PACIFIC JOURNAL OF MATHEMATICS
Vol. 168, No. 1, 1995

METRICS FOR SINGULAR ANALYTIC SPACES

CAROLINE GRANT AND PIERRE MILMAN

Recent work of Saper, Zucker, and others indicates that Kähler metrics with appropriate growth rates on the nonsingular set of a compact Kähler variety are useful in describing the geometry of such a variety. It has been conjectured that for every complex algebraic variety X there exists a Kähler metric on the nonsingular set of X whose L_2-cohomology is isomorphic to the intersection cohomology of X. Saper proved this conjecture for varieties with isolated singularities, using a complete Kähler metric. Similar results have been obtained by others using incomplete metrics. We give natural and explicit constructions of three types of Kähler metrics on the nonsingular set $X - X_{\text{sing}}$ of a subvariety X of a compact Kähler manifold. No restrictions on the type of singularities of X are assumed. Similar constructions can be done for nonembedded compact Kähler varieties. The first metric is Hodge if X is algebraic but is not complete on $X - X_{\text{sing}}$ if X is singular. The completion of $X - X_{\text{sing}}$ under this metric is a desingularization of X. The second metric is complete and generalizes Saper's metric for varieties with isolated singularities. Moreover each incomplete metric of the first type is naturally associated with a complete metric of the second type. The third metric is a sum of the first two and has Poincaré-type growth near the singular locus of X.

§**1. Introduction.** If X is a smooth algebraic variety over **C**, or more generally, a compact Kähler manifold, then the cohomology of X with complex coefficients $H^*(X)$ is isomorphic to the de Rham cohomology $H^*_{\text{DR}}(X)$ and satisfies a collection of cohomology theorems including Poincaré duality, the Lefschetz hyperplane theorem, and the hard Lefschetz theorem. These cohomology theorems do not always hold for $H^*(X)$ when X is singular. In this case the appropriate cohomology is the intersection cohomology of Goresky-MacPherson. It has been conjectured that the appropriate replacement for the de Rham cohomology is L_2-cohomology, or more precisely, that the intersection cohomology of a singular algebraic variety X is always isomorphic to the L_2-cohomology of X with respect to a suitable Kähler metric on the nonsingular set $X - X_{\text{sing}}$ of X. Saper proved this conjecture for varieties with isolated singularities, using a complete Kähler metric [**S2**]. Recently Ohsawa [**O**] announced this conjecture for complex projective varieties using the induced incomplete metric. Complete Kähler metrics are of particular interest because they yield Hodge decompositions.

We construct three types of Kähler metrics on the nonsingular set $X - X_{\text{sing}}$ of a subvariety X of a compact Kähler manifold. Our constructions are explicit and use in a natural way the geometry of a sequence of blow-ups used to resolve the singularities of X. No restrictions on the type of singularities are assumed. These metrics were introduced in [**GrM**]. The first metric is induced by a metric which we construct on a desingularization $\tilde{X}$ of X and has the property that it is Hodge if X is algebraic. If X is singular, this metric is not complete on $X - X_{\text{sing}}$ and the completion of $X - X_{\text{sing}}$ with respect to this metric is the desingularization $\tilde{X}$.

The second metric is complete on $X - X_{\text{sing}}$ and generalizes Saper's metric for varieties with isolated singularities. We show that each incomplete metric of the first type is naturally associated with a complete metric of this type. We also prove one of the conditions needed to apply Goresky-MacPherson's theorem [**GM**] characterizing intersection cohomology: we show that for this metric, the associated complex of L_2-bounded sheaves on X is fine. This modified Saper metric seems to be a good candidate for extending Saper's theorem to more general varieties.

The third metric is the sum of our incomplete metric and our modified Saper metric. It is complete and has Poincaré-type growth near X_{sing}. The L_2-cohomology of a true Poincaré metric on the complement of a divisor with normal crossings on $\tilde{X}$ is isomorphic to the intersection cohomology of $\tilde{X}$, not X. We expect the same to be true for our modified Poincaré metric. This metric is bounded below by the induced incomplete metric and the modified Saper metric. Consequently, the associated sheaves of L_2-bounded forms on both X and $\tilde{X}$ are fine.

Although we have restricted ourselves in this paper to subvarieties of compact Kähler manifolds, similar constructions can be done in more general settings.

In an appendix, we include, for lack of a suitable reference, some tubular neighbourhood constructions which are based on a construction of Clemens [**CL**].

We will give further guidance to the paper in §2, after a few preliminaries.

We would like to acknowledge helpful conversations with John Bland, Michael Hoffman, and Boris Youssin.

TABLE OF CONTENTS

(1.1) Resolution of Singularities. Let X be a reduced singular compact analytic subspace of a compact Kähler manifold M. Let X_{sing} be the singular set of X. The singularities of X may be resolved by a finite sequence of blow-ups $\pi_j : M_j \to M_{j-1}$ along smooth centres C_j in M_{j-1}, where $M_0 = M$. We will describe a systematic way of constructing a complete Kähler metric on $X - X_{\text{sing}}$, using the geometry of these blow-ups.

Let X_j be the strict transform of X in M_j, i.e. X_j is the closure of $\pi_j^{-1}(X_{j-1} - C_j)$ in M_j. Let D_j be the exceptional divisor of the composite $\pi_1 \circ \pi_2 \circ ... \circ \pi_j$ of the first j blow-ups, i.e. $D_1 = \pi_1^{-1}(C_1)$ and $D_j = \pi_j^* D_{j-1} + \pi_j^{-1}(C_j)$ for $j \geq 2$. Let $\tilde{M}$ be the final blow-up of M, $\tilde{X}$ the final strict transform of X, D the final exceptional divisor, and $\pi : \tilde{M} \to M = M_0$ the composite of all the blow-ups. By the notation $\tilde{M} - D$ we will always mean $\tilde{M} - \operatorname{supp} D$.

DEFINITION. We say that an analytic subspace Z of a complex manifold M has only *normal crossings* if, at each point p in Z, there is a local coordinate system $(z_1, ..., z_n)$ on M in which Z is given by the vanishing of a collection of monomials $z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n}$.

Hironaka [**H**] showed that the centres C_j may be chosen so that

(1) C_j is contained in the singular locus $(X_{j-1})_{\text{sing}}$ of X_{j-1} if X_{j-1} is singular, or in $X_{j-1} \cap D_{j-1}$ otherwise,

(2) C_j and D_{j-1} simultaneously have only normal crossings,

(3) $\tilde{X}$ is smooth, and

(4) $\tilde{X}$ and D have normal crossings.

It follows from properties (1) - (4) that

(5) $D = \pi^{-1}(X_{\text{sing}})$,

(6) the restriction of π to $\tilde{M} - D$ is a biholomorphism onto $M - X_{\text{sing}}$ (and consequently the restriction to $\tilde{X} - (\tilde{X} \cap D)$ is a biholomorphism onto $X - X_{\text{sing}}$), and

(7) $\pi_{j,0}^{-1}(X_{\text{sing}}) \supset (X_j)_{\text{sing}}$ where $\pi_{j,0}$ is the composite map $\pi_{j,0} = \pi_1 \circ \pi_2 \circ ... \circ \pi_j : M_j \to M_0$.

Note: A simple proof of canonical resolution of singularities can be found in [**BM2**].

We will construct a complete Kähler metric on each noncompact manifold $M_j - D_j$. The restriction of the final metric on $\tilde{M} - D$ to $\tilde{X} - (\tilde{X} \cap D)$ induces a complete Kähler metric on $X - X_{\text{sing}}$.

(1.2) Metrics and their fundamental $(1,1)$-forms. Let h be a hermitian metric on a complex manifold M. In local coordinates $(z_1, z_2, ..., z_n)$ on an open set U in M, the metric h can be written

$$h|_U = \sum_{i,j=1}^{n} h_{ij} dz_i \otimes d\overline{z}_j$$

where (h_{ij}) is a positive definite hermitian matrix whose entries are C^∞ functions on U. Specifying a hermitian metric h on M is equivalent to specifying a positive hermitian C^∞ $(1,1)$-form ω on

M, given locally by

$$\omega|_U = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} h_{ij} dz_i \wedge d\overline{z}_j.$$

We call ω the *fundamental form* of h. The metric h is called *Kähler* and ω is called a *Kähler form* if ω is d-closed, i.e. if $d\omega = 0$. A Kähler metric is called *Hodge* and its Kähler form ω is called a *Hodge form* if ω is integral, i.e. if the de Rham cohomology class of ω lies in the image of $H^*(M, \mathbf{Z})$ under the natural map $H^*(M, \mathbf{Z}) \to H^*_{DR}(M, \mathbf{C})$. The manifold M is called Kähler (respectively Hodge) if it admits a Kähler (respectively Hodge) metric. The following will be useful in constructing Kähler forms.

LEMMA 1.2.1. *If g is a positive real C^∞ function on an open set U in $\mathbf{C}^n$, then the $(1,1)$-form $\omega = \partial\overline{\partial} \log g$ is hermitian and d-closed on U.*

Proof. It is easily checked that $\frac{\partial^2 g}{\partial z_i \partial \overline{z}_j} = \overline{\frac{\partial^2 g}{\partial z_j \partial \overline{z}_i}}$ for any two local coordinates z_i and z_j in $\mathbf{C}^n$ and that $d\partial\overline{\partial}\psi = 0$ for any C^∞ function ψ. □

(1.3) Quasi-isometry. We say that two metrics h_1 and h_2 are *quasi-isometric* and write $h_1 \sim h_2$ if there are positive constants c and C such that $ch_1 \leq h_2 \leq Ch_1$. Similarly, if ω_1 and ω_2 are $(1,1)$-forms or functions, and there exist positive constants c and C such that $c\omega_1 \leq \omega_2 \leq C\omega_1$, we write $\omega_1 \sim \omega_2$. If ω_1 and ω_2 are positive $(1,1)$-forms and $\omega_1 \sim \omega_2$, we say that ω_1 and ω_2 are quasi-isometric. If $\omega_1 \geq \omega_2$ and $\omega_2 \sim \omega_3$ we write $\omega_1 \gtrsim \omega_3$. When a function f_1 is dominated by a constant multiple of a function f_2 near the exceptional divisor we will write $f_1 = O(f_2)$.

REMARK (1.3.1). All metrics on a compact manifold are quasi-isometric. If ω_1 and ω_2 are positive C^∞ $(1,1)$-forms on an open neighbourhood of a point q, then ω_1 and ω_2 are quasi-isometric in a neighbourhood of q. In local coordinates $(z_1, ..., z_n)$ near q, every positive C^∞ $(1,1)$-form is locally quasi-isometric to the Euclidean form

$$\omega_{\text{Eucl}} = \frac{\sqrt{-1}}{\pi} \sum_{i=1}^{n} dz_i \wedge d\overline{z}_i. \tag{1.3.2}$$

(1.4) Poincaré metrics. Recall that the fundamental form of the Poincaré metric on the punctured disc $\Delta^* = \{z \in \mathbf{C} : 0 <\mid z \mid< 1\}$ is

$$\omega_{\Delta^*} = \frac{\sqrt{-1}}{\pi} \frac{dz \wedge d\bar{z}}{\mid z \mid^2 (\log \mid z \mid^2)^2} = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\log \mid z \mid^2)^2. \tag{1.4.1}$$

We are interested only in the asymptotic behaviour of ω_{Δ^*} near the puncture in Δ^*.

More generally, let M be a compact complex manifold and D a divisor with normal crossings. If q is a point of D at which k components intersect, then q has a neighbourhood whose intersection with $M - D$ is of the form $(\Delta^*)^k \times \Delta^{n-k}$. A metric on $M - D$ is said to be *Poincaré* if it is quasi-isometric near each such q to a product of Poincaré metrics on $(\Delta^*)^k$ and Euclidean metrics on Δ^{n-k}. It is always possible to construct a Poincaré metric on $M - D$ by patching together local forms with C^∞ partitions of unity. If M is Kähler, it is easy to construct Kähler Poincaré metrics on $M - D$ by a global construction (see §5.4).

(1.5) Normal coordinates. Let M and D be as above, let $E_1, ..., E_k$ be the components of D passing through a point q, and let $(z_1, ..., z_n)$ be local coordinates on a neighbourhood U of q such that no other component of D intersects $\overline{U}$. We call the coordinates $(z_1, ..., z_n)$ *normal coordinates for* $E_1, ..., E_k$ if E_i is given locally near q by the equation $z_i = 0$ for $1 \leq i \leq k$.

The fundamental form ω_{Poinc} of a Poincaré metric on $M - D$ may be described locally in any system of normal coordinates by the quasi-isometry

$$\begin{aligned} \omega_{\text{Poinc}} &\sim \frac{\sqrt{-1}}{\pi} \left(\sum_{i=1}^{k} \frac{dz_i \wedge d\bar{z}_i}{\mid z_i \mid^2 (\log \mid z_i \mid^2)^2} + \sum_{i=k+1}^{n} dz_i \wedge d\bar{z}_i \right) \\ &\sim \frac{\sqrt{-1}}{\pi} \left(\sum_{i=1}^{k} \frac{dz_i \wedge d\bar{z}_i}{\mid z_i \mid^2 (\log \mid z_i \mid^2)^2} + \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i \right). \end{aligned} \tag{1.5.1}$$

Note that we are concerned only with the asymptotic behaviour of these forms near $z_1 z_2 ... z_k = 0$. We may assume that U is small enough that singularities of the forms on the right at $| z_i |= 1$ may be ignored.

(1.6) Modified Poincaré metrics. We call a metric on $M - D$ a *modified Poincaré metric* if its fundamental form ω_P may be described locally in normal coordinates by the quasi-isometry

$$(1.6.1) \qquad \omega_P \sim \sum_{i=1}^{m} \tau_i^* \omega_{\Delta^*} + \frac{\sqrt{-1}}{\pi} \sum_{i=1}^{n} dz_i \wedge d\overline{z}_i$$

where each τ_i is a nonconstant monomial map from $(\Delta^*)^k \times \Delta^{n-k}$ to Δ^* of the form $\tau_i(z_1, ..., z_n) = z_1^{\lambda_{i1}} z_2^{\lambda_{i2}} ... z_k^{\lambda_{ik}}$ and the $m \times k$ matrix $\Lambda = (\lambda_{ij})$ has nonnegative integer entries and at least one positive entry in each row and column. We note that there may be many different non-quasi-isometric modified Poincaré metrics on $M - D$. We show in Corollary (5.2.6) that the description (1.6.1) is independent of the choice of normal coordinates. The matrix (λ_{ij}) determines the local quasi-isometry class of the metric, but not vice-versa.

We will say that a modified Poincaré metric is a *homogeneous* Poincaré metric if its fundamental form $\omega_{P,\mathrm{hom}}$ may be described locally in normal coordinates by the quasi-isometry

(1.6.2)

$$\omega_{P,\mathrm{hom}} \sim \frac{\sqrt{-1}}{\pi} \left(\frac{1}{(\log | z_1 z_2 ... z_k |^2)^2} \sum_{i=1}^{k} \frac{dz_i \wedge d\overline{z}_i}{| z_i |^2} + \sum_{i=1}^{n} dz_i \wedge d\overline{z}_i \right).$$

This description is also independent of the choice of normal coordinates (Corollary (5.3.5)).

A little linear algebra shows that a modified Poincaré metric is homogeneous if the matrix Λ has rank k and all the entries of Λ are positive (Lemma (5.3.3)). If the matrix Λ can be chosen to be the identity matrix of suitable dimensions, the modified Poincaré metric is Poincaré. We construct Kähler modified Poincaré metrics and Kähler homogeneous Poincaré metrics in §5.4.

(1.7) Saper's distinguished metrics. Saper [S2] calls a metric on $M - D$ *distinguished* if its fundamental form ω_{Sap} may be described locally in normal coordinates by the quasi-isometry

$$(1.7.1)\quad \omega_{\text{Sap}} \sim \frac{\sqrt{-1}}{\pi}\left(\frac{1}{(\log|z_1 z_2 \ldots z_k|^2)^2}\sum_{i=1}^{k}\frac{dz_i\wedge d\overline{z}_i}{|z_i|^2} + \frac{1}{|\log|z_1 z_2\ldots z_k|^2|}\sum_{i=1}^{n} dz_i\wedge d\overline{z}_i\right).$$

This description is independent of the choice of normal coordinates (Corollary (5.3.5)). We will usually refer to such a metric simply as a Saper metric.

If ω is the fundamental form of a metric on the compact manifold M, and ω_{Sap} and $\omega_{P,\text{hom}}$ are, respectively, fundamental forms of Saper and homogeneous Poincaré metrics on the noncompact manifold $M - D$, then

$$(1.7.2)\qquad \omega_{\text{Sap}} + \omega \sim \omega_{P,\text{hom}}.$$

(1.8) Modified Saper metrics. Let M and $\tilde{M}$ be compact complex manifolds, let D be an effective divisor on $\tilde{M}$ with normal crossings, and let $\pi : \tilde{M} \to M$ be a holomorphic map such that the restriction of π to $\tilde{M} - D$ is a biholomorphism onto its image. The divisor D may be expressed as $D = \sum_{i=1}^{m} \lambda_i E_i$, where $E_1, E_2, \ldots, E_m$ are smooth, reduced, irreducible divisors on $\tilde{M}$ which simultaneously have only normal crossings, and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are positive integers. Let $L = [D]$ be the line bundle on $\tilde{M}$ associated with D and let h be a hermitian metric on L. Let $s : \tilde{M} \to L$ be a global holomorphic section of L such that $(s) = D$. Such a section always exists since D is effective (see §3.4). We denote by $\| s \|$ the norm of s under the metric h. Since $\tilde{M}$ is compact, we may also choose s so that $\| s \| < 1$ everywhere on $\tilde{M}$. The Chern form of L with respect to the metric h may be written as

$$c_1(L, h) = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\| s \|^2$$

(cf. §3.5). We define on $\tilde{M} - D$ a Poincaré-type $(1,1)$-form ν associated with the divisor D, the section s, and the metric h by

$$(1.8.1)\qquad \nu = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(\log\| s \|^2\right)^2.$$

The crucial property of the Poincaré-type form ν is that it splits as $\nu = \mu + \eta$ where

$$\mu = \frac{2}{\log \| s \|^2} c_1(L, h)$$

and the form η near D is essentially the pullback of the Poincaré form ω_{Δ^*} under a monomial map, plus low order terms. More precisely, suppose that $z_1, ..., z_n$ are normal coordinates for the components of D in a neighbourhood of a point $p \in D$ such that D is given locally by the vanishing of the monomial

$$\tau(z) = z_1^{\lambda_1} z_2^{\lambda_2} ... z_k^{\lambda_k}.$$

We treat τ as a map from a neighbourhood U of p to the disc $\Delta = \{z \in \mathbf{C} : | z | < 1\}$. Let ω_{Eucl} be the local Euclidean $(1,1)$-form

$$\omega_{\mathrm{Eucl}} = \frac{\sqrt{-1}}{\pi} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$$

and let $\beta = -\log \| s \|^2$. We will show in §5 that

$$\frac{1}{\beta}\omega_{\mathrm{Eucl}} + \eta \sim \frac{1}{\beta}\omega_{\mathrm{Eucl}} + \tau^* \omega_{\Delta^*}$$

near p. Note that $\beta \neq 0$ on $\tilde{M} - D$ and $\beta \to \infty$ as we approach D.

Let $D_1, ..., D_r$ be divisors of the form

$$D_i = \sum_{j=1}^{m} b_{ij} E_j \qquad (1.8.2)$$

such that the matrix (b_{ij}) has nonnegative integer entries and at least one positive entry in each row and column, i.e. the divisors D_i are effective and their sum has the same support as D. We may choose sections s_i and metrics h_i for the line bundles $[D_i]$ as above, and use them to construct Poincaré-type forms ν_i. Let ω be the fundamental form of a hermitian metric on M and let $l_0, l_1, ..., l_r$ be positive integers.

DEFINITION (1.8.3). We will call any expression of the form

$$\omega_S = l_0 \pi^* \omega + \sum_{i=1}^{r} l_i \nu_i$$

a *modified Saper form which is distinguished with respect to* π. We will call a metric on $\tilde{M} - D$ a *modified Saper metric* if its fundamental form is quasi-isometric to a modified Saper form.

If Y is a smooth submanifold of $\tilde{M}$ which has normal crossings with D, we also call the induced metric on $Y - (Y \cap D)$ a modified Saper metric. We are particularly interested in the case in which the closure X of $\pi(Y - (Y \cap D))$ in M is singular, so that Y is an embedded resolution of a singular analytic subspace $X \subset M$.

Every positive modified Saper form ω_S determines a modified Saper metric. If in addition ω is Kähler, then so is ω_S. We will show that the sum of a modified Saper metric on $\tilde{M} - D$ and the restriction of a metric on $\tilde{M}$ is a modified Poincaré metric on $\tilde{M} - D$.

As with modified Poincaré metrics, there may be many different non-quasi-isometric modified Saper metrics on $\tilde{M} - D$.

Let ω_{Sap} be the fundamental form of a distinguished Saper metric on $\tilde{M} - D$, as defined in §1.7.

DEFINITION (1.8.4). We will say that a modified Saper metric and its fundamental form $\omega_{S,\text{hom}}$ are *homogeneous* if $\omega_{S,\text{hom}}$ satisfies

$$\omega_{S,\text{hom}} \sim \pi^*\omega + \omega_{\text{Sap}}.$$

The sum of a homogeneous Saper metric on $\tilde{M} - D$ and the restriction of a metric on $\tilde{M}$ is a homogeneous Poincaré metric on $\tilde{M} - D$.

(1.9) L_2-cohomology of Poincaré and Saper metrics. Let X be a reduced compact analytic subspace of a compact Kähler manifold M. Let $\pi : \tilde{M} \to M$ be a composite of blow-ups of the type described in §1.1 which resolves the singularities of X, let $\tilde{X}$ be the strict transform of X in $\tilde{M}$, and let D be the exceptional divisor of π in $\tilde{M}$. Then $\tilde{X}$ is smooth and has normal crossings with D, and $\tilde{X} - (\tilde{X} \cap D)$ is biholomorphic to $X - X_{\text{sing}}$. A metric on $\tilde{M} - D$ induces a metric on $X - X_{\text{sing}}$. Since $\tilde{X}$ has normal crossings with D, the restriction to $\tilde{X} - (\tilde{X} \cap D)$ of a distinguished Saper metric on $\tilde{M} - D$ is a distinguished Saper metric on $\tilde{X} - (\tilde{X} \cap D)$, and similarly for Poincaré, modified Poincaré, homogeneous Poincaré, modified Saper, and homogeneous Saper metrics.

It is always possible to construct distinguished metrics on $\tilde{M} - D$ using C^∞ partitions of unity, but these metrics may not be Kähler.

Saper [**S2**] constructed Kähler distinguished metrics in the case that X is a compact Kähler variety with isolated singularities, and used Goresky-MacPherson's characterization of intersection cohomology to prove that the L_2-cohomology of X with respect to such a metric is isomorphic to the intersection cohomology of X.

In contrast, the L_2-cohomology of $\tilde{X}$ with respect to a Poincaré metric on $\tilde{X} - (\tilde{X} \cap D)$ is isomorphic to the cohomology of $\tilde{X}$ ([**Z1**], [**Z2**]). It is easy to construct a Kähler Poincaré metric on $\tilde{M} - D$ and hence on $\tilde{X} - (\tilde{X} \cap D)$ (see §5.4).

Although we have restricted ourselves in this paper to subvarieties of compact Kähler manifolds, similar constructions can be done in more general situations. In particular, if X is any compact Kähler variety with isolated singularities, we may make use of local embeddings of neighbourhoods of the singular points of X into domains in $\mathbf{C}^N$, to construct our metrics on $X - X_{\text{sing}}$. Our modified Saper metrics on varieties with isolated singularities are exactly Saper distinguished metrics.

Let h be a hermitian metric on $X - X_{\text{sing}} \cong \tilde{X} - (\tilde{X} \cap D)$. We define associated complexes of L_2 sheaves $\mathcal{S}$ on X and $\tilde{\mathcal{S}}$ on $\tilde{X}$ as follows. Let $\mathcal{S}_0$ be the complex of presheaves on X whose sections over any open set U in X are smooth measurable differential forms ϕ on $U \cap (X - X_{\text{sing}})$ such that both ϕ and $d\phi$ are L_2-bounded with respect to h. Let $\mathcal{S}$ be the associated complex of sheaves on X. Similarly, we define a complex of presheaves $\tilde{\mathcal{S}}_0$ on $\tilde{X}$, whose sections over any open set U in $\tilde{X}$ are smooth measurable differential forms ϕ on $U - (U \cap D)$ such that both ϕ and $d\phi$ are L_2-bounded with respect to h, and let $\tilde{\mathcal{S}}$ be the associated complex of sheaves on $\tilde{X}$. One of the conditions needed to apply Goresky-MacPherson's theorem to $\mathcal{S}$ is that $\mathcal{S}$ must be a complex of fine sheaves. We will show that if h is a modified Saper or modified Poincaré metric then $\mathcal{S}$ on X is fine, and that if h is a modified Poincaré metric then $\tilde{\mathcal{S}}$ on $\tilde{X}$ is also fine. We prove corresponding statements for L_2 sheaves on M and $\tilde{M}$.

§2. Main results and guidance through the paper.

(2.1) Theorem I. Our first main result concerns the existence of complete Kähler modified Saper metrics. We prove parts (i) and (ii) in §8, part (iii) in §9, and part (iv) in §10.

THEOREM I. *Let M_0 be a compact Kähler manifold with Kähler form ω_0. Suppose that $\{\pi_j : M_j \to M_{j-1}\}$ is a finite sequence of blow-ups along smooth centres $C_j \subset M_{j-1}$, chosen so that C_j has normal crossings with the total exceptional divisor D_{j-1} of the composite of the first $j-1$ blow-ups. Then*

i. *There exist complete Kähler modified Saper metrics with fundamental forms $\omega_{S,j}$, constructed inductively on the manifolds $M_j - D_j$. These metrics are bounded below by the sum of a Saper distinguished metric and an induced metric from M_0, and are quasi-isometric to this lower bound in the case that for $2 \le i \le j$ either $C_i \subset D_{i-1}$ or C_i is disjoint from D_{i-1}. If, in addition, $\dim C_1 = 0$ and $\dim C_i = 0$ for each i such that C_i is disjoint from D_{i-1} (i.e. the image of D_j in M_0 consists of isolated points in M_0), these metrics are exactly Saper distinguished metrics.*
ii. *The sum of a modified Saper metric on $M_j - D_j$ and the restriction of a metric on M_j is a modified Poincaré metric on $M_j - D_j$, which in turn is bounded above by a true Poincaré metric.*
iii. *There also exist complete Kähler homogeneous Saper metrics on $M_j - D_j$.*
iv. *The complex of L_2 sheaves on M_0 associated with any modified Saper or modified Poincaré metric on $M_j - D_j$ is fine. The complex of L_2 sheaves on M_j associated with any modified Poincaré metric on $M_j - D_j$ is also fine.*

The bounds of (i) *and* (ii) *are written more concisely in terms of the composite $\pi_{j,0}$ of the first j blow-ups and the fundamental forms ω_j of a metric on M_j, $\omega_{S,j}$ and $\omega_{P,j}$ of the modified Saper and modified Poincaré metrics, ω_{Sap} of a Saper distinguished metric, and ω_{Poinc} of a true Poincaré metric on $M_j - D_j$:*

$$\pi_{j,0}^*\omega_0 + \omega_{\mathrm{Sap}} \overset{\sim}{\le} \omega_{S,j} < \omega_{S,j} + \omega_j = \omega_{P,j} \overset{\sim}{\le} \omega_{\mathrm{Poinc}}.$$

The fundamental form ω_S of the metric in (iii) *satisfies the quasi-isometry $\omega_S \sim \pi_{j,0}^*\omega_0 + \omega_{\mathrm{Sap}}$.*

Suppose that X is a reduced compact analytic subspace of M_0 and the maps π_j determine a desingularization of X of the type described in §1. Let $\pi : \tilde{M} \to M_0$ be the composite of all the blow-ups, let D be the exceptional divisor of π, and let $\tilde{X}$ be the (smooth) strict transform of X in $\tilde{M}$. Since $\tilde{X}$ and D have normal crossings, the metrics we construct on $\tilde{M} - D$ induce metrics with the corresponding properties on $\tilde{X} - (\tilde{X} \cap D) \cong X - X_{\text{sing}}$.

(2.2) Theorem II. Our second main theorem states that there is a natural relationship between incomplete metrics which determine an embedded resolution of singularities, and complete modified Saper and modified Poincaré metrics.

DEFINITION (2.2.1). Let X be a reduced compact analytic subspace of a compact Kähler manifold M. Let h be an incomplete hermitian metric on $M' = M - X_{\text{sing}}$. We will say that h *determines an embedded resolution of the singularities of* X if the following conditions hold:

i. The completion $\tilde{M}$ of M' under h is a compact Kähler manifold and the completion $\tilde{X}$ of X in $\tilde{M}$ is also smooth.
ii. If $\iota : M' \hookrightarrow \tilde{M}$ is the natural embedding of M' into its completion then $\tilde{M} - \iota(M')$ is the support of an effective divisor D on $\tilde{M}$ with only normal crossings, and the divisor D has only normal crossings with $\tilde{X}$.
iii. The map $\iota^{-1} : \tilde{M} - D \to M'$ extends to a holomorphic map

$$\pi : \tilde{M} \to M$$

which is a biholomorphism from $\tilde{M} - D$ to $M' = M - X_{\text{sing}}$.

THEOREM II. *Let X be a reduced compact analytic subspace of a compact Kähler manifold M and let ω be a Kähler form on M. Suppose that f is a positive C^∞ function on $M - X_{\text{sing}}$ such that $f < 1$ and such that for all sufficiently large integers l, the* (1,1)*-form*

$$\tilde{\omega} = l\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log f$$

is the Kähler form of an incomplete metric on $M - X_{\text{sing}}$ which determines an embedded resolution $\tilde{X} \hookrightarrow \tilde{M}$ of the singularities of X. Let D be the associated divisor of the map $\pi : \tilde{M} \to M$ described above and let s be a section of the line bundle $[D]$ such that the divisor (s) determined by s is D. Suppose also that the function $\pi^* f$ on $\tilde{M} - D$ extends to the norm-squared $\| s \|^2$ of s on

$\tilde{M}$ under some hermitian metric on $[D]$. Then for all sufficiently large integers l, the $(1,1)$-form

$$\omega_S = l\omega - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\log f)^2$$

on $M - X_{\text{sing}}$ is the Kähler form of a complete modified Saper metric on $M - X_{\text{sing}} \cong \tilde{M} - D$. Moreover the associated complexes of L_2 sheaves on M and X are fine.

We construct Kähler forms $\tilde{\omega}$ with the required property in §4. The proof of Theorem II is given in §9.2 (Theorem (9.2.1)) and §10.

REMARK. In practice, the generating function f for the Poincaré-type $(1,1)$-form $-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\log f)^2$ may be very constructive in nature, reflecting the blow-ups used to resolve X. There is a procedure for constructing such a function f explicitly, locally near $p \in X_{\text{sing}}$, as an expression of the form $\sum_{i=1}^{r} \mid w_i \mid^2$, where w_i is a holomorphic function on a neighbourhood U of p which vanishes on $U \cap X_{\text{sing}}$. An algorithm for this construction is not included in this paper but will be given elsewhere.

(2.3) Guidance through the paper. This paper is organized as follows. In §3 we introduce local coordinates for the blow-up $\pi : \tilde{M} \to M$ of a complex manifold M along a smooth centre C. We use these coordinates to describe the exceptional divisor $E = \pi^{-1}(C)$ and the associated line bundle $L = [E]$. If $s : \tilde{M} \to L$ is a nonzero global holomorphic section of L and h is a hermitian metric on L, then the Chern form of the metric h on L is the $(1,1)$-form

$$c_1(L,h) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log \| s \|^2.$$

We may always choose the section s so that its associated divisor (s) is E. We conclude §3 with some linear algebra lemmas on the accumulation of exceptional divisors under repeated blow-ups. These lemmas will be used in §8 in estimating the asymptotic rates of growth of our metrics near the singular locus of X.

In §4 we use the tubular neighbourhood construction of the appendix to show that if ω is a Kähler form on M, then for a suitable metric h on L, the $(1,1)$-form

$$\tilde{\omega} = l\pi^*\omega - c_1(L,h) \tag{2.3.1}$$

is the fundamental form of a Kähler metric on $\tilde{M}$ for all sufficiently large integers l (Theorem (4.2.2)). We construct metrics inductively on repeated blow-ups in this way.

In §5 we study Poincaré-type $(1,1)$-forms on the complement of an effective divisor D with normal crossings in a compact complex manifold M. Given a hermitian metric h on the line bundle $L = [D]$ and a global holomorphic section s of L such that $(s) = D$ and $\| s \| < 1$ on M, we construct a Poincaré-type $(1,1)$-form

$$\nu = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\log \| s \|^2)^2. \tag{2.3.2}$$

The form ν splits into two essential parts $\nu = \mu + \eta$ where

$$\mu = \frac{2}{\log \| s \|^2} c_1(L, h)$$

and where, in local normal coordinates near D, the form η is the pullback of the Poincaré form ω_{Δ^*} under a monomial map τ associated with D, plus low order terms. If p is a point in D at which k components of D intersect, and if $z_1, ..., z_n$ are local normal coordinates in which the components of D through p are given by the equations $z_i = 0$ for $1 \leq i \leq k$, then the monomial map τ is given by

$$\tau(z_1, ..., z_n) = z_1^{\lambda_1} z_2^{\lambda_2} ... z_k^{\lambda_k}$$

where λ_i is the multiplicity of the ith component of D through p.

This splitting leads to the crucial idea in our construction of modified Saper metrics: if we choose appropriate divisors D_j and corresponding Poincaré-type forms ν_j, we can use the forms μ_j to obtain the positivity we need and the forms η_j to get completeness. Using this splitting, it is easy to construct Poincaré and modified Poincaré metrics (Proposition (5.4.1)). We also show that the sum of a modified Saper metric on $M - D$ and a Kähler metric on M is always a modified Poincaré metric on $M - D$ (Proposition (5.5.1)).

In §6 we show that whenever a metric on $M - D$ is bounded below, locally near each point of D, by the pullback of the Poincaré form ω_{Δ^*} under a monomial map τ associated with D, then the metric is complete.

In §7 we introduce our complete metrics for a single blow-up $\pi : \tilde{M} \to M$ of a compact Kähler manifold M along a submanifold

$C \subset M$. We start with a Kähler form ω on M and choose a metric h on the line bundle L associated with π, such that the $(1,1)$-form $\tilde{\omega} = l\pi^*\omega - c_1(L,h)$ of (2.3.1) above is the fundamental form of a Kähler metric on $\tilde{M}$ for sufficiently large l. Using an appropriate section s of L, we construct a Poincaré-type $(1,1)$-form ν, as in (2.3.2) above, on the complement of the exceptional divisor $E = \pi^{-1}(C)$ in $\tilde{M}$. The fundamental form of our modified Saper metric on $\tilde{M} - E$ is

$$\omega_S = l\pi^*\omega + \nu \tag{2.3.3}$$

where l is a sufficiently large positive integer. The $(1,1)$-form

$$\omega_P = \omega_S + \tilde{\omega}$$

is the fundamental form of a Poincaré metric on $\tilde{M} - E$. In Proposition (7.2.1) we use the results of §5 to give the precise asymptotic behaviour of our metrics near the exceptional divisor E. If $(z_1, ..., z_n)$ are normal coordinates near a point in E, such that E is given locally by the equation $z_1 = 0$, then locally

$$\omega_S \sim \pi^*\omega + \frac{\sqrt{-1}}{\pi}\left(\frac{1}{|\log|z_1|^2|}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i + \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2(\log|z_1|^2)^2}\right),$$

i.e. our modified Saper metric for one blow-up is locally quasi-isometric to the sum of the metric induced from M and a Saper distinguished metric. Suppose that $z_2, ..., z_k$ are fibre coordinates for the map $E \to C$. Then locally

$$\omega_S \sim \frac{\sqrt{-1}}{\pi}\left(\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2(\log|z_1|^2)^2} + \frac{1}{|\log|z_1|^2|}\sum_{i=2}^{k} dz_i \wedge d\bar{z}_i + \sum_{i=k+1}^{n} dz_i \wedge d\bar{z}_i\right).$$

These results are generalized in §8 and §9.

In §8 we introduce our complete metrics $\omega_{S,j}$ inductively for a sequence of blow-ups and prove parts (i) and (ii) of Theorem I. Let $\{\pi_j : M_j \to M_{j-1}\}$ be a sequence of blow-ups along smooth centres C_j of the type described in §1.1. We take as $\omega_{S,1}$ the modified Saper form ω_S of §7. If $\omega_{S,j-1}$ is the modified Saper form obtained after $j-1$ blow-ups, we take as $\omega_{S,j}$ the $(1,1)$-form

$$\omega_{S,j} = l\pi_j^*\omega_{S,j-1} + \nu_j$$

where l is a sufficiently large positive integer and ν_j is a Poincaré-type $(1,1)$-form associated with the total exceptional divisor D_j of the composite of the first j blow-ups. In Theorem (8.4.1), we use the results of §3, §5, and §7 to obtain the asymptotic behaviour of our modified Saper metrics near any point p in D_j. We may choose local normal coordinates $(z_1, ..., z_n)$ for M_j in which D_j is given locally by the equation $z_1^{\lambda_1} z_2^{\lambda_2} ... z_k^{\lambda_k} = 0$ for some positive integers $\lambda_1, ..., \lambda_k$. We let τ_j be the corresponding monomial map to the punctured disc Δ^*, given by

$$\tau_j(z_1, ..., z_n) = z_1^{\lambda_1} z_2^{\lambda_2} ... z_k^{\lambda_k}.$$

We may also assume that the map from the exceptional divisor $E_j = \pi_j^{-1}(C_j)$ to the jth centre C_j has fibre coordinates $\{z_i\}_{i\in I}$ for some subset I of $\{1, ..., n\}$. We let $dz_f \wedge d\bar{z}_f = \sum_{i\in I} dz_i \wedge d\bar{z}_i$. In terms of such coordinates, the asymptotic behaviour of $\omega_{S,j}$ is given locally by

$$\omega_{S,j} \sim \pi_j^*\omega_{S,j-1} + \frac{\sqrt{-1}}{\pi}\left(\frac{1}{|\log|z_1 z_2 ... z_k|^2|} dz_f \wedge d\bar{z}_f\right) + \tau_j^*\omega_{\Delta^*}.$$

The last two terms correspond to the terms μ_j and η_j in the decomposition of the Poincaré-type form ν_j. This asymptotic behaviour is one of the key points of our results. The monomial map τ_j is closely related to the linear algebra of the accumulation of exceptional divisors given in §3. The powers λ_i occurring in the map τ_j are the multiplicities of the components of the total exceptional divisor D_j passing through p.

We construct homogeneous Saper metrics in §9 and prove Theorem I (iii) and Theorem II, except for the statement about fine sheaves. In §10 we complete the proofs of Theorems I and II by proving the results on fine sheaves.

§3. Background on geometry of blow-ups.

(3.1) Local coordinates for blow-ups. Let M be a compact complex manifold of dimension n, let C be a submanifold of codimension $k > 1$, and let

$$\pi : \tilde{M} \to M$$

be the blow-up of M along the centre C. The blow-up can be described in local coordinates as follows. Let p be a point in C and let V be a coordinate neighbourhood of p in M, centered at p, with holomorphic coordinates $(Z_1, Z_2, ..., Z_n)$ such that

i. $C \cap V$ is given by $Z_1 = Z_2 = ... = Z_k = 0$ and

ii. $Z_{k+1}, ..., Z_n$ are local coordinates on C.

Let $\xi_1, \xi_2, ..., \xi_k$ be homogeneous coordinates on $\mathbf{P}^{k-1}$. The set $\pi^{-1}(V)$ in $\tilde{M}$ is biholomorphic to the subset $\tilde{V}$ of $V \times \mathbf{P}^{k-1}$ given by

$$\tilde{V} = \{(Z, [\xi]) : Z_i\xi_j = Z_j\xi_i, \quad 1 \le i, j \le k\}.$$

We cover $\tilde{V}$ by open sets $U_j = \{\xi_j \neq 0\}$, for $j = 1, 2, ..., k$, with coordinates $(z_{j1}, ..., z_{jn})$, where

$$\begin{aligned} z_{ji} &= \frac{\xi_i}{\xi_j} = \frac{Z_i}{Z_j}, && \text{if } \ 1 \le i \le k \quad \text{and} \quad i \neq j \\ z_{jl} &= Z_l && \text{if } l = j \text{ or } k+1 \le l \le n. \end{aligned}$$

The restriction of π to U_j is given by the equations

$$\begin{aligned} Z_i &= z_{jj}z_{ji} && \text{if } \ 1 \le i \le k \quad \text{and} \quad i \neq j \\ Z_l &= z_{jl} && \text{if } l = j \text{ or } k+1 \le l \le n. \end{aligned}$$

We will sometimes find it convenient to refer separately to C coordinates and the remaining coordinates in the directions normal to C, setting

$$Z_N = (Z_1, ..., Z_k) \qquad \text{and} \qquad Z_C = (Z_{k+1}, ..., Z_n).$$

Similarly,

$$z_{jN} = (z_{j1}, ..., z_{jk}) \qquad \text{and} \qquad z_C = (z_{j,k+1}, ..., z_{jn}).$$

If q is any point in $\pi^{-1}(p)$ and if $1 \le j \le k$, then by a suitable choice of homogeneous coordinates on $\pi^{-1}(p) \cong \mathbf{P}^{k-1}$, corresponding to a linear change in the coordinates $Z_1, ..., Z_k$, we may assume that q is the origin, given by $z_{j1} = ... = z_{jn} = 0$, in the open set U_j.

(3.2) The exceptional divisor E. The exceptional divisor of the blow-up is $E = \pi^{-1}(C)$. In the open set U_j in $\tilde{M}$, E is given by $z_{jj} = 0$. The map π is a biholomorphism of $\tilde{M} - E$ onto $M - C$ and maps E onto C. Let $\tau : E \to C$ be the restriction of π to E. In local coordinates on U_j and C we have

$$\tau(z_{j1}, ..., z_{jn}) = (z_{j,k+1}, ..., z_{jn})$$

or

$$\tau(z_{jN}, z_C) = z_C.$$

The fibres E_p of the map $\tau : E \to C$ are isomorphic to $\mathbf{P}^{k-1}$. Let $N_{C/M}$ be the normal bundle of C in M and let N be its restriction to $V \cap C$:

$$N = N_{C/M} \mid_{V\cap C} \cong (V \cap C) \times \mathbf{C}^k.$$

There is a natural isomorphism $\psi : E \tilde{\to} \mathbf{P}(N_{C/M})$, given locally as a map $\tilde{V} \cap E \to \mathbf{P}(N) \cong (V \cap C) \times \mathbf{P}^{k-1}$ by

$$\psi(Z_N, Z_C, [\xi]) = (Z_C, [\xi]).$$

In local coordinates on U_j and $\mathbf{P}(N)$, the map ψ is given by

$$\psi(z_{jN}, z_C) = (z_C, [\zeta(j)])$$

where $\zeta(j)$ is defined by

$$\zeta(j)_i = z_{ji} \qquad 1 \le i \le k, \quad i \ne j, \quad \text{and} \quad \zeta(j)_j = 1$$

so that $[\zeta(j)] = [\xi]$. The functions $\zeta(j)_i$, for $i \ne j$, are nonhomogeneous coordinates for an open set in $\mathbf{P}^{k-1}$ and

$$z_{jj}\zeta(j) = (Z_1, ..., Z_k) = Z_N \tag{3.2.1}$$

for all j.

(3.3) The associated line bundle L. Let $L = [E]$ be the line bundle on $\tilde{M}$ associated with the exceptional divisor E. The restriction of L to E is isomorphic to the normal bundle $N_{E/\tilde{M}}$ of E in $\tilde{M}$. The restriction of L to each fibre $E_p \cong \mathbf{P}^{k-1}$ of the map $\tau : E \to C$ is isomorphic to the universal bundle $\mathcal{O}_{\mathbf{P}^{k-1}}(-1)$ on $\mathbf{P}^{k-1}$. If v is a nonzero point in the fibre $N_{C/M,p}$ of $N_{C/M}$ over p, $[v]$

is the corresponding point in $\mathbf{P}(N_{C/M})_p$, and $\psi : E \xrightarrow{\sim} \mathbf{P}(N_{C/M})$ is the natural isomorphism, then $\psi^{-1}(p, [v]) \in E_p$ and the fibre of L over $\psi^{-1}(p, [v])$ corresponds, via the isomorphism $L|_{E_p} \cong \mathcal{O}_{E_p}(-1)$, to the line in $N_{C/M,p}$ spanned by v. We have the following commutative diagram relating $L|_E$ and $N_{C/M}$:

$$\begin{array}{ccccc} L|_E \cong N_{E/\tilde{M}} & \xrightarrow{\iota} & \tau^* N_{C/M} & \xrightarrow{\tau_*} & N_{C/M} \\ \downarrow & & \downarrow & & \downarrow \\ E & = & E & \xrightarrow{\tau} & C \end{array} \tag{3.3.1}$$

where ι denotes inclusion.

(3.4) Transition functions and sections of L. We may cover C by coordinate neighbourhoods V_α of the form described above and then enlarge this collection of coordinate neighbourhoods to a covering $\mathcal{V}$ of all of M. In $\tilde{M}$, the collection of coordinate neighbourhoods $U_{\alpha,j}$ covers E and we enlarge it to a covering $\mathcal{U}$ of $\tilde{M}$. For each $U \in \mathcal{U}$, let f_U be a holomorphic defining function for E on U. For U and $U' \in \mathcal{U}$, let $g_{UU'}$ be the nonvanishing holomorphic function on $U \cap U'$ given by

$$g_{UU'} = \frac{f_U}{f_{U'}}.$$

The functions $\{g_{UU'}\}$ are transition functions for a line bundle L on $\tilde{M}$ associated with E. If $\{F_U\}$ is another collection of defining functions for E, the functions $\{G_{UU'} = F_U/F_{U'}\}$ may be different from the functions $\{g_{UU'}\}$, but the line bundle determined by the functions $\{G_{UU'}\}$ is isomorphic to L.

In particular, z_{ii} is a defining function for E on U_i, so we may use the functions

$$g_{ij} = \frac{z_{ii}}{z_{jj}} = \frac{Z_i}{Z_j}$$

as transition functions for $L = [E]$ on the set $U_i \cap U_j$.

A meromorphic section $s : \tilde{M} \to L$ of L may be given by a collection of meromorphic functions s_U on U satisfying $s_U = g_{UU'} s_{U'}$ on $U \cap U'$. Each meromorphic section s of L defines a divisor (s) which is linearly equivalent to E. The functions f_U themselves give a global holomorphic section s of L for which $(s) = E$. If s' is another section of L such that $(s') = E$ then s and s' differ by a meromorphic function with no zeros or poles, i.e. by a constant. In particular, if $(s) = E$ then we may assume that $s_i = az_{ii}$ on $U_\imath$ for all i and for some constant $a \in \mathbf{C}$.

(3.5) The Chern form of a metric on L. Let h be a hermitian metric on L, let $s : \tilde{M} \to L$ be a nonzero global holomorphic section of L, and let $\| s \|$ be the norm of s under h at any point in $\tilde{M}$. The Chern form of the metric h is the $(1,1)$-form

$$c_1(L,h) = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log \| s \|^2.$$

The following local calculation shows that $c_1(L,h)$ is well-defined and is independent of the section s used to calculate it. On any set $U \in \mathcal{U}$ we have $\| s \|^2 = h_U | s_U |^2$, for some positive C^∞ function h_U. Then

$$\begin{aligned}\partial\overline{\partial}\log \| s \|^2 &= \partial\overline{\partial}\log(h_U | s_U |^2)\\ &= \partial\overline{\partial}\log h_U + \partial\overline{\partial}\log s_U + \partial\overline{\partial}\log \overline{s_U}\\ &= \partial\overline{\partial}\log h_U\end{aligned}$$

since $\overline{\partial}\log s_U = \partial\log\overline{s_U} = 0$.

Note also that $c_1(L,h)$ is hermitian and d-closed, by Lemma (1.2.1). The Chern class of a line bundle is always integral (see [**GH**] or [**W**]).

(3.6) Pullbacks of divisors with normal crossings. Let D be a smooth connected analytic hypersurface in M, i.e. D is a smooth, reduced, irreducible divisor on M. Assume that D has normal crossings with C. Then for every $p \in C \cap D$ we may choose local coordinates as in §3.1 and such that D is given locally by the equation $Z_\alpha = 0$ for some α. In the open set U_j in $\tilde{M}$, the pullback divisor $\pi^* D$ is given by the equation

$$\begin{aligned} z_{jj}z_{j\alpha} &= 0 \qquad \text{if } \; 1 \le \alpha \le k \quad \text{and} \quad \alpha \ne j\\ z_{j\alpha} &= 0 \qquad \text{if } \alpha = j \text{ or } k+1 \le \alpha \le n.\end{aligned}$$

The strict transform $\tilde{D}$ of D is the closure of $\pi^{-1}(D - C)$ in $\tilde{M}$ and is given locally in the open set U_j by the equation $z_{j\alpha} = 0$ if

$j \neq \alpha$. If $j = \alpha$ then $\tilde{D}$ does not intersect U_j. The divisor $\tilde{D}$ is smooth, reduced, and irreducible and has normal crossings with the exceptional divisor E. We may express π^*D globally in terms of $\tilde{D}$ and E as

$$\pi^*D = \tilde{D} + \delta E \tag{3.6.1}$$

where $\delta = 1$ if $C \subset D$ and $\delta = 0$ otherwise. If $L_D = [D]$ is the line bundle on M associated with D, then $\pi^*L_D = [\tilde{D}] \otimes L^\delta$, where $L = [E]$.

Now let D be a divisor on M of the form $D = \sum_{i=1}^r a_i E_i$, where the irreducible components E_i of D are smooth and each a_i is a positive integer, and such that D and C simultaneously have only normal crossings. Then

$$\pi^*D = \sum_{i=1}^r a_i \pi^* E_i = \sum_{i=1}^r a_i \tilde{E}_i + \left(\sum_{i=1}^r a_i \delta_i \right) E \tag{3.6.2}$$

where $\delta_i = 1$ if $C \subset E_i$ and $\delta_i = 0$ otherwise.

(3.7) Pullbacks of divisors under a sequence of blow-ups. Consider a finite sequence of blow-ups $\{\pi_j : M_j \to M_{j-1}\}$ along smooth centres $C_j \subset M_{j-1}$. For $0 \leq k < j$, let $\pi_{j,k} = \pi_{k+1} \circ \pi_{k+2} \circ \ldots \circ \pi_j : M_j \to M_k$. Let E_j be the exceptional divisor of π_j, i.e. $E_j = \pi_j^{-1}(C_j)$. Let D_j be the exceptional divisor of the composite $\pi_{j,0}$ of the first j blow-ups, given inductively by the equations $D_1 = E_1$ and $D_j = \pi_j^* D_{j-1} + E_j$ for $j \geq 2$. Assume that the centres C_j have been chosen so that C_j and D_{j-1} simultaneously have only normal crossings. For $1 \leq j < m$ let $E_{m,j}$ be the pullback of E_j to M_m, i.e. $E_{m,j}$ is the total transform $E_{m,j} = \pi_{m,j}^* E_j$. Let $E_{m,m} = E_m$. We may write D_m as the sum

$$D_m = \sum_{j=1}^m E_{m,j}.$$

For $1 \leq k < m$, let $\tilde{E}_{m,k}$ be the smooth, reduced, irreducible divisor in M_m obtained by taking repeated strict transforms of E_k under the blow-ups $\pi_{k+1}, \ldots, \pi_m$. Let $\tilde{E}_{m,m} = E_m$. The divisors $\{\tilde{E}_{m,k}\}_{k=1}^m$ have only normal crossings. We may express the total transforms $\{E_{m,j}\}$ in terms of the strict transforms $\{\tilde{E}_{m,k}\}$ as

$$E_{m,j} = \sum_{k=1}^{m} a_{jk}\tilde{E}_{m,k}$$

for some nonnegative integers a_{jk}. Let A be the $m \times m$ matrix (a_{jk}). We will sometimes write this transition matrix as $A = T_m(E, \tilde{E})$. Repeated use of formula (3.6.2) shows that A is an upper triangular matrix of the form

$$A = T_m(E,\tilde{E}) = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1,m-1} & a_{1m} \\ 0 & 1 & a_{23} & \dots & a_{2,m-1} & a_{2m} \\ 0 & 0 & 1 & \dots & a_{3,m-1} & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_{m-1,m} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Let $\hat{A} = T_{m-1}(E, \tilde{E})$ be the matrix which describes the collection of total transforms $\{E_{m-1,j}\}_{j=1}^{m-1}$ in terms of the strict transforms $\{\tilde{E}_{m-1,j}\}_{j=1}^{m-1}$ on M_{m-1}. For $1 \le k \le j-1$ define

$$\delta_{kj} = \begin{cases} 1 & \text{if } C_j \subset \tilde{E}_{j-1,k} \\ 0 & \text{if } C_j \not\subset \tilde{E}_{j-1,k}. \end{cases}$$

Using (3.6.2) we obtain the following relationships between the entries of A and $\hat{A}$:

LEMMA 3.7.1.

i. $\quad a_{jk} = \hat{a}_{jk} \qquad$ *if* $1 \le j, k \le m-1$.

ii. $$a_{jm} = \sum_{k=1}^{m-1} \hat{a}_{jk}\delta_{km} \qquad \textit{if } 1 \le j \le m-1$$

$$= \begin{cases} \delta_{jm} + \sum_{k=j+1}^{m-1} a_{jk}\delta_{km} & \textit{if } 1 \le j \le m-2 \\ \delta_{m-1,m} & \textit{if } j = m-1. \end{cases}$$

Property (i) means that we may omit the superscripts $\hat{}$ on the a's without any ambiguity: a_{jk} is the multiplicity of $\tilde{E}_{m,k}$ in $E_{m,j}$ and is the same for all $m \ge k$. Property (ii) implies that if $C_m \subset \tilde{E}_{m-1,j}$ then $a_{jm} > 0$.

Similarly, we define the total transforms $D_{m,i}$ of the divisors D_i by the equations $D_{m,i} = \pi^*_{m,i} D_i$ for $1 \leq i \leq m-1$ and $D_{m,m} = D_m = D_{m,m-1} + E_m$. Then

$$D_{m,i} = \sum_{j=1}^{i} E_{m,j},$$

and the matrix $T_m(D,E)$ for the total transforms $\{D_{m,i}\}$ in terms of the total transforms $\{E_{m,j}\}$ is the $m \times m$ lower triangular matrix

$$T_m(D,E) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

The irreducible components of $D_m = D_{m,m}$ are the strict transforms $\{\tilde{E}_{m,j}\}_{j=1}^m$. The irreducible components of $D_{m,i}$ for $1 \leq i \leq m-1$ are also contained in the set $\{\tilde{E}_{m,j}\}$. We may express the divisors $\{D_{m,i}\}$ in terms of the strict transforms $\{\tilde{E}_{m,j}\}$ as

$$D_{m,i} = \sum_{j=1}^{m} t_{ij} \tilde{E}_{m,j}$$

for some nonnegative integers t_{ij}. The matrix $T = (t_{ij}) = T_m(D, \tilde{E})$ is given by

(3.7.2)

$$T = T_m(D,E) T_m(E,\tilde{E})$$

$$= \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1,m-1} & a_{1m} \\ 1 & a_{12}+1 & a_{13}+a_{23} & \dots & a_{1,m-1}+a_{2,m-1} & a_{1m}+a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_{12}+1 & a_{13}+a_{23}+1 & \dots & \sum_{k=1}^{m-2} a_{k,m-1}+1 & \sum_{k=1}^{m-1} a_{k,m} \\ 1 & a_{12}+1 & a_{13}+a_{23}+1 & \dots & \sum_{k=1}^{m-2} a_{k,m-1}+1 & \sum_{k=1}^{m-1} a_{k,m}+1 \end{pmatrix}.$$

Let $\hat{T} = T_{m-1}(D, \tilde{E})$ be the matrix for the divisors $\{D_{m-1,i}\}$ in terms of the strict transforms $\{\tilde{E}_{m-1,j}\}$.

LEMMA 3.7.3. *The matrices* $T = (t_{ij}) = T_m(D, \tilde{E})$ *and* $\hat{T} = (\hat{t}_{ij}) = T_{m-1}(D, \tilde{E})$ *are related by the following equations:*

i. $t_{ij} = \hat{t}_{ij}$ *for* $1 \leq i, j \leq m-1$,

ii. $t_{im} = \sum_{k=1}^{m-1} t_{ik}\delta_{km}$ *for* $1 \leq i \leq m-1$,

iii. $t_{mj} = t_{m-1,j} = \hat{t}_{m-1,j}$ *for* $1 \leq j \leq m-1$, *and*

iv. $t_{mm} = t_{m-1,m} + 1 = \sum_{k=1}^{m-1} t_{m-1,k}\delta_{km} + 1 = \sum_{k=1}^{m-1} t_{mk}\delta_{km} + 1$.

Proof. Apply equation (3.6.2) to the pullbacks $\pi_m^* D_{m-1,i} = D_{m,i}$ for $1 \leq i \leq m-1$ to obtain equations (i) and (ii). The exceptional divisor $D_{m,m}$ is just

$$D_m = \pi_m^* D_{m-1,m-1} + E_m = D_{m,m-1} + \tilde{E}_{m,m}.$$

Applying (3.6.2) again gives equations (iii) and (iv). □

Using equation (3.7.2) and Lemma (3.7.3) we obtain the following description of $T = T_m(D, \tilde{E})$ for $m \geq 2$.

PROPOSITION 3.7.4. *The matrix* $T = (t_{ij})$ *given by the equations*

$$D_{m,i} = \sum_{j=1}^{m} t_{ij}\tilde{E}_{m,j}$$

has the following properties:

i. t_{ij} *is a nonnegative integer and* $t_{ij} > 0$ *if* $i \geq j$,

ii. $t_{ij} \geq t_{kj}$ *if* $i \geq k$,

iii. $t_{mj} = t_{m-1,j} = \ldots = t_{j+1,j} = t_{jj}$,

iv. $\det T = 1$,

v. $t_{j-1,j} > 0$ *if and only if* $C_j \subset D_{j-1}$, *and*

vi. *the entries of* T *are all positive if and only if* $C_j \subset D_{j-1}$ *for* $2 \leq j \leq m$.

Proof. Properties (i) - (iii) are immediate consequences of equation (3.7.2). To prove (iv), note that $\det T_m(D, E) = \det T_m(E, \tilde{E}) = 1$. For (v), use Lemma (3.7.3ii) to write $t_{j-1,j}$ as

$$t_{j-1,j} = \sum_{k=1}^{j-1} t_{j-1,k}\delta_{kj}$$

where the coefficients $t_{j-1,k}$ are positive for $1 \leq k \leq j-1$ by property (i). Recall that $\delta_{kj} > 0$ if and only if $C_j \subset \tilde{E}_{j-1,k}$. Then $t_{j-1,j} > 0$ if and only if $C_j \subset \tilde{E}_{j-1,k}$ for some k, $1 \leq k \leq j-1$. The divisors $\{\tilde{E}_{j-1,k}\}_{k=1}^{j-1}$ are the irreducible components of D_{j-1}, so $t_{j-1,j} > 0$ if and only if $C_j \subset D_{j-1}$.

It follows from (v) that if $C_j \not\subset D_{j-1}$ for some j then not all entries of T are positive. If $m = 2$ we have

$$T_2 = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} 1 & a_{12} \\ 1 & a_{12}+1 \end{pmatrix}$$

where t_{12} is positive if and only if $C_2 \subset D_1$, by (v). Suppose that for some $j \geq 3$ the matrix T_{j-1} has positive entries and $C_i \subset D_{i-1}$ for $2 \leq i \leq j$. The matrix T_j is of the form

$$T_j = \left(\begin{array}{ccc|c} & & & t_{1j} \\ & T_{j-1} & & \vdots \\ & & & t_{j-1,j} \\ \hline t_{j1} & \cdots & t_{j,j-1} & t_{jj} \end{array}\right)$$

where $(t_{j1}, t_{j2}, ..., t_{j,j-1}) = (t_{j-1,1}, t_{j-1,2}, ..., t_{j-1,j-1})$, by property (iii), and $t_{ij} \geq t_{kj}$ for $i \geq k$ by property (ii). From these properties and our inductive assumption, it is sufficient to show that $t_{1j} > 0$. By Lemma (3.7.3ii), $t_{1j} = \sum_{k=1}^{j-1} t_{1k}\delta_{kj}$. The coefficients t_{1k} are positive for $1 \leq k \leq j-1$ by our inductive assumption. If $C_j \subset D_{j-1}$ then $C_j \subset \tilde{E}_{j-1,k}$ and $\delta_{kj} = 1$ for some k, $1 \leq k \leq j-1$. Consequently $t_{1j} > 0$. □

When working in local coordinates near a point q in M_m we will consider only the submatrix Λ of T_m corresponding to those divisors $\tilde{E}_{m,\alpha}$ which pass through q.

(3.8) Isolated singularities. If X has only isolated singular points, then we may resolve each singular point in turn and write a separate matrix for the exceptional divisor of each, since each blow-up map π_i will be a biholomorphism on all connected components of D_i except the one containing E_i. The exceptional divisors for distinct points will be mutually disjoint. For our purposes it is enough to describe the matrix corresponding to a single singular point of X.

PROPOSITION 3.8.1. *Let X be a singular compact analytic subspace of M_0 with isolated singularities. Let $\{\pi_m : M_m \to M_{m-1}\}$ be a sequence of blow-ups of the type described in §1.1 which resolves one of the singular points of X. Then, for each m, all entries of the matrix $T = T_m(D, \tilde{E})$ are positive, i.e. the multiplicity of the strict transform $\tilde{E}_{m,i}$ of E_i in the total transform $D_{m,j}$ of D_j is positive for all i, j.*

Proof. Suppose that the blow-ups have the properties described in §1.1. Then the image of D_j in M_0 always lies in X_{sing}, so $C_1 \subset X_{\text{sing}}$ and $C_j \subset D_{j-1}$ for $j \geq 2$. Apply Proposition (3.7.4). □

(3.9) Line bundles for repeated pullbacks. We list here, for further reference, the relationships among certain divisors, line bundles, sections, and metrics.

Let h_j be a hermitian metric for the line bundle $L_j = [E_j]$ on M_j and let $s_j : M_j \to L_j$ be a section of L_j such that $(s_j) = E_j$. Such a section always exists (§3.4). For $m > j$ let $\pi_{m,j}$ be the composite map

$$\pi_{m,j} = \pi_{j+1} \circ \pi_{j+2} \circ \ldots \circ \pi_m : M_m \to M_j.$$

Recall that the total exceptional divisor of $\pi_{j,0}$ is $D_j = \sum_{i=1}^{j} E_{j,i}$. Let $\hat{L}_j = [D_j]$. The hermitian metrics h_j and the sections s_j induce hermitian metrics $\hat{h}_j$ and sections $\hat{s}_j$ of the line bundles $\hat{L}_j$. We use the notation

$$L_{m,j} = \pi^*_{m,j} L_j, \quad E_{m,j} = \pi^*_{m,j} E_j, \quad h_{m,j} = \pi^*_{m,j} h_j, \qquad s_{m,j} = \pi^*_{m,j} s_j$$
$$\hat{L}_{m,j} = \pi^*_{m,j} \hat{L}_j, \quad D_{m,j} = \pi^*_{m,j} D_j, \quad \hat{h}_{m,j} = \pi^*_{m,j} \hat{h}_j, \text{ and } \quad \hat{s}_{m,j} = \pi^*_{m,j} \hat{s}_j.$$

Then

$$\hat{L}_j = \bigotimes_{i=1}^{j} L_{j,i}, \qquad \hat{h}_j = \prod_{i=1}^{j} h_{j,i}, \qquad \hat{s}_j = \prod_{i=1}^{j} s_{j,i}, \qquad (\hat{s}_j) = D_j,$$
$$L_{m,j} = [E_{m,j}], \quad (s_{m,j}) = E_{m,j}, \quad \hat{L}_{m,j} = [D_{m,j}], \text{ and } \quad (\hat{s}_{m,j}) = D_{m,j}.$$

For consistency we sometimes write $L_{m,m} = L_m$, $E_{m,m} = E_m$, $D_{m,m} = D_m$, and so on.

§4. An Incomplete Metric on $M - X_{\text{sing}}$ which determines an Embedded Resolution of X by Blow-ups. Let M be a compact complex manifold of dimension n, let C be a submanifold of codimension $k > 1$, and let $\pi : \tilde{M} \to M$ be the blow-up of M along C. Let $E = \pi^{-1}(C)$ be the exceptional divisor of the blow-up and let $L = [E]$ be its associated line bundle on $\tilde{M}$. We begin with a hermitian metric h_M on M, or equivalently a C^∞ positive hermitian $(1,1)$-form ω. We will show that if $c_1(L, h)$ is the Chern form of L with respect to a suitable hermitian metric h on L and l is a sufficiently large integer, then the $(1,1)$-form

$$\tilde{\omega} = l\pi^*\omega - c_1(L, h)$$

is positive and determines a hermitian metric $h_{\tilde{M}}$ on $\tilde{M}$. If h_M is Kähler then so is $h_{\tilde{M}}$. If M is algebraic and h_M is Hodge, then $h_{\tilde{M}}$ is also Hodge.

Applying this construction inductively to a sequence of blow-ups $\pi_j : M_j \to M_{j-1}$, we obtain C^∞ positive hermitian $(1,1)$-forms

$$\omega_j = l_j \pi_j^* \omega_{j-1} - c_1(L_j, h_j)$$

which determine hermitian metrics h_{M_j} on M_j. Suppose that X is an analytic subspace of M_0 and the blow-ups π_j are of the type described in §1.1. Then the restrictions of the metrics h_{M_j} to the strict transforms X_j of X induce hermitian metrics on $X - X_{\text{sing}}$ which are Kähler if h_{M_0} is Kähler but are incomplete unless X is nonsingular. Moreover the completion of $M_j - D_j$ in the metric h_{M_j} is M_j and the completion of $X - X_{\text{sing}}$ is X_j.

(4.1) A metric for L. Let $\tau : E \to C$ be the restriction of the map $\pi : \tilde{M} \to M$ to E. For any subset $W \subset \tilde{M}$, let L_W be the restriction of the line bundle L to W.

PROPOSITION 4.1.1. *There exists a hermitian metric h on L whose Chern form $c_1(L, h)$ is negative along the fibres of the map $\tau : E \to C$, i.e. the restriction of $c_1(L, h)$ to the tangent bundle $T(E_p)$ of each fibre E_p of τ is a negative definite $(1,1)$-form.*

Proof. Let $N = N_{C/M}$ be the normal bundle of C in M. Any hermitian metric on N induces a hermitian metric on L_E by first pulling back to τ^*N and then restricting (see diagram (3.3.1)). A metric on L_E may be extended in a C^∞ way to a metric on L over all of $\tilde{M}$, for example by using the tubular neighbourhood construction of Proposition (A.3). We will show that any hermitian metric constructed in this way has the required property.

Let p be any point in C. Choose local coordinates $(Z_1, ..., Z_n)$ in a coordinate neighbourhood V in M centered at p and local coordinates $(z_{j1}, ..., z_{jn})$ on sets U_j in $\tilde{M}$, as described in §3.1. We will denote points in N over $V \cap C$ by (Z_C, ξ) where $Z_C = (Z_{k+1}, ..., Z_n) \in C$ and

$$\xi = \sum_{i=1}^{k} \xi_i \frac{\partial}{\partial Z_i}.$$

Choose any hermitian metric $|| \quad ||$ on N. In local coordinates,

$$|| (Z_C, \xi) ||^2 = \sum_{\mu,\nu=1}^{k} h_{\mu\nu}(Z_C)\xi_\mu \overline{\xi_\nu}$$

for some C^∞ functions $h_{\mu\nu}$ such that the matrix $(h_{\mu\nu})$ is positive definite hermitian. We may make a linear change of variables in $Z_1, ..., Z_k$ (and hence in $\xi_1, ..., \xi_k$) so that at $p = (0, 0, ..., 0)$, the matrix $(h_{\mu\nu}(0))$ is the $k \times k$ identity and

$$|| (0, \xi) ||^2 = \sum_{\mu=1}^{k} | \xi_\mu |^2.$$

Next we describe the induced metric on L_E. We denote points in L over U_j by $((z_{jN}, z_C), t_j)$ and note that on $U_i \cap U_j$ the fibre coordinate transforms by the rule

$$t_i = g_{ij} t_j$$

where $g_{ij} = z_{ii}/z_{jj}$ is the transition function for L on $U_i \cap U_j$. The natural map $L_E \to N$ is given locally by

$$((z_{jN}, z_C), t_j) \to (z_C, t_j \zeta(j))$$

where $\zeta(j)$ is as defined in §3.2, i.e. $\zeta(j)_i = z_{ji}$ for $1 \le i \le k$, $i \ne j$, and $\zeta(j)_j = 1$. The hermitian metric on L_E induced from N is given locally by

$$|| ((z_{jN}, z_C), t_j) ||^2 = | t_j |^2 \sum_{\mu,\nu=1}^{k} h_{\mu\nu}(z_C)\zeta(j)_\mu \overline{\zeta(j)_\nu}.$$

On the fibre E_p of the map $\tau : E \to C$, we have $z_C = 0$ and

$$\| ((z_{jN}, 0), t_j) \|^2 = | t_j |^2 \sum_{\mu=1}^{k} | \zeta(j)_\mu |^2.$$

Let h be any C^∞ extension of this metric to a metric on L over all of $\tilde{M}$, for example by using the tubular neighbourhood construction of Proposition (A.3). To finish the proof of the proposition, we need only look at the description of h on E. The restriction of $c_1(L, h)$ to the tangent bundle TE of E may be calculated using the formulas of §3.5:

$$c_1(L, h) |_{T(E \cap U_j)} = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left(\sum_{\mu,\nu=1}^{k} h_{\mu\nu}(z_C) \zeta(j)_\mu \overline{\zeta(j)_\nu} \right).$$

The restriction of $c_1(L, h)$ to the tangent bundle of the fibre E_p is

$$c_1(L, h) |_{T(E_p)} = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \sum_{\mu=1}^{k} | \zeta(j)_\mu |^2$$

which is a negative $(1, 1)$-form, the negative of the $(1, 1)$-form associated to the Fubini-Study metric on $\mathbf{P}^{k-1}$. Since p was any point of C, the Chern form $c_1(L, h)$ is negative along every fibre of the map $\tau : E \to C$. □

(4.2) A finite metric for $\tilde{M}$.

PROPOSITION 4.2.1. *If ω is a positive $(1, 1)$-form on M, and h is a hermitian metric on L whose Chern form is negative on the fibres of the map $\tau : E \to C$, then the $(1, 1)$-form*

$$\tilde{\omega} = l\pi^* \omega - c_1(L, h)$$

is positive on $\tilde{M}$ for all sufficiently large integers l.

Proof. Let Y be the subspace of $T\tilde{M}|_E$ consisting of all vectors which are tangent to fibres of the map $\tau : E \to C$, i.e.

$$Y = \bigcup_{c \in C} T(\pi^{-1}(c)).$$

If ω is a positive $(1,1)$-form on M, then $\pi^*\omega$ is positive semi-definite on $\tilde{M}$. If $v \in T\tilde{M}$ is a nonzero tangent vector then $\pi^*\omega(v,v) = 0$ if and only if $v \in Y$. The Chern form $c_1(L,h)$ is negative on Y by construction. These properties can be expressed in terms of functions on the projectivized tangent bundle $P = \mathbf{P}T\tilde{M}$.

Let h′ be any hermitian metric on $T\tilde{M}$ and for v in $T\tilde{M}$ let $\| v \|$ be the norm of v under h′. For $v \neq 0$, let $[v]$ be the image of v in P. The $(1,1)$-forms $\pi^*\omega$ and $-c_1(L,h)$ determine well-defined C^∞ functions on P given by

$$f([v]) = \frac{\pi^*\omega(v,v)}{\| v \|^2} \qquad \text{and} \qquad g([v]) = \frac{-c_1(L,h)(v,v)}{\| v \|^2}$$

with the properties $f > 0$ on $P - \mathbf{P}Y$, $f = 0$ on $\mathbf{P}Y$, and $g > 0$ on $\mathbf{P}Y$. Since $\mathbf{P}Y$ is closed, $g > 0$ on some neighbourhood U of $\mathbf{P}Y$. Then $lf + g > 0$ on U for all $l > 0$. Since P is compact, we may choose $l > 0$ such that $lf + g > 0$ on all of P, and consequently $l\pi^*\omega - c_1(L,h) > 0$ on $\tilde{M}$ for all sufficiently large l. □

Proposition (4.1.1) tells us that there is always a metric h on L with the properties required in Proposition (4.2.1) and Theorem (4.2.2).

THEOREM 4.2.2. *Let ω be the fundamental form of a hermitian metric on M and let h be a hermitian metric on L whose Chern form is negative on the fibres of the map $\tau : E \to C$. Then*

$$\tilde{\omega} = l\pi^*\omega - c_1(L,h)$$

is the fundamental form of a hermitian metric on $\tilde{M}$ for all sufficiently large integers l. If ω is Kähler then so is $\tilde{\omega}$ and if ω is Hodge then so is $\tilde{\omega}$.

Proof. A C^∞ $(1,1)$-form is the fundamental form of a hermitian metric if it is positive and hermitian. The positivity of $\tilde{\omega}$ was proved above (Proposition (4.2.1)). The Chern form of a line bundle is always hermitian (§3.5), so $\tilde{\omega}$ is also hermitian.

Recall that a positive hermitian C^∞ $(1,1)$-form ω determines a Kähler metric if $d\omega = 0$. If ω is also integral, then the metric is Hodge. The identity $d(\pi^*\omega) = \pi^*d\omega$ shows that $\pi^*\omega$ is d-closed if ω is d-closed and $\pi^*\omega$ is integral if ω is integral. Finally, the Chern form of a line bundle is always d-closed and integral (§3.5). □

(4.3) Finite metrics for successive blow-ups. We apply Theorem (4.2.2) to a sequence of blow-ups to obtain the following.

THEOREM 4.3.1. *Let* $\{\pi_j : M_j \to M_{j-1}\}$ *be a finite sequence of blow-ups of a compact complex manifold* M_0 *along smooth centres* $C_j \subset M_{j-1}$. *Let* $E_j = \pi_j^{-1}(C_j)$ *be the exceptional divisor of* π_j *and let* $L_j = [E_j]$ *be the associated line bundle. Let* ω_0 *be the fundamental form of a hermitian metric on* M_0. *There exist hermitian metrics* h_j *on* L_j *and positive integers* l_j *such that the* $(1,1)$*-forms defined inductively by the equation*

$$\omega_j = l_j \pi_j^* \omega_{j-1} - c_1(L_j, h_j)$$

are all positive and determine hermitian metrics on the manifolds M_j. *Moreover if* ω_0 *is Kähler then so are all the forms* ω_j. *If* M_0 *is algebraic and* ω_0 *is Hodge then the forms* ω_j *are also Hodge.*

REMARK. Since a sum of Chern forms equals the Chern form of a product of line bundles, we may write ω_j as

$$\omega_j = l \pi_{j,0}^* \omega_0 - c_1(\mathcal{L}, H)$$

where $\pi_{j,0} : M_j \to M_0$ is the composite of the first j blow-ups, $\mathcal{L}$ is some line bundle on M_j, and H is an appropriate metric on $\mathcal{L}$. Furthermore, $\mathcal{L}$ is of the form $\mathcal{L} = [\mathcal{D}]$ for some effective divisor $\mathcal{D}$ with the same support as D_j.

We may also obtain Kähler metrics inductively using the divisors D_j as follows. Corollary (4.3.2) will be used in §8.6 in the proof of our first main theorem.

COROLLARY 4.3.2. *Let* D_j *be the exceptional divisor of the composite* $\pi_1 \circ \pi_2 \circ \ldots \circ \pi_j$ *of the first* j *blow-ups, i.e.* $D_1 = E_1$ *and* $D_j = \pi_j^* D_{j-1} + E_j$ *for* $j \geq 2$. *There exist hermitian metrics* $\hat{h}_j$ *on the line bundles* $[D_j]$ *and positive integers* l_j *such that the* $(1,1)$*-forms defined inductively by* $\hat{\omega}_0 = \omega_0$ *and*

$$\hat{\omega}_j = l_j \pi_j^* \hat{\omega}_{j-1} - c_1([D_j], \hat{h}_j) \qquad \textit{for } j \geq 1$$

are all positive and determine hermitian metrics on the manifolds M_j. *Moreover if* ω_0 *is Kähler (resp. Hodge) then* $\hat{\omega}_j$ *is Kähler (resp. Hodge) for all* j.

Proof. We need only prove positivity. Let $\hat{L}_j = [D_j]$, let h_j be a hermitian metric on $L_j = [E_j]$ whose Chern form is negative on the fibres of the map $E_j \to C_j$, and define metrics $\hat{h}_j$ on $\hat{L}_j$ inductively by the equations $\hat{h}_1 = h_1$ and $\hat{h}_j = (\pi_j^* \hat{h}_{j-1})h_j$. For $j = 1$ we have $[D_1] = [E_1] = L_1$ and positivity follows from Theorem (4.3.1). Assume that $\hat{\omega}_{j-1} > 0$ on M_{j-1} for some $j \geq 2$. Using the inductive descriptions of D_j and $\hat{h}_j$ we obtain

$$\begin{aligned} c_1([D_j], \hat{h}_j) &= c_1(\hat{L}_j, \hat{h}_j) \\ &= c_1\left((\pi_j^* \hat{L}_{j-1}) \otimes L_j, (\pi_j^* \hat{h}_{j-1})h_j\right) \\ &= \pi_j^*\left(c_1(\hat{L}_{j-1}, \hat{h}_{j-1})\right) + c_1(L_j, h_j). \end{aligned}$$

Now $l'\hat{\omega}_{j-1} - c_1(\hat{L}_{j-1}, \hat{h}_{j-1}) > 0$ for $l' \gg 0$ because $\hat{\omega}_{j-1} > 0$ and $c_1(\hat{L}_{j-1}, \hat{h}_{j-1})$ is bounded on M_{j-1}. Therefore

$$\pi_j^*\left(l'\hat{\omega}_{j-1} - c_1(\hat{L}_{j-1}, \hat{h}_{j-1})\right) \geq 0$$

on M_j. By Proposition (4.2.1), our inductive assumption that $\hat{\omega}_{j-1} > 0$, and our choice of the metric h_j, we have

$$l''\pi_j^*\hat{\omega}_{j-1} - c_1(L_j, h_j) > 0$$

for $l'' \gg 0$. Let $l_j = l' + l''$. □

(4.4) More finite metrics. In this section we construct a family of Kähler forms $\psi_1, ..., \psi_m$ on M_m which will be used in §9 to construct Kähler homogeneous Saper metrics.

PROPOSITION 4.4.1. *There exist divisors $\mathcal{D}_1, ..., \mathcal{D}_m$ on M_m of the form*

$$\mathcal{D}_j = \sum_{k=1}^{m} b_{jk}\tilde{E}_{m,k},$$

and metrics H_j on the line bundles $[\mathcal{D}_j]$, such that the transition matrix $T(\mathcal{D}, \tilde{E}) = (b_{jk})$ is a nonsingular matrix of positive integers and such that the forms

$$\psi_j = r\pi_{m,0}^*\omega_0 - c_1([\mathcal{D}_j], H_j)$$

are Kähler forms on M_m for all sufficiently large integers r.

Proof. First we write the Kähler forms ω_j of §4.3 in terms of the pullback to M_j of ω_0 and the Chern forms of certain line bundles on M_j. Recall that in §3.9 we defined

$$\pi_{j,i} = \pi_{i+1} \circ \pi_{i+2} \circ ... \circ \pi_j : M_j \to M_i$$

and the corresponding pullbacks

$$L_{j,i} = \pi_{j,i}^* L_i = [E_{j,i}] = [\pi_{j,i}^* E_i]$$

for $j > i$. We set $L_{j,j} = L_j$ and $E_{j,j} = E_j$. The induced metric on $L_{j,i}$ is $h_{j,i} = \pi_{j,i}^* h_i$ but for simplicity we will omit the metric and write $c_1(L_{j,i})$ or $c_1([E_{j,i}])$ for the Chern form $c_1(L_{j,i}, h_{j,i})$. Using the integers l_i of Theorem (4.3.1) and letting $r_j = l_1 l_2 ... l_j$, $r_{ji} = l_{i+1} l_{i+2} ... l_j$ for $1 \leq i \leq j-1$, and $r_{jj} = 1$ we may write ω_j as

$$\text{(4.4.2)} \qquad \begin{aligned} \omega_j &= r_j \pi_{j,0}^* \omega_0 - \sum_{i=1}^{j} r_{ji} c_1(L_{j,i}) \\ &= r_j \pi_{j,0}^* \omega_0 - c_1\left(\left[\sum_{i=1}^{j} r_{ji} E_{j,i}\right]\right). \end{aligned}$$

There is a similar formula for $\hat{\omega}_j$.

For $m > j$ the pullback of ω_j to M_m is a positive semi-definite form given by

$$\begin{aligned} \pi_{m,j}^* \omega_j &= r_j \pi_{m,0}^* \omega_0 - \sum_{k=1}^{j} r_{jk} c_1(L_{m,k}) \\ &= r_j \pi_{m,0}^* \omega_0 - c_1\left(\left[\sum_{k=1}^{j} r_{jk} E_{m,k}\right]\right). \end{aligned}$$

The form ω_m is positive and has a similar formula. Let $D_j' = \sum_{k=1}^{j} r_{jk} E_{m,k}$ for $1 \leq j \leq m$ and let $c_1([D_j'])$ be the Chern form with respect to the induced metric on the line bundle $[D_j']$. Then

$$\pi_{m,j}^* \omega_j = r_j \pi_{m,0}^* \omega_0 - c_1([D_j'])$$

for $m > j$ and

$$\omega_m = r_m \pi_{m,0}^* \omega_0 - c_1([D_m']).$$

The forms

$$\begin{aligned} \psi_j &= \sum_{i=j}^{m-1} \pi_{m,i}^* \omega_i + \omega_m \\ &= \left(\sum_{i=j}^{m} r_i\right) \pi_{m,0}^* \omega_0 - \sum_{i=j}^{m} c_1\left([D_i']\right) \end{aligned}$$

are all positive because ω_m is. Let

$$R_j = \sum_{i=j}^{m} r_i \quad \text{and} \quad \mathcal{D}_j = \sum_{i=j}^{m} D_i'$$

and let H_j be the induced metric on the line bundle $[\mathcal{D}_j]$. Then

$$\psi_j = R_j \pi_{m,0}^* \omega_0 - c_1([\mathcal{D}_j], H_j).$$

The form ψ_j remains positive if we replace R_j by $r > R_j$ since $\pi_{m,0}^* \omega_0 \geq 0$.

The matrix $T(\mathcal{D}, \tilde{E})$ for the divisors $\mathcal{D}_j$ in terms of the irreducible divisors $\tilde{E}_{m,k}$ is

$$T(\mathcal{D}, \tilde{E}) = T(\mathcal{D}, D')T(D', E)T(E, \tilde{E}).$$

The matrix $T(\mathcal{D}, D')$ is an upper triangular matrix of 1's, the matrix $T(D', E)$ is a lower triangular matrix of r_{ik}'s which are defined in terms of the integers l_j of Theorem (4.3.1), and the matrix $T(E, \tilde{E})$ is the matrix A described in §3.7. Writing $T(\mathcal{D}, \tilde{E})$ as the product

$$\begin{pmatrix} 1\,1\,1 \ldots 1 \\ 0\,1\,1 \ldots 1 \\ 0\,0\,1 \ldots 1 \\ \vdots\,\vdots\,\vdots\,\ddots\,\vdots \\ 0\,0\,0 \ldots 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \ldots 0 \\ l_2 & 1 & 0 & \ldots 0 \\ l_2 l_3 & l_3 & 1 & \ldots 0 \\ \vdots & \vdots & \vdots & \ddots \vdots \\ l_2 \ldots l_m & l_3 \ldots l_m & l_4 \ldots l_m & \ldots 1 \end{pmatrix} \begin{pmatrix} 1 & a_{12} & a_{13} & \ldots & a_{1m} \\ 0 & 1 & a_{23} & \ldots & a_{2m} \\ 0 & 0 & 1 & \ldots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

it is easy to see that all entries of $T(\mathcal{D}, \tilde{E})$ are positive and $\det T(\mathcal{D}, \tilde{E}) = 1$. □

§5. Splitting of a Poincaré-Type $(1,1)$-Form into a Sum of its Two Essential Parts. Let M be a compact complex manifold of dimension n and let D be an effective divisor on M with only normal crossings. In §5.1 we construct a Poincaré-type $(1,1)$-form ν on $M - D$ by replacing the expression $| z |^2$ in the formula (1.4.1) for the Poincaré form ω_{Δ^*} by the square of the norm of a section of the line bundle $[D]$. In §5.2 we study ν in local coordinates near points of D. We may choose coordinates $(z_1, z_2, ..., z_n)$ in which D is given locally by the vanishing of a monomial $z_1^{\lambda_1} z_2^{\lambda_2} ... z_k^{\lambda_k}$. The growth of ν near D may be described in terms of a Chern form of $[D]$ and the pullback of ω_{Δ^*} under the monomial map τ given by $\tau(z_1, z_2, ..., z_n) = z_1^{\lambda_1} z_2^{\lambda_2} ... z_k^{\lambda_k}$. In §5.3 we describe the quasi-isometry classes of sums of forms of the type $\tau^*\omega_{\Delta^*}$. We use these descriptions in §5.4 to show that Poincaré and modified Poincaré metrics may be constructed by adding Poincaré-type forms ν_i to multiples of the fundamental form ω of a hermitian metric on M. If ω is Kähler, the resulting metrics are also Kähler. We conclude §5 by describing the relationship between modified Saper and modified Poincaré metrics.

(5.1) Definition and decomposition of Poincaré-type (1, 1)-**forms.** An effective divisor D on M, with only normal crossings, may be expressed as $D = \sum_{i=1}^{m} \lambda_i E_i$, where $E_1, E_2, ..., E_m$ are smooth, reduced, irreducible divisors on M which simultaneously have only normal crossings, and $\lambda_1, \lambda_2, ..., \lambda_m$ are positive integers. Let $L = [D]$ be the line bundle on M associated with D and let h be a hermitian metric on L. Let $s : M \to L$ be a global holomorphic section of L such that $(s) = D$. Such a section always exists since D is effective. We denote by $\| s \|$ the norm of s under the metric h. Since M is compact, we may also choose s so that $\| s \| < 1$ everywhere on M. We define on $M - D$ a Poincaré-type $(1,1)$-form ν associated with the divisor D, the section s, and the metric h by

$$\nu = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} \log\left(\log \| s \|^2\right)^2. \tag{5.1.1}$$

Note that $\log \| s \|^2 \neq 0$ on $M - D$ and $\log \| s \|^2 \to -\infty$ as we approach D. Let

$$\beta = -\log \| s \|^2$$

on $M - D$. We may decompose ν as $\nu = \mu + \eta$ where

$$\mu = -\frac{\sqrt{-1}}{\pi}\frac{\partial\bar{\partial}\beta}{\beta} \quad \text{and} \quad \eta = \frac{\sqrt{-1}}{\pi}\frac{\partial\beta \wedge \bar{\partial}\beta}{\beta^2}. \tag{5.1.2}$$

Recall from §3.5 that $c_1(L, h) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} \log \| s \|^2$. Then

$$\mu = -\frac{2}{\beta} c_1(L, h). \tag{5.1.3}$$

The Chern form $c_1(L, h)$ depends on h but is independent of the choice of section s of L. The properties of η will be explored in

Sections 5.2 and 5.3.

If s' is another global holomorphic section of L such that $(s') = D$ and $\| s' \| < 1$, then $s' = cs$ for some positive constant c. Let $\beta' = -\log \| s' \|^2$. Then $\beta' = \beta - 2\log c$, $\partial\beta' = \partial\beta$, and $\overline{\partial}\beta' = \overline{\partial}\beta$. From these relationships and our decomposition of ν, we see that although ν is not independent of s, the order of growth of ν near D is the same for all sections s of L for which $(s) = D$ and $\| s \| < 1$.

Note that the form μ is dominated by any positive C^∞ $(1,1)$-form on M. Hence

LEMMA 5.1.4. *If ω is the fundamental form of a hermitian metric on M and ν is a Poincaré-type $(1,1)$-form with the decomposition $\nu = \mu + \eta$ of (5.1.2) then*

$$l\omega + \nu \sim l\omega + \eta$$

for all sufficiently large integers l.

(5.2) Poincaré-type $(1,1)$-forms in local coordinates. We wish to describe the Poincaré-type $(1,1)$-form ν in local coordinates near points of $D = \sum_{i=1}^m \lambda_i E_i$. In particular, we wish to examine the growth of the form η in the decomposition $\nu = \mu + \eta$ of (5.1.2) and compare it to the growth of the Poincaré form ω_{Δ^*} on the punctured disc.

Let q be a point in M at which k of the components E_i of D, say $E_1, E_2, ..., E_k$, intersect. Since the collection $\{E_i\}$ has normal crossings, there exist local coordinates $z_1, ..., z_n$ in a neighbourhood U of q, such that E_i is given locally by the equation $z_i = 0$ for $i = 1, ..., k$ and such that E_i does not intersect $\overline{U}$ for $i > k$. Recall that we call $(z_1, ..., z_n)$ *normal coordinates for* $E_1, ..., E_k$. Locally, D is given by the equation $z_1^{\lambda_1} z_2^{\lambda_2} ... z_k^{\lambda_k} = 0$, where λ_i is the multiplicity of E_i in D. We will use the notation $z = (z_1, z_2, ..., z_n)$ and $z^\Lambda = z_1^{\lambda_1} z_2^{\lambda_2} ... z_k^{\lambda_k}$. The norm of s is given in local coordinates by $\| s \|^2 = | z^\Lambda |^2 g$ for some bounded positive C^∞ function g on U, and the function $\beta = -\log \| s \|^2$ is given locally by

$$\beta = -\log(| z^\Lambda |^2 g). \tag{5.2.1}$$

For the rest of §5.2 we will assume only that β is a locally defined function of the form of (5.2.1) where g is some bounded positive C^∞ function on U.

Let τ be the monomial map $\tau : U \to \Delta$ given by $\tau(z_1, ..., z_n) = z^\Lambda$. We wish to compare the $(1,1)$-form $\eta = \frac{\sqrt{-1}}{\pi}\frac{\partial\beta\wedge\overline{\partial}\beta}{\beta^2}$ to the pullback of the Poincaré form ω_{Δ^*} under the map τ. The pullback is

(5.2.2)

$$\begin{aligned}
\tau^*\omega_{\Delta^*} &= \frac{\sqrt{-1}}{\pi}\frac{d(z^\Lambda)\wedge d(\overline{z}^\Lambda)}{\mid z^\Lambda \mid^2(\log\mid z^\Lambda\mid^2)^2} \\
&= \frac{\sqrt{-1}}{\pi}\frac{1}{(\log\mid z^\Lambda\mid^2)^2}\sum_{i,j=1}^{k}\frac{\lambda_i\lambda_j}{z_i\overline{z}_j}dz_i\wedge d\overline{z}_j \\
&\sim \frac{\sqrt{-1}}{\pi}\frac{1}{\beta^2}\sum_{i,j=1}^{k}\frac{\lambda_i\lambda_j}{z_i\overline{z}_j}dz_i\wedge d\overline{z}_j.
\end{aligned}$$

Using the expansion $\beta = -\log z^\Lambda - \log\overline{z}^\Lambda - \log g$, we express η in local coordinates as

(5.2.3)

$$\begin{aligned}
\eta &= \frac{\sqrt{-1}}{\pi}\frac{1}{\beta^2}\left(\frac{d(z^\Lambda)}{z^\Lambda}+\frac{\partial g}{g}\right)\wedge\left(\frac{d(\overline{z}^\Lambda)}{\overline{z}^\Lambda}+\frac{\overline{\partial}g}{g}\right) \\
&= \frac{\sqrt{-1}}{\pi}\left(\frac{d(z^\Lambda)\wedge d(\overline{z}^\Lambda)}{\mid z^\Lambda\mid^2(\log\mid z^\Lambda\mid^2 g)^2}+\frac{d(z^\Lambda)\wedge\overline{\partial}g}{\beta^2 z^\Lambda g}\right. \\
&\quad \left.+\frac{\partial g\wedge d(\overline{z}^\Lambda)}{\beta^2 g\overline{z}^\Lambda}+\frac{\partial g\wedge\overline{\partial}g}{\beta^2 g^2}\right) \\
&\sim \tau^*\omega_{\Delta^*}+\frac{\sqrt{-1}}{\pi}\left(\sum_{i=1}^{k}\sum_{j=1}^{n}\left(o\left(\frac{\lambda_i}{z_i\beta^2}\right)dz_i\wedge d\overline{z}_j\right.\right. \\
&\quad \left.\left.+o\left(\frac{\lambda_i}{\overline{z}_i\beta^2}\right)dz_j\wedge d\overline{z}_i\right)+\frac{\partial g\wedge\overline{\partial}g}{\beta^2 g^2}\right).
\end{aligned}$$

Let ω be the Euclidean $(1,1)$-form

$$\omega = \frac{\sqrt{-1}}{\pi}\sum_{i=1}^{n}dz_i\wedge d\overline{z}_i.$$

We now show that, locally, when we add to η any positive multiple of $\frac{1}{\beta}\omega$, all terms but $\tau^*\omega_{\Delta^*}$ in the expansion of η may be ignored. The following lemma is valid for any expression of the form $\eta = \frac{\sqrt{-1}}{\pi}\frac{\partial\beta\wedge\overline{\partial}\beta}{\beta^2}$

where $\beta = -\log(|\ z^{\Lambda}\ |^2 g)$ and g is a bounded positive C^∞ function on U.

PROPOSITION 5.2.4. *Let a be any positive constant. Then the $(1,1)$-form $\frac{a}{\beta}\omega + \eta$ is positive and*

$$\frac{a}{\beta}\omega + \eta \sim \frac{1}{\beta}\omega + \tau^*\omega_{\Delta^*}$$

for $z_1, z_2, ..., z_k$ close enough (but not equal) to 0.

Proof. Clearly the term $\frac{\sqrt{-1}}{\pi}\frac{\partial g \wedge \bar{\partial} g}{\beta^2 g^2}$ in the expression (5.2.3) for η is dominated by $\frac{a}{\beta}\omega$ for $z_1, z_2, ..., z_k$ close enough to 0, since $\frac{1}{\beta} \to 0$ as $(z_1, z_2, ..., z_k) \to (0, 0, ..., 0)$. Next consider terms of type $o\left(\frac{\lambda_i}{z_i \beta^2}\right) dz_i \wedge d\bar{z}_j$ and $o\left(\frac{\lambda_i}{\bar{z}_i \beta^2}\right) dz_j \wedge d\bar{z}_i$ in η. If $i = j$ these terms are dominated by the corresponding terms of $\tau^*\omega_{\Delta^*}$. If $i \neq j$, we compare these terms to the $dz_i \wedge d\bar{z}_i$ term of $\tau^*\omega_{\Delta^*}$ and the $dz_j \wedge d\bar{z}_j$ term of $\frac{a}{\beta}\omega$. It is convenient to write these terms from the expansion of $\frac{a}{\beta}\omega + \eta$ in the form of a chart, showing the matrix of coefficients:

$\frac{\sqrt{-1}}{\pi}\wedge$	$d\bar{z}_i$	$d\bar{z}_j$
dz_i	$\frac{\lambda_i^2}{\mid z_i \mid^2 \beta^2}$	$o\left(\frac{\lambda_i}{z_i \beta^2}\right)$
dz_j	$o\left(\frac{\lambda_i}{\bar{z}_i \beta^2}\right)$	$\frac{a}{\beta}$

Factoring out powers of z_i, $\bar{z}_i$, and β we obtain

$\frac{\sqrt{-1}}{\pi}\wedge$	$\frac{1}{\bar{z}_i \beta} d\bar{z}_i$	$\frac{1}{\sqrt{\beta}} d\bar{z}_j$
$\frac{1}{z_i \beta} dz_i$	λ_i^2	$\lambda_i o\left(\frac{1}{\sqrt{\beta}}\right)$
$\frac{1}{\sqrt{\beta}} dz_j$	$\lambda_i o\left(\frac{1}{\sqrt{\beta}}\right)$	a

$$\begin{array}{c|cc} \frac{\sqrt{-1}}{\pi}\wedge & \frac{1}{\overline{z}_i\beta}d\overline{z}_i & \frac{1}{\sqrt{\beta}}d\overline{z}_j \\ \hline \sim \quad \frac{1}{z_i\beta}dz_i & \lambda_i^2 & 0 \\ \frac{1}{\sqrt{\beta}}dz_j & 0 & a \end{array}$$

for $z_1, z_2, ..., z_k$ close enough to 0. Then $\frac{a}{\beta}\omega + \eta \sim \frac{a}{\beta}\omega + \tau^*\omega_{\Delta^*}$ for $z_1, z_2, ..., z_k$ close enough to 0. Since $\omega > 0$ and $\tau^*\omega_{\Delta^*} \geq 0$, we may replace $\frac{a}{\beta}\omega$ by $\frac{1}{\beta}\omega$ and preserve quasi-isometry. □

An easy consequence of Proposition (5.2.4) is the following corollary which states that the local quasi-isometry class of a modified Poincaré metric looks the same in every system of normal coordinates. Recall (Definition (1.6.1)) that a metric on $M-D$ with fundamental form ω_P is called a modified Poincaré metric if ω_P has the following property: near each point $q \in M$ at which k components of D intersect, there exist normal coordinates $(z_1, ..., z_n)$ and nonconstant monomial maps $\tau_1, ..., \tau_m$ of the form $\tau_i(z_1, ..., z_n) = z_1^{\lambda_{i1}} z_2^{\lambda_{i2}} ... z_k^{\lambda_{ik}}$ such that the matrix (λ_{ij}) has nonnegative integer entries and at least one positive entry in each row and column, and such that locally

$$\omega_P \sim \sum_{i=1}^{m} \tau_i^*\omega_{\Delta^*} + \frac{\sqrt{-1}}{\pi}\sum_{i=1}^{n} dz_i \wedge d\overline{z}_i. \tag{5.2.5}$$

Given such a matrix (λ_{ij}) and any other system of normal coordinates $(y_1, ..., y_n)$ in a neighbourhood of q, let $T_1, ..., T_m$ be the monomial maps given by $T_i(y_1, ..., y_n) = y_1^{\lambda_{i1}} y_2^{\lambda_{i2}} ... y_k^{\lambda_{ik}}$.

COROLLARY 5.2.6. *If ω_P is the fundamental form of a modified Poincaré metric on M whose quasi-isometry class is given locally in normal coordinates $(z_1, ..., z_n)$ by (5.2.5), and if $(y_1, ..., y_n)$ is any other system of normal coordinates, then*

$$\omega_P \sim \sum_{i=1}^{m} T_i^*\omega_{\Delta^*} + \frac{\sqrt{-1}}{\pi}\sum_{i=1}^{n} dy_i \wedge d\overline{y}_i.$$

Proof. The Euclidean form in y is locally quasi-isometric to the Euclidean form ω in z, i.e.

$$(5.2.7)\qquad \frac{\sqrt{-1}}{\pi}\sum_{i=1}^{n} dy_i \wedge d\overline{y}_i \sim \frac{\sqrt{-1}}{\pi}\sum_{i=1}^{n} dz_i \wedge d\overline{z}_i = \omega,$$

so to prove the corollary it is sufficient to show that $\omega + \tau_i^*\omega_{\Delta^*} \sim \omega + T_i^*\omega_{\Delta^*}$ for all i. Since y and z are normal coordinates, there exist bounded nonvanishing holomorphic functions f_i such that $z_i = y_i f_i$ for $1 \leq i \leq k$. Let $\beta_i = -\log |\, z^{\Lambda_i} \,|^2$. Then

$$\beta_i = -\log |\, y^{\Lambda_i} f^{\Lambda_i} \,|^2$$

where the function $g_i = |\, f^{\Lambda_i} \,|^2 = |\, f_1^{\lambda_{i1}} f_2^{\lambda_{i2}} ... f_k^{\lambda_{ik}} \,|^2$ is bounded, positive, and C^∞. Let

$$\eta_i = \frac{\sqrt{-1}}{\pi}\frac{\partial\beta_i \wedge \overline{\partial}\beta_i}{\beta_i^2},$$

i.e. $\eta_i = \tau_i^*\omega_{\Delta^*}$. By Proposition (5.2.4) and quasi-isometry (5.2.7),

$$\frac{1}{\beta_i}\omega + \eta_i \sim \frac{1}{\beta_i}\omega + T_i^*\omega_{\Delta^*}.$$

But $\omega \underset{\sim}{\geq} \frac{1}{\beta_i}\omega$ near D, so we have $\omega + \tau_i^*\omega_{\Delta^*} = \omega + \eta_i \sim \omega + T_i^*\omega_{\Delta^*}$. □

Similarly, we will show in Corollary (5.3.5) that the local quasi-isometry class of a Saper distinguished metric (1.7.1) looks the same in any system of normal coordinates.

(5.3) Pullbacks of Poincaré-type forms under monomial maps. When constructing modified Saper metrics, we will use $(1,1)$-forms of the type $\frac{1}{\beta}\omega + \eta$. Proposition (5.2.4) allows us to replace η by a pullback $\tau^*\omega_{\Delta^*}$ of the Poincaré form ω_{Δ^*} under a monomial map τ. In this section we describe the quasi-isometry classes of sums of such pullbacks. For the purposes of later calculations, in which we consider a collection of divisors $\{D_j\}$, it is useful to consider monomials $z_1^{\lambda_1} z_2^{\lambda_2} ... z_k^{\lambda_k}$ in which some of the powers λ_i may be 0.

For $1 \leq i \leq m$ let $\Lambda_i = (\lambda_{i1}, ..., \lambda_{ik})$ be a vector of nonnegative integers with at least one nonzero entry. Let $z^{\Lambda_i} = z_1^{\lambda_{i1}} z_2^{\lambda_{i2}} ... z_k^{\lambda_{ik}}$ and let τ_i be the monomial map

$$\tau_i : \Delta^k \to \Delta$$

given by $\tau_i(z_1, ..., z_k) = z^{\Lambda_i}$. Let $\beta_i = -\log | z^{\Lambda_i} |^2$ and let $\phi_i = \beta_i^2 \tau_i^* \omega_{\Delta^*}$. Then

$$\phi_i = \frac{\sqrt{-1}}{\pi} \frac{d\left(z^{\Lambda_i}\right) \wedge d\left(\overline{z}^{\Lambda_i}\right)}{| z^{\Lambda_i} |^2} = \frac{\sqrt{-1}}{\pi} \sum_{j=1}^{k} \sum_{l=1}^{k} \frac{\lambda_{ij}\lambda_{il}}{z_j \overline{z}_l} dz_j \wedge d\overline{z}_l.$$

The Poincaré form ω_{Δ^*} is positive definite on Δ^* so all the pullbacks $\tau_i^* \omega_{\Delta^*}$ and all the forms ϕ_i are positive semi-definite on $(\Delta^*)^k$. The form

$$\phi = \frac{\sqrt{-1}}{\pi} \sum_{i=1}^{k} \frac{dz_i \wedge d\overline{z}_i}{| z_i |^2}$$

is positive definite on $(\Delta^*)^k$.

LEMMA 5.3.1. *If k of the vectors Λ_i, say $\Lambda_1, ..., \Lambda_k$, are linearly independent, then*

$$\sum_{i=1}^{k} C_i \phi_i \sim \phi > 0$$

for all positive constants $C_1, ..., C_k$ and for $z_1, ..., z_k$ close enough (but not equal) to 0.

Proof. The matrix of ϕ_i with respect to $\{\xi_j = \frac{dz_j}{z_j}\}_{j=1}^k$ is $A_i = \Lambda_i^T \Lambda_i$ which is positive semi-definite. If $\Lambda_1, ..., \Lambda_k$ are linearly independent then the spans of the matrices $A_1, ..., A_k$ are also linearly independent and $\sum_{i=1}^k C_i A_i$ is positive definite for any positive constants $C_1, ..., C_k$. This implies that $\sum_{i=1}^k C_i \phi_i \sim \phi$ since the matrix of ϕ with respect to $\{\xi_j\}$ is I. □

It follows easily from Lemma (5.3.1) that

LEMMA 5.3.2.

i. $\phi \gtrsim \phi_i$ *for each $1 \leq i \leq m$ and for $z_1, ..., z_k$ close enough (but not equal) to 0.*

ii. *If the matrix* (λ_{ij}) *has rank* k *then* $\sum_{i=1}^{m} C_i\phi_i \sim \phi > 0$ *for all positive constants* $C_1, ..., C_m$ *and for* $z_1, ..., z_k$ *close enough (but not equal) to* 0.

Next we show that a sum of pullbacks of the Poincaré form ω_{Δ^*} under monomial maps is bounded below by the homogeneous Poincaré form

$$\psi_h = \frac{\sqrt{-1}}{\pi} \frac{1}{(\log | z_1 z_2 ... z_k |^2)^2} \sum_{i=1}^{k} \frac{dz_i \wedge d\overline{z}_i}{| z_i |^2}.$$

LEMMA 5.3.3. *If* k *of the vectors* Λ_i, *say* $\Lambda_1, ..., \Lambda_k$, *are linearly independent then*

$$\text{(i)} \qquad \sum_{i=1}^{m} C_i \tau_i^* \omega_{\Delta^*} \mathrel{\tilde{\geq}} \psi_h > 0$$

for all positive constants $C_1, ..., C_m$ *and for* $z_1, ..., z_k$ *close enough (but not equal) to* 0. *If, in addition, the integers* λ_{ij} *are all positive then*

$$\text{(ii)} \qquad \sum_{i=1}^{m} C_i \tau_i^* \omega_{\Delta^*} \sim \psi_h.$$

Proof. Let $\beta = -\log | z_1 z_2 ... z_k |^2$. There exists a positive constant a such that $a\beta \geq \beta_i$ for $1 \leq i \leq m$. Then

$$\text{(5.3.4)} \qquad \sum_{i=1}^{m} C_i \tau_i^* \omega_{\Delta^*} = \sum_{i=1}^{m} \frac{C_i}{\beta_i^2} \phi_i \geq \frac{1}{a^2\beta^2} \sum_{i=1}^{m} C_i \phi_i.$$

By Lemma (5.3.2) and the definitions of β, ϕ, and ψ_h,

$$\frac{1}{a^2\beta^2} \sum_{i=1}^{m} C_i \phi_i \sim \frac{1}{\beta^2} \phi = \psi_h$$

for $z_1, z_2, ..., z_k$ near 0. This proves part (i). If $\lambda_{ij} > 0$ for all i and j, then $\beta_i \sim \beta$ and $\tau_i^* \omega_{\Delta^*} \sim \frac{1}{\beta^2}\phi_i$ for all i. In this case the inequality $\geq$ in equation (5.3.4) becomes quasi-isometry and we obtain part (ii). □

Recall that a metric on $M - D$ is called a Saper distinguished metric if its fundamental form ω_{Sap} may be described locally in normal coordinates by the quasi-isometry

$$\omega_{\text{Sap}} \sim \psi_h + \frac{1}{\beta}\omega_{\text{Eucl}}$$

where ω_{Eucl} is the Euclidean $(1,1)$-form

$$\omega_{\text{Eucl}} = \frac{\sqrt{-1}}{\pi}\sum_{i=1}^{n} dz_i \wedge d\overline{z}_i.$$

A homogeneous Poincaré metric has a fundamental form $\omega_{\text{P, hom}}$ given locally by

$$\omega_{\text{P,hom}} \sim \psi_h + \omega_{\text{Eucl}}.$$

We can now easily show that the local quasi-isometry class of ω_{Sap} (respectively $\omega_{\text{P,hom}}$) looks the same in any system of normal coordinates.

COROLLARY 5.3.5. *The quasi-isometry class of a Saper distinguished metric (respectively a homogeneous Poincaré metric) on $M - D$ is independent of the choice of normal coordinates.*

Proof. Use part (ii) of Lemma (5.3.3) and Proposition (5.2.4). □

Finally, we show that pullbacks of the Poincaré form ω_{Δ^*} under monomial maps are bounded above by the Poincaré form

$$\psi_P = \frac{\sqrt{-1}}{\pi}\sum_{i=1}^{k} \frac{dz_i \wedge d\overline{z}_i}{|\, z_i\,|^2(\log|\, z_i\,|^2)^2}.$$

LEMMA 5.3.6. *Let $\Lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ be a vector of nonnegative integers with at least one nonzero entry and let τ be the corresponding map $\tau : \Delta^k \to \Delta$ given by $\tau(z_1, z_2, ..., z_k) = z_1^{\lambda_1} z_2^{\lambda_2} ... z_k^{\lambda_k}$. Then*

$$\tau^*\omega_{\Delta^*} \quad \stackrel{\sim}{\leq} \quad \psi_P$$

for $z_1, ..., z_k$ close enough (but not equal) to 0.

Proof. Let l be the number of entries of Λ which are nonzero. Reindex the variables z_i so that $\lambda_i > 0$ for $1 \leq i \leq l$ and $\lambda_i = 0$

otherwise. Let $\beta = -\log | z_1^{\lambda_1} z_2^{\lambda_2} ... z_l^{\lambda_l} |^2$. Then $\beta \tilde{\geq} -\log | z_i |^2$ for $1 \leq i \leq l$. Let

$$\phi' = \frac{\sqrt{-1}}{\pi} \sum_{i=1}^{l} \frac{dz_i \wedge d\bar{z}_i}{| z_i |^2}$$

and let $\phi_\Lambda = \beta^2 \tau^* \omega_{\Delta^*}$. The form ϕ_Λ is positive semi-definite and by Lemma (5.3.2), $\phi_\Lambda \tilde{\leq} \phi'$. Dividing ϕ_Λ by β^2 and using the indicated lower bounds for β we obtain

$$\begin{aligned} \tau^* \omega_{\Delta^*} &\tilde{\leq} \frac{1}{\beta^2} \phi' \\ &\tilde{\leq} \frac{\sqrt{-1}}{\pi} \sum_{i=1}^{l} \frac{dz_i \wedge d\bar{z}_i}{| z_i |^2 (\log | z_i |^2)^2} \\ &\leq \psi_P. \end{aligned}$$

□

(5.4) Modified Poincaré metrics. It is easy to construct Poincaré and modified Poincaré metrics on $M - D$ using the results of Sections (5.1) - (5.3). The components $E_1, ..., E_m$ of D are smooth, reduced, irreducible divisors which simultaneously have only normal crossings. Let $D_1, ..., D_r$ be effective divisors of the form

$$D_i = \sum_{j=1}^{m} \lambda_{ij} E_j$$

such that the matrix (λ_{ij}) has nonnegative integer entries and at least one positive entry in each row and column, i.e. the divisors D_i are effective and their sum has the same support as D. Let $s_i : M \to [D_i]$ be a global holomorphic section of the line bundle $L_i = [D_i]$ such that $(s_i) = D_i$ and let h_i be a hermitian metric on L_i. Let

$$\nu_i = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\log || s_i ||^2)^2.$$

The following proposition is an easy consequence of Lemmas (5.1.4) and (5.3.3) and Proposition (5.2.4).

PROPOSITION 5.4.1. *If ω is the fundamental form of a hermitian metric on M and $l_1, ..., l_r$ are positive integers, then the form*

$$\omega_P = l\omega + \sum_{i=1}^{r} l_i \nu_i$$

determines a modified Poincaré metric on the noncompact manifold $M - D$ *for all sufficiently large integers* l. *Moreover*

i. *If* M *is Kähler and* ω *is a Kähler form on* M *then* ω_P *is also Kähler.*

ii. *If* $r = m$ *and* $D_i = E_i$ *for* $1 \leq i \leq m$ *then the metric determined by* ω_P *is a true Poincaré metric.*

iii. *If the matrix* $\Lambda = (\lambda_{ij})$ *satisfies the condition*

(*) *the entries are all positive and the columns are linearly independent*

then the metric determined by ω_P *is a homogeneous Poincaré metric. More generally, if* D *has several connected components, the metric determined by* ω_P *is a homogeneous Poincaré metric if the submatrix of* Λ *corresponding to each connected component of* D *satisfies condition* (*).

(5.5) Modified Saper metrics. Suppose that M_0 is a compact complex manifold and $\pi : M \to M_0$ is a holomorphic map whose restriction to $M - D$ is a biholomorphism onto its image. Let ω_0 be the fundamental form of a hermitian metric on M_0. Let $D_1, ..., D_r$ be divisors on M of the type described above in §5.4 and let $\nu_1, ..., \nu_r$ be Poincaré-type (1,1)-forms for these divisors. Let $l_0, l_1, ..., l_r$ be positive integers. Recall from §1.8 that if the modified Saper form

$$\omega_S = l_0 \pi^* \omega_0 + \sum_{i=1}^{r} l_i \nu_i$$

is positive then we say it determines a *modified Saper metric which is distinguished with respect to* π. If in addition ω_0 is Kähler, then so is ω_S.

Let ω be the fundamental form of a hermitian metric on M, as above. Note that any positive C^∞ (1,1)-form on M dominates $\pi^*\omega_0$. The next result follows immediately from Proposition (5.4.1).

PROPOSITION 5.5.1. *If* ω *is the fundamental form of a hermitian metric on* M *and* ω_S *is a modified Saper form on* $M - D$, *then the form*

$$\omega_P = \omega_S + l\omega$$

determines a modified Poincaré metric on $M - D$ *for all sufficiently* *large integers l. Moreover if ω_S is positive we may use $l = 1$ and we have*

i. *If ω_S is homogeneous so is ω_P.*

ii. *If M_0 and M are Kähler and if ω_0 and ω are Kähler forms, then ω_S and ω_P are also Kähler.*

In §7 we construct Kähler modified Saper metrics, distinguished with respect to a single blow-up of a compact Kähler manifold M. In §8 we construct Kähler modified Saper metrics inductively on successive blow-ups and exhibit in local coordinates the relationship of each new metric to the pullback of the previous. When each centre is either contained in the total exceptional divisor of the previous blow-ups or disjoint from it, these metrics are homogeneous. When the image of D in M is of dimension 0, the metrics are exactly Saper distinguished metrics. In §9 we construct Kähler homogeneous Saper metrics without the restriction that each centre lie in the previous total exceptional divisor.

§6. Main ingredient of completeness of our metrics. Let M be a compact Kähler manifold and let D be an effective divisor on M with only normal crossings. In this section we show that if a metric on $M - D$ is bounded below locally, near D, by pullbacks of the Poincaré metric on the punctured disc under appropriate monomial maps, then the metric is complete. In particular, modified Poincaré metrics are complete. When we construct our modified Saper metrics in Sections 7, 8, and 9, we will use Proposition (5.2.4) and the results of this section to show that they are also complete.

We first consider the case in which a single monomial map suffices.

PROPOSITION 6.1.1. *Let ω_{M-D} be a Kähler form on $M - D$. Suppose that ω_{M-D} satisfies the following condition near each point $q \in D$: there exist normal coordinates $z_1, ..., z_n$ on a neighbourhood U of q such that the components of D are given locally by the equations $z_i = 0$ for $i = 1, ..., k$, and there exists a monomial map $\tau : U \to \Delta$ given by*

$$\tau(z_1, ..., z_n) = z_1^{\lambda_1} ... z_k^{\lambda_k}$$

for some positive integers $\lambda_1, ..., \lambda_k$, such that

$$\omega_{M-D} \stackrel{\sim}{\geq} \tau^* \omega_{\Delta^*}$$

on U. Then the metric on $M-D$ determined by ω_{M-D} is complete.

Proof. We will use the completeness of the Poincaré metric on the punctured disc Δ^*. We are only concerned with the behaviour of this metric near $0 \in \Delta$.

Let x and y be points in M and let $d_M(x, y)$ be the distance between them in the metric determined by some Kähler form ω_M on all of M. Similarly, for $x, y \in M - D$, let $d_{M-D}(x, y)$ be the distance between x and y in the metric determined by ω_{M-D}. Let $\{x_n\}$ be a sequence in $M - D$ which is Cauchy with respect to d_{M-D}. Since M is compact, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges with respect to d_M to a point $q \in M$.

Suppose that $q \in M - D$. Then there exists a neighbourhood U of q in $M - D$ containing all but finitely many of the points $\{x_{n_i}\}$ and such that ω_M and ω_{M-D} are quasi-isometric on U. Then $\{x_{n_i}\}$ converges to q in the metric determined by ω_{M-D} and so $\{x_n\}$ also converges to q in this metric.

Now suppose that $q \in D$ and let U be a coordinate neighbourhood of q in M on which there is a monomial map τ of the form described above. Let d^* be the Poincaré distance function on Δ^*. After shrinking U if necessary, there is a positive constant c such that for any $x, y \in U - U \cap D$ we have

$$d_{M-D}(x, y) \geq cd^*(\tau(x), \tau(y)).$$

Similarly, if d is the usual Euclidean distance function on Δ then there is a positive constant C such that for any $x, y \in U$ we have

$$d_M(x, y) \geq Cd(\tau(x), \tau(y)).$$

Since the sequence $\{x_{n_i}\}$ is Cauchy with respect to d_{M-D}, the sequence $\{\tau(x_{n_i})\}$ is Cauchy with respect to d^*. Therefore $\{\tau(x_{n_i})\}$ converges to a point $p \in \Delta^*$. Similarly, since the sequence $\{x_{n_i}\}$ converges with respect to d_M to $q \in D$, the sequence $\{\tau(x_{n_i})\}$ converges to $0 \in \Delta$ with respect to the usual distance function d on Δ. This is impossible, since in a neighbourhood of $p \in \Delta^*$ the distance functions d and d^* are quasi-isometric. Therefore $q \in M - D$ and the metric determined by ω_{M-D} is complete. □

A similar result applies to a collection of monomial maps:

PROPOSITION 6.1.2. *Let ω_{M-D} be a Kähler form on $M - D$. Suppose that ω_{M-D} satisfies the following condition near each point $q \in D$: there exist normal coordinates $z_1, ..., z_n$ on a neighbourhood U of q such that the components of D are given locally by the equations $z_i = 0$ for $i = 1, ..., k$, and there exist a monomial maps $\tau_1, ..., \tau_r$ from U to Δ of the form*

$$\tau_i(z_1, ..., z_n) = z_1^{\lambda_{i1}} ... z_k^{\lambda_{ik}}$$

such that the matrix (λ_{ij}) has nonnegative integer entries and at least one positive entry in each row and column, and such that

$$\omega_{M-D} \tilde{\geq} \sum_{i=1}^{r} \tau_i^* \omega_{\Delta^*}$$

on U. Then the metric on $M - D$ determined by ω_{M-D} is complete.

Proof. We proceed as before, replacing the map $\tau : U \to \Delta$ of the previous proposition by the map

$$\tau = (\tau_1, ..., \tau_r) : U \to \Delta^r$$

and replacing the Poincaré form ω_{Δ^*} on Δ^* by a product of Poincaré forms $\omega_{(\Delta^*)^r}$ on $(\Delta^*)^r$. Note that $\omega_{(\Delta^*)^r}$ determines a complete metric on $(\Delta^*)^r$ and that

$$\tau^* \omega_{(\Delta^*)^r} = \sum_{i=1}^{r} \tau_i^* \omega_{\Delta^*}.$$

We complete the proof in the same way as above. □

The next corollary follows immediately from the definition of modified Poincaré metrics.

COROLLARY 6.1.3. *Modified Poincaré metrics are complete.*

When we construct our modified Saper metrics we will show that they also have the required lower bound.

§7. Modified Saper and Poincaré metrics for a single blow-up.

(7.1) Introduction. Let M be a compact Kähler manifold with Kähler form ω. Let $\pi : \tilde{M} \to M$ be the blow-up of M along a submanifold $C \subset M$, with exceptional divisor $E = \pi^{-1}(C)$ and associated line bundle $L = [E]$ on $\tilde{M}$. Let h be a hermitian metric on L with the property that the $(1,1)$-form

$$\tilde{\omega} = l\pi^*\omega - c_1(L, h)$$

on $\tilde{M}$ is positive and consequently Kähler for all sufficiently large integers l. In §4 we showed that such metrics h exist. The Chern form $c_1(L, h)$ may be written as

$$c_1(L, h) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log || s ||^2$$

where $s : \tilde{M} \to L$ is a global holomorphic section of L such that $(s) = E$ and such that the norm $|| s ||$ of s under the metric h satisfies $|| s ||< 1$ everywhere on $\tilde{M}$. The Kähler form $\tilde{\omega}$ determines an incomplete metric on $\tilde{M} - E \cong M - C$ and the completion of $M - C$ with respect to this metric is $\tilde{M}$.

We now define a Poincaré-type $(1,1)$-form ν on $\tilde{M} - E$ by the equation

$$\nu = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\log || s ||^2\right)^2. \tag{7.1.1}$$

We will show in this section that the $(1,1)$-form

$$\omega_S = l\pi^*\omega + \nu$$

on $\tilde{M} - E$ is positive and hence Kähler for all sufficiently large integers l. The corresponding Kähler metric is a complete modified Saper metric, distinguished with respect to the blow-up map π. When $\dim C = 0$, i.e. when $\tilde{M}$ is the blow-up of an isolated point in M, our modified Saper metric is precisely a Saper distinguished metric.

It follows from Proposition (5.5.1) that the $(1,1)$-form

$$\omega_P = \omega_S + \tilde{\omega}$$

is the Kähler form of a Poincaré metric on $\tilde{M} - E$, i.e. the sum of our modified Saper metric on $\tilde{M} - E$ and the restriction to $\tilde{M} - E$ of a Kähler metric on $\tilde{M}$ is Poincaré. We may replace $\tilde{\omega}$ by any other Kähler form on $\tilde{M}$ since all Kähler metrics on a compact manifold are quasi-isometric.

We proved in §6 that Poincaré metrics are complete. Completeness of our modified Saper metric follows from the local description of the quasi-isometry class of ω_S and Proposition (6.1.1).

The constructions which we will give in §§8 - 9 of modified Saper and Poincaré metrics for successive blow-ups $M_j \to M_{j-1}$ are similar, but the estimates used are more delicate.

(7.2) Main results. Let M and $\tilde{M}$ be as above and let ω_{Δ^*} be the Poincaré form on the punctured disc. For any point $q \in E$, we may choose local coordinates $(z_1, ..., z_n)$ on a neighbourhood U of q and $(Z_1, ..., Z_n)$ on a neighbourhood of $\pi(q) \in C$ such that

i. E is given locally by the equation $z_1 = 0$,
ii. C is given locally by the equations $Z_1 = ... = Z_k = 0$, and
iii. π is given locally by the equations:

$$\begin{aligned} Z_1 &= z_1 \\ Z_i &= z_1 z_i \quad \text{for} \quad 2 \le i \le k \\ Z_i &= z_i \quad\quad \text{for} \quad k+1 \le i \le n. \end{aligned}$$

We will call the coordinates $(z_1, ..., z_n)$ *normal blow-up coordinates for π corresponding to $(Z_1, ..., Z_n)$*. Let $\tau : U \to \Delta$ be the monomial map given by $\tau(z_1, ..., z_n) = z_1$.

PROPOSITION 7.2.1. (*Modified Saper metrics for a single blow-up.*) *The $(1,1)$-form*

$$\omega_S = l\pi^*\omega + \nu$$

determines a complete Kähler metric on $\tilde{M} - E$ for all sufficiently large integers l. This metric is a modified Saper metric which is distinguished with respect to the map π. If $q \in E$ and if $(z_1, ..., z_n)$ are normal blow-up coordinates on a neighbourhood U of q then $\omega_S|_U$

is locally quasi-isometric to all of the following:

(i)

$$\pi^*\omega|_U + \frac{\sqrt{-1}}{\pi}\frac{1}{|\log|z_1|^2|}\sum_{i=2}^{k} dz_i \wedge d\bar{z}_i + \tau^*\omega_{\Delta^*}$$

(ii)

$$\pi^*\omega|_U + \frac{\sqrt{-1}}{\pi}\left(\frac{1}{|\log|z_1|^2|}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i + \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2(\log|z_1|^2)^2}\right)$$

(iii)

$$\frac{\sqrt{-1}}{\pi}\left(\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2(\log|z_1|^2)^2} + \frac{1}{|\log|z_1|^2|}\sum_{i=2}^{k} dz_i \wedge d\bar{z}_i + \sum_{i=k+1}^{n} dz_i \wedge d\bar{z}_i\right).$$

Note that completeness follows directly from quasi-isometry (i) by Proposition (6.1.1).

Quasi-isometries (i) and (ii) describe the relationship between our modified Saper metric on $\tilde{M} - E$ and the metric on $\tilde{M} - E$ induced from M. Quasi-isometry (i) means that locally, near E, our modified Saper metric on $\tilde{M} - E$ looks like the metric induced from M plus a small term in the fibre directions plus a term with Poincaré-type growth. We will prove a similar statement for successive blow-ups. Quasi-isometry (ii) means that our modified Saper metric is locally quasi-isometric to the sum of the incomplete metric induced from M and a Saper distinguished metric. In (iii) we describe the quasi-isometry class completely in local coordinates. Descriptions (ii) and (iii) of ω_S in local coordinates are more difficult to generalize to successive blow-ups except in special cases, and will be replaced by upper and lower bounds for ω_S.

Let ω_{Sap} be the fundamental form of a Saper distinguished metric on $\tilde{M} - E$, not necessarily Kähler. As a corollary to the local description (ii) we obtain the following global quasi-isometry.

COROLLARY 7.2.2. *The Kähler modified Saper metric on $\tilde{M} - E$ associated with ω_S is quasi-isometric to the sum of the metric induced from M and a Saper distinguished metric on $\tilde{M} - E$. More concisely:*

$$\omega_S \sim \pi^*\omega + \omega_{\text{Sap}}.$$

We can say even more in the case $\dim C = 0$. In this case, $k = n$ and the last term of (iii) disappears, giving us the following:

COROLLARY 7.2.3. (*Blow-ups of isolated points.*) *If* $\dim C = 0$ *then* ω_S *is the fundamental form of a Kähler metric which is distinguished in the sense of Saper.*

Given a Kähler form ω on M, we may construct a Kähler form $\tilde{\omega}$ on $\tilde{M}$, by Theorem (4.2.2). In local coordinates $(z_1, ..., z_n)$ near any point $q \in \tilde{M}$, every positive C^∞ $(1,1)$-form on $\tilde{M}$ is locally quasi-isometric to the Euclidean form $\frac{\sqrt{-1}}{\pi}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$. Comparison to description (iii) of our modified Saper metric gives us

COROLLARY 7.2.4. (*Kähler Poincaré metrics for a single blow-up.*) *Let*

$$\omega_P = \omega_S + \tilde{\omega}$$

where $\tilde{\omega}$ *is the Kähler form of a metric on* $\tilde{M}$. *Then* ω_P *is a Kähler Poincaré metric on* $\tilde{M} - E$ *and* $\omega_P > \omega_S$. *If* $q \in E$ *and if* $(z_1, ..., z_n)$ *are normal blow-up coordinates on a neighbourhood* U *of* q *then*

$$\omega_P|_U \sim \frac{\sqrt{-1}}{\pi}\left(\frac{dz_1 \wedge d\bar{z}_1}{| z_1 |^2(\log | z_1 |^2)^2} + \sum_{i=2}^n dz_i \wedge d\bar{z}_i\right).$$

The corresponding metrics for successive blow-ups are *modified* Poincaré metrics.

(7.3) Proof of Proposition 7.2.1. To show that ω_S is a Kähler form, we must show that it is positive, hermitian, and d-closed. The form ν is hermitian and d-closed by Lemma (1.2.1), so we need only show that ω_S is positive. Since $\tilde{M}$ is compact, it is enough to show that for each point $q \in \tilde{M}$ there is a neighbourhood U of q on which $l\pi^*\omega + \nu$ is positive for $l \gg 0$. This is clear for $q \in \tilde{M} - E$, since in this case there is a neighbourhood U of q on which $\pi^*\omega$ is positive and ν is bounded.

For $q \in E$ we will write $\pi^*\omega$ and ν in normal blow-up coordinates near q and examine the local quasi-isometry class of $\pi^*\omega + \nu$. Let $(z_1, ..., z_n)$ be normal blow-up coordinates in a neighbourhood U of q, corresponding to coordinates $(Z_1, ..., Z_n)$ in a neighbourhood of $\pi(q) \in C$, as in §7.2. We write $z_f = (z_2, ..., z_k)$ (fibre coordinates of the map $E \to C$), $Z_f = (Z_2, ..., Z_k)$, $z_C = (z_{k+1}, ..., z_n)$, and $Z_C = (Z_{k+1}, ..., Z_n)$ (coordinates on C). Then π is given locally by the equations

$$Z_1 = z_1, \qquad Z_f = z_1 z_f, \qquad \text{and} \qquad Z_C = z_C.$$

We will also use the convention that the repeated index f is summed over the set $\{2, ..., k\}$ and the repeated index C over the set $\{k+1, ..., n\}$. In particular,

$$dz_f \wedge d\overline{z}_f = \sum_{i=2}^{k} dz_i \wedge d\overline{z}_i \text{ and } dz_C \wedge d\overline{z}_C = \sum_{i=k+1}^{n} dz_i \wedge d\overline{z}_i.$$

Consider the local expressions for ω and $\pi^*\omega$. Since ω is positive on M, ω is locally quasi-isometric to $\frac{\sqrt{-1}}{\pi} \sum_{i=1}^{n} dZ_i \wedge d\overline{Z}_i$. Then locally

(7.3.1)

$$\pi^*\omega \sim \frac{\sqrt{-1}}{\pi} \left(dz_1 \wedge d\overline{z}_1 + (z_f dz_1 + z_1 dz_f) \wedge (\overline{z}_f d\overline{z}_1 + \overline{z}_1 d\overline{z}_f) + dz_C \wedge d\overline{z}_C\right).$$

Keeping track only of the dominant terms, we write this information in the form of a chart:

$\pi^*\omega \sim$

$\frac{\sqrt{-1}}{\pi}\wedge$	$d\overline{z}_1$	$d\overline{z}_f$	$d\overline{z}_C$
dz_1	1	$\overline{z}_1 z_f$	0
dz_f	$z_1 \overline{z}_f$	$\mid z_1 \mid^2$	0
dz_C	0	0	1

where the 1 in the $dz_C \wedge d\overline{z}_C$ spot represents the identity matrix I_C and the $| z_1 |^2$ in the $dz_f \wedge d\overline{z}_f$ spot represents $| z_1 |^2 I_f$.

Next consider the Poincaré-type $(1,1)$-form $\nu = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\beta^2$ where $\beta = -\log || s ||^2$. The local expression for β is

$$\beta = -\log(| z_1 |^2 g)$$

where g is a bounded positive C^∞ function on U, giving the metric h on L locally. We decompose ν as $\nu = \mu + \eta$, where

$$\mu = -\frac{\sqrt{-1}}{\pi}\frac{\partial\overline{\partial}\beta}{\beta} \qquad \text{and} \qquad \eta = \frac{\sqrt{-1}}{\pi}\frac{\partial\beta \wedge \overline{\partial}\beta}{\beta^2}.$$

In the following lemma we describe the quasi-isometry class of $\pi^*\omega + \mu$ in local coordinates. Then we will use Proposition (5.2.4) to describe the quasi-isometry class of $\pi^*\omega + \nu = \pi^*\omega + \mu + \eta$.

LEMMA 7.3.2. *The restriction to U of the $(1,1)$-form $\pi^*\omega + \mu$ is positive and quasi-isometric to all of the following for z_1 close enough (but not equal) to 0:*

i. $\pi^*\omega + \frac{\sqrt{-1}}{\pi}\frac{1}{\beta}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$,

ii. $\frac{\sqrt{-1}}{\pi}\left(dz_1 \wedge d\bar{z}_1 + \frac{1}{\beta}dz_f \wedge d\bar{z}_f + dz_C \wedge d\bar{z}_C\right)$, *and*

iii. $\pi^*\omega + \frac{\sqrt{-1}}{\pi}\frac{1}{\beta}dz_f \wedge d\bar{z}_f$.

Proof. Recall from (5.1.3) that μ may be expressed in terms of a Chern form as

$$\mu = -\frac{2}{\beta}c_1(L,h).$$

By our choice of metric h on L, the Chern form $c_1(L,h)$ is negative on the fibres of the map $E \to C$. The form

$$\lambda\pi^*\omega + \beta\mu = \lambda\pi^*\omega - 2c_1(L,h)$$

is positive for large λ, by Proposition (4.2.1), and is thus locally quasi-isometric to the Euclidean $(1,1)$-form $\frac{\sqrt{-1}}{\pi}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$. Writing $\pi^*\omega + \mu$ as

$$\left(1 - \frac{\lambda}{\beta}\right)\pi^*\omega + \frac{1}{\beta}\left(\lambda\pi^*\omega + \beta\mu\right)$$

and observing that $\frac{1}{\beta} = -\frac{1}{\log(|z_1|^2 g)} \to 0$ as $z_1 \to 0$, we obtain quasi-isometry (i).

To prove (ii), we write $\pi^*\omega + \mu$ in local coordinates in chart format. Using part (i) and factoring $\frac{1}{\sqrt{\beta}}$ out of the second row and column, we obtain

$$\pi^*\omega + \mu \;\sim\; \begin{array}{c|ccc} \frac{\sqrt{-1}}{\pi}\wedge & d\bar{z}_1 & \frac{1}{\sqrt{\beta}}d\bar{z}_f & d\bar{z}_C \\ \hline dz_1 & 1+\frac{1}{\beta} & \bar{z}_1 z_f\sqrt{\beta} & 0 \\ \frac{1}{\sqrt{\beta}}dz_f & z_1\bar{z}_f\sqrt{\beta} & \mid z_1 \mid^2\beta+1 & 0 \\ dz_C & 0 & 0 & 1+\frac{1}{\beta} \end{array}$$

Noting that $z_1 \log \mid z_1 \mid^2 \to 0$ as $z_1 \to 0$, we see that

$$\pi^*\omega + \mu \;\sim\; \begin{array}{c|ccc} \frac{\sqrt{-1}}{\pi}\wedge & d\bar{z}_1 & \frac{1}{\sqrt{\beta}}d\bar{z}_f & d\bar{z}_C \\ \hline dz_1 & 1 & 0 & 0 \\ \frac{1}{\sqrt{\beta}}dz_f & 0 & 1 & 0 \\ dz_C & 0 & 0 & 1 \end{array}$$

i.e.

$$\pi^*\omega + \mu \;\sim\; \frac{\sqrt{-1}}{\pi}\left(dz_1 \wedge d\bar{z}_1 + \frac{1}{\beta}dz_f \wedge d\bar{z}_f + dz_C \wedge d\bar{z}_C\right)$$

for z_1 close enough (but not equal) to 0. The term in the fibre direction comes from the form $\mu = -\frac{2}{\beta}c_1(L,h)$. Part (iii) follows directly. □

We now calculate the local quasi-isometry class of $\pi^*\omega + \nu = \pi^*\omega + \mu + \eta$.

LEMMA 7.3.3. *Let $\tau : U \to \Delta$ be the map given by $\tau(z_1, ..., z_n) = z_1$ and let ω_{Δ^*} be the Poincaré form on the punctured disc. Then on U, for z_1 close enough (but not equal) to 0:*

$$\pi^*\omega + \nu \quad \sim \quad \pi^*\omega + \mu + \tau^*\omega_{\Delta^*}.$$

Proof. Let ω_E be the Euclidean $(1,1)$-form $\frac{\sqrt{-1}}{\pi}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$. From part (i) of the previous lemma, $\pi^*\omega+\mu \overset{\sim}{\geq} \frac{1}{\beta}\omega_E$. By Proposition (5.2.4), the $(1,1)$-form $\frac{1}{\beta}\omega_E + \eta$ is positive and quasi-isometric to $\frac{1}{\beta}\omega_E + \tau^*\omega_{\Delta^*}$ for z_1 close enough (but not equal) to 0. □

This completes the proof that $l\pi^*\omega + \nu$ is positive for sufficiently large l. To obtain quasi-isometry (i) of the Proposition, we use expression (iii) of Lemma (7.3.2) and note that $\beta \sim \mid \log \mid z_1 \mid^2 \mid$. Then we write out $\tau^*\omega_{\Delta^*}$ as

$$\tau^*\omega_{\Delta^*} \;=\; \frac{\sqrt{-1}}{\pi}\left(\frac{dz_1 \wedge d\bar{z}_1}{\mid z_1 \mid^2 (\log \mid z_1 \mid^2)^2}\right)$$

and add it to expressions (i) and (ii) of Lemma (7.3.2) to obtain the second and third quasi-isometries of the proposition.

§8. Modified Saper and Poincaré metrics for successive blow-ups and proof of Theorem I.

This section contains the proof of parts (i) and (ii) of our first main result, Theorem I of §2.1. Part (iii) of Theorem I is proved in §9.1 and part (iv) in §10.

(8.1) Introduction. Let M_0 be a compact Kähler manifold with Kähler form ω_0. Suppose that $\{\pi_j : M_j \to M_{j-1}\}$ is a finite sequence of blow-ups along smooth centres $C_j \subset M_{j-1}$, chosen so that C_j has normal crossings with the total exceptional divisor D_{j-1} of the composite $\pi_1 \circ \ldots \circ \pi_{j-1}$ of the first $j-1$ blow-ups. Let $L_j = [E_j]$ be the line bundle associated with the exceptional divisor $E_j = \pi_j^{-1}(C_j)$. In §4 we showed that there are hermitian metrics h_j on L_j and positive integers k_j such that the $(1,1)$-forms defined inductively on the compact manifolds M_j by the equations

$$\omega_j = k_j \pi_j^* \omega_{j-1} - c_1(L_j, h_j)$$

are Kähler forms. Metrics h_j on the line bundles L_j induce metrics $\hat{h}_j$ on the line bundles $\hat{L}_j = [D_j]$, as described in §3.9. In §4 we also constructed Kähler forms $\hat{\omega}_j$ on M_j, using Chern forms of the line bundles $\hat{L}_j$ instead of L_j.

Let $s_j : M_j \to L_j$ be a global holomorphic section of L_j such that $(s_j) = E_j$ and such that the norm $\| s_j \|$ of s_j under the metric h_j satisfies $\| s_j \| < 1$ everywhere on M_j. Let $\hat{s}_j : M_j \to \hat{L}_j$ be the induced holomorphic section of $\hat{L}_j$ and $|| \hat{s}_j ||$ the norm of $\hat{s}_j$ under the metric $\hat{h}_j$. We define a Poincaré-type $(1,1)$-form ν_j on $M_j - D_j$ by the equation

$$\nu_j = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\log || \hat{s}_j ||^2\right)^2 . \tag{8.1.1}$$

We will show in §8 that there are positive integers l_j such that the $(1,1)$-forms defined inductively on the noncompact manifolds $M_j - D_j$ by the equations

$$\omega_{S,j} = l_j \pi_j^* \omega_{S,j-1} + \nu_j$$

are positive and consequently Kähler. The Kähler metric on $M_j - D_j$ corresponding to the form $\omega_{S,j}$ is a complete modified Saper metric, distinguished with respect to the map $\pi_1 \circ ... \circ \pi_j : M_j \to M_0$. If the image of D_j in M_0 consists of isolated points in M_0, this modified Saper metric is precisely a Saper distinguished metric.

It follows from Proposition (5.5.1) that the $(1,1)$-form

$$\omega_{P,j} = \omega_{S,j} + \omega_j$$

is the Kähler form of a modified Poincaré metric on $M_j - D_j$. We may replace the Kähler form ω_j by $\hat{\omega}_j$ or any other Kähler form on M_j since all such forms are quasi-isometric.

The advantage of constructing our metrics inductively is that we can describe precisely how the metrics change with each successive blow-up. The disadvantage is that these metrics are difficult to describe totally in local coordinates, except in special cases. In §9 we will use a non-inductive method to construct Kähler homogeneous Saper metrics which can be described more precisely in local coordinates.

(8.2) Normal coordinates for successive blow-ups. In order to describe the local quasi-isometry classes of our metrics on $M_j - D_j$, it is useful to define normal blow-up coordinates on neighbourhoods of points in D_j.

Let q be a point in M_j at which k components $\tilde{E}_1, ..., \tilde{E}_k$ of D_j intersect. Each component $\tilde{E}_i$ is the strict transform $\tilde{E}_{j,\alpha_i}$ of some exceptional divisor E_{α_i}. We order the components so that $\alpha_1 < \alpha_2 < ... < \alpha_k$. Recall that local coordinates $(z_1, ..., z_n)$ are called normal for $\tilde{E}_1, ..., \tilde{E}_k$ if $\tilde{E}_i$ is given locally by the equation $z_i = 0$ for $1 \leq i \leq k$.

If $q \notin E_j$ then π_j is a biholomorphism on a neighbourhood of q. Let $(Z_1, ..., Z_n)$ be normal coordinates for $\pi_j(\tilde{E}_1), ..., \pi_j(\tilde{E}_k)$. We will say that local coordinates $(z_1, ..., z_n)$ near q are *normal blow-up coordinates for π_j corresponding to* $(Z_1, ..., Z_n)$ if π_j is given locally by the equations $Z_i = z_i$ for $1 \leq i \leq n$.

If $q \in E_j$ then $\tilde{E}_k = E_j$ and $\pi_j(\tilde{E}_k) = C_j$. The remaining components $\tilde{E}_i = \tilde{E}_{j,\alpha_i}$ map to the corresponding divisors $\tilde{E}_{j-1,\alpha_i}$ in M_{j-1}. Let $\hat{E}_i = \tilde{E}_{j-1,\alpha_i}$ for $1 \leq i \leq k-1$. Since C_j was chosen to have normal crossings with D_{j-1}, we may choose normal coordinates $(z_1, ..., z_n)$ for the divisors $\tilde{E}_1, ..., \tilde{E}_k$ on a neighbourhood U of q, and normal coordinates $(Z_1, ..., Z_n)$ for the divisors $\hat{E}_1, ..., \hat{E}_{k-1}$ on a neighbourhood V of $\pi_j(q)$, such that

i. C_j is given locally by the equations $Z_\gamma = 0$ for $\gamma \in \Gamma$, where Γ is a subset of $\{1, 2, ..., n\}$ containing k and

ii. π_j is given locally by the equations

$$\begin{aligned} Z_\gamma &= z_k z_\gamma && \text{if } \gamma \in \Gamma - \{k\} \\ Z_i &= z_i && \text{if } i = k \text{ or if } i \notin \Gamma. \end{aligned}$$

We will call the coordinates $(z_1, ..., z_n)$ *normal blow-up coordinates for π_j corresponding to* $(Z_1, ..., Z_n)$.

In the case $q \in E_j$, there may be additional divisors $\tilde{E}_{j-1,l}$ passing through the point $\pi_j(q)$ whose strict transforms $\tilde{E}_{j,l}$ in M_j do not pass through q. Such divisors will not be important for our local calculations near q. Local coordinates W which are normal for the set of all divisors $\tilde{E}_{j-1,l}$ passing through a point $p \in D_{j-1}$ do not necessarily correspond to any normal blow-up coordinates z in a neighbourhood of a given point $q \in \pi_j^{-1}(p)$.

The geometry of the map π_j near $q \in E_j$ is easily described in normal blow-up coordinates. Local coordinates for C_j are $Z_C = (Z_i)_{i \in \Gamma^c}$ where Γ^c is the complement of Γ in $\{1, 2, ..., n\}$. Fiber coordinates of the map $E_j \to C_j$ are $\{z_\gamma\}_{\gamma \in \Gamma - \{k\}}$. We use the convention that the repeated index f is summed over the set $\Gamma - \{k\}$ and the repeated index C over the set Γ^c. In particular,

$$dz_f \wedge d\bar{z}_f = \sum_{\gamma \in \Gamma - \{k\}} dz_\gamma \wedge d\bar{z}_\gamma \text{ and } dz_C \wedge d\bar{z}_C = \sum_{i \in \Gamma^c} dz_i \wedge d\bar{z}_i.$$

(8.3) Exceptional divisors and Poincaré-type forms in normal coordinates. Before stating our main results, we will describe the divisor D_j and the Poincaré-type form ν_j in normal blow-up coordinates, and compare them to the divisors and forms obtained from previous blow-ups.

For each space M_j and for $0 \leq i \leq j-1$, let $\pi_{j,i}$ be the composite map

$$\pi_{j,i} = \pi_{i+1} \circ \pi_{i+2} \circ \ldots \circ \pi_j : M_j \to M_i.$$

Let $D_{j,i} = \pi_{j,i}^* D_i$ be the total transform of D_i in M_j for $1 \leq i \leq j-1$ and let $D_{j,j} = D_j$. Let $\nu_{j,i} = \pi_{j,i}^* \nu_i$ be the pullback of ν_i to M_j for $1 \leq i \leq j-1$, and let $\nu_{j,j} = \nu_j$. The form $\nu_{j,i}$ is a Poincaré-type $(1,1)$-form associated with the divisor $D_{j,i}$.

Recall from §3.9 that metrics h_i on the line bundles $L_i = [E_i]$ and sections s_i of L_i satisfying $(s_i) = E_i$ induce metrics $\hat{h}_{j,i}$ on $\hat{L}_{j,i} = [D_{j,i}]$ and sections $\hat{s}_{j,i}$ of $\hat{L}_{j,i}$ satisfying $(\hat{s}_{j,i}) = D_{j,i}$. We also write $\hat{s}_j = \hat{s}_{j,j}$ so that $(\hat{s}_j) = D_j$. Let $\| \hat{s}_{j,i} \|$ be the norm of $\hat{s}_{j,i}$ under $\hat{h}_{j,i}$. By our choice of the sections s_i, we have $\| \hat{s}_{j,i} \| < 1$ on M_j. The Poincaré-type $(1,1)$-form $\nu_{j,i}$ may be written as

$$\nu_{j,i} = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\log \| \hat{s}_{j,i} \|^2)^2.$$

We now describe the divisors $D_{j,i}$ and the corresponding Poincaré-type $(1,1)$-forms $\nu_{j,i}$ near $q \in D_j$, using normal blow-up coordinates for π_j as described in §8.2. Recall that we may write $D_{j,i}$ in terms of the irreducible divisors $\tilde{E}_{j,l}$ as

$$D_{j,i} = \sum_{l=1}^{j} t_{il} \tilde{E}_{j,l}$$

where t_{il} is the multiplicity of $\tilde{E}_{j,l}$ in $D_{j,i}$. The properties of the matrix $T = (t_{il}) = T_j(D, \tilde{E})$ were described in §3.7. Let $\Lambda = (\lambda_{il})$ be the $j \times k$ submatrix of T corresponding to the irreducible divisors $\tilde{E}_1, \ldots, \tilde{E}_k$ passing through q. The integer λ_{il} is the multiplicity of $\tilde{E}_l$ in $D_{j,i}$. Since T is nonsingular, by Proposition (3.7.4iv), the matrix Λ has rank k. Let $\Lambda_i = (\lambda_{i1}, \ldots, \lambda_{ik})$ be the ith row of Λ and let

$$z^{\Lambda_i} = z_1^{\lambda_{i1}} z_2^{\lambda_{i2}} \ldots z_k^{\lambda_{ik}}.$$

The divisor $D_{j,i}$ is given locally by the equation $z^{\Lambda_i} = 0$ and the $(1,1)$-form $\nu_{j,i}$ by

$$\nu_{j,i} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\log(\mid z^{\Lambda_i}\mid^2 g_i)\right)^2$$

for some bounded positive C^∞ function g_i.

Next we compare the divisors $D_{j,i}$ near q to the divisors $D_{j-1,i}$ near $\pi_j(q)$ for $1 \leq i \leq j-1$. If $q \notin E_j$ then π_j is a biholomorphism on a neighbourhood of q. Suppose that $q \in E_j$ and let $\hat{\Lambda} = (\hat{\lambda}_{il})$ be the $(j-1)\times(k-1)$ submatrix of the matrix $T_{j-1}(D,\tilde{E})$ corresponding to the divisors $\hat{E}_1, \hat{E}_2, ..., \hat{E}_{k-1}$. The strict transform of $\hat{E}_l$ is $\tilde{E}_l$ and the multiplicity of $\hat{E}_l$ in $D_{j-1,i}$ is $\hat{\lambda}_{il}$. By Lemma (3.7.3), $\hat{\Lambda}$ consists of the first $j-1$ rows and $k-1$ columns of Λ, i.e. $\hat{\lambda}_{il} = \lambda_{il}$ for $1 \leq i \leq j-1$ and $1 \leq l \leq k-1$. Let $Z^{\hat{\Lambda}_i} = Z_1^{\lambda_{i1}} Z_2^{\lambda_{i2}} ... Z_{k-1}^{\lambda_{i,k-1}}$. The $(1,1)$-form $\nu_{j-1,i}$ is given locally by

$$\nu_{j-1,i} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\log(\mid Z^{\hat{\Lambda}_i}\mid^2 G_i)\right)^2$$

for some bounded nonnegative C^∞ function G_i. The function G_i vanishes on those components $\tilde{E}_{j-1,l}$ of $D_{j-1,i}$ whose strict transforms $\tilde{E}_{j,l}$ do not pass through q. Locally near q the pullback $\pi_j^* G_i$ is of the form $\mid z_k \mid^{2\lambda} g_i$ for some nonnegative integer λ.

We are particularly interested in the asymptotic behaviour of the forms $\nu_{j,j} = \nu_j$ and $\nu_{j,j-1} = \pi_j^*\nu_{j-1}$ near D_j. We may write ν_j, ν_{j-1}, and $\pi_j^*\nu_{j-1}$ as $\nu_j = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\beta^2)$, $\nu_{j-1} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(B^2)$, and $\pi_j^*\nu_{j-1} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\beta')^2$ where $\beta = -\log\mid\mid \hat{s}_j \mid\mid^2$ and $B = -\log\mid\mid \hat{s}_{j-1}\mid\mid^2$ are given in local coordinates by

$$\beta = -\log(\mid z^{\Lambda_j}\mid^2 g_j) \qquad \text{and} \qquad B = -\log(\mid Z^{\hat{\Lambda}_{j-1}}\mid^2 G_{j-1})$$

and $\beta' = \pi_j^* B = \pi_j^*(-\log\mid\mid \hat{s}_{j-1}\mid\mid^2) = -\log\mid\mid \hat{s}_{j,j-1}\mid\mid^2$ by

$$\beta' = -\log(\mid z^{\Lambda_{j-1}}\mid^2 g_{j-1}).$$

Using properties of the matrices Λ and $\hat{\Lambda}$ derived from Lemmas (3.7.3) and (3.7.4) we obtain the following local descriptions of β and β'.

LEMMA 8.3.1. *If q is a point in D_j and $(z_1, ..., z_n)$ are normal coordinates for the irreducible components $\tilde{E}_1, ..., \tilde{E}_k$ of D_j which pass through q, then the rate of growth of the function $\beta = -\log || \hat{s}_j ||^2$ near q is given by*

i. $\beta \sim -\log | z_1 z_2 ... z_k |^2$.

If, in addition, $q \in E_j$, then the rate of growth of

$$\beta' = \pi_j^*(-\log || \hat{s}_{j-1} ||^2)$$

is given by

$$\text{ii.} \quad \beta' \sim \begin{cases} -\log | z_1 z_2 ... z_k |^2 & \text{if } C_j \subset D_{j-1}. \\ -\log | z_1 z_2 ... z_{k-1} |^2 & \text{if } C_j \not\subset D_{j-1} \text{ and } k \geq 2. \\ 1 & \text{if } C_j \not\subset D_{j-1} \text{ and } k = 1. \end{cases}$$

Proof. Recall that the entries of Λ_j are all positive, i.e. the multiplicity $\lambda_{j,i}$ of $\tilde{E}_i$ in D_j is positive for all i, by Proposition (3.7.4 i). This gives us quasi-isometry (i).

Now assume that $q \in E_j$. The first $k-1$ entries of Λ_j, $\hat{\Lambda}_{j-1}$, and Λ_{j-1} are identical (Lemma (3.7.3 iii)) since these entries are the multiplicities of the strict transforms of $E_{\alpha_1}, E_{\alpha_2}, ..., E_{\alpha_{k-1}}$ in the divisors D_j, D_{j-1}, and $D_{j,j-1} = \pi_j^* D_{j-1}$. The kth entries of Λ_j and Λ_{j-1} are related by the equation $\lambda_{j,k} = \lambda_{j-1,k} + 1$ (Lemma (3.7.3 iv)) since $D_j = D_{j,j-1} + E_j$. The entry $\lambda_{j-1,k}$ is positive if and only if $C_j \subset D_{j-1}$ by Proposition (3.7.4 v). This gives us quasi-isometry (ii). □

In §5 we compared Poincaré-type forms ν to pullbacks of the Poincaré form ω_{Δ^*} by suitable monomial maps. For each i such that Λ_i has at least one positive entry, let $\tau_i : U \to \Delta$ be the monomial map given by

$$\tau_i(z_1, z_2, ..., z_n) = z^{\Lambda_i}.$$

If $\Lambda_i = 0$, let τ_i be a constant map to a point in Δ^* so that $\tau_i^* \omega_{\Delta^*} = 0$.

(8.4) Main results. We can now generalize the results of §7 to successive blow-ups. Our first theorem describes how complete Kähler modified Saper metrics may be constructed inductively on the noncompact spaces $M_j - D_j$ by adding a Poincaré-type form on $M_j - D_j$ to a large enough multiple of the pullback of the Kähler form of the previous metric. The proof of the following theorem is to be found in §§8.5 and 8.6.

THEOREM 8.4.1. (*Complete Kähler modified Saper metrics for successive blow-ups.*) *There exist positive integers l_j such that the $(1,1)$-forms defined inductively on the manifolds $M_j - D_j$ by the equations*

$$\omega_{S,0} = \omega_0 \text{ and } \omega_{S,j} = l_j \pi_j^* \omega_{S,j-1} + \nu_j \qquad \text{for } j \geq 1$$

are Kähler forms. The Kähler metric on $M_j - D_j$ associated with $\omega_{S,j}$ is a complete modified Saper metric which is distinguished with respect to the map $\pi_{j,0} = \pi_1 \circ \pi_2 \circ \ldots \circ \pi_j$.

The quasi-isometry class of $\omega_{S,j}$ may be described as follows:
If $q \notin E_j$, then on a neighbourhood U of q, $\omega_{S,j} \sim \pi_j^* \omega_{S,j-1}$.
If $q \in E_j$, if k is the number of components of D_j intersecting at q, and if $(z_1, \ldots, z_n)$ are normal blow-up coordinates for π_j on a neighbourhood U of q, then locally

$$\text{(i)} \quad \omega_{S,j} \sim \pi_j^* \omega_{S,j-1} + \frac{\sqrt{-1}}{\pi} \left(\frac{1}{|\log | z_1 z_2 \ldots z_k |^2 |} dz_f \wedge d\overline{z}_f \right) + \tau_j^* \omega_{\Delta^*}, \text{ and}$$

$$\text{(ii)} \quad \omega_{S,j} \overset{\sim}{\geq} \pi_{j,0}^* \omega_0 + \frac{\sqrt{-1}}{\pi} \left(\frac{1}{(\log | z_1 z_2 \ldots z_k |^2)^2} \sum_{i=1}^{k} \frac{dz_i \wedge d\overline{z}_i}{| z_i |^2} + \frac{1}{|\log | z_1 z_2 \ldots z_k |^2 |} \sum_{i=1}^{n} dz_i \wedge d\overline{z}_i \right).$$

Note that completeness follows directly from quasi-isometry (i) by Proposition (6.1.1), since the monomial map τ_j is given by

$$\tau_j(z_1, \ldots, z_n) = z^{\Lambda_j} = z_1^{\lambda_{j1}} z_2^{\lambda_{j2}} \ldots z_k^{\lambda_{jk}}$$

and all the integers $\lambda_{j1}, \ldots, \lambda_{jk}$ are positive.

Quasi-isometry (i) means that locally near the exceptional divisor E_j of π_j, the jth modified Saper metric looks like the pullback of the $(j-1)$st metric plus a small term in the fibre directions plus a term with Poincaré-type growth in the variables $z_1, \ldots, z_k$. Inequality (ii) means that the modified Saper metric on $M_j - D_j$ is locally bounded below by the sum of the incomplete metric induced from M_0 and a Saper distinguished metric on $M_j - D_j$.

We noted in Proposition (5.5.1) that the sum of a modified Saper metric on $M_j - D_j$ and the restriction to M_j of a Kähler metric on M_j is a modified Poincaré metric. The following theorem also describes the relationship between our modified Poincaré metrics on $M_j - D_j$ and $M_{j-1} - D_{j-1}$. The proof is in §8.7.

THEOREM 8.4.2. (*Complete Kähler modified Poincaré metrics for successive blow-ups.*) *The Kähler forms*

$$\omega_{P,j} = \omega_{S,j} + \omega_j$$

determine complete modified Poincaré metrics on the manifolds $M_j - D_j$.

The metric associated with $\omega_{P,1}$ *is a true Poincaré metric. For* $j \geq 2$ *the quasi-isometry class of* $\omega_{P,j}$ *may be described as follows:*
If $q \notin E_j$, *then on a neighbourhood* U *of* q, $\omega_{P,j} \sim \pi_j^* \omega_{P,j-1}$.
If $q \in E_j$ *and if* $(z_1, ..., z_n)$ *are normal blow-up coordinates for* π_j *on a neighbourhood* U *of* q, *then locally*

$$\text{(i)} \qquad \omega_{P,j} \sim \pi_j^* \omega_{P,j-1} + \frac{\sqrt{-1}}{\pi} dz_f \wedge d\overline{z}_f + \tau_j^* \omega_{\Delta^*}, \quad \textit{and}$$

$$\text{(ii)} \qquad \omega_{P,j} \sim \sum_{i=1}^{j} \tau_i^* \omega_{\Delta^*} + \frac{\sqrt{-1}}{\pi} \sum_{i=1}^{n} dz_i \wedge d\overline{z}_i.$$

Notice that the modified Saper and Poincaré metrics differ only in the order of growth of the fiber terms $dz_f \wedge d\overline{z}_f$. These terms correspond to multiples of a Chern form of the line bundle $\hat{L}_j = [D_j]$. Quasi-isometry (ii) simply states that $\omega_{P,j}$ is a modified Poincaré metric in the sense of (1.6.1).

Let ω_{Sap} and ω_{Poinc} be the fundamental forms, respectively, of any Saper distinguished metric and any Poincaré metric on $M_j - D_j$, neither necessarily Kähler. Using the local descriptions of $\omega_{S,j}$ and $\omega_{P,j}$, we obtain

THEOREM 8.4.3. (*Global bounds for modified Saper and Poincaré metrics.*) *The modified Saper metric on* $M_j - D_j$ *associated with*

$\omega_{S,j}$ is bounded below by the sum of the metric induced from M_0 and a Saper distinguished metric, and is bounded above by the modified Poincaré metric associated with $\omega_{P,j}$. This modified Poincaré metric is bounded above by a true Poincaré metric. More concisely:

$$\pi_{j,0}^*\omega_0 + \omega_{\mathrm{Sap}} \overset{\sim}{\leq} \omega_{S,j} < \omega_{P,j} \overset{\sim}{\leq} \omega_{\mathrm{Poinc}}.$$

Proof. The first inequality follows directly from part (ii) of Theorem (8.4.1) and the second from the definition of $\omega_{P,j}$. The local quasi-isometry class of the form ω_{Poinc} is given by

$$\omega_{\mathrm{Poinc}} \sim \frac{\sqrt{-1}}{\pi}\left(\sum_{i=1}^{k} \frac{dz_i \wedge d\overline{z}_i}{|\, z_i\,|^2(\log|\, z_i\,|^2)^2} + \sum_{i=1}^{n} dz_i \wedge d\overline{z}_i\right),$$

which dominates $\omega_{P,j}$, by Lemma (5.3.6). □

When more is known about the relationships between the centres C_i and the exceptional divisors D_{i-1}, we can be more precise in our descriptions of the metrics. Let $\omega_{P,\mathrm{hom}}$ be the fundamental form of a modified Poincaré metric on $M_j - D_j$ which is homogeneous in the sense of (1.6.2). The following theorem is proved in §8.8. We allow for the possibility that D_j may have several connected components, but we need only prove the theorem for the case of a single connected component, since each blow-up map π_i is a biholomorphism away from its own exceptional divisor E_i.

THEOREM 8.4.4. (*Metrics for the case $C_i \subset D_{i-1}$.*) *Suppose that for $2 \leq i \leq j$, the centre C_i for the ith blow-up is either contained in the total exceptional divisor D_{i-1} of the first $i-1$ blow-ups or is disjoint from D_{i-1}. Then the Kähler metric associated with $\omega_{S,j}$ is a homogeneous Saper metric in the sense of* (1.8.4) *and the Kähler metric associated with $\omega_{P,j}$ is a homogeneous Poincaré metric. More concisely:*

$$\omega_{S,j} \sim \pi_{j,0}^*\omega_0 + \omega_{\mathrm{Sap}} \quad \textit{and} \quad \omega_{P,j} \sim \omega_{P,\mathrm{hom}}.$$

If, in addition, $\dim C_1 = 0$ and $\dim C_i = 0$ for each i such that C_i is disjoint from D_{i-1} (i.e. the image of D_j in M_0 consists of *isolated points in* M_0*), then the modified Saper metric is exactly a Saper distinguished metric, i.e.* $\omega_{S,j} \sim \omega_{\mathrm{Sap}}$.

Outline of proofs. We proved Theorems (8.4.1) - (8.4.4) for a single blow-up in §7. In that case, $\omega_{S,1} \sim \pi_1^*\omega_0 + \omega_{\mathrm{Sap}}$ and $\omega_{P,1} \sim \omega_{\mathrm{Poinc}} \sim \omega_{P,\mathrm{hom}}$.

Now fix $m \geq 2$ and assume that Theorems (8.4.1) - (8.4.4) are true for $1 \leq j \leq m-1$. For the rest of §8 we will let $\pi = \pi_m$. Consider the forms

$$\omega_{S,m} = l_m \pi^* \omega_{S,m-1} + \nu_m \quad \text{and} \quad \omega_{P,m} = \omega_{S,m} + \omega_m.$$

The form ν_m is hermitian and d-closed so to show that $\omega_{S,m}$ and $\omega_{P,m}$ are Kähler, we need only show that $\omega_{S,m}$ is positive on $M_m - D_m$.

The $(1,1)$-form $\nu = \nu_m$ may be written as $\nu = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\beta^2)$ where $\beta = -\log \| \hat{s}_m \|^2$ and $\| \hat{s}_m \|$ is the norm of the section $\hat{s}_m : M_m \to \hat{L}_m$ described in §8.1. As in §5.1 we decompose ν as $\nu = \mu + \eta$ where

(8.4.5)
$$\mu = -\frac{\sqrt{-1}}{\pi}\frac{\partial\overline{\partial}\beta}{\beta} = -\frac{2}{\beta}c_1([D_m], \hat{h}_m) \text{ and } \eta = \frac{\sqrt{-1}}{\pi}\frac{\partial\beta \wedge \overline{\partial}\beta}{\beta^2}.$$

By Lemma (8.3.1), the rate of growth of β near $q \in D_m$ is given locally in normal coordinates by $\beta \sim -\log | z_1 z_2 ... z_k |^2$.

Application of our inductive assumption of Theorem (8.4.1) for $j = m-1$ is complicated by having to deal with two sets of local coordinates near $\pi(q)$ in M_{m-1}: coordinates W which are normal for the collection of *all* irreducible components $\hat{E}_1, ..., \hat{E}_K$ of D_{m-1} passing through $\pi(q)$, and coordinates Z, as described in §8.2, which are normal for C_m and those irreducible components $\hat{E}_i$ of D_{m-1} whose strict transforms in M_m pass through q. Our local coordinates z on M_m are normal blow-up coordinates corresponding to Z. To use our inductive assumption, written in W coordinates, we first convert to Z or z coordinates.

To simplify notation we will let

(8.4.6)
$$\phi = \frac{\sqrt{-1}}{\pi}\sum_{i=1}^{k}\frac{dz_i \wedge d\overline{z}_i}{| z_i |^2} \text{ and } \Phi = \frac{\sqrt{-1}}{\pi}\sum_{i=1}^{K}\frac{dW_i \wedge d\overline{W}_i}{| W_i |^2}.$$

Throughout §8 we will use freely the quasi-isometries

(8.4.7)
$$\omega_m \sim \frac{\sqrt{-1}}{\pi}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i \text{ and } \omega_{m-1} \sim \frac{\sqrt{-1}}{\pi}\sum_{i=1}^{n} dW_i \wedge d\overline{W}_i,$$

where ω_m and ω_{m-1} may be replaced by the positive $(1,1)$-forms corresponding to any hermitian metrics on M_m and M_{m-1} respectively.

It is convenient to write our inductive assumption in terms of the function $B = -\log \| \hat{s}_{m-1} \|^2$. By Lemma (8.3.1) with $j = m-1$, we have

$$B \sim -\log | W_1 W_2 \ldots W_K |^2$$

near $\pi(q)$. Our inductive assumption of (8.4.1ii) for $j = m-1$ may be written as

$$\omega_{S,m-1} \overset{\sim}{\geq} \pi^*_{m-1,0}\omega_0 + \frac{1}{B^2}\Phi + \frac{1}{B}\omega_{m-1} \tag{8.4.8}$$

from which it follows that $\omega_{S,m-1} \overset{\sim}{\geq} \frac{1}{B}\omega_{m-1}$ and

$$\pi^*\omega_{S,m-1} \overset{\sim}{\geq} \frac{1}{\beta'}\pi^*\omega_{m-1} \tag{8.4.9}$$

where $\beta' = \pi^* B$. Lemma (8.3.1) gives us a description in z coordinates of the rate of growth of β' near q.

The proof of positivity and the calculations of the quasi-isometry classes of Theorem (8.4.1) are done for $q \notin E_m$ in §8.5 and for $q \in E_m$ in §8.6 . We finish the proof of Theorem (8.4.2) in §8.7 and Theorem (8.4.4) in §8.8. □

(8.5) Quasi-isometry class of $\omega_{S,m}$ near $q \notin E_m$.

Proof of Theorem (8.4.1) *for* $q \notin E_m$. If q is not in the exceptional divisor $E_m = \pi^{-1}(C_m)$ it is easy to show that the form $\omega_{S,m} = l\pi^*\omega_{S,m-1} + \nu$ is locally quasi-isometric to $\pi^*\omega_{S,m-1}$ for $l \gg 0$. We need only show that $l\pi^*\omega_{S,m-1}$ dominates $\nu = \mu + \eta$ locally. If $q \notin D_m$ then $\pi^*\omega_{S,m-1}$ is positive and ν is bounded on a neighbourhood of q and we are done.

Suppose that $q \in D_m$ but $q \notin E_m$. The map π is a biholomorphism near q so $\pi^*\omega_{S,m-1}$ has the same description in local coordinates as $\omega_{S,m-1}$. By our inductive assumption that $\omega_{S,m-1}$ satisfies quasi-isometry (ii) of Theorem (8.4.1), we have

$$\pi^*\omega_{S,m-1} \overset{\sim}{\geq} \frac{1}{\beta^2}\phi + \frac{1}{\beta}\omega_m$$

where ϕ is as in equation (8.4.6). Recall that $\mu = -\frac{2}{\beta}c_1([D_m], \hat{h}_m)$. Then μ is locally dominated by $l\frac{1}{\beta}\omega_m$ and hence by $l\pi^*\omega_{S,m-1}$ for l large enough. To see that η is also dominated by $l\pi^*\omega_{S,m-1}$, we use Proposition (5.2.4), which tells us that locally $\frac{1}{\beta}\omega_m + \eta \sim \frac{1}{\beta}\omega_m + \tau_m^*\omega_{\Delta^*}$ where $\tau_m(z_1, ..., z_n) = z^{\Lambda_m}$. The form $\phi_m = \beta^2\tau_m^*\omega_{\Delta^*}$ is positive semi-definite, and by Lemma (5.3.2), $\phi \overset{\sim}{\geq} \phi_m$. Then $\frac{1}{\beta^2}\phi \overset{\sim}{\geq} \tau_m^*\omega_{\Delta^*}$ and $\pi^*\omega_{S,m-1} \overset{\sim}{\geq} \tau_m^*\omega_{\Delta^*} + \frac{1}{\beta}\omega_m \sim \eta + \frac{1}{\beta}\omega_m \overset{\sim}{\geq} \eta$. □

(8.6) Quasi-isometry class of $\omega_{S,m}$ near $q \in E_m$.

Proof of Theorem (8.4.1) *for* $q \in E_m$. Recall that

$$\omega_{S,m} = l\pi^*\omega_{S,m-1} + \nu,$$

where $\nu = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\beta^2)$ has the decomposition $\nu = \mu + \eta$ described in (8.4.5) and $\beta \sim -\log |\, z_1 z_2 ... z_k \,|^2$ near q. Our inductive assumption gives us estimate (8.4.9): $\pi^*\omega_{S,m-1} \overset{\sim}{\geq} \frac{1}{\beta'}\pi^*\omega_{m-1}$. By Lemma (8.3.1), $\beta' \overset{\sim}{\leq} \beta$. Hence

$$\pi^*\omega_{S,m-1} \overset{\sim}{\geq} \frac{1}{\beta}\pi^*\omega_{m-1}. \tag{8.6.1}$$

We will first show that $l\frac{1}{\beta}\pi^*\omega_{m-1}+\mu$ and $l\frac{1}{\beta}\pi^*\omega_{m-1}+\nu$ are positive on $M_m - D_m$ for $l \gg 0$. We will then calculate the quasi-isometry class of $l\frac{1}{\beta'}\pi^*\omega_{m-1} + \mu$ in local coordinates to obtain part (i) of Theorem (8.4.1). To prove part (ii), we estimate the size of the terms in $\omega_{S,m}$ which have Poincaré-type growth, using properties of the multiplicity matrix $T_m(D, \tilde{E})$ from §3 and results on Poincaré-type forms from §5.

LEMMA 8.6.2. *For $l \gg 0$ the $(1,1)$-form $l\frac{1}{\beta}\pi^*\omega_{m-1} + \mu$ is positive on $M_m - D_m$ and*

$$l\frac{1}{\beta}\pi^*\omega_{m-1} + \mu \sim \frac{1}{\beta}\omega_m.$$

Proof. From (8.4.5) we have $\mu = -\frac{2}{\beta}c_1([D_m], \hat{h}_m)$. Recall from Corollary (4.3.2) that there is a Kähler form $\hat{\omega}_{m-1}$ on M_{m-1} such that

$$l\pi^*\hat{\omega}_{m-1} - c_1([D_m], \hat{h}_m) > 0$$

on M_m for $l \gg 0$. Then $l\pi^*\hat{\omega}_{m-1} + \beta\mu > 0$ for $l \gg 0$. Since all metrics on a compact manifold are quasi-isometric, we have

$$l\frac{1}{\beta}\pi^*\omega_{m-1} + \mu \sim \frac{1}{\beta}\omega_m$$

for the positive $(1,1)$-forms ω_{m-1} and ω_m of any hermitian metrics on M_{m-1} and M_m respectively. □

Since $\beta' \overset{\sim}{\leq} \beta$ we have

COROLLARY 8.6.3. *For $l \gg 0$*

$$l\frac{1}{\beta'}\pi^*\omega_{m-1} + \mu \sim \frac{1}{\beta'}\pi^*\omega_{m-1} + \frac{1}{\beta}\omega_m.$$

The next lemma is a consequence of Proposition (5.2.4).

LEMMA 8.6.4. *For $z_1, ..., z_k$ close enough (but not equal) to 0, the $(1,1)$-form $\frac{1}{\beta}\omega_m + \eta$ is positive and*

$$\frac{1}{\beta}\omega_m + \eta \sim \frac{1}{\beta}\omega_m + \tau_m^*\omega_{\Delta^*}.$$

Combining Lemmas (8.6.2) and (8.6.4) we obtain the following quasi-isometry which, together with inequality (8.6.1), implies that $\omega_{S,m}$ is positive.

COROLLARY 8.6.5. *For $z_1, ..., z_k$ close enough (but not equal) to 0 and $l \gg 0$*

$$l\frac{1}{\beta}\pi^*\omega_{m-1} + \nu \sim \frac{1}{\beta}\omega_m + \tau_m^*\omega_{\Delta^*}.$$

Using the inequality $\beta' \lesssim \beta$ again gives us a variation of Corollary (8.6.5) which is used in the conclusion of the proof of Theorem (8.4.1i) below.

COROLLARY 8.6.6. *For $z_1, \ldots, z_k$ close enough (but not equal) to 0 and $l \gg 0$*

$$l\frac{1}{\beta'}\pi^*\omega_{m-1} + \nu \sim \frac{1}{\beta'}\pi^*\omega_{m-1} + \frac{1}{\beta}\omega_m + \tau_m^*\omega_{\Delta^*}.$$

Next we describe the $(1,1)$-form of Corollary (8.6.3) in local coordinates. We show that $\frac{1}{\beta'}\pi^*\omega_{m-1}$ dominates all terms in $\frac{1}{\beta}\omega_m$ except possibly those in the direction of fibres of the map $E_m \to C_m$.

LEMMA 8.6.7. *For $z_1, \ldots, z_k$ close enough (but not equal) to 0*

$$\text{(i)} \qquad \frac{1}{\beta'}\pi^*\omega_{m-1} + \frac{1}{\beta}\omega_m \sim \frac{\sqrt{-1}}{\pi}\left(\frac{1}{\beta'}(dz_k \wedge d\bar{z}_k + dz_C \wedge d\bar{z}_C) + \frac{1}{\beta}dz_f \wedge d\bar{z}_f\right)$$

$$\text{(ii)} \qquad \sim \frac{1}{\beta'}\pi^*\omega_{m-1} + \frac{\sqrt{-1}}{\pi}\frac{1}{\beta}dz_f \wedge d\bar{z}_f.$$

Proof. We write out $\pi^*\omega_{m-1}$ in local coordinates, using the notation of §8.2:

(8.6.8)

$$\begin{aligned}\pi^*\omega_{m-1} &\sim \pi^*\left(\frac{\sqrt{-1}}{\pi}\sum_{i=1}^{n} dZ_i \wedge d\bar{Z}_i\right)\\ &= \frac{\sqrt{-1}}{\pi}(dz_k \wedge d\bar{z}_k + dz_C \wedge d\bar{z}_C\\ &\qquad + (z_f dz_k + z_k dz_f) \wedge (\bar{z}_f d\bar{z}_k + \bar{z}_k d\bar{z}_f))\end{aligned}$$

where the repeated index f indicates summation over the set $\Gamma - \{k\}$. Adding $\frac{1}{\beta}\omega_m$ and writing the forms in chart format we have:

$$\frac{1}{\beta'}\pi^*\omega_{m-1} + \frac{1}{\beta}\omega_m$$

$$\sim\quad \begin{array}{c|ccc} \frac{\sqrt{-1}}{\pi}\wedge & d\overline{z}_k & d\overline{z}_f & d\overline{z}_C \\ \hline dz_k & \frac{1}{\beta'} + \frac{1}{\beta} & \overline{z}_k z_f \frac{1}{\beta'} & 0 \\ dz_f & z_k \overline{z}_f \frac{1}{\beta'} & \mid z_k \mid^2 \frac{1}{\beta'} + \frac{1}{\beta} & 0 \\ dz_C & 0 & 0 & \frac{1}{\beta'} + \frac{1}{\beta} \end{array}$$

$$\sim\quad \begin{array}{c|ccc} \frac{\sqrt{-1}}{\pi}\wedge & \frac{1}{\sqrt{\beta'}}d\overline{z}_k & \frac{1}{\sqrt{\beta}}d\overline{z}_f & \frac{1}{\sqrt{\beta'}}d\overline{z}_C \\ \hline \frac{1}{\sqrt{\beta'}}dz_k & 1 + \frac{\beta'}{\beta} & \overline{z}_k z_f \sqrt{\frac{\beta}{\beta'}} & 0 \\ \frac{1}{\sqrt{\beta}}dz_f & z_k \overline{z}_f \sqrt{\frac{\beta}{\beta'}} & \mid z_k \mid^2 \frac{\beta}{\beta'} + 1 & 0 \\ \frac{1}{\sqrt{\beta'}}dz_C & 0 & 0 & 1 + \frac{\beta'}{\beta} \end{array}$$

From Lemma (8.3.1), $\beta \sim \beta'$ if $C_m \subset D_{m-1}$ and $1 \lesssim \frac{\beta}{\beta'} \lesssim -\log \mid z_k \mid^2$ for $z_1, \ldots, z_k$ near 0 if $C_m \not\subset D_{m-1}$. In both cases, $z_k \frac{\beta}{\beta'} \to 0$ and $\frac{\beta'}{\beta}$ is bounded as $z_k \to 0$. This proves quasi-isometry (i). Quasi-isometry (ii) follows because we have showed that the form $\frac{1}{\beta'}\pi^*\omega_{m-1}$ dominates all terms in $\frac{1}{\beta}\omega_m$ except $\frac{\sqrt{-1}}{\pi}\frac{1}{\beta}dz_f \wedge d\overline{z}_f$. □

Conclusion of the proof of Theorem (8.4.1 i). From Corollary (8.6.6) and Lemma (8.6.7) we obtain the quasi-isometry

$$l\frac{1}{\beta'}\pi^*\omega_{m-1} + \nu \sim \frac{1}{\beta'}\pi^*\omega_{m-1} + \frac{\sqrt{-1}}{\pi}\frac{1}{\beta}dz_f \wedge d\overline{z}_f + \tau_m^*\omega_{\Delta^*}$$

for $l \gg 0$ and $z_1, \ldots, z_k$ close enough (but not equal) to 0. Applying inequality (8.4.9), which says that $\pi^*\omega_{S,m-1} \gtrsim \frac{1}{\beta'}\pi^*\omega_{m-1}$, we obtain quasi-isometry (i) of Theorem (8.4.1):

$$\omega_{S,m} = l\pi^*\omega_{S,m-1} + \nu \sim \pi^*\omega_{S,m-1} + \frac{\sqrt{-1}}{\pi}\frac{1}{\beta}dz_f \wedge d\overline{z}_f + \tau_m^*\omega_{\Delta^*}.$$

□

Conclusion of the proof of Theorem (8.4.1 ii). We start with the pullback of the forms of our inductive assumption (8.4.8):

$$\pi^*\omega_{S,m-1} \mathrel{\tilde{\geq}} \pi_{m,0}^*\omega_0 + \pi^*\left(\frac{1}{B^2}\Phi + \frac{1}{B}\omega_{m-1}\right)$$

where Φ is as in (8.4.6). Notice that if $\omega_{\mathrm{Sap},m-1}$ is the fundamental form of a distinguished Saper metric on $M_{m-1} - D_{m-1}$ then

$$\omega_{\mathrm{Sap},m-1} \sim \frac{1}{B^2}\Phi + \frac{1}{B}\omega_{m-1}$$

near $\pi(q)$. Applying Corollary (8.6.5) and the inequality $\pi^*B = \beta' \mathrel{\tilde{\leq}} \beta$ gives

$$l\pi^*\omega_{S,m-1} + \nu \mathrel{\tilde{\geq}} \pi_{m,0}^*\omega_0 + \pi^*\omega_{\mathrm{Sap},m-1} + \frac{1}{\beta}\omega_m + \tau_m^*\omega_{\Delta^*}.$$

Part (ii) of Theorem (8.4.1) can be written as

$$l\pi^*\omega_{S,m-1} + \nu \mathrel{\tilde{\geq}} \pi_{m,0}^*\omega_0 + \frac{1}{\beta^2}\phi + \frac{1}{\beta}\omega_m$$

with ϕ as in (8.4.6), and follows from the next lemma.

LEMMA 8.6.9. *For $z_1, ..., z_k$ close enough (but not equal to) 0,*

$$\pi^*\omega_{\mathrm{Sap},m-1} + \frac{1}{\beta}\omega_m + \tau_m^*\omega_{\Delta^*} \mathrel{\tilde{\geq}} \frac{1}{\beta^2}\phi + \frac{1}{\beta}\omega_m.$$

Proof. The quasi-isometry class of $\omega_{\mathrm{Sap},m-1}$ near $\pi(q)$ was described in terms of local coordinates W which are normal for *all* irreducible components $\hat{E}_1, ..., \hat{E}_K$ of D_{m-1} passing through $\pi(q)$. We will first rewrite some of the terms of $\omega_{\mathrm{Sap},m-1}$ in the Z coordinates of §8.2, which are normal for C_m and those components

$\hat{E}_1, ..., \hat{E}_{k-1}$ of D_{m-1} whose strict transforms in M_m pass through q. We will then use our §3 description of the relationship between the multiplicities of the components of D_{m-1} and D_m and our §5 results on Poincaré-type forms.

Recall (Corollary (5.3.5)) that the quasi-isometry class of a distinguished Saper metric looks the same in any system of normal coordinates W for $\hat{E}_1, ..., \hat{E}_K$. We may choose $W_i = Z_i$ for $1 \leq i \leq k-1$ and then augment the collection $W_1, ..., W_{k-1}$ to a collection of normal coordinates $W_1, ..., W_n$ for $\hat{E}_1, ..., \hat{E}_K$. Then

$$\omega_{\text{Sap},m-1} \sim \frac{1}{B^2}\Phi + \frac{1}{B}\omega_{m-1} \overset{\sim}{\geq} \frac{\sqrt{-1}}{\pi}\frac{1}{B^2}\sum_{i=1}^{k-1}\frac{dZ_i \wedge d\overline{Z}_i}{\mid Z_i \mid^2} + \frac{1}{B}\omega_{m-1}.$$

Let

$$\phi_i = \pi^*\left(\frac{\sqrt{-1}}{\pi}\frac{dZ_i \wedge d\overline{Z}_i}{\mid Z_i \mid^2}\right).$$

Using $\pi^* B = \beta' \overset{\sim}{\leq} \beta$, we obtain

$$\pi^*\omega_{\text{Sap},m-1} \overset{\sim}{\geq} \frac{1}{\beta^2}\sum_{i=1}^{k-1}\phi_i + \frac{1}{\beta}\pi^*\omega_{m-1}.$$

The forms ϕ_i have a simple description in z coordinates, since for $1 \leq i \leq k-1$ the pullback of Z_i under π is

$$\pi^* Z_i = z_i z_k^{\delta_i}$$

where $\delta_i = 1$ if $C_m \subset \hat{E}_i$ and $\delta_i = 0$ otherwise.

Recall that $\tau_m(z_1, ..., z_n) = z^{\Lambda_m} = z_1^{\lambda_{m1}} z_2^{\lambda_{m2}} ... z_k^{\lambda_{mk}}$ where $\lambda_{m,1}, ..., \lambda_{m,k}$ are all positive integers. Let $\beta_m = -\log \mid z^{\Lambda_m} \mid^2$ and $\phi_k = \beta_m^2 \tau_m^* \omega_{\Delta^*}$. Then $\beta_m \sim \beta$ and

$$\frac{1}{\beta^2}\sum_{i=1}^{k-1}\phi_i + \frac{1}{\beta}\omega_m + \tau_m^*\omega_{\Delta^*} \sim \frac{1}{\beta^2}\sum_{i=1}^{k}\phi_i + \frac{1}{\beta}\omega_m.$$

By Lemma (5.3.1), $\sum_{i=1}^{k}\phi_i \sim \phi$, provided that the matrix of multiplicities

$$\Lambda' = \begin{pmatrix} 1 & 0 & \dots & 0 & \delta_1 \\ 0 & 1 & \dots & 0 & \delta_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \delta_{k-1} \\ \lambda_{m,1} & \lambda_{m,2} & \cdots & \lambda_{m,k-1} & \lambda_{m,k} \end{pmatrix}$$

corresponding to the forms $\phi_1, ..., \phi_k$ has nonzero determinant. It is easily shown that

$$\det \Lambda' = \lambda_{m,k} - \sum_{i=1}^{k-1} \lambda_{m,i}\delta_i.$$

We now use our careful calculations of the properties of the matrix $T = T_m(D, \tilde{E})$ in §3. Recall from §8.3 and §8.2 that $\lambda_{m,i} = t_{m,j_i}$ and $\lambda_{m,k} = t_{m,j_k} = t_{mm}$, and note that $\delta_i = \delta_{j_i,m}$. By Lemma (3.7.3),

$$t_{mm} = 1 + \sum_{j=1}^{m-1} t_{mj}\delta_{jm},$$

where the integers t_{mj} and δ_{jm} are all nonnegative. Then

$$\begin{aligned}\det \Lambda' &= t_{mm} - \sum_{i=1}^{k-1} t_{m,j_i}\delta_{j_i,m} \\ &\geq t_{mm} - \sum_{j=1}^{m-1} t_{mj}\delta_{jm} \\ &= 1.\end{aligned}$$

□

(8.7) Modified Poincaré metrics.

Proof of (8.4.2). Part (ii) of this theorem follows from Proposition (5.5.1). We now prove part (i). Recall from (8.6.8) that

$$\pi^*\omega_{m-1} \sim \frac{\sqrt{-1}}{\pi}(dz_k \wedge d\bar{z}_k + dz_C \wedge d\bar{z}_C + (z_f dz_k + z_k dz_f) \wedge (\bar{z}_f d\bar{z}_k + \bar{z}_k d\bar{z}_f))$$

where the repeated index f indicates summation over $\Gamma - \{k\}$. Then $\pi^*\omega_{m-1} + \frac{\sqrt{-1}}{\pi} dz_f \wedge d\bar{z}_f$ is positive and hence locally quasi-isometric to ω_m. Substituting, we obtain

$$\begin{aligned}\omega_{P,m} &= \omega_{S,m} + \omega_m \\ &\sim \pi^*\omega_{S,m-1} + \frac{\sqrt{-1}}{\pi}\frac{1}{| \log | z_1 z_2 ... z_k |^2 |} dz_f \wedge d\bar{z}_f + \tau_m^* \omega_{\Delta^*} \\ &\quad + \pi^*\omega_{m-1} + \frac{\sqrt{-1}}{\pi} dz_f \wedge d\bar{z}_f\end{aligned}$$

$$\sim \pi^*(\omega_{S,m-1} + \omega_{m-1}) + \frac{\sqrt{-1}}{\pi} dz_f \wedge d\bar{z}_f + \tau_m^* \omega_{\Delta^*}$$
$$= \pi^* \omega_{P,m-1} + \frac{\sqrt{-1}}{\pi} dz_f \wedge d\bar{z}_f + \tau_m^* \omega_{\Delta^*}.$$

□

(8.8) Metrics in the case $C_i \subset D_{i-1}$.

Proof of (8.4.4). The statement of Theorem (8.4.4) allowed for the possibility that D_j might have several connected components, but it is sufficient to prove the theorem in the case that D_j is connected, since each blow-up π_i is a biholomorphism away from its exceptional divisor E_i. This simplification allows us to avoid having to work with submatrices of multiplicity matrices corresponding to individual connected components of D_j. For the rest of this proof we will assume that $C_i \subset D_{i-1}$ for $2 \le i \le m$.

We continue to use the notation of §8.3. We may write $\omega_{S,m}$ as

$$\omega_{S,m} = r_0 \pi_{m,0}^* \omega_0 + \sum_{i=1}^{m} r_i \nu_{m,i}$$

for some integers $r_0, r_1, ..., r_m$. The forms

$$\nu_{m,i} = \pi_{m,i}^* \nu_i = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\log || \hat{s}_{m,i} ||^2)^2$$

are Poincaré-type $(1,1)$-forms corresponding to the divisors $D_{m,i} = \pi_{m,i}^* D_i$ and have the usual decompositions $\nu_{m,i} = \mu_i + \eta_i$ as in (5.1.2). In local coordinates, the function $\beta_i = -\log || \hat{s}_{m,i} ||^2$ is given by

$$\beta_i = -\log(| z^{\Lambda_i} |^2 g_i)$$

for some bounded positive C^∞ function g_i. If $C_i \subset D_{i-1}$ for $2 \le i \le m$, the calculation of the quasi-isometry class of $\omega_{S,m}$ is greatly simplified, because in this case all entries of the matrix Λ are positive, by Proposition (3.7.4), and consequently $\beta_i \sim \beta \sim -\log | z_1 z_2 ... z_k |^2$ for all i.

Applying Lemma (8.6.2) inductively in this case, it is easy to show that there are positive integers r_i (r_i a sufficiently large multiple of r_{i+1}), such that

$$r_0 \pi_{m,0}^* \omega_0 + \sum_{i=1}^{m} r_i \mu_i \sim \pi_{m,0}^* \omega_0 + \frac{1}{\beta} \omega_m.$$

By Proposition (5.2.4),

$$\frac{a}{\beta}\omega_m + \eta_i \sim \frac{1}{\beta}\omega_m + \tau_i^* \omega_{\Delta^*}$$

locally, for any positive constant a. Then locally

$$\begin{aligned} r_0 \pi_{m,0}^* \omega_0 + \sum_{i=1}^m r_i \nu_{m,i} &\sim \pi_{m,0}^* \omega_0 + \frac{1}{\beta}\omega_m + \sum_{i=1}^m \tau_i^* \omega_{\Delta^*} \\ &\sim \pi_{m,0}^* \omega_0 + \frac{1}{\beta}\omega_m + \frac{1}{\beta^2}\sum_{i=1}^m \beta_i^2 \tau_i^* \omega_{\Delta^*} \text{ since } \beta_i \sim \beta. \end{aligned}$$

Recall from §8.3 that the $m \times k$ matrix Λ has rank k. We may apply Lemma (5.3.1) to the sum of the forms $\phi_i = \beta_i^2 \tau_i^* \omega_{\Delta^*}$ to obtain

$$\sum_{i=1}^m \beta_i^2 \tau_i^* \omega_{\Delta^*} \sim \phi.$$

This shows that $\omega_{S,m}$ is homogeneous in the sense of (1.8.4).

Recall that $\omega_{S,1} \sim \pi_1^* \omega_0 + \omega_{\mathrm{Sap}}$ (Corollary (7.2.2)) and if $\dim C_1 = 0$ then $\omega_{S,1} \sim \omega_{\mathrm{Sap}}$ (Corollary (7.2.3)). If $\omega_{S,1} \sim \omega_{\mathrm{Sap}}$, then we may neglect the term $\pi_{m,0}^* \omega_0$ in all successive blow-ups, and we obtain the local quasi-isometry

$$\omega_{S,m} \sim \frac{1}{\beta^2}\phi + \frac{1}{\beta}\omega_m$$

which describes a Saper metric.

To obtain the quasi-isometry $\omega_{P,m} \sim \omega_{P,\mathrm{hom}}$, use Proposition (5.5.1). □

§9. Homogeneous Saper metrics and proof of Theorem II. Let M be a compact complex Kähler manifold and let X be a reduced compact analytic subspace of M. In §9.1 we will show how to construct a homogeneous Saper metric on the nonsingular set of X. This metric has the advantage that it is more easily described in local coordinates than the metrics of Theorem (8.4.1), but the disadvantage that there is not a natural decomposition into terms corresponding to each of the blow-up maps used to resolve X.

In §9.2 we will show that an incomplete metric on $M - X_{\mathrm{sing}}$ which determines an embedded resolution of singularities of X is associated in a very simple way with a certain complete Kähler modified

Saper metric on $M - X_{\text{sing}}$. This result, together with Corollary (10.2.4) on fine sheaves, constitutes our second main result, Theorem II of §2.2.

(9.1) Complete Kähler homogeneous Saper metrics. Let $M = M_0$ be a compact complex Kähler manifold with Kähler form ω and let X be a reduced compact analytic subspace of M. Suppose that $\{\pi_j : M_j \to M_{j-1}\}$ is a finite sequence of blow-ups along smooth centres $C_j \subset M_{j-1}$ which resolves the singularities of X and such that C_j has normal crossings with the total exceptional divisor D_{j-1} of the composite map $\pi_{j,0} : M_j \to M_0$. Let $\tilde{M} = M_m$ be the final blow-up, let $\pi = \pi_{m,0}$ be the composite map from $\tilde{M}$ to M, and let $D = D_m$ be the total exceptional divisor of π. The strict transform $\tilde{X}$ of X in $\tilde{M}$ is smooth and has normal crossings with D, and $\tilde{X} - (\tilde{X} \cap D)$ is isomorphic to $X - X_{\text{sing}}$. A metric on $\tilde{M} - D$ induces a metric on $X - X_{\text{sing}}$.

In §8 we constructed complete Kähler modified Saper metrics for a sequence of blow-ups of this type. We showed that if for $2 \leq j \leq m$ the centre C_j is either contained in the total exceptional divisor D_{j-1} or is disjoint from D_{j-1}, our Kähler form $\omega_{S,m}$ on $\tilde{M} - D$ is homogeneous, i.e.

$$\omega_{S,m} \sim \pi^*\omega + \omega_{\text{Sap}}$$

where ω_{Sap} is the fundamental form of a distinguished Saper metric on $\tilde{M} - D$. Now we will show that we may obtain such a metric even when we remove this requirement.

Recall that the irreducible components $\tilde{E}_{m,1}, ..., \tilde{E}_{m,m}$ of D are the strict transforms of the exceptional divisors $E_j = \pi_j^{-1}(C_j)$.

THEOREM 9.1.1. *Let $\mathcal{D}_1, ..., \mathcal{D}_m$ be effective divisors on $\tilde{M}$ with the same support as D, i.e. such that*

$$\mathcal{D}_i = \sum_{j=1}^{m} b_{ij} \tilde{E}_{m,j}$$

for some positive integers b_{ij}. For each divisor $\mathcal{D}_i$, let H_i be a hermitian metric on the line bundle $\mathcal{L}_i = [\mathcal{D}_i]$ and let S_i be a section of $\mathcal{L}_i$ such that $(S_i) = \mathcal{D}_i$ and $|| S_i ||^2 < 1$ on $\tilde{M}$. Let ν_i be the associated Poincaré-type $(1,1)$-form

$$\nu_i = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\log || S_i ||^2)^2.$$

Suppose that

i. *the matrix of multiplicities* (b_{ij}) *is nonsingular, and*

ii. *the forms*

$$\psi_j = l\pi^*\omega - c_1([\mathcal{D}_j], H_j)$$

are positive for all sufficiently large integers l.

Then the form

$$\omega_S = l\pi^*\omega + \sum_{i=1}^{m} \nu_i$$

is the fundamental form of a complete Kähler homogeneous Saper metric on $\tilde{M} - D$.

Proof. The construction is similar to that of §8. The form ω_S is d-closed so we need only show that it is positive and has the required local quasi-isometry class.

We decompose ν_j as $\nu_j = \mu_j + \eta_j$ as usual and note that

$$\mu_j = -\frac{2}{\beta_j} c_1([\mathcal{D}_j], H_j)$$

where $\beta_j = -\log \| S_j \|^2$.

Consider any point $q \in \tilde{M}$. If $q \notin D$ then $\pi^*\omega$ is positive and ν_j is bounded in a neighbourhood of q so $l\pi^*\omega$ dominates ν_j for large l. Suppose that $q \in D$ and let $\tilde{E}_1, ..., \tilde{E}_k$ be the components of D passing through q. Each $\tilde{E}_i$ is one of the strict transforms $\tilde{E}_{m,\alpha}$. The divisors D and $\mathcal{D}_j$ have the same irreducible components so $\tilde{E}_1, ..., \tilde{E}_k$ are also the components of $\mathcal{D}_j$ passing through q. Let λ_{ji} be the multiplicity of $\tilde{E}_i$ in $\mathcal{D}_j$. The matrix $\Lambda = (\lambda_{ji})$ is the $m \times k$ submatrix of the matrix (b_{ij}) corresponding to the divisors $\tilde{E}_1, ..., \tilde{E}_k$. The rank of Λ is k.

Let $(z_1, ..., z_n)$ be normal coordinates for the divisors $\tilde{E}_i$ in a neighbourhood U of q and let $\beta = -\log | z_1 z_2 ... z_k |^2$. Near q we have

$$\beta_j \sim -\log | z_1^{\lambda_{j1}} z_2^{\lambda_{j2}} ... z_k^{\lambda_{jk}} |^2 \sim \beta.$$

By assumption (ii), the forms

$$\psi_j = l\pi^*\omega + \frac{1}{2}\beta_j \mu_j$$

are positive for all sufficiently large integers l. Then

$$l\pi^*\omega + \mu_j \sim \pi^*\omega + \frac{1}{\beta}\frac{\sqrt{-1}}{\pi}\sum_{i=1}^{n} dz_i \wedge d\overline{z}_i$$

for $z_1, ..., z_k$ close enough (but not equal) to 0. Let $\tau_j : U \to \Delta$ be the monomial map given by

$$\tau_j(z_1, ..., z_n) = z_1^{\lambda_{j1}} z_2^{\lambda_{j2}} ... z_k^{\lambda_{jk}}.$$

It follows from Proposition (5.2.4) that

$$l\pi^*\omega + \mu_j + \eta_j \sim \pi^*\omega + \frac{1}{\beta}\frac{\sqrt{-1}}{\pi}\sum_{i=1}^{n} dz_i \wedge d\overline{z}_i + \tau_j^*\omega_{\Delta^*}$$

and from Lemma (5.3.3) that

$$\sum_{j=1}^{m} \tau_j^*\omega_{\Delta^*} \sim \frac{\sqrt{-1}}{\pi}\frac{1}{(\log \mid z_1 z_2 ... z_k \mid^2)^2}\sum_{i=1}^{k}\frac{dz_i \wedge d\overline{z}_i}{\mid z_i \mid^2}$$

because the matrix Λ has positive entries and rank k. Then

$$\begin{aligned}\omega_S &= l\pi^*\omega + \sum_{j=1}^{m} \nu_j \\ &\sim \pi^*\omega + \frac{\sqrt{-1}}{\pi}\left(\frac{1}{(\log \mid z_1 z_2 ... z_k \mid^2)^2}\sum_{i=1}^{k}\frac{dz_i \wedge d\overline{z}_i}{\mid z_i \mid^2}\right. \\ &\qquad \left. + \frac{1}{\mid \log \mid z_1 z_2 ... z_k \mid^2 \mid}\sum_{i=1}^{n} dz_i \wedge d\overline{z}_i\right)\end{aligned}$$

for $z_1, ..., z_k$ close enough (but not equal) to 0. This shows that ω_S is positive for $l \gg 0$ and that $\omega_S \sim \pi^*\omega + \omega_{\mathrm{Sap}}$, i.e. ω_S is homogeneous. Completeness follows from Proposition (6.1.1). □

COROLLARY 9.1.2. *There exists a complete Kähler homogeneous Saper metric on* $\tilde{M} - D$ *and hence on* $X - X_{\mathrm{sing}} \cong \tilde{X} - (\tilde{X} \cap D)$.

Proof. Let $\mathcal{D}_1, ..., \mathcal{D}_m$ and $H_1, ..., H_m$ be the divisors and metrics of Proposition (4.4.1). These divisors and metrics satisfy conditions (i) and (ii) of the previous theorem. For each i we may choose a section S_i of the line bundle $\mathcal{L}_i = [\mathcal{D}_i]$ such that $(S_i) = \mathcal{D}_i$ and such that $\| S_i \|^2 < 1$ on $\tilde{M}$. Let $\nu_1, ..., \nu_m$ be the associated Poincaré-type $(1,1)$-forms. Then the form

$$\omega_S = l\pi^*\omega + \sum_{i=1}^{m} \nu_i$$

is the fundamental form of a complete Kähler homogeneous Saper metric on $\tilde{M} - D$. □

(9.2) Incomplete and complete Kähler metrics. We will now describe a simple relationship between certain incomplete and complete metrics on $X - X_{\text{sing}}$.

First note that in Theorem (9.1.1) we needed only a single Poincaré-type $(1,1)$-form ν_i to obtain completeness because each divisor $\mathcal{D} = \mathcal{D}_i$ has the same support as D. Furthermore, we assumed that the Chern form $c_1([\mathcal{D}], H)$ corresponding to the metric $H = H_i$ had the property that

$$\psi = l\pi^*\omega - c_1([\mathcal{D}], H)$$

was positive on $\tilde{M}$ for all sufficiently large integers l. We may write this assumption in terms of a section S of $[\mathcal{D}]$ such that $(S) = \mathcal{D}$ and $\| S \|^2 < 1$ as

$$\psi = l\pi^*\omega - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(-\log \| S \|^2) > 0$$

on $\tilde{M}$. The form ψ is a Kähler form on $\tilde{M}$ which determines an incomplete metric on $\tilde{M} - D \cong M - X_{\text{sing}}$. The completion of $M - X_{\text{sing}}$ under this metric is $\tilde{M}$. Recall (definition (2.2.1)) that we say that this incomplete metric on $M - X_{\text{sing}}$ determines an embedded resolution of the singularities of X. We will now show that each such metric corresponds to a complete Kähler modified Saper metric on $M - X_{\text{sing}} \cong \tilde{M} - D$.

THEOREM 9.2.1. *Let $\mathcal{D}$ be an effective divisor on $\tilde{M}$ with the same support as D, let S be a section of the line bundle $\mathcal{L} = [\mathcal{D}]$ such that $(S) = \mathcal{D}$ and $\| S \|^2 < 1$, and let H be a metric on $\mathcal{L}$ such that for all sufficiently large integers l*

$$\tilde{\Omega} = l\pi^*\omega - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(-\log \| S \|^2)$$

determines a Kähler metric on $\tilde{M}$. *Then the* $(1,1)$*-form*

$$\Omega_S = l\pi^*\omega - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(-\log || S ||^2)^2$$

determines a complete Kähler modified Saper metric on $\tilde{M} - D$ *such that*

$$\pi^*\omega \overset{\sim}{\leq} \Omega_S \overset{\sim}{\leq} \pi^*\omega + \omega_{\mathrm{Sap}}$$

and

$$\frac{1}{-\log || S ||^2}\tilde{\Omega} \overset{\sim}{\leq} \Omega_S.$$

Proof. Since Ω_S is positive away from D for $l \gg 0$, we need only prove the given quasi-isometries on neighbourhoods of points of D. We write Ω_S as $l\pi^*\omega + \nu = l\pi^*\omega + \mu + \eta$ where

$$\begin{aligned}
\nu &= -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(-\log || S ||^2)^2 \\
\mu &= -\frac{1}{\beta}\frac{\sqrt{-1}}{\pi}\partial\overline{\partial}(-\log || S ||^2) = -\frac{2}{\beta}c_1([\mathcal{D}], H), \\
\eta &= \frac{\sqrt{-1}}{\pi}\frac{\partial\beta \wedge \overline{\partial}\beta}{\beta^2}, \qquad \text{and} \\
\beta &= -\log || S ||^2.
\end{aligned}$$

Then

$$\tilde{\Omega} = l\pi^*\omega + \frac{1}{2}\beta\mu$$

and

$$l\pi^*\omega + \mu \sim l\pi^*\omega + \frac{1}{\beta}\tilde{\Omega} > 0.$$

Next we calculate the local quasi-isometry class of

$$\Omega_S = l\pi^*\omega + \nu = l\pi^*\omega + \mu + \eta.$$

Choose normal coordinates $(z_1, ..., z_n)$ near $q \in \operatorname{supp} D = \operatorname{supp} \mathcal{D}$. Locally the divisor $\mathcal{D}$ is given by the equation $z_1^{\lambda_1} z_2^{\lambda_2} ... z_k^{\lambda_k} = 0$ for some positive integers $\lambda_1, ..., \lambda_k$. Let τ be the local monomial map given by

$$\tau(z_1, ..., z_n) = z_1^{\lambda_1} z_2^{\lambda_2} ... z_k^{\lambda_k}.$$

Using Proposition (5.2.4) we get locally

$$l\pi^*\omega + \mu + \eta \sim \pi^*\omega + \frac{1}{\beta}\tilde{\Omega} + \tau^*\omega_{\Delta^*}$$

which is always positive. This shows that Ω_S determines a Kähler modified Saper metric on $\tilde{M} - D$ and that $\Omega_S \gtrsim (1/\beta)\tilde{\Omega}$. Moreover Ω_S is bounded above by the Kähler form ω_S of a homogeneous Saper metric, i.e. a Kähler form ω_S such that $\omega_S \sim \pi^*\omega + \omega_{\mathrm{Sap}}$. To construct such an ω_S, we proceed as in the proof of Corollary (9.1.2) but note that for some i we may replace $\mathcal{D}_i$ by $\mathcal{D}$ and preserve the properties needed to apply Theorem (9.1.1).

We use Proposition (6.1.1) to get completeness.

□

COROLLARY 9.2.2. *There exist a divisor $\mathcal{D}$ on $\tilde{M}$ with the same support as D, a section S of the line bundle $\mathcal{L} = [\mathcal{D}]$, and a metric H on $\mathcal{L}$ such that for all sufficiently large integers l*

i. *the $(1,1)$-form*

$$\tilde{\Omega} = l\pi^*\omega - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(-\log \| S \|^2)$$

determines a Kähler metric on $\tilde{M}$ and

ii. *the $(1,1)$-form*

$$\Omega_S = l\pi^*\omega - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(-\log \| S \|^2)^2$$

determines a complete Kähler modified Saper metric on $\tilde{M} - D$ such that

$$\pi^*\omega \lesssim \Omega_S \lesssim \pi^*\omega + \omega_{\mathrm{Sap}}$$

and

$$\frac{1}{-\log \| S \|^2}\tilde{\Omega} \lesssim \Omega_S.$$

Proof. We have many choices for $\mathcal{D}$, since all the Kähler forms we constructed on $\tilde{M}$ by the methods of §4 may be written in the form (i). For example, let $\mathcal{D}$ be any of the divisors $\mathcal{D}_j$ of Corollary (9.1.2) and Proposition (4.4.1). Then

$$\tilde{\Omega} = \psi_j$$

is positive on $\tilde{M}$ by Proposition (4.4.1) so it determines a Kähler metric on $\tilde{M}$. Now apply Theorem (9.2.1). □

§**10. Fine sheaves.** Let M be a compact complex manifold and let X be a reduced compact analytic subspace of M. Let $\pi : \tilde{M} \to M$ be the composite of a finite sequence of blow-ups of the type described in §1 which resolve the singularities of X. Let D be the exceptional divisor of π. The strict transform $\tilde{X}$ of X in $\tilde{M}$ is smooth and has normal crossings with D and the restriction of π to $\tilde{X} - (\tilde{X} \cap D)$ is a biholomorphism onto $X - X_{\text{sing}}$.

Let h be a hermitian metric on $X - X_{\text{sing}} \cong \tilde{X} - (\tilde{X} \cap D)$. Let $\mathcal{S}_0$ be the complex of presheaves on X whose sections over any open set U in X are smooth measurable differential forms ϕ on $U \cap (X - X_{\text{sing}})$ such that both ϕ and $d\phi$ are L_2-bounded with respect to h. Let $\mathcal{S}$ be the associated complex of sheaves on X. Similarly, let $\tilde{\mathcal{S}}_0$ be the complex of presheaves on $\tilde{X}$ whose sections over any open set U in $\tilde{X}$ are smooth measurable differential forms ϕ on $U - (U \cap D)$ such that both ϕ and $d\phi$ are L_2-bounded with respect to h, and let $\tilde{\mathcal{S}}$ be the associated complex of sheaves on $\tilde{X}$.

We will show that if h is bounded below by the restriction to $X - X_{\text{sing}}$ of a hermitian metric on M, then $\mathcal{S}$ is a complex of fine sheaves. In particular, if h is a modified Saper or modified Poincaré metric then the complex of sheaves $\mathcal{S}$ on X is fine. Similarly, if h is bounded below by the restriction to $\tilde{X} - (\tilde{X} \cap D)$ of a hermitian metric on $\tilde{X}$, then $\tilde{\mathcal{S}}$ is also fine. In particular, if h is a modified Poincaré metric then the complex of sheaves $\tilde{\mathcal{S}}$ on $\tilde{X}$ is fine.

We first review some technical material concerning L_2-norms of forms.

(10.1) Comparison of L_2-bounded forms in two metrics. Let h_A and h_B be two hermitian metrics on an m-dimensional complex manifold Y. Let W be a coordinate neighbourhood of a point y in Y. After shrinking W if necessary, we may choose C^∞ sections $t_1, \dots, t_m$ of the tangent bundle TW which are linearly independent at each point of W and orthonormal with respect to h_A. We call $t_1, \dots, t_m$ an *orthonormal frame* for h_A. Let $\tau_1, \dots, \tau_m$ be dual sections of the cotangent space T^*W. We call $\tau_1, \dots, \tau_m$ an *orthonormal coframe* for h_A. Let $G = (G_{ij})$ be the C^∞ matrix for h_B with respect to the frame $t_1, \dots, t_m$, i.e. $\langle t_i, t_j \rangle_B = G_{ij}$, where G_{ij} is some C^∞ function on W. Then $\langle \tau_i, \tau_j \rangle_B = (G^{-1})_{ji}$. The matrix G is hermitian positive definite at every point, so there is a (constant) unitary matrix U

such that

$$U\,G(y)\,U^* = D,$$

where $U^* = \overline{U}^t$ and D is a (constant) diagonal matrix with positive diagonal entries d_i.

REMARK 10.1.1. At each point y in Y, we have $h_B \geq h_A$ if and only if $d_i \geq 1$ for all i.

Next we construct an orthonormal frame and coframe for h_A with respect to which the matrix for h_B at the point y is diagonal. Let

$$e_i = \sum_{j=1}^{m} U_{ij} t_j$$

and

$$\xi_i = \sum_{j=1}^{m} \tau_j (U^{-1})_{ji}.$$

Then $e_1, ..., e_m$ is another orthonormal frame for h_A and $\xi_1, ..., \xi_m$ is another orthonormal coframe for h_A. The matrix for h_B with respect to the frame $\{e_i\}$ is

$$\hat{D} = UGU^*$$

where $\langle e_i, e_j \rangle_B = \hat{D}_{ij}$ and $\langle \xi_i, \xi_j \rangle_B = (\hat{D}^{-1})_{ji}$. Note that $\hat{D}(y) = D$. At the point y we have

$$\begin{aligned} \langle e_i(y), e_j(y) \rangle_A &= \delta_{ij}, & \langle e_i(y), e_j(y) \rangle_B &= d_i \delta_{ij}, \\ \langle \xi_i(y), \xi_j(y) \rangle_A &= \delta_{ij}, \quad \text{and} & \langle \xi_i(y), \xi_j(y) \rangle_B &= \frac{1}{d_i} \delta_{ij}, \end{aligned}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. We may extend the collection of tangent vectors $\{\frac{1}{\sqrt{d_i}} e_i(y)\}$ to an orthonormal frame $\{e_i'\}$ for h_B on W and the collection of cotangent vectors $\{\sqrt{d_i}\xi_i(y)\}$ to an orthonormal coframe $\{\xi_i'\}$ for h_B. Let C be the matrix for the change of frame

$$e_i' = \sum_j C_{ij} e_j \qquad \text{and} \qquad \xi_i' = \sum_j \xi_j (C^{-1})_{ji}.$$

Then

$$C\hat{D}C^* = I \qquad \text{and} \qquad C(y) = \sqrt{D^{-1}},$$

i.e. $C(y)$ is the diagonal matrix whose diagonal elements are $\frac{1}{\sqrt{d_i}}$. Volume forms for h_A and h_B are

$$dV_A = \frac{\sqrt{-1}}{2m!}\xi_1 \wedge \overline{\xi}_1 \wedge ... \wedge \xi_m \wedge \overline{\xi}_m$$

and

$$\begin{aligned} dV_B &= \frac{\sqrt{-1}}{2m!}\xi_1' \wedge \overline{\xi}_1' \wedge ... \wedge \xi_m' \wedge \overline{\xi}_m' \\ &= \det C^{-1} \det \overline{C}^{-1}\, dV_A \\ &= \det \hat{D}\, dV_A. \end{aligned}$$

Recall that $\hat{D}(y) = D$, so that $\det \hat{D}(y) = d_1 d_2 ... d_m$.

We now compare L_2-norms with respect to h_A and h_B. Let ψ be a p-form which is locally of the form

$$\psi = \sum_{i_1<...<i_p} \alpha_{(i_1,...,i_p)}\xi_{i_1} \wedge ... \wedge \xi_{i_p} = \sum_I \alpha_I \xi_I = \sum_I \beta_I \xi_I'.$$

Similar calculations can be done for (p, q)-forms. We have

$$\langle \psi, \psi \rangle_A = \sum_I \mid \alpha_I \mid^2 \quad \text{and} \quad \langle \psi, \psi \rangle_B = \sum_I \mid \beta_I \mid^2.$$

Integrating over Y gives the L_2-norms

$$(\psi, \psi)_A = \int_Y \langle \psi, \psi \rangle_A\, dV_A \quad \text{and} \quad (\psi, \psi)_B = \int_Y \langle \psi, \psi \rangle_B\, dV_B$$

where $dV_B = \det \hat{D}\, dV_A$ locally.

To compare L_2-norms with respect to h_A and h_B we will compare the local expressions for $\langle \psi, \psi \rangle_A$ and $\langle \psi, \psi \rangle_B \det \hat{D}$.

For example, if $p = 0$ and $\psi = \alpha$, then

$$\langle \psi, \psi \rangle_A = \langle \psi, \psi \rangle_B = \mid \alpha \mid^2$$

and ψ is locally B-L_2-bounded if and only if $\sqrt{\det \hat{D}}\psi$ is locally A-L_2-bounded.

If $p = m$ and $\psi = \alpha\xi_1 \wedge ... \wedge \xi_m$, then

$$\langle \psi, \psi \rangle_B \det \hat{D} = \frac{\mid \alpha \mid^2}{\det \hat{D}} \det \hat{D} = \langle \psi, \psi \rangle_A$$

and ψ is B-L_2-bounded if and only if ψ is A-L_2-bounded.

If $p = 1$ and $\psi = \sum \alpha_i \xi_i = \sum \beta_i \xi_i'$ then at the point y,

$$\beta_i = \frac{1}{\sqrt{d_i}} \alpha_i.$$

If $h_B \geq h_A$ then $d_i \geq 1$ for all i so

$$\sum_i | \beta_i |^2 = \sum_i \frac{1}{d_i} | \alpha_i |^2 \leq \sum_i | \alpha_i |^2,$$

i.e.

$$\langle \psi, \psi \rangle_B \leq \langle \psi, \psi \rangle_A. \tag{10.1.2}$$

Similarly,

$$\begin{aligned} \langle \psi, \psi \rangle_B \det \hat{D}(y) &= \sum_i \frac{\prod_j d_j}{d_i} | \alpha_i |^2 \\ &\geq \langle \psi, \psi \rangle_A \end{aligned}$$

at each point y so

$$(\psi, \psi)_B \geq (\psi, \psi)_A \tag{10.1.3}$$

if $h_B \geq h_A$.

(10.2) Fine sheaves. Let Y be a reduced compact analytic subspace of an n-dimensional compact Kähler manifold M and let Z be a complex analytic subspace of Y containing the singular set of Y. Let h_M be a hermitian metric on $Y - Z$ induced from a hermitian metric on M. Let h be another hermitian metric on $Y - Z$ and let $\mathcal{S}_0$ be the complex of presheaves on Y whose sections over any open set U in Y are smooth measurable differential forms ϕ on $U \cap (Y - Z)$ such that both ϕ and $d\phi$ are L_2-bounded with respect to h. Let $\mathcal{S}$ be the associated complex of sheaves on Y. We will show that if $h \geq h_M$ on $Y - Z$ then $\mathcal{S}$ is a complex of fine sheaves.

The main technical result we need is the following proposition. Recall that by a C^∞ function on Y we mean the restriction to Y of a C^∞ function on M.

PROPOSITION 10.2.1. *Let U be an open set in Y and let ϕ be a smooth measurable differential form on $U \cap (Y - Z)$ such that*

ϕ is L_2-bounded with respect to h. Let f be a C^∞ function on Y. Suppose that $h \geq h_M$ on $Y - Z$. Then $df \wedge \phi$ is L_2-bounded with respect to h.

We will work in terms of frames and coframes of the type described in §10.1 above, letting $h_A = h_M$ and $h_B = h$. More specifically, for each $y \in Y - Z$ we may choose a coordinate neighbourhood W on which we have an orthonormal frame $\{e_i\}$ and coframe $\{\xi_i\}$ for h_A, and an orthonormal frame $\{e'_i\}$ and coframe $\{\xi'_i\}$ for h_B such that $\langle e_i, e_j \rangle_B = \hat{D}_{ij}$ and $\hat{D}(y) = D$ is a diagonal matrix with positive diagonal entries d_i. The statement that $h_B \geq h_A$ means that every eigenvalue d_i of D satisfies $d_i \geq 1$. Then for every 1-form ψ, we have $\langle \psi, \psi \rangle_B \leq \langle \psi, \psi \rangle_A$. In particular, $\langle df, df \rangle_B \leq \langle df, df \rangle_A$. But $\langle df, df \rangle_A$ is bounded because f is a C^∞ function on Y and h_A comes from a metric on M. Choose $K > 0$ such that $\langle df, df \rangle_A \leq K$ everywhere on Y.

Suppose that ϕ is a p-form.

LEMMA 10.2.2. $\langle df \wedge \phi, df \wedge \phi \rangle_B \leq K\binom{n}{p+1}\langle \phi, \phi \rangle_B$.

Proof. In terms of the coframe $\{\xi'_i\}$, let

$$df = \sum_j f_j \xi'_j \qquad \text{and} \qquad \phi = \sum_I \phi_I \xi'_I.$$

Then

$$df \wedge \phi = \sum_{|J|=p+1} \left(\sum_{\{j,I\}=J} f_j \phi_I (-1)^{\operatorname{sgn}(j,I)} \right) \xi'_J$$

and

$$\begin{aligned}
\langle df \wedge \phi, df \wedge \phi \rangle_B &= \sum_{|J|=p+1} \left| \sum_{\{j,I\}=J} f_j \phi_I (-1)^{\operatorname{sgn}(j,I)} \right|^2 \\
&\leq \sum_{|J|=p+1} \left(\sum_{j \in J} | f_j |^2 \right) \left(\sum_{I \subset J} | \phi_I |^2 \right) \\
&\leq \sum_{|J|=p+1} \langle df, df \rangle_B \ \langle \phi, \phi \rangle_B \\
&= \binom{n}{p+1} \langle df, df \rangle_B \ \langle \phi, \phi \rangle_B
\end{aligned}$$

$$\leq \binom{n}{p+1} \langle df, df \rangle_A \langle \phi, \phi \rangle_B$$
$$\leq K \binom{n}{p+1} \langle \phi, \phi \rangle_B.$$

□

We now conclude the proof of Proposition (10.2.1). The h-L_2-norm of $df \wedge \phi$ on $U' = U \cap (Y - Z)$ is

$$\begin{aligned}(df \wedge \phi, df \wedge \phi)_B &= \int_{U'} \langle df \wedge \phi, df \wedge \phi \rangle_B \, dV_B \\ &\leq K \binom{n}{p+1} \int_{U'} \langle \phi, \phi \rangle_B \, dV_B \\ &< \infty\end{aligned}$$

since the form ϕ on U' is L_2-bounded with respect to $h = h_B$. □

We use Proposition (10.2.1) to prove our main result about fine sheaves.

PROPOSITION 10.2.3. *If $h \geq h_M$ on $Y - Z$, then $\mathcal{S}$ is a complex of fine sheaves.*

Proof. It is sufficient to prove that multiplication by a C^∞ function determines a morphism of presheaves $\mathcal{S}_0 \to \mathcal{S}_0$. Let ϕ be a section of $\mathcal{S}_0$ over an open set U in Y, i.e. ϕ is a smooth measurable differential form on $U \cap (Y - Z)$ such that ϕ and $d\phi$ are L_2-bounded with respect to h. We will show that for any C^∞ function f on Y, the forms $f\phi$ and $d(f\phi)$ are also h-L_2-bounded. Now $d(f\phi) = df \wedge \phi + f\, d\phi$, and $f\phi$ and $f\, d\phi$ are h-L_2-bounded because f is bounded. By Proposition (10.2.1), $df \wedge \phi$ is also h-L_2-bounded.

Thus multiplication by a function which is C^∞ on Y determines a presheaf morphism from $\mathcal{S}_0$ to $\mathcal{S}_0$. The induced map from $\mathcal{S}$ to $\mathcal{S}$ is a sheaf morphism, so $\mathcal{S}$ admits partitions of unity and is a complex of fine sheaves. □

Now we return to the situation of the introduction to §10, namely that X is a reduced compact analytic subspace of M, $\pi : \tilde{M} \to M$ is a composite of blow-ups which resolves the singularities of X, D is the exceptional divisor of π, and $\tilde{X}$ is the (smooth) strict transform of X in $\tilde{M}$.

COROLLARY 10.2.4. *Let h be a modified Saper metric on $\tilde{M}-D$. Then the associated complexes of L_2 sheaves on X and M are fine.*

Proof. Recall that every modified Saper metric on $\tilde{M} - D \cong M - X_{\text{sing}}$ is bounded below by a metric induced from M and then use Proposition (10.2.3). ☐

COROLLARY 10.2.5. *Let h_P be a modified Poincaré metric on $\tilde{M} - D$. Then the associated complexes of L_2 sheaves on X, M, $\tilde{X}$, and $\tilde{M}$ are all fine.*

Proof. Recall that every modified Poincaré metric on $\tilde{M} - D \cong M - X_{\text{sing}}$ is bounded below by a metric on $M - X_{\text{sing}}$ induced from M and also by a metric on $\tilde{M} - D$ induced from $\tilde{M}$. ☐

Appendix: Tubular neighbourhood construction. The following tubular neighbourhood constructions are based on Clemens [**CL**, §5] and are used in the proof of Proposition (4.1.1).

Let X be a complex manifold and let Y be a compact complex submanifold. Let $N = N_{Y/X}$ be the normal bundle of Y in X. Let F be a vector bundle over X and, for any subset Z of X, let F_Z be the restriction of F to Z.

PROPOSITION A.1. *There exist an open neighbourhood U of Y in X and a C^∞ projection $\tau : U \to Y$ such that*

(*) *τ is surjective and the fibres of τ are holomorphic submanifolds of U, transverse to Y.*

PROPOSITION A.2. *There exist an open neighbourhood U of Y in X and C^∞ maps $\tau : U \to Y$ and $\psi : U \to N$ such that τ satisfies (*) of Proposition 1 and such that*

i. *the restriction of ψ to Y is the natural identification of Y with the zero section of N,*

ii. *ψ is a diffeomorphism onto its image, and*

iii. *the restriction of ψ to any fibre of $\tau : U \to Y$ is a biholomorphism onto an open set in the corresponding fibre of N over Y.*

PROPOSITION A.3. *There exist an open neighbourhood U of Y and C^∞ maps $\tau : U \to Y$ and $\rho : F_U \to \tau^* F_Y$ such that τ satisfies* (*) *of Proposition* 1 *and such that*

i. *the restriction of ρ to F_Y is the identity map,*
ii. *ρ is a C^∞ vector bundle isomorphism, and*
iii. *the following diagram commutes.*

$$\begin{array}{ccccc} F_U & \xrightarrow{\rho} & \tau^* F_Y & \xrightarrow{\tau_*} & F_Y \\ \downarrow & & \downarrow & & \downarrow \\ U & = & U & \xrightarrow{\tau} & Y \end{array}$$

All these propositions are proved in a similar manner, by piecing together local maps using a partition of unity on Y and the appropriate transition functions.

Proof of Proposition A.1. Let $\{U_\alpha\}$ be a finite open cover of Y in X with the following properties:

i. The closure $\overline{U_\alpha}$ of U_α in X is compact.
ii. On U_α there are holomorphic coordinates $z_{\alpha,1}, \ldots, z_{\alpha,n}$ such that the set $Y_\alpha = Y \cap U_\alpha$ is given by $z_{\alpha,1} = \ldots = z_{\alpha,k} = 0$ and such that $z_{\alpha,k+1}, \ldots, z_{\alpha,n}$ are holomorphic coordinates on Y_α.

Define a map $v_\alpha : U_\alpha \times Y_\alpha \to \mathbf{C}^{n-k}$ by

$$v_\alpha(x, y) = (z_{\alpha,i}(x) - z_{\alpha,i}(y))_{i=k+1,\ldots,n}$$

where $x \in U_\alpha$, $y \in Y_\alpha$, and $z_{\alpha,i}(p)$ represents the value of the holomorphic coordinate $z_{\alpha,i}$ at the point p.

For each y in Y_α, the restriction of v_α to $Y_\alpha \times \{y\} \subset U_\alpha \times Y_\alpha$ is holomorphic in x and gives a system of holomorphic coordinates on Y_α, centered at y. For y in $Y_\alpha \cap Y_\beta$, let $J_{\alpha,\beta}(y)$ be the Jacobian matrix relating v_α-coordinates to v_β-coordinates at y.

Let $\{\sigma_\alpha\}$ be a C^∞ partition of unity on Y, subordinate to the cover $\{Y_\alpha\}$ of Y. Pick any Riemannian metric on X and let

$$d = \min_\alpha \operatorname{dist}(\operatorname{supp}(\sigma_\alpha), U_\alpha^c).$$

Since $\operatorname{supp}(\sigma_\alpha)$ is closed and contained in U_α, and $\{U_\alpha\}$ is a finite cover of Y with $\overline{U_\alpha}$ compact in X, we have $0 < d < \infty$. Let W_α be the neighbourhood of $Y_\alpha \times Y_\alpha$ in $U_\alpha \times Y_\alpha$ defined by

$$W_\alpha = \left\{(x, y) \in U_\alpha \times Y_\alpha : \mathrm{dist}(x, y) < \frac{d}{2}\right\}.$$

In order for products of the form $\sigma_\gamma v_\gamma J_{\gamma,\alpha}$ to be defined on all of W_α, we make the convention that v_γ is 0 outside $U_\gamma \times Y_\gamma$ and $J_{\gamma,\alpha}$ is 0 outside $Y_\alpha \cap Y_\gamma$.

Define maps

$$A_\alpha : W_\alpha \to \mathbf{C}^{n-k}$$

by

$$A_\alpha(x, y) = \sum_\gamma \sigma_\gamma(y) v_\gamma(x, y) J_{\gamma,\alpha}(y).$$

Notice that if $(x, y) \in W_\alpha$ and $\sigma_\gamma(y) \neq 0$, then

$$\mathrm{dist}(x, \mathrm{supp}(\sigma_\gamma)) < \frac{d}{2}$$

so $x \in U_\gamma$. Then A_α is a C^∞ function and the restriction of A_α to $(U_\alpha \times \{y\}) \cap W_\alpha$ is holomorphic.

On $W_\alpha \cap W_\beta$ we have

$$\begin{aligned} A_\beta(x, y) J_{\beta,\alpha}(y) &= \sum_\gamma \sigma_\gamma(y) v_\gamma(x, y) J_{\gamma,\beta}(y) J_{\beta,\alpha}(y) \\ &= \sum_\gamma \sigma_\gamma(y) v_\gamma(x, y) J_{\gamma,\alpha}(y) \\ &= A_\alpha(x, y) \end{aligned}$$

so $A_\alpha(x, y) = 0$ if and only if $A_\beta(x, y) = 0$.

For each $y_0 \in Y_\alpha$, the restriction of A_α to $\{y_0\} \times Y_\alpha$ gives a holomorphic coordinate system on a neighbourhood of y_0 in Y_α. By the implicit function theorem, there exist C^∞ functions $u_{k+1}, \ldots, u_n$ on a neighbourhood of y_0 in X, such that in a neighbourhood of (y_0, y_0) in W_α, $A_\alpha(x, y) = 0$ if and only if $z_{\alpha,i}(y) = u_i(x)$ for $k+1 \leq i \leq n$. Locally, define a map τ by

$$\tau(x) = (u_i(x))_{k+1 \leq i \leq n}$$

so that $A_\alpha(x, y) = 0$ if and only if $y = \tau(x)$. Since on $W_\alpha \cap W_\beta$, $A_\alpha(x, y) = 0$ if and only if $A_\beta(x, y) = 0$, the map τ may be extended to a C^∞ function on a neighbourhood W of Y in X, with the property: for each γ, $A_\gamma(x,y) = 0$ if and only if $y = \tau(x)$. Since the restriction of A_α to $(U_\alpha \times \{y\}) \cap W_\alpha$ is holomorphic and the restriction to $Y_\alpha \times \{y_0\}$ is nondegenerate in a neighbourhood of (y_0, y_0), there is a neighbourhood $U(y_0)$ of y_0 in W such that for y in $Y \cap U(y_0)$, the fibre $\tau^{-1}(y) \cap U(y_0)$ of τ is a holomorphic manifold of dimension k, transverse to Y. Cover Y by a finite number of such neighbourhoods, let U be their union, and restrict τ to be the projection $\tau : U \to Y$. □

Proof of Proposition A.2. Let $\tau : U \to Y$ be as in Proposition 1. Let $\{U_\alpha\}$ be a finite open covering of Y in X with properties (i) and (ii) of the proof of Proposition 1 and such that:

iii. There are local trivializations of the normal bundle

$$\phi_\alpha : N\,|_{Y_\alpha} \to Y_\alpha \times \mathbf{C}^k$$

$$\phi_\alpha(\xi) = (y, \xi_\alpha)$$

with transition functions $g_{\alpha,\beta}$ on $Y_\alpha \cap Y_\beta$, such that $g_{\alpha,\beta}(y)\xi_\beta = \xi_\alpha$. Note that $g_{\alpha,\beta}$ is a $k \times k$ matrix so ξ_α and ξ_β should be regarded as column vectors. In terms of holomorphic coordinate systems z_α and z_β on U_α and U_β, the matrix $g_{\alpha,\beta}$ is given by

$$g_{\alpha,\beta} = \left(\frac{\partial z_{\alpha,i}}{\partial z_{\beta,j}}\right)_{1 \le i,j \le k}.$$

Let $Y_\alpha = Y \cap U_\alpha$. Shrinking the sets U_α and U if necessary, we may assume that the union of the sets U_α is U and that $\tau^{-1}(Y_\alpha) = U_\alpha$.

To give a map $\psi : U \to N$, it is enough to give maps

$$\psi_\alpha : U_\alpha \to Y_\alpha \times \mathbf{C}^k$$

of the form

$$\psi_\alpha(x) = (\tau(x), \xi_\alpha(x))$$

such that $g_{\alpha,\beta}(\tau(x))\xi_\beta(x) = \xi_\alpha(x)$ for $x \in U_\alpha \cap U_\beta$.

Let $\{\sigma_\alpha\}$ be a partition of unity on Y, subordinate to the cover $\{Y_\alpha\}$. Let $\tilde{z}_\gamma = (z_{\gamma,i})_{i=1,\ldots,k}$. We use the convention that $g_{\alpha,\gamma}(y) = 0$ if $y \notin Y_\alpha \cap Y_\gamma$ and $z_\gamma(x) = 0$ if $x \notin U_\gamma$. Define maps

$$\xi_\alpha : U_\alpha \to \mathbf{C}^k$$

by

$$\xi_\alpha(x) = \sum_\gamma \sigma_\gamma(\tau(x)) g_{\alpha,\gamma}(\tau(x)) \tilde{z}_\gamma(x).$$

Notice that ξ_α is C^∞ since $\operatorname{supp}(\sigma_\gamma) \subset Y_\gamma$. On $U_\alpha \cap U_\beta$ we have

$$\begin{aligned} g_{\alpha,\beta}(\tau(x))\xi_\beta(x) &= \sum_\gamma \sigma_\gamma(\tau(x)) g_{\alpha,\beta}(\tau(x)) g_{\beta,\gamma}(\tau(x)) \tilde{z}_\gamma(x) \\ &= \sum_\gamma \sigma_\gamma(\tau(x)) g_{\alpha,\gamma}(\tau(x)) \tilde{z}_\gamma(x) \\ &= \xi_\alpha(x) \end{aligned}$$

so that the maps $\psi_\alpha(x) = (\tau(x), \xi_\alpha(x))$ do indeed define a C^∞ map $\psi : U \to N$.

Next we check that ψ has the desired properties. If $y \in Y$ then $\tau(y) = y$ and $\tilde{z}_\gamma(y) = 0$ for all γ, so the restriction of ψ to Y is the natural map of Y onto the zero section of N. The restriction of ψ to the fibre $U_y = \tau^{-1}(y)$ of τ is given by

$$\psi(x) = (y, \xi_\alpha(x))$$

where

$$\xi_\alpha(x) = \sum_\gamma \sigma_\gamma(y) g_{\alpha,\gamma}(y) \tilde{z}_\gamma(x),$$

so $\psi \mid_{U_y}$ is a holomorphic map to the fibre N_y of N. To see that (after shrinking U if necessary) ψ is a diffeomorphism onto its image and the restriction of ψ to any fibre of τ is a local biholomorphism, note that in a neighbourhood of y on the fibre U_y of τ, ξ_α is a nondegenerate map to $\mathbf{C}^k$. □

Proof of Proposition A.3. Let $\tau : U \to Y$ be as in Proposition 1 and let $r = \operatorname{rank}(F)$. Let $\{U_\alpha\}$ be a finite open covering of Y in X with properties (i) and (ii) of the proof of Proposition 1 and such that:

iii. There are local trivializations of F

$$\phi_\alpha : F \mid_{U_\alpha} \to U_\alpha \times \mathbf{C}^r$$

$$\phi_\alpha(\xi) = (x, \xi_\alpha)$$

with transition functions $g_{\alpha,\beta}$ on $U_\alpha \cap U_\beta$ such that $g_{\alpha,\beta}(x)\xi_\beta = \xi_\alpha$.

Let $Y_\alpha = Y \cap U_\alpha$. Shrinking the sets U_α and U if necessary, we may assume that the union of the sets U_α is U and that $\tau^{-1}(Y_\alpha) = U_\alpha$. Transition functions for $\tau^* F$ on $U_\alpha \cap U_\beta$ are $g_{\alpha,\beta} \circ \tau$. To give a map $\rho : F_U \to \tau^* F_Y$, it is enough to give maps

$$\rho_\alpha : U_\alpha \times \mathbf{C}^r \to U_\alpha \times \mathbf{C}^r$$

of the form

$$\rho_\alpha(x, \xi_\alpha) = (x, f_\alpha(x, \xi_\alpha))$$

such that $g_{\alpha,\beta}(\tau(x)) f_\beta(x, \xi_\beta) = f_\alpha(x, \xi_\alpha)$ for $x \in U_\alpha \cap U_\beta$.

Let $\{\sigma_\alpha\}$ be a partition of unity on Y, subordinate to the covering $\{Y_\alpha\}$. We use the convention that $g_{\alpha,\gamma}(x) = 0$ if $x \notin U_\alpha \cap U_\gamma$. Define maps

$$f_\alpha : U_\alpha \times \mathbf{C}^r \to \mathbf{C}^r$$

by

$$f_\alpha(x, \xi_\alpha) = \sum_\gamma \sigma_\gamma(\tau(x)) g_{\alpha,\gamma}(\tau(x)) g_{\gamma,\alpha}(x) \xi_\alpha.$$

Since $g_{\gamma,\alpha}(x)\xi_\alpha = \xi_\gamma$ we have

$$f_\alpha(x, \xi_\alpha) = \sum_\gamma \sigma_\gamma(\tau(x)) g_{\alpha,\gamma}(\tau(x)) \xi_\gamma.$$

On the restriction of F to $U_\alpha \cap U_\beta$ we have

$$\begin{aligned} g_{\alpha,\beta}(\tau(x)) f_\beta(x, \xi_\beta) &= \sum_\gamma \sigma_\gamma(\tau(x)) g_{\alpha,\beta}(\tau(x)) g_{\beta,\gamma}(\tau(x)) \xi_\gamma \\ &= \sum_\gamma \sigma_\gamma(\tau(x)) g_{\alpha,\gamma}(\tau(x)) \xi_\gamma \\ &= f_\alpha(x, \xi_\alpha). \end{aligned}$$

Notice that f_α is C^∞ since $\operatorname{supp}(\sigma_\gamma) \subset Y_\gamma$.

Finally, we check that the map ρ defined by the functions f_α has the required properties. If $y \in Y_\alpha$ then $\tau(y) = y$ and

$$\begin{aligned} f_\alpha(y, \xi_\alpha) &= \sum_\gamma \sigma_\gamma(y) g_{\alpha,\gamma}(y) \xi_\gamma \\ &= \sum_\gamma \sigma_\gamma(y) g_{\alpha,\gamma}(y) g_{\gamma,\alpha}(y) \xi_\alpha \\ &= \left(\sum_\gamma \sigma_\gamma(y)\right) \xi_\alpha \\ &= \xi_\alpha \end{aligned}$$

so ρ is the identity map on F_Y. After shrinking U if necessary, the map $\rho : F_U \to \tau^* F_Y$ will be a C^∞ vector bundle isomorphism. By construction, ρ commutes with projection to U. □

REFERENCES

[BM1] E. Bierstone and P. Milman, *Uniformization of analytic spaces*, J. Amer. Math. Soc., **2** (1989), 801-836.

[BM2] ______, *A simple constructive proof of Canonical Resolution of Singularities*, Effective Methods in Algebraic Geometry, Birkhäuser, Progr. Math., vol. **94**, Boston, (1991), 11-30.

[CL] C. H. Clemens, *Degeneration of Kähler Manifolds*, Duke Math. J., **44** (1977), 214-290.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[GM] M. Goresky and R. MacPherson, *Intersection Homology II*, Invent. Math., **71** (1983), 77-129.

[GrM] C. Grant and P. Milman, *Metrics for singular analytic spaces*, preprint (Dec. 1990), 1-33.

[H] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero:* I, II, Ann. Math., **79** (1964), 109-326.

[O] T. Ohsawa, *On the L^2 cohomology of complex spaces*, Math. Z., **209** (1992), 519-530.

[S1] L. Saper, *L_2-cohomology and intersection homology of certain algebraic varieties with isolated singularities*, Invent. Math., **82** (1985) 207-255.

[S2] ______, *L_2-cohomology of Kähler varieties with isolated singularities*, J. Diff. Geom., **36** (1992), 89-161.

[W] R. O. Wells, *Differential Analysis on Complex Manifolds*, Springer-Verlag, New York, 1980.

[Z1] S. Zucker, *Hodge theory with degenerating coefficients: L_2 cohomology in the Poincaré metric*, Ann. of Math., **109** (1979), 415-476.

[Z2] ______, *L_2 Cohomology of Warped Products and Arithmetic Groups*, Invent. Math., **70** (1982), 169-218.

Received September 24, 1992. The first author was partially supported by a grant from the Naval Academy Research Council. The second author wes partially supported by a grant from NSERC.

U.S. NAVAL ACADEMY
572 HOLLOWAY RD.

Annapolis, MD 21402-5002
E-mail address: cgg@sma.usna.navy.MIL

and

University of Toronto
Toronto, Ontario M53 1A1
E-mail address: milman@math.toronto.edu