PACIFIC JOURNAL OF MATHEMATICS
Vol. 167, No. 2, 1995

DESINGULARIZATIONS OF SOME UNSTABLE ORBIT CLOSURES

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Let σ be a semisimple automorphism of a connected reductive group G, and let G_σ be the fixed points of σ. We consider the G_σ-orbits on the space of nilpotent elements in an eigenspace of $d\sigma$. We give a desingularization of the orbit closures and relate the G_σ-orbits to the G-orbits. Along the way, we describe the fixed points of σ on a flag variety G/P where P is a σ-stable parabolic subgroup of G.

I. Introduction. In this note we observe some consequences of Richardson's theorems on orbits of reductive groups, in the following situation. Let G be a simply-connected reductive algebraic group over an algebraically closed field F whose characteristic is either zero or sufficiently large (as specified below). Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathcal{N}$ be the variety of nilpotent elements in $\mathfrak{g}$. Let σ be a semisimple automorphism of G, fix a nonzero element $q \in F^\times$, and consider the variety

$$\mathcal{N}_{\sigma,q} = \{x \in \mathcal{N} : \ d\sigma(x) = qx\}.$$

If q is not a root of unity then $\mathcal{N}_{\sigma,q}$ is the whole q-eigenspace of $d\sigma$, hence is a linear subspace of $\mathfrak{g}$. If q is a root of unity, the variety $\mathcal{N}_{\sigma,q}$ may even be reducible.

It was shown by Steinberg that the group of σ-fixed points G_σ is also a connected reductive F-group ([**S**]). The adjoint action of G_σ preserves each eigenspace of $d\sigma$, and $\mathcal{N}_{\sigma,q}$ consists of those G_σ-orbits in the q-eigenspace of $d\sigma$ which are "unstable", in the sense of geometric invariant theory ([**H2**]). According to a theorem of Kac and Richardson ([**Ri3**]), the G_σ-orbits on $\mathcal{N}_{\sigma,q}$ are exactly the irreducible components of sets of the form $\mathcal{N}_{\sigma,q} \cap \tilde{\mathcal{O}}$, where $\tilde{\mathcal{O}}$ is a nilpotent G orbit. Richardson also proved (with our assumptions on the characteristic of F, see [**Ri1**]) that there are only finitely many nilpotent G-orbits, hence there are only finitely many G_σ-orbits in $\mathcal{N}_{\sigma,q}$.

Let $\mathcal{O} \subseteq \mathcal{N}_{\sigma,q}$ be one such G_σ-orbit. Then the Zariski closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ in $\mathcal{N}_{\sigma,q}$ is an affine variety which is generally singular. The main purpose of this note is to resolve the singularities of $\overline{\mathcal{O}}$ (Prop. (3.2)). More precisely, we construct a vector bundle E over a partial flag variety of G_σ, and define a closed morphism $\pi : E \longrightarrow \overline{\mathcal{O}}$ such that $\pi : \pi^{-1}(\mathcal{O}) \longrightarrow \mathcal{O}$ is an isomorphism. The idea is simply that (σ, q^{-1}) acts on the known resolution of the closure of $G\cdot\mathcal{O}$, the nilpotent G-orbit containing $\mathcal{O}$. The desired resolution of $\overline{\mathcal{O}}$ is then found by taking fixed points under (σ, q^{-1}). Moreover these fixed points separate the various G_σ-orbits in $\mathcal{N}_{\sigma,q} \cap G\cdot\mathcal{O}$, as described in Prop. (4.1) below. For example if σ is induced by an automorphism of the Dynkin diagram, (4.1) implies that every nilpotent G-orbit meets $\mathcal{N}_{\sigma,q}$ in at most one G_σ-orbit. Taking $q = 1$, we recover the well-known fact that nilpotent orbits in $\mathfrak{so}(2n+1)$ and $\mathfrak{sp}(n)$ are determined by elementary divisors.

The method requires a precise description (Prop. (2.3)) of the fixed points of σ in a flag variety G/P where P is a σ-stable parabolic subgroup of G. Richardson ([**Ri3**]) has already proven that there are finitely many orbits and they are all closed. Here we count and describe the orbits explicitly, using Steinberg's work in [**S**].

Hesselink ([**H2**]) has constructed desingularizations of closures of "strata" for more general group actions. Each stratum is a union of orbits, and in the case of nilpotent G-orbits, Kraft proved that the strata and orbits coincide. For our varieties $\mathcal{N}_{\sigma,q}$ it is not known if the orbits are strata, and even if they are (it is an interesting question), the proof is likely to be a more difficult route to a desingularization than the one taken here.

The group action $(G_\sigma, \mathcal{N}_{\sigma,q}, Ad)$ arises in many settings. Suppose that q is not a root of unity, so that $\mathcal{N}_{\sigma,q}$ is a linear space, and in particular an irreducible variety. There is a unique Zariski dense orbit in $\mathcal{N}_{\sigma,q}$, so by definition, the triple $(G_\sigma, \mathcal{N}_{\sigma,q}, Ad)$ is a "prehomogeneous vector space", hereafter abbreviated as PV. The complex PV's which are also irreducible representations were classified by Sato and Kimura ([**S-K**]). The PV's occuring as some $(G_\sigma, \mathcal{N}_{\sigma,q}, Ad)$ are called "PV's of parabolic type" ([**Ru**]), because they also arise as subspaces of nilradicals of parabolic subalgebras which are invariant under a Levi subgroup. (G_σ is a Levi subgroup in some reductive subgroup $H < G$ whose Lie algebra contains $\mathcal{N}_{\sigma,q}$.) Most irreducible PV's with finitely many orbits are of parabolic type. However, parabolic PV's can easily be reducible, and there seems to be no classification of reducible PV's.* We remark that the big group G is not to be discarded, as it greatly clarifies the structure of $(G_\sigma, \mathcal{N}_{\sigma,q}, Ad)$.

Suppose now that $q \in F^\times$ is arbitrary, but that σ either has finite order or is conjugation by some element in a one-parameter subgroup of G. Then $\mathfrak{g}$ has a grading $\mathfrak{g} = \oplus \mathfrak{g}_i$ such that G_σ has Lie algebra $\mathfrak{g}_0$ and V is the set of nilpotent elements in $\mathfrak{g}_1$. These spaces were initially studied by Vinberg ([**V**]), who proposed a scheme for classifying the orbits.

My initial interest in such orbit closures $\overline{\mathcal{O}}$ came from the representation theory of p-adic groups ([**K-L**]). In this setting, G is a complex group (the "Langlands dual group"), σ is inner, and q is the cardinality of the residue field, hence not a root of unity. The intersection cohomology of $\overline{\mathcal{O}}$ is apparently related to multiplicities in unramified principal series representations of the p-adic group dual to G, just as with Schubert varieties and Verma modules (see [**G**], [**R**], [**Z**]). In this context, Zelevinsky found desingularizations of $\overline{\mathcal{O}}$ for $G = GL_n$ and used them to compute the intersection cohomology of some special $\overline{\mathcal{O}}$'s, for which his resolution was "small" in the sense of [**G-M**]. The resolutions constructed in this paper are not always small.

An earlier version of this paper had σ being inner, as above. However, in addition to other helpful comments, the referee pointed out Sekiguchi's paper [**Se**], which discusses $\mathcal{N}_{\sigma,-1}$ when σ is an involution. Among other results, Sekiguchi gives a resolution of the orbit closures of maximal dimension in $\mathcal{N}_{\sigma,-1}$, so following the referee's suggestion, I modified this paper to include semisimple automorphisms, thus extending that part of Sekiguchi's work. In this setting, the G_σ-orbits in $\mathcal{N}_{\sigma,-1}$ are of interest in the representation theory of real Lie groups (see [**Vo**]).

Thanks are due to Gary Seitz, for telling me about Richardson's paper [**Ri3**].

*Added in proof: See Kasei-Kimura-Yasukura, Amer. J. Math., **108** (1986), 643-692.

II. Fixed point varieties in flag manifolds. Here we use some of Steinberg's results in [**S**] to describe the fixed point subvariety of a flag manifold under the action of a semisimple automorphism. In this section there is no restriction on the characteristic of the ground field F.

Let G be a connected, simply-connected and semisimple algebraic group over an algebraically closed field F, and let σ be a semisimple automorphism of G. That is, σ is an automorphism of G whose differential $d\sigma$ acts diagonalizably on the Lie algebra $\mathfrak{g}$ of G. We say a subgroup $H \subseteq G$ is "σ-stable" if $\sigma H = H$. If $K \subseteq H \subseteq G$ are closed σ-stable subgroups of G, then σ acts on the variety H/K by the rule $\sigma(hK) = (\sigma h)K$, for $h \in H$. We write $(H/K)_\sigma$ for the fixed points of σ in H/K. According to [**S**, (8.2)], G_σ is a connected reductive group. By [**S**, (7.5)], there exists a σ-stable Borel subgroup $B \subseteq G$, and a σ-stable maximal torus $T \subseteq B$. Let U be the unipotent radical of B, and let $\mathfrak{t} \subseteq \mathfrak{t} \oplus \mathfrak{u} = \mathfrak{b}$ be the corresponding Lie algebras. Let Δ, Δ^+, Σ be the roots of $\mathfrak{t}$ in $\mathfrak{g}$, $\mathfrak{u}$ and $\mathfrak{u}/[\mathfrak{u},\mathfrak{u}]$, respectively. Since σ preserves T and B, it acts on Δ, preserving Σ and Δ^+, and on the normalizer N of T in G, and hence on the Weyl group $W = N/T$. Let $W_\sigma^1 = N_\sigma/T_\sigma$. This is the subgroup of W_σ consisting of elements which can be represented in N_σ.

Let $V \subset \mathfrak{t}^*$ be the real span of the roots in Δ, and let V_σ be the fixed points of σ in V. For any root α, let $\bar{\alpha}$ denote its orthogonal projection into V_σ, with respect to a W-invariant inner product on V. Then W_σ preserves and acts faithfully on V_σ ([**S**, (1.32)]). Let $\overline{W_\sigma}$ and $\overline{W_\sigma^1}$ denote the restrictions of W_σ and W_σ^1 to V_σ. We have $\overline{W_\sigma} \simeq W_\sigma$ as abstract groups, but the latter is a reflection group with respect to a new root system. We describe this more precisely.

The projections of all roots form a non-reduced root system in V_σ. We get a reduced root system as follows ([**S**, §1]). Let S_α be the collection of positive roots whose projection to V_σ is proportional to $\bar{\alpha}$. There are two possibilities for S_α ([**S**, (8.2)]):

(1) $S_\alpha = \{\alpha, \sigma\alpha, \sigma^2\alpha, \dots\}$ no two of which sum to a root, or

(2) $S_\alpha = \{\alpha, \sigma\alpha, \beta = \alpha + \sigma\alpha\}$ is a σ-stable positive system of type A_2.

For $\alpha \in \Delta^+$, let $[\alpha] \in V_\sigma$ be the longest projection to V_σ of a root in S_α. Likewise, for any subset $J \subseteq \Delta^+$, let $[J] = \{[\alpha] : \alpha \in J\}$.

Set $[\Delta] := W_\sigma[\Sigma]$. Then by [**S**, (1.32)], $[\Delta]$ is a root system in V_σ with base $[\Sigma]$, positive roots $[\Delta]^+ = [\Delta^+]$ and Weyl group $\overline{W_\sigma}$. (The root system of G_σ turns out to be a subsystem of $[\Delta]$.) According to [**S**, (8.2)], the reflection $s_{[\alpha]} \in \overline{W_\sigma}$ corresponding to $[\alpha] \in [\Delta^+]$ is given by

(1) $s_{[\alpha]} = s_\alpha s_{\sigma\alpha} \cdots |_{V_\sigma}$ or

(2) $s_{[\alpha]} = s_\beta|_{V_\sigma}$,

according to the two cases for S_α described above.

All of this is related to the structure of fixed point group G_σ in the following way. For each positive root α, let $U_\alpha \subseteq U$ be the corresponding root subgroup. We have $\sigma U_\alpha = U_{\sigma\alpha}$. Consider the product $\prod_{\beta\in S_\alpha} U_\beta$. In the two cases for S_α, either the root groups in the product commute (case (1)), or the whole product is the three dimensional Heisenberg group (case (2)). Hence the product is a group. Since $\sigma S_\alpha = S_\alpha$, the product is also invariant under σ, and we set

$$U[\alpha] = \left(\prod_{\beta\in S_\alpha} U_\beta\right)_\sigma .$$

By [**S**, (8.2)] again, $U[\alpha]$ is either trivial, or a one-parameter group. Finally, [**S**, (8.2)(4)-(7)] combine to give

LEMMA 2.1. *$U[\alpha]$ is nontrivial if and only if the reflection $s_{[\alpha]}$ belongs to $\overline{W_\sigma^1}$. Moreover, $\overline{W_\sigma^1}$ is generated by such reflections.*

This allows us to prove

LEMMA 2.2. *There exists a set $Y(\sigma)$ of coset representatives for $W_\sigma^1\backslash W_\sigma$ such that if $n_w \in N$ represents $w \in Y(\sigma)$, then*

$$n_w^{-1} B_\sigma n_w \subseteq B.$$

Proof. Since U_σ is the product of the $U[\alpha]$'s, this amounts to having $n_w^{-1}U[\alpha]n_w \subseteq U$ for all $\alpha \in \Delta^+$. Since

$$n_w^{-1}U[\alpha]n_w \subseteq \prod_{\beta\in S_\alpha} U_{w^{-1}\beta},$$

it is enough to find coset representatives $w \in W_\sigma$ such that $w^{-1}S_\alpha \subseteq \Delta^+$ whenever $U[\alpha]$ is nontrivial. Let $<$ be the Bruhat order on the

Weyl group $\overline{W_\sigma}$, with respect to the positive system $[\Delta^+]$ of $[\Delta]$. This satisfies the rule ([**J**, (2.19)])

$$s_{[\alpha]}\bar{w} > \bar{w} \Leftrightarrow \bar{w}^{-1}[\alpha] \in [\Delta^+].$$

In each coset in $\overline{W^1_\sigma}\backslash\overline{W_\sigma}$, choose one element $\bar{w} \in \overline{W_\sigma}$ which is not $<$ any other member of its coset $\overline{W^1_\sigma}\bar{w}$. Let $w \in W_\sigma$ be the unique element whose restriction to V_σ is $\bar{w}$, and let $Y(\sigma)$ be the collection of w's so obtained. Then $Y(\sigma)$ is a set of coset representatives for $W^1_\sigma\backslash W_\sigma$.

For any $x \in W_\sigma$, we have $x\sigma = \sigma x$, as automorphisms of Δ. Let $\alpha \in \Delta^+$. Since σ preserves Δ^+ and S_α is spanned over the positive integers by the σ-translates of one of its members, we have either $xS_\alpha \subseteq \Delta^+$ or $xS_\alpha \subseteq -\Delta^+$. Moreover, the former possibility holds if and only if $\bar{x}[\alpha] \in [\Delta]^+$. Now take $x = w^{-1}$, where $w \in Y(\sigma)$. By (2.1) and the Bruhat-minimality of w, we have $x[\alpha] \in [\Delta]^+$ whenever $U[\alpha]$ is nontrivial, so w has the required properties. $\square$

Now let P be a σ-stable parabolic subgroup of G. By [**S**, (7.5)], there is a maximal torus T and a Borel subgroup B, both σ-stable, such that $T \subseteq B \subseteq P$. With notation and results as above, P corresponds to a σ-stable subset of $\Sigma_P \subseteq \Sigma$. More precisely, Σ_P consists of those simple roots which are roots of $\mathfrak{t}$ in the Lie algebra of the unique Levi subgroup L of P containing T. Since $\sigma T = T$ and $\sigma P = P$, we also have $\sigma L = L$, so Σ_P is indeed σ-stable. Let W_P be the subgroup of W generated by the reflections in Σ_P. Then W_P is also preserved by σ. The following result may be viewed as the determination of all σ-stable parabolic subgroups in the G-conjugacy class of P.

PROPOSITION 2.3. *Let G be a semisimple simply connected algebraic group. Let σ be a semisimple automorphism of G, and let P be a σ-stable parabolic subgroup of G. Then with notation as above,*

(1) $(G/P)_\sigma = \coprod_{w\in W^1_\sigma\backslash W_\sigma/(W_P)_\sigma} G_\sigma wP/P$ *(disjoint union). In particular, each connected component of $(G/P)_\sigma$ is a G_σ-orbit.*

(2) *Each double coset $W^1_\sigma w(W_P)_\sigma$ meets $Y(\sigma)$ (see (2.2)).*

(3) *Each G_σ-orbit in $(G/P)_\sigma$ is complete. In other words, the group of σ-fixed points in any σ-stable parabolic subgroup of G is parabolic in G_σ.*

(4) *If* $P = B$ *is a* σ*-stable Borel subgroup and* $y \in Y(\sigma)$*, then* $(yBy^{-1})_\sigma = B_\sigma$*, so* $(G/B)_\sigma$ *is a disjoint union of* $[W_\sigma : W_\sigma^1]$ *copies of the flag variety* G_σ/B_σ*.*

Proof. Assertion (2) follows immediately from (2.2). For (1) we recall the Bruhat decomposition. For each $w \in W$, let U_w be the product, taken in some fixed order, of the root groups U_α with $w^{-1}\alpha \in -\Delta^+$. Let W^P be the set of $w \in W$ such that $w\Sigma_P \subseteq \Delta^+$. Then every point in G/P may be uniquely written as uwP for some $w \in W^P$, $u \in U_w$. Since Σ_P and Δ^+ are σ-stable, so are W^P and U_w. It follows that

$$(G/P)_\sigma = \bigcup_{w \in W^P} (U_w wP/P)_\sigma \subseteq \bigcup_{w \in W_\sigma} U_\sigma wP/P \subseteq \bigcup_{w \in W_\sigma} G_\sigma wP/P.$$

Let $n \in N$ represent some $w \in W_\sigma$. Then $\sigma(n) = nt$ for some $t \in T$, so $\sigma(wP) = \sigma(n)P = ntP = nP = wP$. Hence $w \in W_\sigma$ implies $G_\sigma wP/P \subseteq (G/P)_\sigma$, so we have

$$(G/P)_\sigma = \bigcup_{w \in W_\sigma} G_\sigma wP/P = \bigcup_{y \in Y(\sigma)} G_\sigma yP/P,$$

where $Y(\sigma)$ is as in (2.2).

We next show disjointness. Since $G = UNB = \coprod_{w \in W} UwB$, we have

$$G_\sigma = U_\sigma N_\sigma B_\sigma = \coprod_{x \in W_\sigma^1} U_\sigma x B_\sigma.$$

Let $y \in Y(\sigma)$ and consider the orbit $G_\sigma yP/P$ in $(G/P)_\sigma$. By (2.2) we have

$$G_\sigma yP/P = \bigcup_{x \in W_\sigma^1} U_\sigma x B_\sigma yP/P = \bigcup_{x \in W_\sigma^1} U_\sigma xyP/P.$$

It follows that for two elements $y, y' \in Y(\sigma)$ we have $G_\sigma yP/P = G_\sigma y'P/P$ if and only if we have equality of double cosets

$$W_\sigma^1 y W_P = W_\sigma^1 y' W_P.$$

Since $Y(\sigma) \subseteq W_\sigma$, this is the same as

$$W_\sigma^1 y (W_P)_\sigma = W_\sigma^1 y' (W_P)_\sigma,$$

proving (1).

Now $(G/P)_\sigma$, being closed in G/P, is complete. Hence each connected component of $(G/P)_\sigma$ is complete, proving (3).

Finally, let $y \in Y(\sigma)$. Then $y^{-1}B_\sigma y \subseteq B$, so $B_\sigma \subseteq (yBy^{-1})_\sigma$. Both sides are solvable, and are parabolic subgroups of G_σ by (3). Hence they are Borel subgroups of G_σ, and must be equal. □

2.5. REMARKS. (1) Simple examples (eg. [**Ri3**, 3.3]) show that (2.3)(4) fails for nonminimal parabolic subgroups. See also (2.6) below.

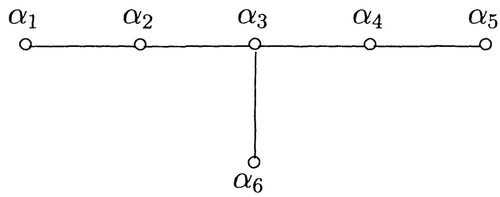
(2) We could have given a more conceptual, though less explicit, construction of $Y(\sigma)$ by invoking [**Ri3**, 10.2.1] (our (2.3)(3)) at the outset. For, if $w \in W_\sigma$ and B is a σ-stable Borel subgroup of G, then by [**Ri3**, 10.2.1], B_σ and $(wBw^{-1})_\sigma$ are Borel subgroups of G_σ containing the same maximal torus T_σ, so there exists $x \in W_\sigma^1$ such that $x^{-1}(B_\sigma)x = (wBw^{-1})_\sigma$, from which it follows that $(xw)^{-1}B_\sigma(xw) \subseteq B$.

(3) Matsuki (over $\mathbb{C}$, [**M**]) and Springer ($char(F) \neq 2$, [**Sp**]) have described all of the G_σ-orbits in G/B when σ is an involution. They prove in particular that $(G/B)_\sigma$ is exactly the union of the closed G_σ-orbits in G/B. Let P be a parabolic subgroup containing B, and let $\pi : G/B \longrightarrow G/P$ be the natural projection. If Y is a closed G_σ-orbit in G/P, then $Y = \pi(Y')$ for any closed G_σ-orbit $Y' \subseteq \pi^{-1}(Y)$. Hence the result of Matsuki and Springer implies that $(G/P)_\sigma$ is the union of the closed G_σ-orbits. The same assertion holds when σ is inner but not necessarily involutive. Indeed, σ is then conjugation by some element of a torus $S \subseteq G_\sigma$. Let $Y \subseteq G/P$ be a closed G_σ-orbit as above. By Borel's fixed-point theorem [**B**, (10.4)], the fixed point set Y_S of S in Y is nonempty, so Y meets $(G/P)_\sigma$, hence $Y \subseteq (G/P)_\sigma$.

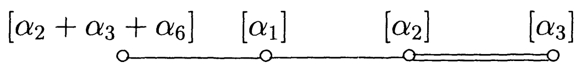
2.6. EXAMPLES. There are two extreme cases of (2.3). If σ is conjugation by an element $s \in T$, then $W_\sigma = W$ and W_σ^1 is generated by the reflections about roots which are trivial on s. On the other hand if σ is induced by an automorphism ρ of the Dynkin diagram, then $W_\sigma^1 = W_\sigma$ ([**S**, (8.2)(5)]), so $(G/P)_\sigma$ is connected.

Here is an example of a mixed case. Take $G = E_6$ (simply connected version), and let σ be the involution of G whose fixed point set has type C_4. Explicitly, take ρ to be the automorphism of E_6

induced by the nontrivial diagram symmetry as in the previous paragraph, and let $s = \check{\alpha}_0(-1)$, where α_0 is the highest root and $\check{\alpha}_0(t)$ is the corresponding one parameter subgroup of T. We can take $\sigma = i_s\rho$, where i_s is conjugation by s. Then W_σ, which only depends on ρ, has type F_4. Meanwhile W_σ^1, being the Weyl group of G_σ, has type C_4. Hence $[W_\sigma : W_\sigma^1] = 3$. We compute $Y(\sigma)$. Number the simple roots of E_6 as follows:



Using [**S**, (8.2)], one gets explicit conditions for each $U[\alpha]$ to be nontrivial, and the simple roots of T_σ in B_σ are then found to be



If $w \in Y(\sigma)$, we must have $w^{-1}\alpha > 0$ for $\alpha = \alpha_1, \alpha_2, \alpha_3, \alpha_2+\alpha_3+\alpha_6$. (See the proof of (2.2).) Since w is σ-invariant, we must also have $w^{-1}\alpha_4 > 0$ and $w^{-1}\alpha_5 > 0$. It is now easy to see that w can only be one of $\{1, s_6, s_6s_1\}$, where $s_i \in W$ is the simple reflection about α_i. Hence this set must be $Y(\sigma)$. Using (2.3), we find that $(G/P)_\sigma$ has $3 - |\Sigma_P \cap \{\alpha_1, \alpha_6\}|$ connected components. The components are not isomorphic in general. For example, let P be a σ-stable parabolic subgroup with Levi component of type A_5. Then P_σ has Levi of type C_3, while $(s_6Ps_6^{-1})_\sigma$ has Levi of type A_3, so $(G/P)_\sigma$ has two components: $G_\sigma/P_\sigma \simeq \mathbb{P}^7$ and $G_\sigma/(s_6Ps_6^{-1})_\sigma$, which is the variety of Lagrangian 4-planes in F^8.

III. The desingularization. We retain the notation and hypotheses of the previous section, except we now assume that the characteristic of the ground field F is either zero or $p > 2\kappa$, where κ is the maximum of the Coxeter numbers of the simple factors of G. For any variety S on which σ or $d\sigma$ acts, S_σ denotes the fixed points. If H is any σ-stable subgroup of G with Lie algebra $\mathfrak{h}$, it follows from [**B**, III.9.1] that the Lie algebra of H_σ is $\mathfrak{h}_\sigma$. We abbreviate the adjoint action of G by $g\cdot Y := Ad(g)Y$ for $g \in G$ and any subset $Y \subseteq \mathfrak{g}$. Let $\mathcal{N}$ be the variety of nilpotent elements in the Lie algebra $\mathfrak{g}$ of G. Fix $q \in F^\times$, and set

$$\mathcal{N}_{\sigma,q} = \{x \in \mathcal{N} : \; d\sigma(x) = qx\}.$$

We need the analogue of a normal $\mathfrak{sl}_2$-triple ([**K-R**, (I.2)]).

LEMMA 3.1. *Let $e \in \mathcal{N}_{\sigma,q}$. Then there exists $h \in \mathfrak{g}_\sigma$, $f \in \mathfrak{g}$ such that*

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

Proof. The proof is a straightforward modification of ([**K-R**, Prop. 4]). By the version of the Jacobson-Morozov theorem given in [**Ca**, p. 152], there exists a triple (e,h',f') satisfying the above relations. Write $h' = \sum h'_\lambda$ where $d\sigma(h'_\lambda) = \lambda h'_\lambda$, $\lambda \in F^\times$. Then $2e = [h',e] = \sum[h'_\lambda, e]$. Comparing $d\sigma$ eigenvalues, we get $2e = [h'_1, e]$. Likewise, we write $f' = \sum f'_\lambda$, and the relation $[e,f'] = h'$ implies $h'_1 = [e, f'_{q^{-1}}] \in [e,\mathfrak{g}]$. Set $h = h'_1$. By [**Ko**, Cor. 3.5] (which is shown in [**Ca**, p. 141] to apply when $\mathrm{char}(F) > 2\kappa$), there exists f such that (e,h,f) satisfy the above relations. □

Let $e \in \mathcal{N}_{\sigma,q}$ and fix a triple (e,h,f) as in (3.1). We have a grading $\mathfrak{g} = \oplus_i \mathfrak{g}(i)$, where $\mathfrak{g}(i) = \{x \in \mathfrak{g} : \; [hx] = ix\}$. Each $\mathfrak{g}(i)$ is also $d\sigma$-stable, since $\mathrm{ad}(h)$ and $d\sigma$ commute. Hence the parabolic subalgebra

$$\mathfrak{p} = \bigoplus_{i\geq 0} \mathfrak{g}(i)$$

is $d\sigma$-stable, so the corresponding parabolic subgroup P of G is σ-stable, and we are in the situation of (2.3). In particular, P_σ is a parabolic subgroup of the connected reductive group G_σ, and the Lie algebra of P_σ is

$$\mathfrak{p}_\sigma = \bigoplus_{i\geq 0} \mathfrak{g}(i)_\sigma.$$

We next set

$$\mathfrak{p}^2 = \bigoplus_{i\geq 2} \mathfrak{g}(i), \quad \mathfrak{q} = \mathcal{N}_{\sigma,q} \cap \mathfrak{p}^2.$$

The latter is an $\mathrm{Ad}(P_\sigma)$-invariant linear subspace of $\mathcal{N}_{\sigma,q}$, even if q is a root of unity, because $\mathfrak{p}^2$ consists of nilpotent elements. Let

$$\mathfrak{p}_0^2 \subset \mathfrak{p}^2$$

be the P-orbit of e. By [**Ca**, 5.7.3], $\mathfrak{p}_0^2$ is the unique open dense P-orbit in $\mathfrak{p}^2$. It follows that $\mathfrak{p}_0^2$ and hence $G{\cdot}e$ are $d\sigma$ stable.

Define the incidence variety

$$E = \{(gP_\sigma, x) \in G_\sigma/P_\sigma \times \mathfrak{g} : \ x \in g{\cdot}\mathfrak{q}\}.$$

This is a vector bundle over G_σ/P_σ via projection onto the first factor. One could view E as the fiber product $G_\sigma \times_{P_\sigma} \mathfrak{q}$, but the former picture is more convenient here. Let

$$\pi : E \longrightarrow \mathcal{N}_{\sigma,q}$$

be projection onto the second factor. Then π is G_σ-equivariant, and closed since G_σ/P_σ is complete, by (2.3)(3).

PROPOSITION 3.2. *Let $\mathcal{O}$ be the G_σ-orbit of $e \in \mathcal{N}_{\sigma,q}$. Then the image of π is $\overline{\mathcal{O}}$ and $E \xrightarrow{\pi} \overline{\mathcal{O}}$ is a resolution of singularities.*

Proof. We first show that the P_σ-orbit of e is dense in $\mathfrak{q}$. From the representation theory of SL_2 ([**Ca**, 5.4]) we see that the map

$$\mathrm{ad}(e) : \mathfrak{g}(i) \longrightarrow \mathfrak{g}(i+2)$$

is surjective for $i \geq 0$. Hence, given $x \in \mathfrak{q}$, there exists $y \in \mathfrak{p}$ with $[y, e] = x$. We can write $y = \Sigma y^\lambda$ where $y^\lambda \in \mathfrak{p}$ and $d\sigma(y^\lambda) = \lambda y^\lambda$ for $\lambda \in F^\times$. Comparing eigenvalues of $d\sigma$, we get $x = [y^1, e]$, and $y^1 \in \mathfrak{p}_\sigma$. This shows that $\mathrm{ad}(e) : \mathfrak{p}_\sigma \longrightarrow \mathfrak{q}$ is surjective, so $P_\sigma \cdot e$ is open and dense in $\mathfrak{q}$.

From this it follows easily that the image of π is $\overline{\mathcal{O}}$. Indeed, since the image of π is closed and contains e, we have $\overline{\mathcal{O}} \subseteq \pi(E)$. On the other hand,

$$\mathfrak{q} = \overline{P_\sigma{\cdot}e} \subseteq \overline{\mathcal{O}}.$$

Since $\pi(E) = G_\sigma \cdot \mathfrak{q}$ and $\overline{\mathcal{O}}$ is preserved by G_σ, we have the other containment.

We next show that π is bijective over $\mathcal{O}$. We recall the well-known resolution of $\overline{G{\cdot}e}$ (c.f. [**H**, p. 108]). Consider the vector bundle

$$\tilde{E} = \{(gP, x) \in G/P \times \mathfrak{g} : \ x \in g{\cdot}\mathfrak{p}^2\}.$$

Then the map

$$\tilde{E} \xrightarrow{\tilde{\pi}} \mathfrak{g}$$

given by projection onto the second factor is G-equivariant, closed, has image $\overline{G\cdot e}$, and is bijective over $G\cdot e$.

We define an automorphism σ_q of $\tilde{E}$ by

$$\sigma_q(gP, x) = \big(\sigma(g)P, q^{-1}d\sigma(x)\big).$$

The fixed point space of σ_q is

$$\tilde{E}_{\sigma_q} = \{(gP, x) \in (G/P)_\sigma \times \mathfrak{g} : \ x \in \mathcal{N}_{\sigma,q} \cap g\cdot\mathfrak{p}^2\}.$$

This is a disjoint union of vector bundles over the connected components of $(G/P)_\sigma$. In the notation of (2.3) these components are given by

$$(G/P)_\sigma = \coprod_{w\in W_\sigma^1\backslash W_\sigma/(W_P)_\sigma} G_\sigma wP/P,$$

and each component is a flag variety for G_σ.

The component of $\tilde{E}_{\sigma_q}$ lying over $G_\sigma P/P$ is

$$\{(gP, x) \in G_\sigma P/P \times \mathfrak{g} : \ x \in \mathcal{N}_{\sigma,q} \cap g\cdot\mathfrak{p}^2\}.$$

Since $G_\sigma/P_\sigma = G_\sigma P/P$ and for $g \in G_\sigma$, we have $\mathcal{N}_{\sigma,q} \cap g\cdot\mathfrak{p}^2 = g\cdot(\mathcal{N}_{\sigma,q} \cap \mathfrak{p}^2) = g\cdot\mathfrak{q}$, this component is none other than E.

Note that $\tilde{\pi}$ restricted to E is just our original $\pi : E \longrightarrow \overline{\mathcal{O}}$. Since the fibers of $\tilde{\pi}$ over $G\cdot e$ are singletons, the same must be true of the fibers of π over $\mathcal{O}$. The proof of (3.2) is now complete if the characteristic of F is zero.

In sufficiently large characteristic, we use an argument from [**Ri1**] to prove that π is separable. The assumption that F has characteristic $p > 2\kappa$ is, by a wide margin, enough to ensure (see [**Ri1**, §5]) that the orbit maps under the adjoint action of G are separable. Hence the tangent space to $G\cdot e$ is $T_eG\cdot e = [\mathfrak{g}, e]$. Decompose $\mathfrak{g} = \oplus\mathfrak{g}^\lambda$ into $d\sigma$-eigenspaces, so $\mathcal{O} \subseteq \mathfrak{g}^q$. Then $T_e\mathcal{O} \subseteq \mathfrak{g}^q \cap T_eG e = \mathfrak{g}^q \cap \oplus_\lambda[\mathfrak{g}^\lambda, e] = [\mathfrak{g}_\sigma, e]$. It follows that the differential of π at $(P_\sigma, e) \in E$ is surjective, so π is separable by [**B**, p. 41]. Hence $\pi : \pi^{-1}(\mathcal{O}) \longrightarrow \mathcal{O}$ is an isomorphism by Zariski's Main Theorem. □

COROLLARY 3.3.
(1) $\dim \mathcal{O} = \dim G_\sigma - \dim P_\sigma + \dim \mathfrak{q}$.

(2) $C_{G_\sigma}(e) = C_{P_\sigma}(e)$, *where* $C_H(e)$ *denotes centralizer in* H *of* e.

(3) $\mathcal{O} \cap \mathfrak{q}$ *is exactly the dense* P_σ*-orbit on* $\mathfrak{q}$. *(See the proof of (3.2).)*

Proof. The first two assertions are immediate from (3.2). We use them to compute

$$\begin{aligned}\dim P_\sigma \cdot e &= \dim P_\sigma - \dim C_{P_\sigma}(e)\\ &= \dim P_\sigma - \dim C_{G_\sigma}(e)\\ &= \dim P_\sigma + \dim \mathcal{O} - \dim G_\sigma\\ &= \dim P_\sigma + [\dim G_\sigma + \dim \mathfrak{q} - \dim P_\sigma] - \dim G_\sigma\\ &= \dim \mathfrak{q}.\end{aligned}$$

It follows that $P_\sigma \cdot e$ contains an open subset of $\mathfrak{q}$ which must meet, hence equal, the dense P_σ orbit on $\mathfrak{q}$. □

According to [**K**], a result like (3.2) has the following consequence in characteristic zero. Let $\mathcal{O}$ be an G_σ-orbit in $\mathcal{N}_{\sigma,q}$, and let E be the bundle over G_σ/P_σ as in (3.2). Let $S^d\check{\mathcal{E}}$ be the graded sheaf of sections of the d^{th} symmetric power of the dual bundle of E. Then $\oplus_{d\geq 0} S^d\check{\mathcal{E}}$ is the structure sheaf $\mathcal{O}_E$ of E, and the global sections $H^0(G_\sigma/P_\sigma, S^d\check{\mathcal{E}})$ are the regular functions on E which are polynomial of degree d on each fiber. Let $F[\overline{\mathcal{O}}]$ be the coordinate ring of the affine variety $\overline{\mathcal{O}}$. The map π induces an injection of G_σ-modules

$$\pi^* : F[\overline{\mathcal{O}}] \hookrightarrow \bigoplus_{d\geq 0} H^0(G_\sigma/P_\sigma, S^d\check{\mathcal{E}})$$

given by $\pi^*(f)(mP_\sigma) = f|_{m\cdot\mathfrak{q}}$.

COROLLARY 3.4. *Assume the characteristic of* F *is zero and that* $\mathfrak{q}$ *is a completely reducible* P_σ*-module. Then*

(1) π^* *is an isomorphism of* G_σ*-modules.*

(2) $\overline{\mathcal{O}}$ *is a normal variety.*

(3) $H^i(E, \mathcal{O}_E) = 0$ *for* $i > 0$. *(One says that* $\overline{\mathcal{O}}$ *has "rational singularities").*

The proof of the analogous result for nilpotent orbits in [**H**] carries over without change. Note that $\mathfrak{q}$ is a completely reducible P_σ-module if and only if the unipotent radical of P_σ acts trivially on $\mathfrak{q}$.

IV. From G-orbits to G_σ-orbits. We retain the notation from the proof of (3.2), and investigate the other components of $\tilde{E}_{\sigma_q} = (G \times_P \mathfrak{p}^2)_{\sigma_q}$ occurring there. For $w \in W_\sigma$, we put

$$\mathfrak{q}_w = \mathcal{N}_{\sigma,q} \cap w \cdot \mathfrak{p}^2,$$

so $\mathfrak{q} = \mathfrak{q}_1$. Note that $\mathfrak{q}_w$ is an $\mathrm{Ad}(wPw^{-1})_\sigma$-invariant linear subspace of $\mathcal{N}_{\sigma,q}$. It follows from (2.3) that for each $w \in W_\sigma^1 \backslash W_\sigma/(W_P)_\sigma$ we have a component

$$E_w = \{(gwP, x) \in G_\sigma wP/P \times \mathfrak{g} : \ x \in g \cdot \mathfrak{q}_w\},$$

and

$$\tilde{E}_{\sigma_q} = \coprod_{w \in W_\sigma^1 \backslash W_\sigma/(W_P)_\sigma} E_w.$$

One could also view E_w as the fiber product

$$E_w = G_\sigma \times_{P_\sigma} \mathfrak{q}_w.$$

Since E_w is a vector bundle over a smooth variety and $\tilde{\pi}$ is G_σ-equivariant and closed, we see that $\pi(E_w) = G_\sigma \cdot \mathfrak{q}_w$ is an G_σ-stable, closed, irreducible subvariety of $\mathcal{N}_{\sigma,q} \cap \overline{G \cdot e}$. Hence $\pi(E_w) = \overline{\mathcal{O}_w}$ for a unique G_σ-orbit $\mathcal{O}_w \subseteq G \cdot e$. Note that $\pi(E_w)$ meets $G \cdot e$ if and only if $\mathcal{N}_{\sigma,q} \cap w \cdot \mathfrak{p}_0^2 \neq \varnothing$. (Recall that $\mathfrak{p}_0^2 = \mathfrak{p}^2 \cap G \cdot e$ is the unique dense P-orbit in $\mathfrak{p}^2$.) By [**Ri3**] the G_σ-orbits in $\mathcal{N}_{\sigma,q} \cap G \cdot e$ are exactly the irreducible components of $\mathcal{N}_{\sigma,q} \cap G \cdot e$. Thus, the closure of each such G_σ-orbit is an irreducible component of $\mathcal{N}_{\sigma,q} \cap \overline{G \cdot e}$. The main result in this section ties these considerations together and, in tandem with (2.3), reduces the computation of the G_σ-orbits in $G \cdot e$ to knowing the dense orbit in the prehomogeneous vector space $(P, \mathfrak{p}^2, \mathrm{Ad})$. We note that for each G there are only finitely many of the latter to consider.

PROPOSITION 4.1. (1) *The map $w \mapsto \mathcal{O}_w$ described above is a bijection from*

$$\{w \in W_\sigma^1 \backslash W_\sigma/(W_P)_\sigma : \ \mathcal{N}_{\sigma,q} \cap w \cdot \mathfrak{p}_0^2 \neq \varnothing\}$$

to the set of G_σ-orbits in $\mathcal{N}_{\sigma,q} \cap G \cdot e$. The inverse of this bijection sends a G_σ-orbit $\mathcal{O} \subseteq G \cdot e$ to the $w \in W_\sigma^1 \backslash W_\sigma/(W_P)_\sigma$ such that E_w

is the unique component of $\tilde{E}_{\sigma_q}$ *meeting a* $\tilde{\pi}$*-fiber over some point in* $\mathcal{O}$.

(2) *The map* $\pi : E_w \longrightarrow \overline{\mathcal{O}_w}$ *is a desingularization of* $\overline{\mathcal{O}_w}$.

(3) *If* $\mathcal{N}_{\sigma,q} \cap w \cdot \mathfrak{p}_0^2$ *is nonempty, then* $\mathcal{O}_w = G_\sigma \cdot (\mathcal{N}_{\sigma,q} \cap w \cdot \mathfrak{p}_0^2) = \overline{\mathcal{O}_w} \cap G \cdot e$.

Proof. Since $\tilde{\pi}$ is bijective over $G \cdot e$, it is clear that any G_σ-orbit in $\mathcal{N}_{\sigma,q} \cap G \cdot e$ can meet $\pi(E_w)$ for at most one $w \in W_\sigma^1 \backslash W_\sigma / (W_P)_\sigma$. Hence $w \mapsto \mathcal{O}_w$ is injective.

Take $x \in \mathcal{N}_{\sigma,q} \cap G \cdot e$, and look at the fiber $\tilde{\pi}^{-1}(x) = \{(gP, x)\}$, where $x \in g \cdot \mathfrak{p}^2$. We have $qx = d\sigma(x) \in d\sigma(g \cdot \mathfrak{p}^2) = \sigma(g) \cdot \mathfrak{p}^2$, so $x \in \sigma(g) \cdot \mathfrak{p}^2$. But then $(\sigma(g)P, x) \in \tilde{\pi}^{-1}(x)$, so $gP \in (G/P)_\sigma$. This shows that $\mathcal{N}_{\sigma,q} \cap G \cdot e$ is contained in the union of the $\pi(E_w)$'s. Now consider a G_σ-orbit $\mathcal{O}_0 \subseteq \mathcal{N}_{\sigma,q} \cap G \cdot e$, and suppose $\mathcal{O}_0$ meets $\pi(E_w)$. Then $\mathcal{O}_0 \subseteq \pi(E_w)$ so $\overline{\mathcal{O}_0} \subseteq \pi(E_w)$ since the latter is closed. Recall however, that $\overline{\mathcal{O}_0}$ is an irreducible component of $\mathcal{N}_{\sigma,q} \cap \overline{G \cdot e}$, so we must have $\overline{\mathcal{O}_0} = \pi(E_w)$, since the latter is also irreducible.

Assertion (2) follows from the birationality of $\tilde{\pi}$, as we have argued before.

Let $x \in \overline{\mathcal{O}_w} \cap Ge = G_\sigma \mathfrak{q}_w \cap Ge$. Then there exist $m \in G_\sigma$, $g \in G$ and $v \in \mathfrak{p}^2$ such that $x = mw \cdot v = g \cdot e$, so $v = (mw)^{-1} g \cdot e \in \mathfrak{p}^2 \cap G \cdot e = \mathfrak{p}_0^2$. Hence $x \in \mathcal{N}_{\sigma,q} \cap mw \cdot \mathfrak{p}_0^2$. This proves the second equality in (3), the other direction there being trivial.

We now know $\mathcal{O}_w \subseteq G_\sigma \cdot (\mathcal{N}_{\sigma,q} \cap w \cdot \mathfrak{p}_0^2) = \overline{\mathcal{O}_w} \cap G \cdot e$. Moreover, $\mathcal{N}_{\sigma,q} \cap w \cdot \mathfrak{p}_0^2$, being open in the linear space $\mathcal{N}_{\sigma,q} \cap w \cdot \mathfrak{p}^2$, is irreducible. It follows that $G_\sigma \cdot (\mathcal{N}_{\sigma,q} \cap w \cdot \mathfrak{p}_0^2)$ is an irreducible subset of $\mathcal{N}_{\sigma,q} \cap G \cdot e$ containing the irreducible component $\mathcal{O}_w$ of $\mathcal{N}_{\sigma,q} \cap G \cdot e$. Hence $\mathcal{O}_w = G_\sigma \cdot (\mathcal{N}_{\sigma,q} \cap w \cdot \mathfrak{p}_0^2)$. □

COROLLARY 4.2. *Let* $\tilde{\mathcal{O}} \subset \mathfrak{g}$ *be a nilpotent* G*-orbit. Then* $\tilde{\mathcal{O}} \cap \mathcal{N}_{\sigma,q}$ *is a union of at most* $|W_\sigma^1 \backslash W_\sigma / (W_P)_\sigma|$ *orbits under* G_σ.

COROLLARY 4.3. *Assume* σ *is induced by a symmetry of the Dynkin diagram of* G. *Let* $\tilde{\mathcal{O}} \subset \mathfrak{g}$ *be a nilpotent* G*-orbit. Then* $\tilde{\mathcal{O}}$ *meets* $\mathcal{N}_{\sigma,q}$ *in at most one* G_σ*-orbit.*

Proof. In this situation, $W_\sigma^1 = W_\sigma$ as remarked in (2.6), so the result is a special case of (4.3). □

When $q = 1$, we are counting the number of nilpotent G_σ-orbits in $\mathfrak{g}_\sigma$ which are contained in a single G-orbit. It is well-known that nilpotent orbits in $\mathfrak{so}(2n+1)$ and $\mathfrak{sp}(n)$ are determined by elementary divisors. The above results say this is due to the connectedness of the varieties $(G/P)_\sigma$, which is in turn deduced from the equality $W^1_\sigma = W_\sigma$. Likewise, every $\mathfrak{e}_6$ nilpotent orbit meets $\mathfrak{f}_4$ in at most one F_4-orbit, and similarly for $G_2 \subset Spin(8)$. In the case $G = SL(2n)$, $G_\sigma = SO(2n)$, one checks that $|Y(\sigma)| = 2$, so (4.2) reduces to the well-known result that at most two nilpotent orbits in $\mathfrak{so}(2n)$ have the same elementary divisors, and these orbits are conjugate in $O(2n)$.

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Received September 17, 1992 and in revised form January 12, 1993. Research partially supported by NSF grant DMS-9104183.

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