is the projection from Ω to M which is defined at least near M, and $D(p, q)$ is the nonisotropic distance between p and q. Also *dist* refers to the Euclidean distance. The regions $C(\gamma, \alpha, p)$ are the usual conical approach regions. Since D is roughly Euclidean in the complex tangential directions, the approach regions $\mathcal{A}$ allow quadratic approach to M along these directions. Since D^2 is roughly Euclidean in the totally real directions, these regions allow only nontangential approach in these directions. Thus the regions $\mathcal{A}(\gamma, \alpha, p)$ are the analogues of admissible approach regions for strictly pseudoconvex domains.

It is easy to see that if Ω is a domain with edge M, and if $F \in H^p_{\text{loc}}(\Omega, M)$, then F has polynomial growth locally near M. To be precise, given $p > 0$, and given a point $q \in M$ and a compact cone $\gamma < \Gamma_q$, there is a real number s, a neighborhood $\omega \subset M$ of q and an $\epsilon > 0$ so that for every $F \in H^p_{loc}(\Omega, M)$, there exists a constant C so that

$$|F(z)| \leq C \ \text{dist}(z, M)^{-s}$$

for $z \in \omega + \gamma_\epsilon$. This polynomial growth in turn implies that every $F \in H^p_{loc}(\Omega, M)$ has a distributional limit F^* along M in the sense that if $\varphi \in C_0^\infty(M)$ then

$$\lim_{\epsilon \to 0} \int_M F(\zeta + \epsilon z)\, \varphi(\zeta)\, d\sigma(\zeta) = F^*(\varphi) \quad \text{for } z \in \gamma$$

(see [**BCT**]). Our main results deal with the existence of pointwise and dominated limits, rather than distributional limits, and with a partial characterization of the boundary value distributions.

THEOREM 5.3. *Suppose M is a type 2 - CR submanifold of $\mathbb{C}^n$ and suppose Ω is a domain with edge M. Let $f \in H^p_{loc}(\Omega, M)$, $0 < p \leq \infty$. For almost all $q \in M$, the following holds: Given $\gamma < \Gamma_q$ and $\alpha > 0$, then*

$$\lim_{\substack{z \to q \\ z \in \mathcal{A}(\gamma, \alpha, q)}} f(z, w) \quad \textit{exists.}$$

This limit defines an element of $L^p_{loc}(M)$ and the convergence is dominated. Conversely, if $1 \leq p \leq \infty$ and $f \in L^p_{loc}(M)$ is a CR distribution, then f has a unique holomorphic extension F belonging to $H^p_{loc}(\Omega, M)$ for some open set Ω with edge M.

Proof. First we recall the definition of the *maximal function* $\mathcal{M}$ associated to the family of nonisotropic balls defined in Section 1. For a locally integrable function f on M, define

$$\mathcal{M}(f)(q) = \sup_{\delta > 0} \frac{1}{|B(q, \delta)|} \int_{B(q,\delta)} |f(\zeta)| d\sigma(\zeta) \quad \text{for } q \in M.$$

□

LEMMA 5.4. [**NSW**] *For* $1 < p < \infty$, $\mathcal{M} : L^p(M) \mapsto L^p(M)$ *is continuous.*

This lemma holds for any submanifold of finite type. Its proof is similar to the proof of the analogous fact for the Euclidean maximal function [**SW**]. The key ingredient of the proof is property (2) of Lemma 1.8 which is used to obtain the required convering lemma for our family of balls. We shall need Lemma 5.4 only for submanifolds of type 2 and only for the case $p = 2$.

The next lemma contains the key estimate for the proof of the theorem. It follows easily from the estimate on plurisubharmonic functions given in Theorem 1.

LEMMA 5.5 (MAXIMAL FUNCTION ESTIMATE). *Suppose* M *is a generic type 2 submanifold of* $\mathbb{C}^n$ *and suppose* Ω *is a domain with edge* M. *Given* $q_0 \in M$ *and any sufficiently small open set* $W \subset \mathbb{C}^n$ *which contains* q_0, *there is an open set* W' *in* $\mathbb{C}^n$ *with* $q_0 \in W' \subset W$ *such that for each* $\alpha > 0$ *and* $\gamma < \Gamma_{q_0}$, *there are constants* $0 < C_1, C_2 < \infty$ *which depend only on* α, γ, W, W' *such that for* $q \in W' \cap M$ *and* $\zeta \in A(\gamma, \alpha, q) \cap W'$ *the estimate*

$$u(\zeta) \le \frac{C_1}{|B(q, C_2\delta)|} \int_{B(q,C_2\delta)} u \, d\sigma$$

holds for each nonnegative plurisubharmonic function u *on* $W \cap \Omega$ *which is continuous up to* $M \cap W$, *and where*

$$\delta = (\text{dist}\{\zeta, M\})^{\frac{1}{2}}.$$

In particular, this estimate implies

$$\sup_{\mathcal{A}(\gamma,\alpha,q)\cap W'} u \le C_1 \mathcal{M}(u)(q)$$

for a nonnegative plurisubharmonic function u *defined on* $W \cap \Omega$ *which is continuous up to* M.

We proceed with the proof of Theorem 2. Fix any $q_0 \in M$, $\alpha > 0$ and $\gamma < \Gamma_{q_0}$. We assume $0 < p < \infty$ (the case $p = \infty$ is similar and in fact easier). Since $f \in H^p_{\text{loc}}(\Omega, M)$, for any ν in the interior of Γ_{q_0}

with $|\nu| = 1$, there exist $\epsilon_0 > 0$ and an open set $\omega \subset M$ containing q_0 with

$$\sup_{0<\epsilon<\epsilon_0} \int_{\zeta\in\omega} |f_\epsilon(\zeta)|^p \, d\sigma(\zeta) \ < \ \infty$$

where $f_\epsilon(\zeta) = f(\zeta + \epsilon\nu)$. The collection $\{|f_\epsilon|^{\frac{p}{2}};\ \epsilon > 0\}$ forms a bounded set in $L^2(\omega)$, and so there exists an element $g \in L^2(\omega)$ such that for some subsequence $\epsilon_k \mapsto 0$

$$|f_{\epsilon_k}|^{\frac{p}{2}} \mapsto g \quad \text{weakly in} \quad L^2(\omega).$$

We extend g to all of M by letting $g(\zeta) = 0$ for $\zeta \in M - \omega$.

Choose an open set $W \subset \mathbb{C}^n$ which contains q_0 so that $W \cap M \subset \omega$. Then choose an open set $W' \subset \mathbb{C}^n$ containing q_0 so that W' satisfies the conclusion of the maximal function estimate. We shall assume that W' is small enough so that the following property holds: there is an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ and all $\zeta \in W' \cap \Omega$ and all $q \in M \cap W'$ we have $B(q, C_2\delta_\epsilon) \subset W \cap M \subset \omega$ where $\delta_\epsilon = (\text{dist}\{\zeta + \epsilon\nu, M\})^{\frac{1}{2}}$, and where C_2 is the constant appearing in the maximal function estimate.

LEMMA 5.6. *Given $\gamma < \Gamma_{q_0}$ with $\nu \in \gamma$ as above. Suppose $\alpha > 0$ is given. There is a constant $C > 0$ such that for all $q \in M \cap W'$*

$$\sup_{A(q,\alpha,\gamma)\cap W'} |f_\epsilon|^{\frac{p}{2}} \le C\mathcal{M}(g)(q) \quad \text{for } 0 < \epsilon < \epsilon_0.$$

Proof. The maximal function estimate applied to $|f_{\epsilon_k}|^{\frac{p}{2}}$ and $\zeta \in A(\gamma, \alpha, q) \cap W'$ yields

$$|f_{\epsilon_k}(\zeta + \epsilon\nu)|^{\frac{p}{2}} \le \frac{C_1}{|B(q, C_2\delta_\epsilon)|} \int_{B(q,C_2\delta_\epsilon)} |f_{\epsilon_k}|^{\frac{p}{2}} d\sigma$$

where $\delta_\epsilon = [\text{dist}(\zeta + \epsilon\nu, M)]^{\frac{1}{2}}$. For $\zeta \in A(\gamma, \alpha, q) \cap W'$, $q \in M \cap W'$ and $0 < \epsilon < \epsilon_0$, we have $B(q, C_2\delta_\epsilon) \subset M \cap W \subset \omega$ by the choice of ϵ_0, W and W'. Since $|f_{\epsilon_k}|^{\frac{p}{2}} \mapsto g$ weakly in $L^2(\omega)$ and $f_{\epsilon_k}(\zeta + \epsilon\nu) \mapsto f(\zeta + \epsilon\nu) = f_\epsilon(\zeta)$ as $\epsilon_k \mapsto 0$, the lemma easily follows. □

LEMMA 5.7. *For almost all $q \in W' \cap M$, $\lim_{\epsilon \mapsto 0} f_\epsilon(q)$ exists.*

Proof. Since g is an element of $L^2(\omega)$, Lemmas 5.6 and 5.4 imply that f is almost everywhere nontangentially bounded in Ω near M. But then it follows that f has almost everywhere limits within the conical approach regions $C(\gamma, \alpha, q)$ from Ω on M. In particular, $\lim_{\epsilon \mapsto 0} f_\epsilon(q)$ exists for almost all $q \in W' \cap M$. The classical version of this result dealing with harmonic functions on a half plane or a tube over a cone is discussed in [**SW**]. The ideas and techniques of Rosay [**R**] can then be used to handle nontangentially bounded holomorphic functions on Ω.

We denote by f_0 the pointwise almost everywhere limit given in Lemma 5.7. □

LEMMA 5.8. $\lim_{\epsilon_k \mapsto 0} ||f_{\epsilon_k} - f_0||_{L^p(W' \cap M)} = 0$.

Proof. This lemma follows from Lemmas 5.6, 5.7, 5.4 and the dominated convergence theorem. □

Now we complete the proof of the first part of the Theorem 2. We first extend f (and therefore f_ϵ) by zero outside of W. Since f_ϵ is continuous up to $M \cap W$, we can apply the maximal function estimate to $|f_{\epsilon_j} - f_{\epsilon_k}|^{\frac{p}{2}}$ to obtain

$$\int_{M \cap W'} \left(\sup_{A(\gamma,\alpha,q) \cap W'} |f_{\epsilon_j} - f_{\epsilon_k}|^{\frac{p}{2}} \right)^2 d\sigma(q)$$
$$\leq C_1^2 \int_{M \cap W'} \left(\mathcal{M}(|f_{\epsilon_j} - f_{\epsilon_k}|^{\frac{p}{2}}) \right)^2 (q) d\sigma(q)$$
$$\leq \tilde{C} \int_{M \cap W'} |f_{\epsilon_j} - f_{\epsilon_k}|^p d\sigma(q) \quad \text{(by Lemma 5.4 with } p = 2\text{)}$$

where $\tilde{C}$ is a uniform constant. By letting $k \mapsto \infty$, we obtain

$$(5.9) \quad \int_{M \cap W'} \left(\sup_{A(\gamma,\alpha,q) \cap W'} |f_{\epsilon_j} - f|^p \right) d\sigma(q) \leq \tilde{C} \int_{M \cap W'} |f_{\epsilon_j} - f_0|^p d\sigma.$$

This latter integral converges to zero as $\epsilon_j \mapsto 0$ by Lemma 5.8.

For a real valued function $u : \Omega \mapsto \mathbb{R}$ and for $\delta > 0$, let

$$\Omega(u,\delta) = \left\{ q \in M \cap W';\ \limsup_{\substack{\zeta \mapsto q \\ \zeta \in A(\gamma,\alpha,q)\cap W'}} u(\zeta) - \liminf_{\substack{\zeta \mapsto q \\ \zeta \in A(\gamma,\alpha,q)\cap W'}} u(\zeta) > \delta \right\}.$$

We wish to show that for each $\delta > 0$, $|\Omega(\operatorname{Re} f, \delta)| = 0 = |\Omega(\operatorname{Im} f, \delta)|$. Since f_{ϵ_j} is continuous up to M, $\Omega(\operatorname{Re} f, \delta) = \Omega(\operatorname{Re}(f - f_{\epsilon_j}), \delta)$. Moreover

$$\Omega(\operatorname{Re}(f - f_{\epsilon_j}), \delta) \subset \left\{ q \in M \cap W';\ \sup_{A(\gamma,\alpha,q)\cap W'} |f - f_{\epsilon_j}| > \frac{\delta}{2} \right\}$$

and the measure of this latter set converges to zero as $\epsilon_j \mapsto 0$ in view of (5.9). Therefore, $|\Omega(\operatorname{Re} f, \delta)| = 0$ and similar arguments show $|\Omega(\operatorname{Im} f, \delta)| = 0$. It follows that for each $q_0 \in M$ and each $\gamma < \Gamma_0$, $\alpha > 0$, there is a neighborhood W' of q_0 in $\mathbb{C}^n$ such that for almost all $q \in M \cap W'$,

$$\lim_{\substack{\zeta \mapsto q \\ \zeta \in A(\gamma,\alpha,q)\cap W'}} f(\zeta) \quad \text{exists.}$$

The proof of the first part of the Theorem 2 is now complete.

To prove the converse, we use an approximation theorem from Baouendi and Treves (see [**BT**]). Their techniques can be easily modified to show that if f_0 is a CR distribution in $L^p(M \cap W)$, $1 \le p \le \infty$, then there is an open set W in $\mathbb{C}^n$ containing q_0 and a sequence F_j of entire functions such that $F_j \mapsto f_0$ in $L^p(M \cap W)$. Fix any $\gamma < \Gamma_{q_0}$. An application of the maximal function estimate to $|F_j - F_k|^p$ implies that the sequence F_j converges uniformly on the compact subsets of $\{M + \gamma\} \cap W'$ to a holomorphic limit denoted by F which is the unique holomorphic extension of f. Here, W' is an open set in $\mathbb{C}^n$ which contains q_0 and which depends on q_0 and γ. Let Ω be the union of all sets of the form $\{M + \gamma\} \cap W'$ where the union is over all $\gamma < \Gamma_{q_0}$ and all $q_0 \in M$. By piecing together all the local extensions we obtain a holomorphic function F on Ω which extends f. Fix any $q_0 \in M$ and fix any $\gamma < \Gamma_{q_0}$. Another application of our maximal function estimate to $|F_j|^p$ yields: $|F_j(\zeta + \epsilon\nu)|^p \le$

$C[\mathcal{M}(\chi|F_j|^{\frac{p}{2}})]^2$ for $\zeta \in M \cap W'$, $0 < \epsilon < \epsilon_0$ and $\nu \in \gamma$, where χ is the characteristic function on $M \cap W$. An easy limit argument (as $j \mapsto \infty$) yields

$$\sup_{\substack{\nu\in\gamma,|\nu|=1\\0<\epsilon<\epsilon_0}} \int_{\zeta\in M\cap W'} |F(\zeta+\epsilon\nu)|^p d\sigma(\zeta) \leq \int_{M\cap W} |f_0(q)|^p d\sigma(\zeta) < \infty,$$

and so F is an element of $H^p_{\text{loc}}(\Omega, M)$ as desired.

References

[BCT] M. S. Baouendi, C. H. Chang, and F. Treves, *Microlocal hypoanalyticity and extension of CR functions*, J. Differential Geom., **18** (1983), 331-391.

[BT] M. S. Baouendi and F. Treves, *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields* Ann. Math., **113** (1981), 387-421.

[B] A. Boggess, *CR Manifolds and the Tangential Cauchy-Riemann Complex*, CRC Press, Boca Raton FL, 1991.

[BDN] A. Boggess, R. Dwilewicz, and A. Nagel, *The hull of holomorphy of a nonisotropic ball in a real hypersurface of finite type*, Trans. Amer. Math. Soc., **323** (1991), 209-232.

[BN] A. Boggess and A. Nagel *Representing measures for holomorphic functions on type* 2 *wedges*, to appear in Proceedings of the Fourier Analysis Conference in honor of E.M. Stein, Princeton University Press.

[FS] G. Folland and E.M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math., **27** (1974), 429-522.

[NSW] A. Nagel, E. M. Stein and S. Wainger, *Balls and metrics defined by vector fields,I: basic properties*, Acta. Math., **155** (1985), 105-147.

[R] J.-P. Rosay, *On the radial maximal function and the Hardy-Littlewood maximal function in wedges*, to appear in Proceedings of the Madison Symposium in Complex Analysis, Contemporary Mathematics, Amer. Math. Soc.

[RS] L. Rothschild and E.M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math., **137** (1976), 247-320.

[SW] E. M. Stein and G. Weiss, *Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton NJ, 1971.

Received August 12, 1992. Supported in part by an NSF grant at the University of Wisconsin-Madison.

TEXAS A&M UNIVERSITY

College Station, TX 77845
E-mail address: boggess@math.tamu.edu

and

University of Wisconsin-Madison
Madison, WI 53706
E-mail address: nagel@math.wisc.edu