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FOURIER COEFFICIENTS OF AN ORTHOGONAL EISENSTEIN SERIES

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This paper defines a nonholomorphic Eisenstein series for a totally real algebraic number field F and the special orthogonal group with respect to a bilinear form $S = \begin{pmatrix} T & & \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix}$, where $T \in M_n(F)$ and its embedded images $T^v \in M_n(\mathbb{R})$ under archimedean places v of F have signature $(1, n-1)$. This group has an associated product of tube domains $\mathcal{H}^{\mathbf{a}} = \prod_{v \in \mathbf{a}} \mathcal{H}_v$, the product taken over archimedean places of F and each $\mathcal{H}_v \subset \mathbb{C}^n$. The series is denoted $E(z, s; k, \psi, \mathfrak{b})$ or simply $E(z, s)$, with $z \in \mathcal{H}^{\mathbf{a}}$, $s \in \mathbb{C}$ a complex parameter, $k \in \mathbb{Z}$ the weight, ψ a Hecke character on the ideles of F, and the level $\mathfrak{b}$ an integral ideal in F. E has the Fourier expansion

$$E(z,s) = (-1)^{dk} 2^{d(k+2s)} \sum_{h \in L'} a(h, y, s)\, \mathbf{e}\left(\sum_{v \in \mathbf{a}} T^v(x_v, h_v) \right),$$

where $d = [F : \mathbb{Q}]$, L' is the lattice dual to $\mathfrak{o}_F^n$ under T, $\mathbf{e}(x) = e^{2\pi i x}$, and $z = (x_v + i y_v)_{v \in \mathbf{a}} \in \mathcal{H}^{\mathbf{a}}$. The Fourier coefficient $a(h, y, s)$ is the product $(N\mathfrak{d})^{-\frac{n}{2}} a_{\mathbf{a}}(h, y, s) a_{\mathbf{f}}(h, s)$ with $N\mathfrak{d}$ the norm of the different of F over $\mathbb{Q}$. The archimedean factor is $a_{\mathbf{a}}(h, y, s) = \prod_{v \in \mathbf{a}} \xi(y_v, h_v; k+s, s; T^v)$ with ξ a certain confluent hypergeometric function studied by Shimura. The nonarchimedean factor $a_{\mathbf{f}}(h, s)$ is essentially a product and quotient of Hecke L-functions, depending on the parity of n and the nature of h. Specializing to $s = 0$ gives holomorphic and in special cases nearly holomorphic behavior.

1. Introduction and notation.

Introduction. This paper defines an Eisenstein series $E(z, s)$ of weight k for z in a tube domain and s a complex parameter, and computes its Fourier expansion explicitly. The series is of interest as a special case of the nearly holomorphic functions studied by Shimura and Bluher.

Section 2 describes the action of a subgroup of the adelization of a certain orthogonal group on an associated complex domain. A tube domain $\mathcal{H}$ is associated to a bilinear form S of signature $(2, n)$ on $\mathbb{R}^{n+2}$, and the identity component of $\mathrm{SO}(S, \mathbb{R})$, the special orthogonal group over $\mathbb{R}$ with respect to S, acts on $\mathcal{H}$. Take a totally real algebraic number field F, a symmetric matrix S all of whose embedded images S^v in $\mathrm{M}_{n+2}(\mathbb{R})$ under archimedean places v of F have signature $(2, n)$, and the algebraic group $G = \mathrm{SO}(S, F)$. Then $G_{\mathbf{A}+}$, a suitable subgroup of the adelization of G, acts on $\mathcal{H}^{\mathfrak{a}}$, a product of tube domains $\mathcal{H}_v$ over the archimedean places v of F.

Section 3 defines an Eisenstein series $E(z, s)$ for $z \in \mathcal{H}^{\mathfrak{a}}$ and $s \in \mathbb{C}$, and shows that it has a Fourier expansion. The series agrees with a series studied by Indik in the case $F = \mathbb{Q}$. $E(z, s)$ has an associated series $\widetilde{E}(y, s)$ for y in a certain subset of $G_{\mathbf{A}+}$. Harmonic analysis gives a Fourier expansion of $\widetilde{E}(y, s)$ with coefficients $b(h, w_y, s)$, where h runs through a lattice in F^n and w_y depends on $y = \mathrm{Im}(z)$. This transforms back to a Fourier expansion of $E(z, s)$.

Section 4 expresses the global Fourier coefficient $a(h, y, s)$ of $E(z, s)$ as a simple factor multiplied by a product of local coefficents $a_v(h, y, s)$, the product being taken over all places of F. For archimedean v, $a_v(h, y, s)$ is equal to a certain confluent hypergeometric function ξ studied by Shimura.

Section 5 continues to study the local coefficients of $E(z, s)$. The coefficients at finite places v dividing $\mathfrak{b}$ (where $\mathfrak{b}$, an integral ideal of F, is the level of $E(z, s)$) are equal to 1. The coefficients at finite places v not dividing $\mathfrak{b}$ are power series $\alpha_v(h_v, X) = \sum_\lambda S_v(\lambda, h_v) X^\lambda$ evaluated at certain values of X, where the coefficients $S_v(\lambda, h_v)$ are sums of exponentials.

Section 6 expresses the power series $\alpha_v(h_v, X)$ as a simple rational expression of Euler factors of Hecke L-functions, which depend on the v-adic nature of the lattice vector h. In some cases $\alpha_v(h_v, X)$ is not expressed precisely, but then it is a polynomial of bounded degree. Taking the product of $\alpha_v(h_v, X)$ over finite places v not dividing $\mathfrak{b}$ expresses the finite part of $a(h, y, s)$ as essentially a product and quotient of Hecke L-functions. Thus the Fourier coefficients of $E(z, s)$ are explicit expressions in well understood functions, up to some polynomial factors. The methods in this section are from Indik.

Section 7 specializes the Eisenstein series to $s = 0$ to obtain holomorphic and in special cases nearly holomorphic behavior. Also, for certain values of k and s, $E(z, s)$ is either finite or exhibits a simple pole with residue that is holomorphic up to a factor.

My warmest thanks to Goro Shimura for suggesting this problem as a Ph.D thesis and for all his generous help as my advisor.

Notation. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbf{T}$ denote the integers, the rational, real and complex numbers, and the unit circle $\{ z \in \mathbb{C} : |z| = 1 \}$. For an associative ring A with identity, A^* denotes the group of invertible elements of A. When A is commutative, $\mathrm{M}_n(A)$ denotes the ring of n-by-n matrices with entries in A, $\mathrm{GL}_n(R)$ means $\mathrm{M}_n(R)^*$, and $\mathrm{SL}_n(R)$ denotes the elements of $\mathrm{GL}_n(R)$ with determinant 1. $\left(\frac{-}{-}\right)$ denotes the Jacobi symbol, and for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer n such that $n < x$.

2. Archimedean and adelic preliminaries.

The quadratic forms T and S and the complex domain $\mathcal{H}$. Let $n > 2$ be an integer, and let T, a symmetric element of $M_n(\mathbb{R})$, define a quadratic form of signature $(1, n-1)$ on $\mathbb{R}^n$. Write $T(x, y) = {}^t xTy$ and $T[x] = T(x, x)$ for $x, y \in \mathbb{C}^n$. Set

$$S = \begin{pmatrix} T & & \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix}, \tag{2.1}$$

defining a quadratic form of signature $(2, n)$ on $\mathbb{R}^{n+2}$, and write $S(x, y) = {}^t xSy$, $S[x] = S(x, x)$ for $x, y \in \mathbb{C}^{n+2}$.

Fix $\varepsilon \in \mathbb{R}^n$ such that $T[\varepsilon] = 1$. Define a set $\mathcal{P}$ of "positive" elements in $\mathbb{R}^n$ by

$$\mathcal{P} = \{ y \in \mathbb{R}^n : T[y] > 0 \text{ and } T(y, \varepsilon) > 0 \}$$

and a complex domain $\mathcal{H}$ by

$$\mathcal{H} = \{ z = x + iy \in \mathbb{C}^n : y \in \mathcal{P} \}.$$

$\mathcal{P}$ and $\mathcal{H}$ are connected.

The action of $\mathrm{SO}(S,\mathbb{R})^\circ$ on $\mathcal{H}$. Let $\mathcal{G} = \mathrm{SO}(S,\mathbb{R})^\circ$, where "$\circ$" denotes the identity component and

$$\mathrm{SO}(S,\mathbb{R}) = \left\{ \alpha \in \mathrm{SL}_{n+2}(\mathbb{R}) : {}^t\alpha S \alpha = S \right\}.$$

Thus for $x, y \in \mathbb{C}^{n+2}$ and $\alpha \in \mathcal{G}$, $S(\alpha x, \alpha y) = S(x,y)$ and $S[\alpha x] = S[x]$.

For $z \in \mathbb{C}^n$, define $w(z) = \begin{pmatrix} z \\ \frac{1}{2}T[z] \\ 1 \end{pmatrix} XS \in \mathbb{C}^{n+2}$. Any S-isotropic $w \in \mathbb{C}^{n+2}$ with bottom entry 1 is of this form. If $z \in \mathcal{H}$ and $\alpha \in \mathcal{G}$ then $\{ \mathrm{Re}(\alpha w(z)), \mathrm{Im}(\alpha w(z)) \}$ forms an orthogonal basis $\{ u, v \}$ (with $S[u] = S[v]$) of a subspace in $\mathbb{R}^{n+2}$ where S is positive definite. Set $j(\alpha, z) = \alpha w(z)_{n+2}$, which is nonzero, and define $\alpha(z) \in \mathbb{C}^n$ by

$$w(\alpha(z)) = j(\alpha,z)^{-1}\alpha w(z). \tag{2.2}$$

Since $j(\alpha,z)^{-1}\alpha w(z)$ is S-isotropic and has bottom entry 1, such an $\alpha(z)$ indeed exists.

To show that $\alpha(z) \in \mathcal{H}$, first note that $0 < T[\mathrm{Im}(\alpha(z))] = S[\mathrm{Im}(w(\alpha(z)))]$ follows from (2.2) and the properties of $\{ u, v \}$. Also, $T(\mathrm{Im}(\alpha(z)), \varepsilon) > 0$: because $T(\mathrm{Im}(\alpha(z)), \varepsilon)$ can not vanish as T is negative definite on $\{ x \in \mathbb{R}^n : T(x, \varepsilon) = 0 \}$ but positive at $\mathrm{Im}(\alpha(z))$, it suffices to show $T(\mathrm{Im}(\alpha(z)), \varepsilon) > 0$ for one α from the connected group $\mathcal{G}$, and taking $\alpha = I_{n+2}$ completes the proof.

Not all of $\mathrm{SO}(S,\mathbb{R})$ acts on $\mathcal{H}$ because while $\mathcal{G}$ fixes $\mathcal{H}$ and $-\mathcal{H}$, the other component interchanges them. Taking $\alpha = \begin{pmatrix} I_n & \\ & -I_2 \end{pmatrix}$, so that $\alpha(z) = -z$, shows this. From (2.2), the action of $\mathcal{G}$ on $\mathcal{H}$ is associative and j is a factor of automorphy. The action is well known to be transitive.

The field F and the group G. Let F denote a totally real algebraic number field of degree d, $\mathfrak{o}_F$ the ring of algebraic integers in F, and $\boldsymbol{a} = \{ v_1, \dots, v_d \}$ the set of archimedean places of F. Each $v \in \boldsymbol{a}$ is an embedding $v : F \hookrightarrow \mathbb{R}$. Take T a symmetric element of $M_n(\mathfrak{o}_F)$ such that T^v defines a form of signature $(1, n-1)$ on $\mathbb{R}^n$ for each $v \in \boldsymbol{a}$. Define S as in (2.1), so that the S^v for all $v \in \boldsymbol{a}$ define forms of signature $(2, n)$. For each $v \in \boldsymbol{a}$ take an $\varepsilon_v \in \mathbb{R}^n$ such that $T^v[\varepsilon_v] = 1$. Set

$$G = \mathrm{SO}(S,F) = \left\{ \alpha \in \mathrm{SL}_{n+2}(F) : {}^t\alpha S \alpha = S \right\}.$$

The action of $G_{\mathbf{A}+}$ on $\mathcal{H}^{\boldsymbol{a}}$. Let $\boldsymbol{f}$ and $\boldsymbol{a}$ denote the set of nonarchimedean and archimedean places of F, respectively. For $v \in \boldsymbol{f} \cup \boldsymbol{a}$ denote by F_v the v-completion of F and, if $v \in \boldsymbol{f}$, by $\mathfrak{o}_v$ the v-closure of $\mathfrak{o}_F$ in F_v; if $v \in \boldsymbol{a}$, identify F_v with $\mathbb{R}$. Denote the adeles and ideles of F as $F_{\mathbf{A}}$ and $F_{\mathbf{A}}^*$ and identify F with its embedded images in $F_{\mathbf{A}}$ and F_v for any v. $F_{\boldsymbol{f}}$ denotes the adeles $(a_v)_{v \in \boldsymbol{f} \cup \boldsymbol{a}}$ such that $a_v = 0$ for $v \notin \boldsymbol{f}$, $F_{\boldsymbol{a}}$ is defined similarly, and $\mathfrak{o}_{\boldsymbol{f}}$ denotes the elements of $F_{\boldsymbol{f}}$ such that $a_v \in \mathfrak{o}_v$ for all $v \in \boldsymbol{f}$; $F_{\boldsymbol{f}}^*$, $F_{\boldsymbol{a}}^*$ and $\mathfrak{o}_{\boldsymbol{f}}^*$ are the similarly defined subgroups of $F_{\mathbf{A}}^*$. The image of $\mathfrak{o}_F$ in $\mathbb{R}^d$ under $x \mapsto (x^v)_{v \in \boldsymbol{a}}$ is a lattice Λ of volume $(N\mathfrak{d})^{\frac{1}{2}}$, where N denotes the norm from F to $\mathbb{Q}$ and $\mathfrak{d}$ denotes the different of F over $\mathbb{Q}$.

Define G_v to be the v-completion of G for $v \in \boldsymbol{f} \cup \boldsymbol{a}$. Thus if $v \in \boldsymbol{a}$, G_v can be identified with $\mathrm{SO}(S^v, \mathbb{R})$. Take the adelization $G_{\mathbf{A}}$ of G; put $G_{\boldsymbol{f}} = \prod_{v \in \boldsymbol{f}} G_v \cap G_{\mathbf{A}}$, $G_{\boldsymbol{a}} = \prod_{v \in \boldsymbol{a}} G_v$. Identify G with its embedded image in $G_{\mathbf{A}}$ and the same convention holds for other groups defined below. For $x \in G_{\mathbf{A}}$ define $x_{\boldsymbol{f}} \in G_{\boldsymbol{f}}$ and $x_{\boldsymbol{a}} \in G_{\boldsymbol{a}}$ by $x = x_{\boldsymbol{f}} x_{\boldsymbol{a}}$. Define

$$G_{\mathbf{A}+} = \{\, x \in G_{\mathbf{A}} : x_v \in \mathrm{SO}(S^v, \mathbb{R})^\circ \text{ for all } v \in \boldsymbol{a} \,\}$$

and $G_{\boldsymbol{a}+} = G_{\boldsymbol{a}} \cap G_{\mathbf{A}+}$, $G_+ = G \cap G_{\mathbf{A}+}$.

For each $v \in \boldsymbol{a}$, let $\mathcal{H}_v$ be the complex domain of the previous section associated to T^v and ε_v. Denote $\prod_{v \in \boldsymbol{a}} \mathcal{H}_v$ as $\mathcal{H}^{\boldsymbol{a}}$ and define the action of $G_{\boldsymbol{a}+}$ on $\mathcal{H}^{\boldsymbol{a}}$ componentwise. The action extends to $G_{\mathbf{A}+}$ by defining $x \in G_{\mathbf{A}+}$ to act as $x_{\boldsymbol{a}}$.

3. The Eisenstein series $E(z, s; k, \psi, \mathfrak{b})$ and its Fourier expansion.

The series E on $\mathcal{H}^{\boldsymbol{a}}$. Fix an integer k. Take a Hecke character $\psi : F_{\mathbf{A}}^* \to \mathbf{T}$ ($\psi(F^*) = 1$) with $\psi(a) = \prod_{v \in \boldsymbol{a}} \mathrm{sgn}(a_v)^k$ for $a \in F_{\boldsymbol{a}}^*$; let $\mathfrak{c}$ denote the finite part of its conductor, ψ_v the v-component of ψ, and $\psi_\iota = \prod_{v | \iota} \psi_v$ for any integral ideal ι. Let $\mathfrak{b} \subset F$ be an integral ideal divisible by $\mathfrak{c}$, by 2, and by $\det T$. Define $\mathcal{U} = \{\, u \in F^{n+2} : S[u] = 0 \,\}$, and for $u \in \mathcal{U}$, $z \in \mathcal{H}^{\boldsymbol{a}}$, set $S(u, w(z)) = \prod_{v \in \boldsymbol{a}} S^v(u_v, w_v(z))$, where $w_v(z) = \begin{pmatrix} z_v \\ \frac{1}{2} T^v[z_v] \\ 1 \end{pmatrix}$. Our

Eisenstein series is defined as follows:

$$E(z,s;k,\psi,\mathfrak{b}) = \sum_{(u,t)\in\mathcal{U}\times F_{\boldsymbol{f}}^*/\sim} c(tu)\psi(t)^{-1}|t|^{k+2s}S(u,w(z))^{-k}|S(u,w(z))|^{-2s}$$

for $z\in\mathcal{H}^{\boldsymbol{a}}$ and $s\in\mathbb{C}$, where $(u,t)\sim(u',t')$ means that for some $b\in F^*$, $u'=bu$ and $t'\mathfrak{o}_F=b^{-1}t\mathfrak{o}_F$ (so that $t'=eb_{\boldsymbol{f}}^{-1}t$ with $e\in\mathfrak{o}_{\boldsymbol{f}}^*$). Here $c:F_{\mathbf{A}}^{n+2}\to\mathbb{C}$ is the locally constant function

$$c(x)=\begin{cases}\psi_{\mathfrak{b}}(x_{n+2}), & \text{if } x_{\boldsymbol{f}}\in\mathfrak{o}_{\boldsymbol{f}}^{n+2} \text{ and } x_{n+2} \text{ is prime to } \mathfrak{b}\\ 0, & \text{otherwise.}\end{cases}$$

This series is also denoted simply $E(z)$ or $E(z,s)$.

E is readily seen to be well-defined. The series converges for sufficiently large $\mathrm{Re}(s)$ and has an analytic continuation, as shown in [**Sh80**]. In the special case $F=\mathbb{Q}$, E reduces to the series studied by Indik in [**In**].

Transformation of E. Define subgroups of $G_{\mathbf{A}+}$ by

$$P_{\mathbf{A}}=\left\{\gamma\in G_{\mathbf{A}+}:\gamma=\begin{pmatrix}*&*&*\\ &*&*\\ 0&0&*\end{pmatrix}\right\};$$

$$C=\prod_v C_v,\ \text{where } C_v=\begin{cases}\mathrm{SO}(S,\mathfrak{o}_v) & \text{if } v\in\boldsymbol{f},\\ \text{stabilizer of } i\varepsilon_v & \text{if } v\in\boldsymbol{a};\end{cases}$$

$$D=\left\{\gamma\in C:\gamma\equiv\begin{pmatrix}*&*&*\\ &*&*\\ 0&0&d_\gamma\end{pmatrix}\pmod{\mathfrak{b}}\right\};$$

and $\Gamma_0(\mathfrak{b})\subset G_+$ by

$$\begin{aligned}\Gamma_0(\mathfrak{b})&=G_+\cap DG_{\boldsymbol{a}}\\ &=\left\{\gamma\in G_+\cap\mathrm{SO}(S,\mathfrak{o}_F):\gamma\equiv\begin{pmatrix}*&*&*\\ &*&*\\ 0&0&d_\gamma\end{pmatrix}\pmod{\mathfrak{b}}\right\}.\end{aligned}$$

For $\gamma\in G_{\mathbf{A}+}$ and $z\in\mathcal{H}^{\boldsymbol{a}}$ define

$$J(\gamma,z)=j(\gamma,z)^k|j(\gamma,z)|^{2s}\qquad\text{where } j(\gamma,z)=\prod_{v\in\boldsymbol{a}}j(\gamma_v,z_v),$$

$$J_\psi(\gamma,z)=\psi_{\mathfrak{b}}(d_\gamma)J(\gamma,z).$$

The relation $J(\alpha\beta, z) = J(\alpha, \beta z)J(\beta, z)$ holds for all $\alpha, \beta \in G_{\mathbf{A}+}$, and the same relation holds for J_ψ when $\alpha, \beta \in DG_{\mathfrak{a}+}$.

For $\gamma \in \Gamma_0(\mathfrak{b})$ and $z \in \mathcal{H}^{\mathfrak{a}}$ one easily verifies that

$$E(\gamma(z)) = J_\psi(\gamma, z)E(z).$$

If, in particular,

$$\gamma \in \Gamma_0(\mathfrak{b}) \cap N, \text{where} N = \left\{ \left(\begin{smallmatrix} 1 & 0 & b \\ & 1 & * \\ 0 & 0 & 1 \end{smallmatrix}\right) : b \in F^n \right\},$$

then $b \in \mathfrak{o}_F^n$, $\gamma(z) = z + b$, and $J_\psi(\gamma, z) = 1$. Thus, $E(z+b) = E(z)$ for $b \in \mathfrak{o}_F^n$.

The series $\widetilde{E}$ on $G_+DG_{\mathfrak{a}+}$. Define $\widetilde{E}(y, s)$ for $y \in G_+DG_{\mathfrak{a}+}$ and $s \in \mathbb{C}$ by

$$\widetilde{E}(y, s) = E(x(i\varepsilon), s)J_\psi(x, i\varepsilon)^{-1} \qquad \text{for } y = \alpha x \text{ with } \alpha \in G_+, x \in DG_{\mathfrak{a}+}.$$

Here $i\varepsilon$ means $(i\varepsilon_v)_{v\in\mathfrak{a}} \in \mathcal{H}^{\mathfrak{a}}$. $\widetilde{E}(y, s)$ is well defined. Denote this series also $\widetilde{E}(y)$. Then

$$\widetilde{E}(\alpha y w) = \widetilde{E}(y)J_\psi(w, i\varepsilon)^{-1} \text{for } \alpha \in G_+,\ y \in G_+DG_{\mathfrak{a}+},\ w \in D.$$

To write $\widetilde{E}$ explicitly, first note that

$$S(u, w(x(i\varepsilon))) = j(x, i\varepsilon)^{-1}S(x^{-1}u, w(i\varepsilon)).$$

So for $\alpha \in G_+$, $x \in DG_{\mathfrak{a}+}$,

$$\begin{aligned}
&\widetilde{E}(\alpha x) \\
&\quad = \sum_{(u,t)} c(tu)\psi(t)^{-1}|t|^{k+2s}J(x, i\varepsilon)S(x^{-1}u, w(i\varepsilon))^{-k} \\
&\qquad\qquad \cdot |S(x^{-1}u, w(i\varepsilon))|^{-2s}J_\psi(x, i\varepsilon)^{-1} \\
&\quad = \sum \psi_{\mathfrak{b}}(d_{x^{-1}})c(tu)\psi(t)^{-1}|t|^{k+2s}S(x^{-1}u, w(i\varepsilon))^{-k} \\
&\qquad\qquad \cdot |S(x^{-1}u, w(i\varepsilon))|^{-2s} \\
&\quad = \sum c(x^{-1}tu)\psi(t)^{-1}|t|^{k+2s}S(x^{-1}u, w(i\varepsilon))^{-k} \\
&\qquad\qquad \cdot |S(x^{-1}u, w(i\varepsilon))|^{-2s}.
\end{aligned}$$

The Fourier expansions of $\widetilde{E}$ and E. Let $V = F^n$ and $V_{\mathbf{A}} = F^n_{\mathbf{A}}$. For $x, y \in V_{\mathbf{A}}$ define a complex number $\chi(T(x,y))$:

$$\chi(T(x,y)) = \prod_{v \in \mathfrak{f} \cup \mathfrak{a}} \mathbf{e}_v(T(x_v, y_v))$$
$$= \prod_{v \in \mathfrak{f}} \mathbf{e}_p(Tr_{F_v/\mathbb{Q}_p}(T(x_v, y_v))) \prod_{v \in \mathfrak{a}} \mathbf{e}(T^v(x_v, y_v)),$$

where $v \mid p$, $\mathbf{e}_p(t) = \mathbf{e}$(the fractional part of $-t$) for $t \in \mathbb{Q}_p$, and $\mathbf{e}(s) = e^{2\pi i s}$ for $s \in \mathbb{C}$. Define

$$\tau(v) = \begin{pmatrix} 1 & 0 & v \\ & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in G_+ D G_{\mathfrak{a}+} \text{for} v \in V_{\mathbf{A}} (\tau(v) \in G_+ D G_{\mathfrak{a}+}$$

since $v = v' + w$ with $v' \in V$, $w \in \prod_{\mathfrak{f}} \mathfrak{o}^n_{\mathfrak{f}} \times F^n_{\mathfrak{a}}$), and fix a Haar measure μ on $V_{\mathbf{A}}$ so that $\mu(V_{\mathbf{A}}/V) = 1$.

Consider $\widetilde{E}(\tau(v)w)$ with $v \in V_{\mathbf{A}}$ and $w \in G_{\mathfrak{a}+}$ as a function on $V_{\mathbf{A}}$. Then for $u \in V$, $\widetilde{E}(\tau(v+u)w) = \widetilde{E}(\tau(u)\tau(v)w) = \widetilde{E}(\tau(v)w)$, so $\widetilde{E}$ is a function on $V_{\mathbf{A}}/V$. This gives the expansion

$$\widetilde{E}(\tau(v)w, s) = \sum_{h \in V} b(h, w, s)\chi(T(v,h)) \quad \text{for } v \in V_{\mathbf{A}}, w \in G_{\mathfrak{a}+},$$

where

$$b(h, w, s) = \int_{v \in V_{\mathbf{A}}/V} \widetilde{E}(\tau(v)w, s)\chi(-T(v,h))d\mu(v) \quad \text{for } h \in V.$$

Define lattices $L = \mathfrak{o}^n_F \subset V$ and $L_v = \mathfrak{o}^n_v \subset V_v$ for $v \in \mathfrak{f}$. For $u \in L_v$, $\widetilde{E}(\tau(v+u)w) = \widetilde{E}(\tau(v)w\tau(u)) = \widetilde{E}(\tau(v)w)J_\psi(\tau(u), i\varepsilon)^{-1} = \widetilde{E}(\tau(v)w)$. Hence $b(h,w,s) = \int_{v \in V_{\mathbf{A}}/V} \widetilde{E}(\tau(v+u)w, s)\chi(-T(v+u,h))d\mu(v) = \chi(-T(u,h))b(h,w,s)$; this shows that $b(h,w,s) \neq 0$ only when $\chi(-T(u,h)) = 1$, i.e., when $h \in L'$ with L' = the dual lattice to L under T, defined by $L' = \{ h \in V : T(h, L) \subset \mathfrak{d}^{-1} \}$, where $\mathfrak{d}$ is the different of F over $\mathbb{Q}$. Thus,

$$\widetilde{E}(\tau(v)w, s) = \sum_{h \in L'} b(h, w, s)\chi(T(v,h)) \text{for } v \in V_{\mathbf{A}}, w \in G_{\mathfrak{a}+}.$$

To express this on $\mathcal{H}^{\mathfrak{a}}$ for $z = (z_v)_{v \in \mathfrak{a}}$ with $z_v = x_v + iy_v$, put $w_y = (w_{y_v})_{v \in \mathfrak{a}}$ with

$$w_{y_v} = \begin{pmatrix} A_v & & \\ & \sqrt{T[y_v]} & \\ & & \sqrt{T[y_v]}^{-1} \end{pmatrix},$$

where $A_v\varepsilon_v = y_v/\sqrt{T^v[y_v]}$ and $T^v(A_vx, A_vy) = T^v(x,y)$ for $x, y \in \mathbb{R}^n$, so that $w_{y_v}(i\varepsilon_v) = iy_v$ and hence $w_y(i\varepsilon) = iy$. Then

$$\begin{aligned} E(z,s) &= \widetilde{E}(\tau(x)w_y, s)J_\psi(\tau(x)w_y, i\varepsilon) \\ &= \widetilde{E}(\tau(x)w_y, s)J(w_y, i\varepsilon), \end{aligned}$$

so

$$E(z,s) = J(w_y, i\varepsilon)\sum_{h\in L'} b(h, w_y, s)\,\boldsymbol{e}\Big(\sum_{v\in\boldsymbol{a}} T^v(x_v, h_v)\Big).$$

4. Fourier coefficients of E: reduction to the local case.

The coefficient $b(h, w_y, s)$**.** For $h \in L'$ and $x + iy \in \mathcal{H}^{\boldsymbol{a}}$ we have $b(h, w_y, s) = \int_{v\in V_{\mathbf{A}}/V} \widetilde{E}(\tau(v)w_y, s)\chi(-T(v,h))d\mu(v)$. Choosing representatives v of $V_{\mathbf{A}}/V$ such that $\tau(v) \in DG_{\boldsymbol{a}}$ gives

$$\begin{aligned} &b(h, w_y, s) \\ &= \int_{v\in V_{\mathbf{A}}/V}\Big\{\sum_{(u,t)\in\mathcal{U}\times F_{\boldsymbol{f}}^*/\sim} c\Big((\tau(v)w_y)^{-1}tu\Big)\psi(t)^{-1}|t|^{k+2s} \\ &\qquad\cdot S\Big((\tau(v)w_y)^{-1}u, w(i\varepsilon)\Big)^{-k} \\ &\qquad\cdot \Big|S\Big((\tau(v)w_y)^{-1}u, w(i\varepsilon)\Big)\Big|^{-2s}\chi(-T(v,h))\Big\}d\mu(v). \end{aligned}$$

If $u_{n+2} = 0$ then $((\tau(x)w_y)^{-1}tu)_{n+2} = 0$ at $\boldsymbol{f}$ since $(\tau(x)w_y)_{\boldsymbol{f}} \in P_{\mathbf{A}}$. So normalize $u_{n+2} = 1$ and sum over $\big\{\,(w(v'), t) : v' \in V,\ t \in F_{\boldsymbol{f}}^*/\mathfrak{o}_{\boldsymbol{f}}^*\,\big\}$. This gives

$$\begin{aligned} &b(h, w_y, s) \\ &= \int_{v\in V_{\mathbf{A}}/V}\Big\{\sum_{v'\in V} S\Big((\tau(v)w_y)^{-1}w(v'), w(i\varepsilon)\Big)^{-k} \\ &\qquad\cdot\Big|S\Big((\tau(v)w_y)^{-1}w(v'), w(i\varepsilon)\Big)\Big|^{-2s} \\ &\qquad\cdot\sum_{t\in F_{\boldsymbol{f}}^*/\mathfrak{o}_{\boldsymbol{f}}^*} c\Big((\tau(v)w_y)^{-1}tw(v')\Big) \\ &\qquad\cdot\psi(t)^{-1}|t|^{k+2s}\chi(-T(v,h))\Big\}d\mu(v) \end{aligned}$$

$$= \int_{v \in V_{\mathbf{A}}/V} \Big\{ \sum_{v' \in V} S\Big(w_y^{-1} w(v'-v), w(i\varepsilon)\Big)^{-k} \cdot \Big|S\Big(w_y^{-1} w(v'-v), w(i\varepsilon)\Big)\Big|^{-2s} \cdot \sum_{t \in F_f^*/\mathfrak{o}_f^*} c\Big(t w_y^{-1} w(v'-v)\Big)\psi(t)^{-1}|t|^{k+2s} \cdot \chi(-T(v-v',h)) \Big\} d\mu(v)$$

$$= \int_{v \in V_{\mathbf{A}}} \Big\{ S\Big(w_y^{-1} w(v), w(i\varepsilon)\Big)^{-k} \Big|S\Big(w_y^{-1} w(v), w(i\varepsilon)\Big)\Big|^{-2s} \cdot \sum_{t \in F_f^*/\mathfrak{o}_f^*} c(tw(v))\psi(t)^{-1}|t|^{k+2s}\chi(T(v,h)) \Big\} d\mu(v)$$

$$= \int_{v \in V_{\mathbf{A}}} \Big\{ S\Big(w(v), j(w_y, i\varepsilon) w(iy)\Big)^{-k} \Big|S\Big(w(v), j(w_y, i\varepsilon) w(iy)\Big)\Big|^{-2s} \cdot \sum_{t \in F_f^*/\mathfrak{o}_f^*} c(tw(v))\psi(t)^{-1}|t|^{k+2s}\chi(T(v,h)) \Big\} d\mu(v)$$

$$= J(w_y, i\varepsilon)^{-1} \int_{v \in V_{\mathbf{A}}} \Big\{ S\Big(w(v), w(iy)\Big)^{-k} \Big|S\Big(w(v), w(iy)\Big)\Big|^{-2s} \cdot \sum_{t \in F_f^*/\mathfrak{o}_f^*} c(tw(v))\psi(t)^{-1}|t|^{k+2s}\chi(T(v,h)) \Big\} d\mu(v).$$

LEMMA. *$S(w(v), w(iy)) = \left(-\frac{1}{2}\right)^d T_{\mathfrak{a}}[-v+iy]$, where $d = [F:\mathbb{Q}]$ and $T_{\mathfrak{a}}[x] = \prod_{v \in \mathfrak{a}} T^v[x_v]$ for $x \in V_{\mathbf{A}}$.*

Proof. Immediate from

$$S(w(v), w(iy)) = \prod_{v \in \mathfrak{a}} \begin{pmatrix} {}^t v_v & \frac{1}{2}T^v[v_v] & 1 \end{pmatrix} \begin{pmatrix} T^v & & \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix} \begin{pmatrix} iy_v \\ \frac{1}{2}T^v[iy_v] \\ 1 \end{pmatrix}.$$

∎

This gives

$$\begin{aligned}
&b(h, w_y, s)\\
&\quad = J(w_y, i\varepsilon)^{-1}(-1)^{dk}2^{d(k+2s)}\\
&\qquad \cdot \int_{v\in V_{\mathbf{A}}} T_{\mathfrak{a}}[-v+iy]^{-k}\,|T_{\mathfrak{a}}[-v+iy]|^{-2s}\,\sigma(v,s)\chi(T(v,h))d\mu(v)\\
&\quad = J(w_y, i\varepsilon)^{-1}(-1)^{dk}2^{d(k+2s)}\\
&\qquad \cdot \int_{v\in V_{\mathbf{A}}} T_{\mathfrak{a}}[v+iy]^{-k}\,|T_{\mathfrak{a}}[v+iy]|^{-2s}\,\sigma(v,s)\chi(-T(v,h))d\mu(v),
\end{aligned}$$

where

$$\sigma(x,s) = \sum_{t\in F_{\boldsymbol{f}}^*/\mathfrak{o}_{\boldsymbol{f}}^*} c(tw(x))\psi(t)^{-1}|t|^{k+2s} \text{for } x\in V_{\mathbf{A}},\ s\in\mathbb{C}.$$

The sum $\sigma(x,s)$**.** For $x \in V_{\mathbf{A}}$ and $v \in \boldsymbol{f}$ define a local ideal $\iota_v(x_v) \subset \mathfrak{o}_v$ by $\iota_v(x_v) = \mathfrak{p}_v^{i_v(x)}$, where $\mathfrak{p}_v$ is the maximal ideal of $\mathfrak{o}_v$ and $i_v(x) = -\min_{1\le i\le n+2}\{\nu_v(w(x)_i)\}$ with ν_v the normalized v-adic valuation on F_v. $\iota_v(x_v)$ is integral since $w(x_v)_{n+2} = 1$, and $\iota_v(x_v) = \mathfrak{o}_v$ for almost all v.

The product ideal $\iota(x) = \prod_{v\in\boldsymbol{f}} \iota_v(x_v) \subset \mathfrak{o}_{\boldsymbol{f}}$ is such that $tw(x) \in \mathfrak{o}_{\boldsymbol{f}}^{n+2}$ for $t \in F_{\boldsymbol{f}}^*$ if and only if $t \in \iota(x)$. Thus $c(tw(x)) \neq 0$ if and only if $t\in\iota(x)$ and $(tw(x))_{n+2} = t$ is prime to $\mathfrak{b}$, in which case $c(tw(x)) = \psi_{\mathfrak{b}}(t)$ and the summand of $\sigma(x,s)$ is $\prod_{v\in\boldsymbol{f},\, v\nmid\mathfrak{b}} \psi(t_v)^{-1}|t_v|_v^{k+2s}$. Thus

$$\begin{aligned}
\sigma(x,s) &= \sum_{\left\{t=\prod_{v\in\boldsymbol{f},\, v\nmid\mathfrak{b}}\mathfrak{p}_v^{j_v}:\iota(x)|t\right\}} \prod_{\substack{v\in\boldsymbol{f}\\ v\nmid\mathfrak{b}}} \psi(\mathfrak{p}_v^{j_v})^{-1}|\mathfrak{p}_v^{j_v}|_v^{k+2s}\\
&= \sum_t\prod_v (\psi(\mathfrak{p}_v)^{-1}|\mathfrak{p}_v|_v^{k+2s})^{j_v}.
\end{aligned}$$

(The sum is empty if $\iota(x)$ is nontrivial at $\mathfrak{b}$.) This has the Euler product expansion $\sigma(x,s) = \prod_{v\in\boldsymbol{f}}\sigma_v(x_v,s)$, where

$$\begin{aligned}
&\sigma_v(x_v,s)\\
&\quad = \begin{cases} \delta_v(x_v), & \text{if } v \mid \mathfrak{b}\\ (1-\psi(\mathfrak{p}_v)^{-1}|\mathfrak{p}_v|_v^{k+2s})^{-1}(\psi(\mathfrak{p}_v)^{-1}|\mathfrak{p}_v|_v^{k+2s})^{i_v(x_v)}, & \text{if } v\nmid\mathfrak{b}. \end{cases}
\end{aligned}$$

Here $\delta_v(x_v) = 1$ if $x \in L_v$ (so that $\iota_v(x_v) = \mathfrak{o}_v$), 0 if $x_v \notin L_v$ (so that $\iota_v(x_v) \neq \mathfrak{o}_v$).

The local coefficient $a_v(h,y,s)$**.** We now have for $z=(z_v)=(x_v+iy_v)\in\mathcal{H}^{\boldsymbol{a}}$,

$$E(z,s)=(-1)^{dk}2^{d(k+2s)}\sum_{h\in L'}a(h,y,s)\,\boldsymbol{e}\left(\sum_{v\in\boldsymbol{a}}T^v(x_v,h_v)\right),$$

where

$$a(h,y,s)=\int_{x\in V_{\mathbf{A}}}T_{\boldsymbol{a}}[x+iy]^{-k}\left|T_{\boldsymbol{a}}[x+iy]\right|^{-2s}\sigma(x,s)\chi(-T(x,h))d\mu(x),$$

with

$$T_{\boldsymbol{a}}[x+iy]=\prod_{v\in\boldsymbol{a}}T^v[x_v+iy_v],\qquad \sigma(x,s)=\prod_{v\in\boldsymbol{f}}\sigma_v(x_v,s),$$
$$\chi(-T(x,h))=\prod_{v}\boldsymbol{e}_v(-T(x_v,h_v)),\qquad d\mu(x)=c_\mu\prod_v d\mu_v(x_v),$$

where $\mu(V_{\mathbf{A}}/V)=1$, $\mu=c_\mu\prod_v\mu_v$, $\mu_v(L_v)=1$ for $v\in\boldsymbol{f}$, and μ_v is Euclidean measure on $\mathbb{R}^n$ for $v\in\boldsymbol{a}$; these determine $c_\mu=N\mathfrak{d}^{-n/2}$. So

$$a(h,y,s)=N\mathfrak{d}^{-n/2}\prod_v a_v(h,y,s),$$

where for $v\in\boldsymbol{a}$,

$$\begin{aligned}a_v(h,y,s)&=\int_{x\in V_v}T^v[x+iy_v]^{-k}\left|T^v[x+iy_v]\right|^{-2s}\boldsymbol{e}(-T^v(x,h_v))d\mu_v(x)\\&=\int_{x\in V_v}T^v[x+iy_v]^{-k-s}T^v[x-iy_v]^{-s}\boldsymbol{e}(-T^v(x,h_v))d\mu_v(x)\\&=\xi(y_v,h_v;k+s,s;T^v),\end{aligned}$$

with ξ the confluent hypergeometric function studied by Shimura in [**Sh82**]. For $v\in\boldsymbol{f}$, the local coefficient does not depend on y and so may be denoted $a_v(h,s)$. Setting $q_v=|\mathfrak{p}_v|_v^{-1}$ and $X_v(s)=\psi(\mathfrak{p}_v)^{-1}q_v^{-k-2s}$ gives

$$\begin{aligned}a_v(h,s)&=\int_{x\in V_v}\sigma_v(x,s)\,\boldsymbol{e}_v(-T(x,h_v))d\mu_v(x)\\&=\begin{cases}\displaystyle\int_{x\in V_v}\delta_v(x_v)\,\boldsymbol{e}_v(-T(x,h_v))d\mu_v(x) & \text{if } v\mid\mathfrak{b}\\ \displaystyle(1-X_v(s))^{-1}\int_{x\in V_v}X_v(s)^{i_v(x_v)}\,\boldsymbol{e}_v(-T(x,h_v))d\mu_v(x) & \text{if } v\nmid\mathfrak{b}.\end{cases}\end{aligned}$$

5. Local Fourier coefficients of E.

The archimedean coefficient $\xi(y_v, h_v; k+s, s; T^v)$. In [**Sh82**], Shimura defines the functions

$$\xi(y,h;\alpha,\beta;T) = \int_{x\in\mathbb{R}^n} T[x+iy]^{-\alpha} T[x-iy]^{-\beta} \boldsymbol{e}(-T(x,h))dx,$$

where $y \in \mathcal{P}$, $h \in \mathbb{R}^n$, $(\alpha,\beta) \in \mathbb{C}^2$, T defines a form of signature $(1, n-1)$ on $\mathbb{R}^n$; and

$$\begin{aligned}\eta^*&(y,h;\alpha,\beta;T)\\ &= T[y]^{\alpha+\beta-\frac{n}{2}} \int_{x\in Q(h)} T[x+h]^{\alpha-\frac{n}{2}} T[x-h]^{\beta-\frac{n}{2}} e^{-T(y,x)} dx,\end{aligned}$$

where $Q(h) = \{ x \in \mathbb{R}^n : x \pm h \in \mathcal{P} \}$. Both integrals converge when $Re(\alpha) > n/2 - 1$, $Re(\beta) > n/2 - 1$. He defines

$$\omega(y,h;\alpha,\beta;T) = \eta^*(y,h;\alpha,\beta;T)$$
$$\cdot \begin{cases} 2^{-2\alpha}\Gamma_n(\beta)^{-1}\delta(hy)^{\frac{n}{2}-\alpha}, & h \in \mathcal{P} \\ 2^{-2\beta}\Gamma_n(\alpha)^{-1}\delta(hy)^{\frac{n}{2}-\beta}, & -h \in \mathcal{P} \\ |\det T|^{\frac{1}{2}} 2^{-2\alpha-2\beta}\Gamma(\alpha - \frac{n-2}{2})^{-1}\Gamma(\beta - \frac{n-2}{2})^{-1} & \\ \quad \cdot\delta_+(hy)^{1-\alpha+\frac{n-2}{4}}\delta_-(hy)^{1-\beta+\frac{n-2}{4}}, & T[h] < 0 \\ |\det T|^{\frac{1}{2}} 2^{-2\alpha-2\beta}\Gamma(\alpha+\beta - \frac{n}{2})^{-1}\Gamma(\beta - \frac{n-2}{2})^{-1} & \\ \quad \cdot\delta(hy)^{\frac{n}{2}-\alpha}, & T[h] = 0, \\ & T(\varepsilon,h) > 0 \\ |\det T|^{\frac{1}{2}} 2^{-2\alpha-2\beta}\Gamma(\alpha+\beta - \frac{n}{2})^{-1}\Gamma(\alpha - \frac{n-2}{2})^{-1} & \\ \quad \cdot\delta(hy)^{\frac{n}{2}-\beta}, & T[h] = 0, \\ & T(\varepsilon,h) < 0 \\ \Gamma_n(\alpha+\beta-\dfrac{n}{2})^{-1}, & h = 0, \end{cases}$$

where ε is as in section 2 and

$$\Gamma_n(s) = |\det T|^{-\frac{1}{2}} 2^{2s-1}\pi^{\frac{n}{2}-1}\Gamma(s)\Gamma\left(s - \frac{n}{2} + 1\right),$$
$$\delta_+(hy) = \text{ the product of all positive roots to } \lambda^2 - 2T(y,h)\lambda + T[y]T[h] = 0,$$
$$\delta_-(hy) = \delta_+((-h)y), \qquad \delta(hy) = \delta_+(hy)\delta_-(hy);$$

and proves the relation

$$
\begin{aligned}
&(5.1)\\
&\xi(y,h;k+s,s;T)\\
&\quad = |\det T|^{-\frac{1}{2}}(-1)^k 2^{n-2k-4s} T[y]^{\frac{n}{2}-k-2s}\omega(2\pi y,h;k+s,s;T)\\
&\quad\cdot\begin{cases}
2^{2k+2s+1}\pi^{2k+2s+1-\frac{n}{2}}\Gamma(k+s)^{-1}\Gamma(k+s+1-\frac{n}{2})^{-1} & \\
\quad\cdot\delta_+(hy)^{k+s-\frac{n}{2}}, & h\in\mathcal{P}\\
2^{2s+1}\pi^{2s+1-\frac{n}{2}}\Gamma(s)^{-1}\Gamma(s+1-\frac{n}{2})^{-1}\delta_-(hy)^{s-\frac{n}{2}}, & -h\in\mathcal{P}\\
2^{k+2s+\frac{n}{2}+1}\pi^{k+2s+1-\frac{n}{2}}\Gamma(k+s)^{-1}\Gamma(s)^{-1} & \\
\quad\cdot\delta_+(hy)^{k+s-1-\frac{n-2}{4}}\delta_-(hy)^{s-1-\frac{n-2}{4}}, & T[h]<0\\
2^{k+s+2+\frac{n}{2}}\pi^{k+s+2-\frac{n}{2}}\Gamma(k+2s-\frac{n}{2})\Gamma(k+s)^{-1} & T(\varepsilon,h)>0\\
\quad\cdot\Gamma(s)^{-1}\Gamma(k+s+1-\frac{n}{2})^{-1}\delta_+(hy)^{k+s-\frac{n}{2}}, & T[h]=0,\\
2^{s+2+\frac{n}{2}}\pi^{s+2-\frac{n}{2}}\Gamma(k+2s-\frac{n}{2})\Gamma(k+s)^{-1} & T(\varepsilon,h)<0\\
\quad\cdot\Gamma(s)^{-1}\Gamma(s+1-\frac{n}{2})^{-1}\delta_-(hy)^{s-\frac{n}{2}}, & T[h]=0,\\
2\pi^{\frac{n}{2}+1}\Gamma(k+2s-\frac{n}{2})\Gamma(k+2s+1-n)\Gamma(k+s)^{-1} & \\
\quad\cdot\Gamma(s)^{-1}\Gamma(k+s+1-\frac{n}{2})^{-1}\Gamma(s+1-\frac{n}{2})^{-1}, & h=0.
\end{cases}
\end{aligned}
$$

The main result of [**Sh82**] is that ω can be continued as a holomorphic function in (α,β) to $\mathbb{C}^2$. Thus, zeros and poles of ξ can be read off from the previous equation.

The next result will be used in Section 7.

PROPOSITION 5.1. (a) $\omega(2\pi y,h;\alpha,0;T)=2^{-n}\boldsymbol{e}(T(iy,h))$ *if* $h\in\mathcal{P}$;
(b) $\omega(2\pi y,h;\alpha,0;T)=\omega(2\pi y,h;n/2,\beta)=2^{-1-n}\pi^{n/2-1}\boldsymbol{e}(T(iy,h))$ *if* $T[h]=0$, $T(h,\varepsilon)>0$;
(c) $\omega(2\pi y,0;\alpha,\beta;T)=1$.

Proof. (a) and part of (b) are shown in [**Sh82**, 4.35.IV]. The remainder of (b) follows from [**Sh82**, 4.12.IV, 4.29, 3.15], where m, n there are n, $n-2$ here, respectively. (c) is [**Sh82**, 4.9]. □

The finite coefficient $a_v(h,s)$ for $v\mid\mathfrak{b}$. For $v\mid\mathfrak{b}$,

$$
\begin{aligned}
a_v(h,s) &= \int_{x\in V_v}\delta_v(x)\,\boldsymbol{e}_v(-T(x,h_v))d\mu_v(x)\\
&= \int_{x\in L_v}\boldsymbol{e}_v(-T(x,h_v))d\mu_v(x)=\int_{x\in L_v}d\mu_v(x)=1.
\end{aligned}
$$

Thus

$$a_v(h,s) = 1 \qquad \text{if } v \mid \mathfrak{b}.$$

The finite coefficient $a_v(h,s)$ for $v \nmid \mathfrak{b}$. For $v \nmid \mathfrak{b}$,

$$a_v(h,s) = (1 - X_v(s))^{-1} \int_{x\in V_v} X_v(s)^{i_v(x)} \boldsymbol{e}_v(-T(x,h_v))d\mu_v(x).$$

Since the integrand is invariant under $x \mapsto x + l$ for $l \in L_v$, this is

$$\begin{aligned} a_v(h,s) &= (1 - X_v(s))^{-1} \sum_{x\in V_v/L_v} X_v(s)^{i_v(x)} \boldsymbol{e}_v(-T(x,h_v)) \\ &= (1 - X_v(s))^{-1} \sum_{\lambda=0}^{\infty} \sum_{\substack{x\in V_v/L_v \\ i_v(x)=\lambda}} X_v(s)^{\lambda} \boldsymbol{e}_v(-T(x,h_v)) \\ &= (1 - X_v(s))^{-1} \sum_{\lambda=0}^{\infty} X_v(s)^{\lambda} \sum_{\substack{x\in V_v/L_v \\ i_v(x)=\lambda}} \boldsymbol{e}_v(-T(x,h_v)). \end{aligned}$$

Now sum by by parts, $\sum_{\lambda=0}^{\nu} a_\lambda b_\lambda = \sum_{\lambda=0}^{\nu-1} A_\lambda(b_\lambda - b_{\lambda+1}) + A_\nu b_\nu$, where $A_\lambda = \sum_{j=0}^{\lambda} a_j$. Letting $a_\lambda = \sum_{\substack{x\in V_v/L_v \\ i_v(x)=\lambda}} \boldsymbol{e}_v(-T(x,h_v))$, $b_\lambda = X_v(s)^\lambda$ gives

$$\begin{aligned} A_\lambda &= \sum_{\substack{x\in V_v/L_v \\ i_v(x)\le\lambda}} \boldsymbol{e}_v(-T(x,h_v)) \\ &= \sum_{\substack{x\in V_v/L_v \\ w(x)\in\mathfrak{p}_v^{-\lambda}\mathfrak{o}_v^{n+2}}} \boldsymbol{e}_v(-T(x,h_v)) \overset{\text{call}}{=} S_v(\lambda,h_v) \end{aligned}$$

and $b_\lambda - b_{\lambda+1} = (1 - X_v(s))X_v(s)^\lambda$. Hence

$$\begin{aligned} &\sum_{\lambda=0}^{\nu} X_v(s)^\lambda \sum_{\substack{x\in V_v/L_v \\ i_v(x)=\lambda}} \boldsymbol{e}_v(-T(x,h_v)) \\ &\qquad = (1 - X_v(s)) \left(\sum_{\lambda=0}^{\nu-1} X_v(s)^\lambda S_v(\lambda,h_v) \right) + X_v(s)^\nu S_v(\nu,h_v). \end{aligned}$$

The last term goes to 0 as $\nu \to \infty$ when $\mathrm{Re}(k+2s) > n$, giving

$$a_v(h,s) = \alpha_v(h_v, X_v(s)) \qquad \text{if } v \nmid \mathfrak{b}$$

where $\alpha_v(h_v, X)$ is the power series

$$\alpha_v(h_v, X) = \sum_{\lambda=0}^{\infty} S_v(\lambda,h_v)X^\lambda.$$

The exponential sum $S_v(\lambda, h_v)$. Let π_v generate the maximal ideal $\mathfrak{p}_v$ of $\mathfrak{o}_v$, and let $y = \pi_v^\lambda x$. Summing over y's, the set of summation for $S_v(\lambda, h_v)$ becomes

$$\left\{ y \in V_v/\mathfrak{p}_v^\lambda L_v : \begin{pmatrix} \pi_v^{-\lambda} y \\ \frac{1}{2}\pi_v^{-2\lambda} T[y] \\ 1 \end{pmatrix} \in \mathfrak{p}_v^{-\lambda}\mathfrak{o}_v^{n+2} \right\}$$
$$= \left\{ y \in L_v/\mathfrak{p}_v^\lambda L_v : \frac{1}{2}T[y] \in \mathfrak{p}_v^\lambda \right\}.$$

Since $2 \mid \mathfrak{b}$ and $v \nmid \mathfrak{b}$ the $\frac{1}{2}$ is irrelevant, so the sum is

$$S_v(\lambda, h_v) = \sum_{\substack{y \in L_v/\mathfrak{p}_v^\lambda L_v \\ T[y] \in \mathfrak{p}_v^\lambda}} \boldsymbol{e}_v\left(\frac{-T(y, h_v)}{\pi_v^\lambda}\right).$$

This is independent of the choice of π_v since the set being summed over is stable under multiplication by units.

6. The power series $\alpha_v(h_v, X)$.

Definitions. The methods in this section are from Indik [**In**].

From now on all work is local at a fixed place $v \nmid \mathfrak{b}$ (so that $v \nmid 2 \det T$), and v's will be suppressed in the notation; for example, F, V, L, $\mathfrak{o}$, $\mathfrak{p}$ and $\mathfrak{d}$ now denote the local objects F_v, V_v, L_v, $\mathfrak{o}_v$, $\mathfrak{p}_v$ and $\mathfrak{d}_v$ (the local different of F_v over $\mathbb{Q}_p$). Locally T^{-1} is integral; so for $y \in V$, $\nu(Ty) = \nu(y)$ and hence $L' = \mathfrak{d}^{-1}L$. To study the sum $S(\lambda, h)$, begin with some definitions.

Extend the v-adic valuation ν on F to a function also called ν on V by

$$\nu(x) = \min_{1 \le i \le n} \{ \nu(x_i) \}, \qquad \text{for } x \in V.$$

For $\lambda \ge 0$ and $a \in \mathfrak{o}$ define the sets

$$\sigma(\lambda, a) = \left\{ y \in L : T[y] \equiv a \pmod{\mathfrak{p}^\lambda} \right\},$$
$$\sigma'(\lambda, a) = \{ y \in \sigma(\lambda, a) : \nu(y) = 0 \},$$
$$\overline{\sigma(\lambda, a)} = \left\{ y \in L/\mathfrak{p}^\lambda L : T[y] \equiv a \pmod{\mathfrak{p}^\lambda} \right\},$$
$$\overline{\sigma'(\lambda, a)} = \{ y \in \overline{\sigma(\lambda, a)} : \nu(y) = 0 \}.$$

When $a = 0$, write $\sigma(\lambda)$ for $\sigma(\lambda, a)$ and so on. We will sometimes use the sets $\sigma(\lambda, a), \ldots$ defined as above but for forms R other than T, in which case they are denoted $\sigma_R(\lambda, a)$, etc.

Extend the definition of S to

$$S(\lambda,h)=\begin{cases}\sum_{y\in\overline{\sigma(\lambda)}} \boldsymbol{e}_v\left(-\frac{T(y,h)}{\pi^\lambda}\right) & \text{if } h\in L'\\ 0 & \text{if } h\notin L',\end{cases}$$

and define

$$S'(\lambda,h)=\sum_{y\in\overline{\sigma'(\lambda)}} \boldsymbol{e}_v\left(-\frac{T(y,h)}{\pi^\lambda}\right) \qquad \text{for } h\in L',$$

i.e., just sum over primitive vectors.

Recall that $q=|\mathfrak{p}|^{-1}=\#(\mathfrak{o}/\mathfrak{p})$.

PROPOSITION 6.1. *For symmetric $R\in M_n(\mathfrak{o}/\mathfrak{p})$ defining a non-degenerate bilinear form on $(\mathfrak{o}/\mathfrak{p})^n$,*

$$\#\overline{\sigma_R(1)}-q^{n-1}=\begin{cases}q^{\frac{n}{2}-1}(q-1)\left(\frac{(-1)^{\frac{n}{2}}\det R}{\mathfrak{p}}\right) & \text{if } n \text{ is even}\\ 0 & \text{if } n \text{ is odd.}\end{cases}$$

Proof. This is a standard textbook exercise. □

Recurrence formula for $\#\overline{\sigma'(\lambda,a)}$. Fix $\lambda\geq 1$ and $a\in\mathfrak{o}$, and recall that $v\nmid 2$.

LEMMA. *For $\tilde{y}\in\sigma'(\lambda,a)$, there exists $d\in L$ such that $T(\tilde{y},d)=\frac{1}{2}$.*

Proof. $(Ty)_i\in\mathfrak{o}^*$ for some i, so take $d_i=\frac{1}{2}(Ty)_i^{-1}$ and $d_j=0$ for $j\neq i$. □

LEMMA. *For $v\in\overline{\sigma'(\lambda+1,a)}$, $\#\{\,l\in L/\mathfrak{p}L : T(v,l)\in\mathfrak{p}\,\}=q^{n-1}$.*

Proof. $(Ty)_i\in\mathfrak{o}^*$ for some i; consequently $T(v,l)\in\mathfrak{p}$ if and only if $l_i=(Tv)_i^{-1}\left(-\sum_{j\neq i}(Tv)_j l_j\right)+k$ with $k\in\mathfrak{p}$. This determines the value of $l_i \pmod{\mathfrak{p}}$ once the l_j for $j\neq i$ have been chosen. □

PROPOSITION 6.2. $\#\overline{\sigma'(\lambda+1,a)}=q^{n-1}\#\overline{\sigma'(\lambda,a)}$. *Consequently, $\#\overline{\sigma'(\lambda,a)}=q^{(n-1)(\lambda-1)}\#\overline{\sigma'(1,a)}$ for $\lambda\geq 1$, and this value depends only on a* (mod $\mathfrak{p}$).

Proof. Let $\pi_\lambda^{\lambda+1}:\overline{L/\mathfrak{p}^{\lambda+1}L}\to\overline{L/\mathfrak{p}^\lambda L}$ be the natural map. We will show that $\pi_\lambda^{\lambda+1}:\overline{\sigma'(\lambda+1,a)}\to\overline{\sigma'(\lambda,a)}$ is surjective with multiplicity q^{n-1}.

Construct a function $\varphi : \overline{\sigma'(\lambda, a)} \to \overline{\sigma'(\lambda+1, a)}$ as follows: Choose any lifting, denoted $\tilde{}$, from $L/\mathfrak{p}^\lambda L$ to L. Given $y \in \overline{\sigma'(\lambda, a)}$, there exists $d \in L$ such that $T(\tilde{y}, d) = \frac{1}{2}$, by the first lemma. Take $\varphi(y) = \tilde{y} + (a - T[y])d \pmod{\mathfrak{p}^{\lambda+1}L}$. Then $T[\varphi(y)] \equiv a \pmod{\mathfrak{p}^{\lambda+1}}$ is easy to check. Thus $\overline{\sigma'(\lambda, a)} \xrightarrow{\varphi} \overline{\sigma'(\lambda+1, a)} \xrightarrow{\pi_\lambda^{\lambda+1}} \overline{\sigma'(\lambda, a)}$, and the composite is the identity since $\varphi(y) \equiv y \pmod{p^\lambda L}$. This shows that $\pi_\lambda^{\lambda+1} : \overline{\sigma'(\lambda+1, a)} \to \overline{\sigma'(\lambda, a)}$ is surjective.

For $v \in \overline{\sigma'(\lambda+1, a)}$ and $v' \in L/\mathfrak{p}^{\lambda+1}L$, $\pi_\lambda^{\lambda+1}(v') = \pi_\lambda^{\lambda+1}(v)$ if and only if $v' = v + \pi^\lambda l$ for some $l \in L/\mathfrak{p}L$, in which case $T[v'] \equiv a + 2\pi^\lambda T(v, l) \pmod{\mathfrak{p}^{\lambda+1}}$. This shows that $v' \in \overline{\sigma'(\lambda+1, a)}$ if and only if $T(v, l) \in \mathfrak{p}$. The number of l satisfying this is q^{n-1} by the second lemma, so $\pi_\lambda^{\lambda+1} : \overline{\sigma'(\lambda+1, a)} \to \overline{\sigma'(\lambda, a)}$ has multiplicity q^{n-1}, proving the proposition. □

Recurrence formula for $S(\lambda, h)$.

LEMMA. $\sigma(\lambda) = \sigma'(\lambda) \cup \mathfrak{p}\sigma(\lambda - 2)$ *for* $\lambda \geq 2$, *a disjoint union.*

Proof. $\sigma(\lambda) \supset \sigma'(\lambda)$ and $\sigma(\lambda) \supset \mathfrak{p}\sigma(\lambda - 2)$ are clear, as is disjointness. Let $y \in \sigma(\lambda) - \sigma'(\lambda)$. Then $y = \pi x$ for some $x \in L$, and $\pi^2 T[x] = T[y] \in \mathfrak{p}^\lambda$ shows that $T[x] \in \mathfrak{p}^{\lambda-2}$, i.e., $x \in \sigma(\lambda - 2)$. □

PROPOSITION 6.3. $S(\lambda, h) = S'(\lambda, h) + q^n S(\lambda - 2, h/\pi)$ *for* $\lambda \geq 2$ *and* $h \in L'$.

Proof.

$$S(\lambda, h) = S'(\lambda, h) + \sum_{\substack{y \in \mathfrak{p}\sigma(\lambda-2) \\ (\text{mod } \mathfrak{p}^\lambda L)}} \boldsymbol{e}_v\left(-\frac{T(y, h)}{\pi^\lambda}\right)$$

by the lemma, so we need to evaluate this last sum, which is equal to

$$\sum_{\substack{y \in \sigma(\lambda-2) \\ (\text{mod } \mathfrak{p}^{\lambda-1} L)}} \boldsymbol{e}_v\left(-\frac{T(y, h)}{\pi^{\lambda-1}}\right) \overset{\text{call}}{=} S.$$

The set $\sigma(\lambda - 2) \pmod{\mathfrak{p}^{\lambda-1}L}$ is stable under translation by any $\pi^{\lambda-2} l \in \mathfrak{p}^{\lambda-2} L$. So

$$S = \sum_{\substack{y \in \sigma(\lambda-2) \\ (\text{mod } \mathfrak{p}^{\lambda-1} L)}} \boldsymbol{e}_v\left(-\frac{T(y + \pi^{\lambda-2} l, h)}{\pi^{\lambda-1}}\right) = \boldsymbol{e}_v\left(-T(l, h/\pi)\right) S.$$

If $\frac{h}{\pi} \in L'$ then

$$S = \sum_{\substack{y \in \sigma(\lambda-2) \\ (\mathrm{mod}\ \mathfrak{p}^{\lambda-1}L)}} \boldsymbol{e}_v\left(-\frac{T(y, h/\pi)}{\pi^{\lambda-2}}\right) = q^n S(\lambda-2, h/\pi).$$

If $\frac{h}{\pi} \notin L'$ then $T(L, h/\pi) \not\subset \mathfrak{d}^{-1}$, so for some $l \in L$ we have $Tr(T(l, h/\pi)) \notin \mathbb{Z}_p$, giving $\boldsymbol{e}_v(-T(l, h/\pi)) \neq 1$, whence $S = 0$. Thus $S(\lambda, h) = S'(\lambda, h) + q^n S(\lambda-2, h/\pi)$ in all cases. □

COROLLARY 6.4. *$S(\lambda, 0) - q^n S(\lambda-2, 0) = q^{(n-1)(\lambda-1)}\#\overline{\sigma'(1)}$ for $\lambda \geq 2$. Equivalently, $\#\overline{\sigma(\lambda)} - q^n \#\overline{\sigma(\lambda-2)} = q^{(n-1)(\lambda-1)}\#\overline{\sigma'(1)}$.*

Proof.

$$S(\lambda, 0) - q^n S(\lambda-2, 0) = S'(\lambda, 0) = \#\overline{\sigma'(\lambda)} = q^{(n-1)(\lambda-1)}\#\overline{\sigma'(1)}$$

by the previous proposition. □

The value of $\alpha(h, X)$ when $h = 0$.

PROPOSITION 6.5.

$$\alpha(0, X) = \frac{1 + (\#\overline{\sigma(1)} - q^{n-1})X - q^{n-1}X^2}{(1 - q^n X^2)(1 - q^{n-1}X)}.$$

Proof. Since $S(\lambda, 0) - q^n S(\lambda-2, 0) = q^{(n-1)(\lambda-1)}\#\overline{\sigma'(1)}$ for $\lambda \geq 2$,

$$\begin{aligned}
(1 - q^n X^2) &\sum_{\lambda=0}^{\infty} S(\lambda, 0)X^\lambda \\
&= 1 + S(1, 0) + \sum_{\lambda=2}^{\infty}(S(\lambda, 0) - q^n S(\lambda-2, 0))X^\lambda \\
&= 1 + \#\overline{\sigma(1)}X + \sum_{\lambda=2}^{\infty} q^{(n-1)(\lambda-1)}\#\overline{\sigma'(1)}X^\lambda,
\end{aligned}$$

and since $\#\overline{\sigma(1)} = 1 + \#\overline{\sigma'(1)}$, this is

$$\begin{aligned}
&= 1 + X + \sum_{\lambda=1}^{\infty} q^{(n-1)(\lambda-1)}\#\overline{\sigma'(1)}X^\lambda \\
&= 1 + X + \frac{\#\overline{\sigma'(1)}X}{1 - q^{n-1}X}.
\end{aligned}$$

The result follows easily. □

DEFINITION. For n even, define a quadratic character θ by $\theta(\mathfrak{p}) = \left(\dfrac{(-1)^{\frac{n}{2}} \det T}{\mathfrak{p}}\right)$.

This gives for n even $\#\overline{\sigma(1)} - q^{n-1} = q^{\frac{n}{2}-1}(q-1)\theta(\mathfrak{p})$, so in the proposition the numerator becomes $(1+q^{\frac{n}{2}}\theta(\mathfrak{p})X)(1-q^{\frac{n}{2}-1}\theta(\mathfrak{p})X)$, and the denominator, $(1+q^{\frac{n}{2}}\theta(\mathfrak{p})X)(1-q^{\frac{n}{2}}\theta(\mathfrak{p})X)(1-q^{n-1}X)$. For n odd, $\#\overline{\sigma(1)} - q^{n-1} = 0$. Thus,

$$\alpha(h, X) = \begin{cases} \dfrac{1 - q^{\frac{n}{2}-1}\theta(\mathfrak{p})X}{(1-q^{\frac{n}{2}}\theta(\mathfrak{p})X)(1-q^{n-1}X)} & \text{if } h = 0,\ n \text{ even} \\ \dfrac{1-q^{n-1}X^2}{(1-q^nX^2)(1-q^{n-1}X)} & \text{if } h = 0,\ n \text{ odd.} \end{cases}$$

Formula for S.

DEFINITION. Let $\nu_{\mathfrak{d}} = \nu(\mathfrak{d})$, the valuation of the different.

PROPOSITION 6.6. *For a set $\sigma \subset L/\mathfrak{p}^\lambda L$ such that $u\sigma = \sigma$ for all $u \in \mathfrak{o}^*$,*

$$\sum_{y\in\sigma} e_v\left(-\frac{T(y,h)}{\pi^\lambda}\right) = \#\{\, y \in \sigma : \nu(T(y,h)) \geq \lambda - \nu_{\mathfrak{d}} \,\} - \frac{1}{q-1}\#\{\, y \in \sigma : \nu(T(y,h)) = \lambda - \nu_{\mathfrak{d}} - 1 \,\}.$$

Proof. We may assume $\lambda \geq 1$. Let $U_\lambda = \mathfrak{o}^*/\mathfrak{p}^\lambda = \mathfrak{o}/\mathfrak{p}^\lambda - \mathfrak{p}/\mathfrak{p}^\lambda$, with $\#U_\lambda = q^\lambda - q^{\lambda-1} = q^{\lambda-1}(q-1)$. Then

$$\begin{aligned} q^{\lambda-1}(q-1)\sum_{y\in\sigma} e_v\left(-\frac{T(y,h)}{\pi^\lambda}\right) &= \sum_{u\in U_\lambda}\sum_{y\in\sigma} e_v\left(-\frac{T(uy,h)}{\pi^\lambda}\right) \\ &= \sum_y\sum_u e_v\left(-\frac{T(uy,h)}{\pi^\lambda}\right) \\ &= \sum_y\left\{\sum_{u\in\mathfrak{o}/\mathfrak{p}^\lambda} e_v\left(-\frac{T(uy,h)}{\pi^\lambda}\right) - \sum_{u\in\mathfrak{o}/\mathfrak{p}^{\lambda-1}} e_v\left(-\frac{T(uy,h)}{\pi^{\lambda-1}}\right)\right\}. \end{aligned}$$

Since the sums over $\mathfrak{o}/\mathfrak{p}^\lambda$ and $\mathfrak{o}/\mathfrak{p}^{\lambda-1}$ are character sums over finite groups, and since $\dfrac{T(uy,h)}{\pi^\lambda} \in \mathfrak{d}^{-1}$ for all u if and only if $\nu(T(y,h)) \geq$

$\lambda - \nu_{\mathfrak{d}}$, the inner sums yield

$$\begin{cases} 0 & \text{if } \nu(T(y,h)) < \lambda - \nu_{\mathfrak{d}} - 1 \\ -q^{\lambda-1} & \text{if } \nu(T(y,h)) = \lambda - \nu_{\mathfrak{d}} - 1 \\ q^{\lambda} - q^{\lambda-1} & \text{if } \nu(T(y,h)) \geq \lambda - \nu_{\mathfrak{d}}, \end{cases}$$

so

$$\begin{aligned} q^{\lambda-1}(q-1)\sum_{y\in\sigma} e_v\left(-\frac{T(y,h)}{\pi^{\lambda}}\right) \\ =&(q^{\lambda} - q^{\lambda-1})\#\{\, y \in \sigma : \nu(T(y,h)) \geq \lambda - \nu_{\mathfrak{d}} \,\} \\ &- q^{\lambda-1}\#\{\, y \in \sigma : \nu(T(y,h)) = \lambda - \nu_{\mathfrak{d}} - 1 \,\}, \end{aligned}$$

giving the result. □

This shows that the coefficients of the power series $\alpha_v(h, X)$ are elements of $\mathbb{Q}$.

The value of $\alpha(h, X)$ when $T[h] = 0$. Now assume that $T[h] = 0$, $h \neq 0$.

DEFINITION. Given a nonzero $h \in L'$, define $\nu_h \in \mathbb{Z}$ and $h' \in L$ by $h = \pi^{\nu_h} h'$, where $\nu_h = \nu(h) \geq -\nu_{\mathfrak{d}}$ and $\nu(h') = 0$. Further define $\nu_{h\mathfrak{d}} = \nu_h + \nu_{\mathfrak{d}} \geq 0$.

There is an $x_0 \in L$ such that $T(x_0, h') = 1$; then setting $x = x_0 - \frac{1}{2}T[x_0]h'$ gives $T[x] = T[h'] = 0$, $T(x, h') = 1$, and $L = \mathfrak{o}h' + \mathfrak{o}x + W$, where $W = \{\, w \in L : T(w, h') = T(w, x) = 0 \,\}$. Define $T' = T|_W$.

PROPOSITION 6.7. *For a nonzero $h \in L'$ such that $T[h] = 0$,*

$$\alpha(h, X) = \frac{1 + (\#\overline{\sigma_{T'}(1)} - q^{n-3})qX - q^{n-1}X^2}{1 - q^n X^2} G_{h,v}(X),$$

where

$$G_{h,v}(X) = \sum_{i=0}^{\nu_{h\mathfrak{d}}} (q^{n-1}X)^i = \frac{1 - (q^{n-1}X)^{\nu_{h\mathfrak{d}}+1}}{1 - q^{n-1}X}.$$

Proof. For $y = ah' + bx + w \in L$, $T[y] = 2ab + T[w]$, so $y \in \sigma(\lambda)$ if and only if $T[w] \equiv -2ab \pmod{\mathfrak{p}^{\lambda}}$. Given $w \in W/\mathfrak{p}^{\lambda}W$ and $b \in \mathfrak{o}/\mathfrak{p}^{\lambda}$, there is an $a \in \mathfrak{o}/\mathfrak{p}^{\lambda}$ such that $T[w] \equiv -2ab \pmod{\mathfrak{p}^{\lambda}}$ if and only if $\nu(T[w]) \geq \nu(b)$, in which case there are $q^{\min(\lambda,\nu(b))}$ such values a. Proposition 6.6 says,

$$S(\lambda, \pi^{\nu_h} h') = \#\left\{ y \in \overline{\sigma(\lambda)} : \nu(T(y,h')) \geq \lambda - \nu_{h\mathfrak{d}} \right\}$$

$$(6.1) \qquad - \frac{1}{q-1} \#\left\{ y \in \overline{\sigma(\lambda)} : \nu(T(y,h')) = \lambda - \nu_{h\mathfrak{d}} - 1 \right\}.$$

Setting $M = \max(0, \lambda - \nu_{h\mathfrak{d}})$ one finds that the first term of (6.1) is

$$\begin{aligned}
&\sum_{m=M}^{\lambda} \#\left\{ b \in \mathfrak{o}/\mathfrak{p}^\lambda : \nu(b) = m \right\} \#\left\{ \sigma_{T'}(m) \pmod{\mathfrak{p}^\lambda L} \right\} q^m \\
&= \sum_{m=M}^{\lambda-1} q^{\lambda-m-1}(q-1) q^{(n-2)(\lambda-m)} \#\overline{\sigma_{T'}(m)} q^m + \#\overline{\sigma_{T'}(\lambda)} q^\lambda \\
&= \sum_{m=M}^{\lambda} q^\lambda q^{(n-2)(\lambda-m)} \#\overline{\sigma_{T'}(m)} - \sum_{m=M}^{\lambda-1} q^{\lambda-1} q^{(n-2)(\lambda-m)} \#\overline{\sigma_{T'}(m)} \\
&= q^\lambda \sum_{m=M}^{\lambda} q^{(n-2)(\lambda-m)} \#\overline{\sigma_{T'}(m)} \\
&\quad - q^\lambda \sum_{m=M+1}^{\lambda} q^{-1} q^{(n-2)(\lambda-m+1)} \#\overline{\sigma_{T'}(m-1)} \\
&= q^\lambda \sum_{m=M+1}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m) + q^\lambda q^{(n-2)(\lambda-M)} \#\overline{\sigma_{T'}(M)} \\
&= \begin{cases} q^\lambda \displaystyle\sum_{m=1}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m) + q^\lambda q^{(n-2)\lambda} & \text{if } \lambda \leq \nu_{h\mathfrak{d}} \\ q^\lambda \displaystyle\sum_{m=\lambda-\nu_{h\mathfrak{d}}+1}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m) & \\ \qquad + q^\lambda q^{(n-2)\nu_{h\mathfrak{d}}} \#\overline{\sigma_{T'}(\lambda - \nu_{h\mathfrak{d}})} & \text{if } \lambda > \nu_{h\mathfrak{d}}, \end{cases}
\end{aligned}$$

where $\Delta(m) = \#\overline{\sigma_{T'}(m)} - q^{n-3}\#\overline{\sigma_{T'}(m-1)}$. The second term of (6.1) is 0 when $\lambda \leq \nu_{h\mathfrak{d}}$, and is

$$\begin{aligned}
&- \frac{q^{\lambda-\nu_{h\mathfrak{d}}-1}}{q-1} \#\left\{ b \in \mathfrak{o}/\mathfrak{p}^\lambda : \nu(b) = \lambda - \nu_{h\mathfrak{d}} - 1 \right\} \\
&\qquad \#\left\{ \sigma_{T'}(\lambda - \nu_{h\mathfrak{d}} - 1) \pmod{\mathfrak{p}^\lambda L} \right\} \\
&= - \frac{q^{\lambda-\nu_{h\mathfrak{d}}-1}}{q-1} q^{\nu_{h\mathfrak{d}}}(q-1) q^{(n-2)(\nu_{h\mathfrak{d}}+1)} \#\overline{\sigma_{T'}(\lambda - \nu_{h\mathfrak{d}} - 1)} \\
&= - q^\lambda q^{(n-2)\nu_{h\mathfrak{d}}} q^{n-3} \#\overline{\sigma_{T'}(\lambda - \nu_{h\mathfrak{d}} - 1)}
\end{aligned}$$

when $\lambda > \nu_{h\mathfrak{d}}$. So

(6.2)

$$S(\lambda,h)=\begin{cases} q^{\lambda}\sum\limits_{m=1}^{\lambda} q^{(n-2)(\lambda-m)}\Delta(m)+q^{\lambda}q^{(n-2)\lambda} & \text{if } \lambda\le\nu_{h\mathfrak{d}} \\ q^{\lambda}\sum\limits_{m=\lambda-\nu_{h\mathfrak{d}}}^{\lambda} q^{(n-2)(\lambda-m)}\Delta(m) & \text{if } \lambda>\nu_{h\mathfrak{d}}. \end{cases}$$

Now, $\Delta(m)$ satisfies

$$\begin{aligned} &\Delta(m+2)-q^{n-2}\Delta(m) \\ &\quad=(\#\overline{\sigma_{T'}(m+2)}-q^{n-2}\#\overline{\sigma_{T'}(m)}) \\ &\qquad-q^{n-3}(\#\overline{\sigma_{T'}(m+1)}-q^{n-2}\#\overline{\sigma_{T'}(m-1)}) \\ &\quad=\#\overline{\sigma'_{T'}(m+2)}-q^{n-3}\#\overline{\sigma'_{T'}(m+1)} \\ &\quad=0, \end{aligned}$$

by Corollary 6.4 and Proposition 6.2 with T' in place of T. This shows that for $\lambda > \nu_{h\mathfrak{d}}$,

$$\begin{aligned} &S(\lambda+2,h)-q^{n}S(\lambda,h) \\ &\quad=q^{\lambda+2}\sum_{m=\lambda-\nu_{h\mathfrak{d}}+2}^{\lambda+2} q^{(n-2)(\lambda+2-m)}\Delta(m) \\ &\qquad-q^{\lambda}\sum_{m=\lambda-\nu_{h\mathfrak{d}}}^{\lambda} q^{n}q^{(n-2)(\lambda-m)}\Delta(m) \\ &\quad=q^{\lambda+2}\sum_{m=\lambda-\nu_{h\mathfrak{d}}}^{\lambda} q^{(n-2)(\lambda-m)}(\Delta(m+2)-q^{n-2}\Delta(m)) \\ &\quad=0. \end{aligned}$$

So for $\nu_{h\mathfrak{d}}=0$,

$$\begin{aligned} &(1-q^nX^2)\sum_{\lambda=0}^{\infty} S(\lambda,h)X^{\lambda} \\ &\quad=1+S(1,h)X+\sum_{\lambda=2}^{\infty}(S(\lambda,h)-q^nS(\lambda-2,h))X^{\lambda} \\ &\quad=1+(\#\overline{\sigma_{T'}(1)}-q^{n-3})qX+(q^2\#\overline{\sigma_{T'}(2)}-q^{n-1}\#\overline{\sigma_{T'}(1)}-q^{n})X^2, \end{aligned}$$

giving the result in this case, as the relations $\#\overline{\sigma_{T'}(2)} = q^{n-2}\#\overline{\sigma_{T'}(0)} + q^{n-3}\#\overline{\sigma'_{T'}(1)}$ and $\#\overline{\sigma_{T'}(1)} = \#\overline{\sigma'_{T'}(1)} + 1$ show that the coefficient of X^2 is $-q^{n-1}$. Also when $\nu_{h\mathfrak{d}} = 0$, (6.2) shows that

$$\sum_{\lambda=0}^{\infty} S(\lambda, h) = 1 + \sum_{m=1}^{\infty} (qX)^m \Delta(m),$$

so that

$$1 + \sum_{m=1}^{\infty} (qX)^m \Delta(m) = \frac{1 + (\#\overline{\sigma_{T'}(1)} - q^{n-3})qX - q^{n-1}X^2}{1 - q^n X^2}.$$

For general $\nu_{h\mathfrak{d}}$, the same formula gives

(6.3)

$$\begin{aligned}\alpha(h, X) &= 1 + \sum_{\lambda=1}^{\nu_{h\mathfrak{d}}} (qX)^\lambda \left(\sum_{m=1}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m) + q^{(n-2)\lambda} \right) \\ &\quad + \sum_{\lambda=\nu_{h\mathfrak{d}}+1}^{\infty} (qX)^\lambda \sum_{m=\lambda-\nu_{h\mathfrak{d}}}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m) \\ &= \sum_{\lambda=0}^{\nu_{h\mathfrak{d}}} (q^{n-1}X)^\lambda + \sum_{\lambda=1}^{\nu_{h\mathfrak{d}}} (qX)^\lambda \sum_{m=1}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m) \\ &\quad + \sum_{\lambda=\nu_{h\mathfrak{d}}+1}^{\infty} (qX)^\lambda \sum_{m=\lambda-\nu_{h\mathfrak{d}}}^{\lambda} q^{(n-2)(\lambda-m)} \Delta(m).\end{aligned}$$

The first sum in (6.3) is $\dfrac{1 - (q^{n-1}X)^{\nu_{h\mathfrak{d}}+1}}{1 - q^{n-1}X}$. The second sum is

$$\begin{aligned}&\sum_{m=1}^{\nu_{h\mathfrak{d}}} q^{-m(n-2)} \Delta(m) \sum_{\lambda=m}^{\nu_{h\mathfrak{d}}} (q^{n-1}X)^\lambda \\ &\quad = \sum_{m=1}^{\nu_{h\mathfrak{d}}} q^{-m(n-2)} \Delta(m) (q^{n-1}X)^m \frac{1 - (q^{n-1}X)^{\nu_{h\mathfrak{d}}+1-m}}{1 - q^{n-1}X} \\ &\quad = \sum_{m=1}^{\nu_{h\mathfrak{d}}} \Delta(m) (qX)^m \frac{1 - (q^{n-1}X)^{\nu_{h\mathfrak{d}}+1-m}}{1 - q^{n-1}X}.\end{aligned}$$

The third sum is

$$\begin{aligned}&\sum_{m=1}^{\nu_{h\mathfrak{d}}} q^{-m(n-2)} \Delta(m) \sum_{\lambda=\nu_{h\mathfrak{d}}+1}^{m+\nu_{h\mathfrak{d}}} (q^{n-1}X)^\lambda \\ &\qquad\qquad + \sum_{m=\nu_{h\mathfrak{d}}+1}^{\infty} q^{-m(n-2)} \Delta(m) \sum_{\lambda=m}^{m+\nu_{h\mathfrak{d}}} (q^{n-1}X)^\lambda,\end{aligned}$$

which splits into

$$\sum_{m=1}^{\nu_{h\mathfrak{v}}} q^{-m(n-2)}\Delta(m)(q^{n-1}X)^{\nu_{h\mathfrak{v}}+1}\frac{1-(q^{n-1}X)^m}{1-q^{n-1}X}$$
$$= \sum_{m=1}^{\nu_{h\mathfrak{v}}} q^{-m(n-2)}\Delta(m)(q^{n-1}X)^m\frac{(q^{n-1}X)^{\nu_{h\mathfrak{v}}+1-m}-(q^{n-1}X)^{\nu_{h\mathfrak{v}}+1}}{1-q^{n-1}X}$$
$$= \sum_{m=1}^{\nu_{h\mathfrak{v}}} \Delta(m)(qX)^m\left(\frac{1-(q^{n-1}X)^{\nu_{h\mathfrak{v}}+1}}{1-q^{n-1}X}-\frac{1-(q^{n-1}X)^{\nu_{h\mathfrak{v}}+1-m}}{1-q^{n-1}X}\right)$$

and

$$\sum_{m=\nu_{h\mathfrak{v}}+1}^{\infty} q^{-m(n-2)}\Delta(m)(q^{n-1}X)^m\frac{1-(q^{n-1}X)^{\nu_{h\mathfrak{v}}+1}}{1-q^{n-1}X}$$
$$= \sum_{m=\nu_{h\mathfrak{v}}+1}^{\infty} \Delta(m)(qX)^m\frac{1-(q^{n-1}X)^{\nu_{h\mathfrak{v}}+1}}{1-q^{n-1}X}.$$

The total is thus

$$\alpha(h,X) = \left(1+\sum_{m=1}^{\infty}\Delta(m)(qX)^m\right)\left(\frac{1-(q^{n-1}X)^{\nu_{h\mathfrak{v}}+1}}{1-q^{n-1}X}\right)$$
$$= \frac{1+(\#\overline{\sigma_{T'}(1)}-q^{n-3})qX-q^{n-1}X^2}{1-q^nX^2}\,G_{h,v}(X),$$

which completes the proof of the proposition. □

For n even, observe that since

$$\det T' = -\det T,\ \theta(\mathfrak{p}) = \left(\frac{(-1)^{\frac{n}{2}-1}\det T'}{\mathfrak{p}}\right),$$

and the first factor becomes

$$\frac{1+q^{\frac{n}{2}-2}(q-1)\theta(\mathfrak{p})qX-q^{n-1}X^2}{(1+q^{\frac{n}{2}}\theta(\mathfrak{p})X)(1-q^{\frac{n}{2}}\theta(\mathfrak{p})X)} = \frac{1-q^{\frac{n}{2}-1}\theta(\mathfrak{p})X}{1-q^{\frac{n}{2}}\theta(\mathfrak{p})X}.$$

For n odd the first factor becomes $\dfrac{1-q^{n-1}X^2}{1-q^nX^2}$. Thus,

$$\alpha(h,X) = \begin{cases} \dfrac{1-q^{\frac{n}{2}-1}\theta(\mathfrak{p})X}{1-q^{\frac{n}{2}}\theta(\mathfrak{p})X}\,G_{h,v}(X) & \text{if } T[h]=0,\ n \text{ even} \\ \dfrac{1-q^{n-1}X^2}{1-q^nX^2}\,G_{h,v}(X) & \text{if } T[h]=0,\ n \text{ odd.} \end{cases}$$

The value of $\alpha(h,X)$ when $\nu(T[h'])=0$.

PROPOSITION 6.8. *For $h \in L'$ such that $\nu(T[h'])=0$, $\alpha(h,X)$ is a polynomial $H_{h,v}(X) \in \mathbb{Q}[X]$ of degree $< 2(\nu_{h\mathfrak{d}}+1)$. If $\nu_{h\mathfrak{d}}=0$ then*

$$\alpha(h,X) = 1 + \frac{1}{q-1}\left(q(\#\overline{\sigma_{T''}(1)} - q^{n-2}) - (\#\overline{\sigma(1)} - q^{n-1})\right) X,$$

where $T'' = T|_W$, with $W = \{\, w \in L : T(w,h) = 0 \,\}$.

Proof. We will compute $S'(\lambda,h)$ for all values of λ. For $\lambda \le \nu_{h\mathfrak{d}}$, $S'(\lambda,h) = \#\overline{\sigma'(\lambda)}$ is clear. Suppose now that $\lambda > \nu_{h\mathfrak{d}}+1$. Any $y \in \sigma(\lambda)$ takes the form $y = ah' + w$, where $a = \dfrac{T(y,h')}{T[h']}$, $w \in W$, $T[w] \equiv -a^2 T[h'] \pmod{\mathfrak{p}^\lambda}$, and $\nu(y)=0$ if and only if $\nu(w)=0$. For $l \ge 0$, $l = \nu(T(y,h)) = \nu(T(ah', \pi^{\nu_h} h')) = \nu(a) + \nu_h$ if and only if $\nu(a) = l - \nu_h$. Thus by Proposition 6.6,

(6.4)
$$\begin{aligned} S'(\lambda,h) &= \#\left\{\, y \in \overline{\sigma'(\lambda)} : \nu(T(y,h)) \ge \lambda - \nu_{\mathfrak{d}} \,\right\} \\ &\quad - \frac{1}{q-1}\#\left\{\, y \in \overline{\sigma'(\lambda)} : \nu(T(y,h)) = \lambda - \nu_{\mathfrak{d}} - 1 \,\right\} \\ &= \sum_{\substack{a \in \mathfrak{o}/\mathfrak{p}^\lambda \\ \nu(a) \ge \lambda - \nu_{h\mathfrak{d}}}} \#\left\{\, w \in \overline{\sigma'_{T''}(\lambda, -a^2 T[h'])} \,\right\} \\ &\quad - \frac{1}{q-1} \sum_{\substack{a \in \mathfrak{o}/\mathfrak{p}^\lambda \\ \nu(a) = \lambda - \nu_{h\mathfrak{d}} - 1}} \#\left\{\, w \in \overline{\sigma'_{T''}(\lambda, -a^2 T[h'])} \,\right\}. \end{aligned}$$

Since $\lambda > \nu_{h\mathfrak{d}}+1$, $\nu(a^2) > 0$ in both sums. By Proposition 6.2, the set cardinalities depend only on $a^2 T[h'] \pmod{\mathfrak{p}}$, which is 0, so for $\lambda > \nu_{h\mathfrak{d}}+1$,

$$\begin{aligned} S'(\lambda,h) &= \#\overline{\sigma'_{T''}(\lambda)}\Bigg(\#\left\{\, a \in \mathfrak{o}/\mathfrak{p}^\lambda : \nu(a) \ge \lambda - \nu_{h\mathfrak{d}} \,\right\} \\ &\qquad\qquad - \frac{1}{q-1}\#\left\{\, a \in \mathfrak{o}/\mathfrak{p}^\lambda : \nu(a) = \lambda - \nu_{h\mathfrak{d}} - 1 \,\right\}\Bigg) \\ &= \#\overline{\sigma'_{T''}(\lambda)}\left(q^{\nu_{h\mathfrak{d}}} - \frac{1}{q-1}(q^{\nu_{h\mathfrak{d}}+1} - q^{\nu_{h\mathfrak{d}}})\right) \\ &= 0. \end{aligned}$$

This bounds the degree, for if $\lambda \geq 2\nu_{h\mathfrak{d}} + 2$ then

$$S(\lambda, h) = \sum_{r=0}^{\nu_{h\mathfrak{d}}} q^{nr} S'(\lambda - 2r, \pi^{-r} h)$$

by repeated application of Proposition 6.3, and the summand is always zero since $\lambda > 2\nu_{h\mathfrak{d}} + 1$ implies $\lambda - 2r > \nu_{\pi^{-r}h,\mathfrak{d}} + 1$ for $r = 0, \dots, \nu_{h\mathfrak{d}}$.

The remaining case is $\lambda = \nu_{h\mathfrak{d}} + 1$. In this instance (6.4) becomes

$$\begin{aligned} S'(\nu_{h\mathfrak{d}} + 1, h) &= q^{\nu_{h\mathfrak{d}}} \#\overline{\sigma'_{T''}(\nu_{h\mathfrak{d}} + 1)} \\ &\quad - \frac{1}{q-1} \sum_{\substack{a \in \mathfrak{o}/\mathfrak{p}^\lambda \\ \nu(a)=0}} \#\overline{\sigma'_{T''}(\nu_{h\mathfrak{d}} + 1, -a^2 T[h'])}. \end{aligned}$$

To simplify this expression, note that

$$\begin{aligned} \#\overline{\sigma'(\nu_{h\mathfrak{d}} + 1)} &= q^{\nu_{h\mathfrak{d}}} \#\overline{\sigma'_{T''}(\nu_{h\mathfrak{d}} + 1)} \\ &\quad + \sum_{\substack{a \in \mathfrak{o}/\mathfrak{p}^\lambda \\ \nu(a)=0}} \#\overline{\sigma'_{T''}(\nu_{h\mathfrak{d}} + 1, -a^2 T[h'])}, \end{aligned}$$

obtained from $\#\overline{\sigma'(\nu_{h\mathfrak{d}} + 1)} = S'(\nu_{h\mathfrak{d}} + 1, 0)$ and analysis of $S'(\nu_{h\mathfrak{d}} + 1, 0)$ similar to the argument above. Combining these gives

$$\begin{aligned} &S'(\nu_{h\mathfrak{d}} + 1, h) \\ &= \frac{1}{q-1} \left(q^{\nu_{h\mathfrak{d}}+1} \#\overline{\sigma'_{T''}(\nu_{h\mathfrak{d}} + 1)} - \#\overline{\sigma'(\nu_{h\mathfrak{d}} + 1)} \right) \\ &= \frac{1}{q-1} \left(q^{\nu_{h\mathfrak{d}}+1} q^{(n-2)\nu_{h\mathfrak{d}}} \#\overline{\sigma'_{T''}(1)} - q^{(n-1)\nu_{h\mathfrak{d}}} \#\overline{\sigma'(1)} \right). \end{aligned}$$

The expression for $\nu_{h\mathfrak{d}} = 0$ follows since in this case the formulae for $S'(\lambda, h)$ give

$$\begin{aligned} \alpha(h, X) &= S(0, h) + X + S'(1, h) X \\ &= 1 + X + \frac{1}{q-1} \left(q \#\overline{\sigma'_{T''}(1)} - \#\overline{\sigma'(1)} \right) X \\ &= 1 + \frac{1}{q-1} \left(q(1 + \#\overline{\sigma'_{T''}(1)}) - (1 + \#\overline{\sigma'(1)}) \right) X \\ &= 1 + \frac{1}{q-1} \left(q \#\overline{\sigma_{T''}(1)} - \#\overline{\sigma(1)} \right) X, \end{aligned}$$

which completes the proof. □

DEFINITION. For n odd and h such that $\nu(T[h']) = 0$, define a quadratic character θ_h by $\theta_h(\mathfrak{p}) = \left(\frac{(-1)^{\frac{n-1}{2}} T[h'] \det T}{\mathfrak{p}}\right)$.

Since $\det T'' = T[h']^{-1} \det T$ and $\left(\frac{T[h']^{-1}}{\mathfrak{p}}\right) = \left(\frac{T[h']}{\mathfrak{p}}\right)$, when $\nu_{h\mathfrak{d}} = 0$ we get

$$\alpha(h, X) = \begin{cases} 1 + q^{\frac{n}{2}-1}\theta(\mathfrak{p})X & \text{if } \nu(T[h']) = 0,\ n \text{ even} \\ 1 + q^{\frac{n-1}{2}}\theta_h(\mathfrak{p})X = \dfrac{1 - q^{n-1}X^2}{1 - q^{\frac{n-1}{2}}\theta_h(\mathfrak{p})X} & \text{if } \nu(T[h']) = 0,\ n \text{ odd.} \end{cases}$$

The value of $\alpha(h, X)$ when $\nu(T[h']) > 0$.

LEMMA. *For $y \in L$, $h \in L'$, $\mu \in \mathbb{Z}$, the following equivalence holds:*

$$Th \equiv aTy \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1} L} \text{ for some } a \in \mathfrak{d}^{-1}$$
$$\Leftrightarrow \Big(T(d, y) \in \mathfrak{p}^\mu \Rightarrow T(d, h) \in \mathfrak{p}^\mu \mathfrak{d}^{-1} \text{ for all } d \in L\Big).$$

Proof. $\Rightarrow$: If $Th \equiv aTy \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1} L}$ then $T(d, h) \equiv aT(d, y) \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1}}$ for all $d \in L$, hence $T(d, y) \in \mathfrak{p}^\mu \Rightarrow aT(d, y) \in \mathfrak{p}^\mu \mathfrak{d}^{-1} \Rightarrow T(d, h) \in \mathfrak{p}^\mu \mathfrak{d}^{-1}$ for all $d \in L$.

$\Leftarrow$: If $(Ty)_i \in \mathfrak{p}^\mu$ then setting $d = e_i$ (the i^{th} basis vector) gives $T(d, y) = (Ty)_i \in \mathfrak{p}^\mu$, so $T(d, h) = (Th)_i \in \mathfrak{p}^\mu \mathfrak{d}^{-1}$. At such i, $(Th)_i \equiv a(Ty)_i \equiv 0 \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1}}$ holds for any $a \in \mathfrak{d}^{-1}$.

If $(Ty)_i \notin \mathfrak{p}^\mu$, setting $d = \pi^{\mu - \nu(Ty)_i} e_i$ gives

$$T(d, y) = \pi^{\mu - \nu(Ty)_i} (Ty)_i \in \mathfrak{p}^\mu,$$

so $T(d, h) = \pi^{\mu - \nu(Ty)_i} (Th)_i \in \mathfrak{p}^\mu \mathfrak{d}^{-1}$, showing $\nu(Th)_i \geq \nu(Ty)_i - \nu_\mathfrak{d}$. We may assume that $(Ty)_1$ has the smallest valuation among the $(Ty)_i$ and define $a = \dfrac{(Th)_1}{(Ty)_1} \in \mathfrak{d}^{-1}$. $(Th)_1 \equiv a(Ty)_1 \pmod{\mathfrak{p}^\mu \mathfrak{d}^{-1}}$ certainly holds. For $i \neq 1$ such that $(Ty)_i \notin \mathfrak{p}^\mu$, set

$$d = \pi^{\nu(Ty)_i}((Ty)_1^{-1} e_1 - (Ty)_i^{-1} e_i) \in L.$$

$T(d,y) = 0 \in \mathfrak{p}^\mu$, hence

$$T(d,h) = \pi^{\nu(Ty)_i}\left(a - \frac{(Th)_i}{(Ty)_i}\right) \in \mathfrak{p}^\mu\mathfrak{d}^{-1},$$

so

$$\pi^{\nu(Ty)_i}a \equiv \frac{\pi^{\nu(Ty)_i}}{(Ty)_i}(Th)_i \pmod{\mathfrak{p}^\mu\mathfrak{d}^{-1}},$$

i.e., $(Th)_i \equiv a(Ty)_i \pmod{\mathfrak{p}^\mu\mathfrak{d}^{-1}}$.

The relation now holds at all i, showing that

$$Th \equiv aTy \pmod{\mathfrak{p}^\mu\mathfrak{d}^{-1}L}.$$ □

LEMMA.

$$S(\lambda,h) = \sum_{y\in\tau(\lambda,h)} \boldsymbol{e}_v\left(-\frac{T(y,h)}{\pi^\lambda}\right),$$

where

$$\tau(\lambda,h) = \left\{ y \in \overline{\sigma(\lambda)} : \nu(Th - aTy) \geq \lfloor \tfrac{\lambda}{2} \rfloor - \nu_{\mathfrak{d}} \textit{ for some } a \in \mathfrak{d}^{-1} \right\}.$$

Proof. Let $\mu = \lfloor \frac{\lambda}{2} \rfloor$ and $\nu = \lambda - \mu$ so that $2\nu \geq \lambda$. For any $y \in \sigma(\lambda)$ and $d \in L$ we have $T[y+\pi^\nu d] \equiv 2\pi^\nu T(y,d) \pmod{\mathfrak{p}^\lambda}$, showing that $\sigma(\lambda) = \{\, y + \pi^\nu d : y \in \sigma(\lambda), d \in L, T(y,d) \in \mathfrak{p}^\mu \,\}$. Projecting mod $\mathfrak{p}^\lambda$, $\overline{\sigma(\lambda)} = \{\, y + \pi^\nu d : y \in \overline{\sigma(\lambda)}, d \in L/\mathfrak{p}^\lambda, T(y,d) \in \mathfrak{p}^\mu/\mathfrak{p}^\lambda \,\}$. To avoid redundancy, take only $y \in \sigma(\lambda) \pmod{\mathfrak{p}^\nu L}$. So

$$\begin{aligned} S(\lambda,h) &= \sum_{\substack{y\in\sigma(\lambda) \pmod{\mathfrak{p}^\nu L} \\ d\in L/\mathfrak{p}^\lambda \\ T(y,d)\in\mathfrak{p}^\mu/\mathfrak{p}^\lambda}} \boldsymbol{e}_v\left(-\frac{T(y+\pi^\nu d,h)}{\pi^\lambda}\right) \\ &= \sum_y \boldsymbol{e}_v\left(-\frac{T(y,h)}{\pi^\lambda}\right)\sum_d \boldsymbol{e}_v\left(-\frac{T(d,h)}{\pi^\mu}\right). \end{aligned}$$

The sum over d vanishes if there exists some $d \in L$ such that $T(y,d) \in \mathfrak{p}^\mu$ and $\boldsymbol{e}_v\left(-\frac{T(d,h)}{\pi^\mu}\right) \neq 1$, since it is then a nontrivial character sum over a finite group. Such d exists if and only if $T(y,d) \in \mathfrak{p}^\mu \not\Rightarrow T(d,h) \in \mathfrak{p}^\mu\mathfrak{d}^{-1}$. So by the previous lemma, we may sum only over y such that $Th \equiv aTy \pmod{\mathfrak{p}^\mu\mathfrak{d}^{-1}L}$ for some

$a \in \mathfrak{d}^{-1}$, thus:

$$S(\lambda,h) = \sum_{\substack{y+\pi^\nu d:\\ y\in\sigma(\lambda) \pmod{\mathfrak{p}^\nu L}\\ d\in L \pmod{\mathfrak{p}^\mu}\\ T(y,d)\in\mathfrak{p}^\mu/\mathfrak{p}^\lambda\\ Th\equiv aTy \pmod{\mathfrak{p}^\mu\mathfrak{d}^{-1}L}\\ (\text{for some } a\in\mathfrak{d}^{-1})}} \boldsymbol{e}_v\left(-\frac{T(y+\pi^\nu d,h)}{\pi^\lambda}\right)$$
$$= \sum_{y\in\tau(\lambda,h)} \boldsymbol{e}_v\left(-\frac{T(y,h)}{\pi^\lambda}\right).$$

□

PROPOSITION 6.9. *If $\nu(T[h']) > 0$, $\alpha(h,X)$ is a polynomial $K_{h,v}(X) \in \mathbb{Q}[X]$ of degree less than $2(\nu'+1+2\nu_{h\mathfrak{d}}+\nu_\mathfrak{d})$, where $\nu' = \nu(T[h'])$.*

Proof. We will prove $\tau(\lambda,h)$ is empty for $\lambda \geq 2(\nu'+1+2\nu_{h\mathfrak{d}}+\nu_\mathfrak{d})$. Suppose $y \in \tau(\lambda,h)$. Then for some $a \in \mathfrak{d}^{-1}$, $Th - aTy \in \mathfrak{p}^{\lfloor\frac{\lambda}{2}\rfloor}\mathfrak{d}^{-1}L \subset \mathfrak{p}^{(\nu'+1+2\nu_\mathfrak{d})}L$, i.e., $Th \equiv aTy \pmod{\mathfrak{p}^{\nu'+1+2\nu_\mathfrak{d}}L}$. Multiplying by T^{-1} gives also $h \equiv ay \pmod{\mathfrak{p}^{\nu'+1+2\nu_\mathfrak{d}}L}$, so $\pi^{2\nu_h}T[h'] = T[h] \equiv a^2T[y] \pmod{\mathfrak{p}^{\nu'+1+2\nu_\mathfrak{d}}}$. But since $y \in \tau(\lambda,h)$, $a^2T[y] \in \mathfrak{p}^\lambda\mathfrak{d}^{-2} \subset \mathfrak{p}^{2(\nu'+1+2\nu_{h\mathfrak{d}})} \subset \mathfrak{p}^{\nu'+1+2\nu_{h\mathfrak{d}}}$, giving the contradiction $T[h'] \in \mathfrak{p}^{\nu'+1+2\nu_\mathfrak{d}}$. □

Summary. We gather the results of this chapter.

THEOREM 6.10. *For n even,*

$$\alpha_v(h_v,X) = \begin{cases} \left(1-q_v^{\frac{n}{2}-1}\theta(\mathfrak{p}_v)X\right)\left(1-q_v^{\frac{n}{2}}\theta(\mathfrak{p}_v)X\right)^{-1} \\ \qquad (1-q_v^{n-1}X)^{-1} & \text{if } h_v = 0\\ \left(1-q_v^{\frac{n}{2}-1}\theta(\mathfrak{p}_v)X\right)\left(1-q_v^{\frac{n}{2}}\theta(\mathfrak{p}_v)X\right)^{-1} \\ \qquad G_{h,v}(X) & \text{if } T[h_v] = 0\\ \left(1-q_v^{\frac{n}{2}-1}\theta(\mathfrak{p}_v)X\right) & \text{if } \nu(T[h_v']) = 0,\ \nu_{h\mathfrak{d}} = 0\\ H_{h,v}(X) & \text{if } \nu(T[h_v']) = 0,\ \nu_{h\mathfrak{d}} > 0\\ K_{h,v}(X) & \text{if } \nu(T[h_v']) > 0. \end{cases}$$

For n odd,

$$\alpha_v(h_v, X) = \begin{cases} (1-q_v^{n-1}X^2)(1-q_v^nX^2)^{-1} & \\ (1-q_v^{n-1}X)^{-1} & \text{if } h_v = 0 \\ (1-q_v^{n-1}X^2)(1-q_v^nX^2)^{-1}G_{h,v}(X) & \text{if } T[h_v]=0 \\ (1-q_v^{n-1}X^2)\left(1-q_v^{\frac{n-1}{2}}\theta_h(\mathfrak{p}_v)X\right)^{-1} & \text{if } \nu(T[h_v'])=0,\ \nu_{h\mathfrak{d}}=0 \\ H_{h,v}(X) & \text{if } \nu(T[h_v'])=0,\ \nu_{h\mathfrak{d}}>0 \\ K_{h,v}(X) & \text{if } \nu(T[h_v'])>0. \end{cases}$$

Recalling that $a_v(h_v,s) = \alpha_v(h_v, X_v(s))$ for $v \in \boldsymbol{f}$, $v \nmid \mathfrak{b}$, where from before $X_v(s) = \psi(\mathfrak{p}_v)^{-1}q_v^{-k-2s}$, and taking the product over such v gives,

THEOREM 6.11. *For* $z = (z_v) = (x_v + iy_v) \in \mathcal{H}^{\boldsymbol{a}}$,

$$E(z,s;k,\psi,\mathfrak{b}) = (-1)^{dk}2^{d(k+2s)} \sum_{h\in L'} a(h,y,s)\, \boldsymbol{e}\left(\sum_{v\in\boldsymbol{a}} T^v(x_v,h_v)\right),$$

with

$$a(h,y,s) = N\mathfrak{d}^{-n/2}a_{\boldsymbol{a}}(h,y,s)\,a_{\boldsymbol{f}}(h,s),$$

where

$$a_{\boldsymbol{a}}(h,y,s) = \prod_{v\in\boldsymbol{a}} \xi(y_v,h_v;k+s,s;T^v);$$

for n even,

(6.5a)

$$a_{\boldsymbol{f}}(h,s) = L_{\mathfrak{b}}\left(k+2s+1-\frac{n}{2},\theta\psi^{-1}\right)^{-1}$$

$$\cdot\begin{cases} L_{\mathfrak{b}}\left(k+2s-\frac{n}{2},\theta\psi^{-1}\right)L_{\mathfrak{b}}(k+2s-n+1,\psi^{-1}) & \text{if } h=0 \\ L_{\mathfrak{b}}\left(k+2s-\frac{n}{2},\theta\psi^{-1}\right)\prod_{v\nmid\mathfrak{b}:\nu_v(h)+\nu_v(\mathfrak{d})>0} G_{h,v}(X_v(s)) & \text{if } T[h]=0 \\ \prod_{\substack{v\nmid\mathfrak{b}:\nu_v(T[h_v'])=0,\\ \nu_v(h)+\nu_v(\mathfrak{d})>0}} \dfrac{H_{h,v}(X_v(s))}{(1-q_v^{\frac{n}{2}-1}\theta(\mathfrak{p}_v)X_v(s))} & \\ \quad\cdot\prod_{v\nmid\mathfrak{b}:\nu_v(T[h_v'])>0}\dfrac{K_{h,v}(X_v(s))}{\left(1-q_v^{\frac{n}{2}-1}\theta(\mathfrak{p}_v)X_v(s)\right)} & \text{if } T[h]\neq 0; \end{cases}$$

and for n odd,

(6.5b)

$$a_f(h,s) = L_{\mathfrak{b}}(2(k+2s)-n+1,\psi^{-2})^{-1}$$

$$\cdot \begin{cases} L_{\mathfrak{b}}(2(k+2s)-n,\psi^{-2})L_{\mathfrak{b}}(k+2s-n+1,\psi^{-1}) & \text{if } h=0 \\ L_{\mathfrak{b}}(2(k+2s)-n,\psi^{-2}) \displaystyle\prod_{v\nmid\mathfrak{b}:\nu_v(h)+\nu_v(\mathfrak{d})>0} G_{h,v}(X_v(s)) & \text{if } T[h]=0 \\ L_{\mathfrak{b}\mathfrak{h}}\left(k+2s-\dfrac{n-1}{2},\theta_h\psi^{-1}\right) & \\ \quad \cdot \prod_{\substack{v\nmid\mathfrak{b}:\nu_v(T[h'_v])=0,\\ \nu_v(h)+\nu_v(\mathfrak{d})>0}} \dfrac{H_{h,v}(X_v(s))}{(1-q_v^{n-1}X_v(s)^2)} & \\ \quad \cdot \prod_{v\nmid\mathfrak{b}:\nu_v(T[h'_v])>0} \dfrac{K_{h,v}(X_v(s))}{(1-q_v^{n-1}X_v(s)^2)} & \text{if } T[h]\neq 0. \end{cases}$$

Here $\mathfrak{h} = \prod_{\substack{v\nmid\mathfrak{b}:\nu_v(T[h'_v])=0,\\ \nu_v(h)+\nu_v(\mathfrak{d})>0}} \mathfrak{p}_v \prod_{v\nmid\mathfrak{b}:\nu_v(T[h'_v])>0} \mathfrak{p}_v$, *$\theta$ and θ_h are the quadratic characters defined in this chapter, and $G_{h,v}$, $H_{h,v}$ and $K_{h,v}$ are the polynomials from Propositions* 6.7, 6.8 *and* 6.9.

7. $E(z,s)$ at special values of s.

The order of $a(h,y,s)$ at $s=0$. For a discussion of near holomorphy and arithmeticity of a class of functions containing $E(z,s)$ the reader is referred to **[Sh86]**, **[Sh87]**, **[Bl90]**, **[Blpp]**. As a special case, we exhibit the Fourier expansion of $E(z,s)$ at $s=0$.

DEFINITION. For $h \in L'$ such that $T[h] \neq 0$, define

$$p_h = \#\{\, v \in \boldsymbol{a} : \quad h_v \in \mathcal{P}_v \,\},$$

$$q_h = \#\{\, v \in \boldsymbol{a} : -h_v \in \mathcal{P}_v \,\},$$

$$r_h = \#\{\, v \in \boldsymbol{a} : T^v[h_v] < 0 \,\}.$$

For nonzero $h \in L'$ with $T[h]=0$, define

$$s_h = \#\{\, v \in \boldsymbol{a} : T^v(h_v,\varepsilon_v) > 0 \,\},$$

$$t_h = \#\{\, v \in \boldsymbol{a} : T^v(h_v,\varepsilon_v) < 0 \,\}.$$

Define $b = \#\{\, v \in \boldsymbol{f} : v \mid \mathfrak{b} \,\}$.

Observe that $p_h + q_h + r_h = s_h + t_h = d$, where $d = [F:\mathbb{Q}]$, and that $b > 0$.

PROPOSITION 7.1. *For n even and $k \geq n/2$, $L_{\mathfrak{b}}(k+2s+1-n/2, \theta\psi^{-1})a(h,y,s)\mid_{s=0}$ has a zero of order at least*

$$\begin{cases} d-1, & \text{if } h=0 \text{ and } k=n/2+1, \psi=\theta \\ d, & \text{if } h=0 \text{ otherwise} \\ d+t_h-1, & \text{if } T[h]=0 \text{ and } k=n/2+1, \psi=\theta \\ d, & \text{if } T[h]=0 \text{ otherwise} \\ 2q_h+r_h, & \text{if } T[h]\neq 0. \end{cases}$$

For n odd and $k \geq (n+1)/2$, $L_{\mathfrak{b}}(2(k+2s)+1-n, \psi^{-2})a(h,y,s)\mid_{s=0}$ has a zero of order at least

$$\begin{cases} d-1, & \text{if } h=0 \text{ or } T[h]=0 \text{ and } k=(n+1)/2, \psi^2=1 \\ d, & \text{if } h=0 \text{ or } T[h]=0 \text{ otherwise} \\ q_h+r_h-1, & \text{if } T[h]\neq 0 \text{ and } k=(n+1)/2, \psi=\theta_h \\ q_h+r_h, & \text{if } T[h]\neq 0 \text{ otherwise}. \end{cases}$$

Proof. This is straightforward from examining the $\Gamma-$ and $L-$ factors that occur in $a(h,y,s)\mid_{s=0}$. For example, consider the case n even, $k \geq n/2$, $h=0$. A d-fold product of the archimedean factor in (5.1) gives a zero of order $2d$ if $k \geq n$; d if $n/2 < k < n$; 0 if $k=n/2$. The term $L_{\mathfrak{b}}(k-n/2, \theta\psi^{-1})$ in (6.5a) gives a zero of order 0 if $k > n/2+1$ or $k=n/2+1$, $\psi \neq \theta$; -1 if $k=n/2+1$, $\psi=\theta$; $d-1+b \geq d$ if $k=n/2$, $\psi=\theta$; d if $k=n/2$, $\psi\neq\theta$. And the term $L_{\mathfrak{b}}(k+1-n, \theta\psi^{-1})$ in (6.5a) gives a zero of order 0 unless $k=n$, $\psi=1$; -1 if $k=n$, $\psi=1$. Combining these gives the result. The other cases are simpler. □

COROLLARY 7.2. *For n even and $k \geq n/2$, $L_{\mathfrak{b}}(k+2s+1-n/2, \theta\psi^{-1})a(h,y,s)\mid_{s=0}$ is finite. It is nonzero only in the cases* (a) *$h \in \mathcal{P}^{\mathfrak{a}}$,* (b) *$F=\mathbb{Q}$, $k=n/2+1, \psi=\theta$, $T[h]=0$, $T(h,\varepsilon)>0$ or $h=0$.*

For n odd and $k \geq (n+1)/2$, excepting the case $k=(n+1)/2$, $\psi=\theta_h$ for some h, $L_{\mathfrak{b}}(2(k+2s)-n+1, \psi^{-2})a(h,y,s)\mid_{s=0}$ is finite. It is nonzero only in the cases (a) *$h \in \mathcal{P}^{\mathfrak{a}}$,* (b) *$F=\mathbb{Q}$, $k=(n+1)/2$, $\psi^2=1$, $T[h]=0$ or $h=0$.*

The Fourier expansion of $E(z,s)$ at $s=0$. From Proposition 5.1 we obtain

$$a_{\mathfrak{a}}(h,y,0) = (-1)^{dk}2^d\pi^{d(2k+1-\frac{n}{2})}\Gamma(k)^{-d}\Gamma(k+1-n/2)^{-d} \cdot |N(\det T)|^{-\frac{1}{2}}N(T[h])^{k-\frac{n}{2}}$$

$\boldsymbol{e}\left(\sum_{v\in\mathfrak{a}} T^v(iy_v,h_v)\right)$ if $h\in\mathcal{P}^{\mathfrak{a}}$. Thus for n even, $k\geq n/2$, excepting the case $F=\mathbb{Q}$, $k=n/2+1$, $\psi=\theta$, specializing to $s=0$ gives the holomorphic function

(7.1)
$$\begin{aligned}
&L_{\mathfrak{b}}\left(k+2s+1-\frac{n}{2},\theta\psi^{-1}\right)E(z,s;k,\psi,\mathfrak{b})\,|_{s=0}\\
&= \pi^{d(2k+1-\frac{n}{2})}|N(\det T)|^{-\frac{1}{2}}N\mathfrak{d}^{-\frac{n}{2}}2^{d(k+1)}\Gamma(k)^{-d}\Gamma\left(k+1-\frac{n}{2}\right)^{-d}\\
&\cdot \sum_{h\in L'\cap\mathcal{P}^{\mathfrak{a}}} N(T[h])^{k-\frac{n}{2}} \prod_{\substack{v\nmid\mathfrak{b}:\nu_v(T[h'_v])=0,\\ \nu_v(h)+\nu_v(\mathfrak{d})>0}} \frac{H_{h,v}(\psi^{-1}(\mathfrak{p}_v)q_v^{-k})}{\left(1-\theta\psi^{-1}(\mathfrak{p}_v)q_v^{\frac{n}{2}-k-1}\right)}\\
&\cdot \prod_{v\nmid\mathfrak{b}:\nu_v(T[h'_v])>0} \frac{K_{h,v}(\psi^{-1}(\mathfrak{p}_v)q_v^{-k})}{\left(1-\theta\psi^{-1}(\mathfrak{p}_v)q_v^{\frac{n}{2}-k-1}\right)}\,\boldsymbol{e}\left(\sum_{v\in\mathfrak{a}} T^v(z_v,h_v)\right),
\end{aligned}$$

with Fourier coefficients in $\pi^{d(2k+1-\frac{n}{2})}|N(\det T)|^{-\frac{1}{2}}\mathbb{Q}(\psi)$, where $\mathbb{Q}(\psi)$ is the extension of $\mathbb{Q}$ generated by values of ψ.

In the case $F=\mathbb{Q}$, $k=n/2+1$, $\psi=\theta$ our function also has nonholomorphic terms at $s=0$. Using Proposition 5.1 gives

(7.2)
$$\begin{aligned}
&\zeta_{\mathfrak{b}}(2+2s)E\left(z,s;\frac{n}{2}+1,\theta,\mathfrak{b}\right)|_{s=0}\\
&= \pi^{\frac{n}{2}+1}|\det T|^{-\frac{1}{2}}\left(1-\frac{n}{2}\right)\prod_{p|\mathfrak{b}}(1-p^{-1})2^{\frac{n}{2}-2}\\
&\cdot\Gamma\left(\frac{n}{2}+1\right)^{-1}L_{\mathfrak{b}}\left(2-\frac{n}{2},\theta\right)T[y]^{-1}\\
&+\pi^{\frac{n}{2}+2}|\det T|^{-\frac{1}{2}}\prod_{p|\mathfrak{b}}(1-p^{-1})2^{\frac{n}{2}+1}\Gamma\left(\frac{n}{2}+1\right)^{-1}\\
&\cdot\sum_{\substack{h\in L':T[h]=0,\\ T(h,\varepsilon)>0}}\ \prod_{p\nmid\mathfrak{b}:\nu_p(h)>0} G_{h,p}(\theta(p)p^{1-\frac{n}{2}})T[y]^{-1}T(y,h)\,\boldsymbol{e}\left(T(z,h)\right)
\end{aligned}$$

$$+\pi^{\frac{n}{2}+3}|\det T|^{-\frac{1}{2}}2^{\frac{n}{2}+2}\Gamma\left(\frac{n}{2}+1\right)^{-1}$$
$$\cdot\sum_{h\in L'\cap\mathcal{P}} T[h] \prod_{\substack{p\nmid\mathfrak{b}:\nu_p(T[h_p'])=0,\\ \nu_p(h)>0}} \frac{H_{h,p}(\theta(p)p^{-\frac{n}{2}-1})}{(1-p^{-2})}$$
$$\cdot \prod_{p\nmid\mathfrak{b}:\nu_p(T[h_p'])>0} \frac{K_{h,p}(\theta(p)p^{-\frac{n}{2}-1})}{(1-p^{-2})}\, \boldsymbol{e}\left(T(z,h)\right).$$

Here the coefficient of $T[y]^{-1}$ in the $h=0$ term is in $\pi^{\frac{n}{2}+1}|\det T|^{-\frac{1}{2}}\mathbb{Q}$ and is nonzero only if $n \equiv 2 \pmod 4$; the coefficients of $T[y]^{-1}T(y,h)\,\boldsymbol{e}(T(z,h))$ in the $T[h]=0$, $T(h,\varepsilon)>0$ terms are in $\pi^{\frac{n}{2}+2}|\det T|^{-\frac{1}{2}}\mathbb{Q}$; and the Fourier coefficients of the holomorphic terms are in $\pi^{\frac{n}{2}+3}|\det T|^{-\frac{1}{2}}\mathbb{Q}$.

Similar calculations show that for n odd, $k \geq (n+1)/2$, excepting the case $F=\mathbb{Q}$, $k=(n+1)/2$, $\psi^2=1$, specializing to $s=0$ gives the holomorphic function

(7.3)
$$L_{\mathfrak{b}}(2(k+2s)+1-n,\psi^{-2})E(z,s;k,\psi,\mathfrak{b})\,|_{s=0}$$
$$=\pi^{d(2k+1-\frac{n}{2})}\Gamma\left(k+1-\frac{n}{2}\right)^{-d}|N(\det T)|^{-\frac{1}{2}}N\mathfrak{d}^{-\frac{n}{2}}2^{d(k+1)}\Gamma(k)^{-d}$$
$$\cdot\sum_{h\in L'\cap\mathcal{P}^{\boldsymbol{a}}} N(T[h])^{k-\frac{n}{2}}L_{\mathfrak{b}\mathfrak{h}}\left(k-\frac{n-1}{2},\theta_h\psi^{-1}\right)$$
$$\cdot\prod_{\substack{v\nmid\mathfrak{b}:\nu_v(T[h_v'])=0,\\ \nu_v(h)+\nu_v(\mathfrak{d})>0}} \frac{H_{h,v}(\psi^{-1}(\mathfrak{p}_v)q_v^{-k})}{(1-\psi^{-2}(\mathfrak{p}_v)q_v^{n-2k-1})}$$
$$\cdot\prod_{v\nmid\mathfrak{b}:\nu_v(T[h_v'])>0} \frac{K_{h,v}(\psi^{-1}(\mathfrak{p}_v)q_v^{-k})}{(1-\psi^{-2}(\mathfrak{p}_v)q_v^{n-2k-1})}\,\boldsymbol{e}\left(\sum_{v\in\boldsymbol{a}}T^v(z_v,h_v)\right).$$

In this case the Fourier coefficients are in

$$\pi^{d(2k-\frac{n-1}{2})}|N(\det T)|^{-\frac{1}{2}}N\mathfrak{d}^{-\frac{n}{2}}\mathbb{Q}_{\mathrm{ab}}(\psi),$$

where $\mathbb{Q}_{\mathrm{ab}}$ denotes the maximal abelian extension of $\mathbb{Q}$ in $\mathbb{C}$.

In the case $F=\mathbb{Q}$, $k=(n+1)/2$, $\psi^2=1$, $\psi\neq\theta_h$ for all h, our function again has nonholomorphic terms at $s=0$. Let

$l = \lim_{s\to 0} L_{\mathfrak{b}}((n+1)/2 - n + 1 + 2s, \psi)/2s$. Then

(7.4)

$$\begin{aligned}
&\zeta_{\mathfrak{b}}(2+4s)E\left(z,s;\frac{n+1}{2},\psi,\mathfrak{b}\right)|_{s=0} \\
&= \pi^{\frac{n}{2}+1}|\det T|^{-\frac{1}{2}}\prod_{p|\mathfrak{b}}(1-p^{-1})(-1)^{\frac{n+1}{2}}2^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right)^{-1} \\
&\cdot \Gamma\left(\frac{n-1}{2}\right)^{-1}\Gamma\left(1-\frac{n}{2}\right)^{-1} lT[y]^{-\frac{1}{2}} \\
&+ \pi^{\frac{n+1}{2}}|\det T|^{-\frac{1}{2}}\prod_{p|\mathfrak{b}}(1-p^{-1})2^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)^{-1} \\
&\cdot \sum_{\substack{h\in L':T[h]=0,\\ T(h,\varepsilon)>0}}\ \prod_{p\nmid\mathfrak{b}:\nu_p(h)>0} G_{h,p}(\psi(p)p^{-\frac{n+1}{2}})T[y]^{-\frac{1}{2}}T(y,h)^{\frac{1}{2}}\,\boldsymbol{e}\left(T(z,h)\right) \\
&+ \pi^{\frac{n+1}{2}+1}|\det T|^{-\frac{1}{2}}2^{\frac{n+1}{2}+2}\Gamma\left(\frac{n+1}{2}\right)^{-1} \\
&\cdot \sum_{h\in L'\cap\mathcal{P}} T[h]^{\frac{1}{2}}L_{\mathfrak{b}\mathfrak{h}}(1,\theta_h\psi)\prod_{\substack{p\nmid\mathfrak{b}:\nu_p(T[h_p'])=0,\\ \nu_p(h)>0}}\frac{H_{h,p}\left(\psi(p)p^{-\frac{n+1}{2}}\right)}{(1-p^{-2})} \\
&\cdot \prod_{p\nmid\mathfrak{b}:\nu_p(T[h_p'])>0}\frac{K_{h,p}\left(\psi(p)p^{-\frac{n+1}{2}}\right)}{(1-p^{-2})}\,\boldsymbol{e}\left(T(z,h)\right).
\end{aligned}$$

The residue of $E(z,s)$ at special values of s. Analysis of (5.1) and (6.5) shows that for n even, $k = n/2 - 1$, $s = 1$, $L_{\mathfrak{b}}(k + 2s + 1 - n/2, \theta\psi^{-1})E(z,s;k,\psi,\mathfrak{b})$ is finite unless $\psi = \theta$, in which case it has a simple pole and

(7.5)

$$\begin{aligned}
&\mathrm{Res}_{s=1}\zeta_{\mathfrak{b}}(2s)E\left(z,s;\frac{n}{2}-1,\theta,\mathfrak{b}\right) \\
&= \pi^{d(\frac{n}{2}+1)}|N(\det T)|^{-\frac{1}{2}}N\mathfrak{d}^{-\frac{n}{2}}2^{d\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)^{-d} \\
&\cdot \mathrm{Res}_{\sigma=1}\zeta_{\mathfrak{b}}(\sigma)T[y]^{-d}\left\{2^{-d}L_{\mathfrak{b}}\left(2-\frac{n}{2},\theta\right)\right. \\
&+ \left.\sum_{\substack{h\in L':T[h]=0,\\ T^v(h_v,\varepsilon_v)>0, v\in\mathfrak{a}}}\ \prod_{v\nmid\mathfrak{b}:\nu_v(h>0)} G_{h,v}\left(\theta(\mathfrak{p}_v)q_v^{-\frac{n}{2}-1}\right)\boldsymbol{e}\left(\sum_{v\in\mathfrak{a}}T^v(z_v,h_v)\right)\right\}.
\end{aligned}$$

Similarly for n odd, $k = (n-1)/2$, $s = 1/2$, excluding the case $\psi = \theta_h$ for some h, $L_{\mathfrak{b}}(2(k+2s)+1-n, \psi^{-2})E(z,s;k,\psi,\mathfrak{b})$ is finite unless $\psi^2 = 1$, in which case it has a simple pole and

(7.6)

$$\begin{aligned}
&\mathrm{Res}_{s=1/2}\zeta_{\mathfrak{b}}(4s)E\left(z,s;\frac{n-1}{2},\psi,\mathfrak{b}\right)\\
&= \pi^{d(\frac{n}{2}+1)}|N(\det T)|^{-\frac{1}{2}}N\mathfrak{d}^{-\frac{n}{2}}2^{d\frac{n-1}{2}-2}\Gamma\left(\frac{n}{2}\right)^{-d}\mathrm{Res}_{\sigma=1}\zeta_{\mathfrak{b}}(\sigma)\\
&\cdot T[y]^{-\frac{d}{2}}\Bigg\{2^{-d}L_{\mathfrak{b}}\left(2-\frac{n+1}{2},\psi^{-1}\right)\\
&\sum_{\substack{h\in L':T[h]=0,\\ T^v(h_v,\varepsilon_v)>0, v\in\mathbf{a}}}\prod_{v\nmid\mathfrak{b}:\nu_v(h)>0}G_{h,v}\left(\psi(\mathfrak{p}_v)q_v^{-\frac{n+1}{2}}\right)e\left(\sum_{v\in\mathbf{a}}T^v(z_v,h_v)\right)\Bigg\}.
\end{aligned}$$

In (7.5) and (7.6), multiplying the residue by $T[y]^{sd}$ gives a holomorphic function.

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has a simple pole and

(7.5)

$$\operatorname{Res}_{s=1}\zeta_{\mathfrak{b}}(2s)E\left(z,s;\frac{n}{2}-1,\theta,\mathfrak{b}\right)$$

$$=\pi^{d(\frac{n}{2}+1)}|N(\det T)|^{-\frac{1}{2}}N\mathfrak{d}^{-\frac{n}{2}}2^{d\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)^{-d}$$

$$\cdot\operatorname{Res}_{\sigma=1}\zeta_{\mathfrak{b}}(\sigma)T[y]^{-d}\Bigg\{2^{-d}L_{\mathfrak{b}}\left(2-\frac{n}{2},\theta\right)$$

$$+\sum_{\substack{h\in L':T[h]=0,\\ T^v(h_v,\varepsilon_v)>0,v\in\boldsymbol{a}}}\prod_{v\nmid\mathfrak{b}:\nu_v(h>0)}G_{h,v}\left(\theta(\mathfrak{p}_v)q_v^{-\frac{n}{2}-1}\right)\boldsymbol{e}\left(\sum_{v\in\boldsymbol{a}}T^v(z_v,h_v)\right)\Bigg\}.$$

Similarly for n odd, $k=(n-1)/2$, $s=1/2$, excluding the case $\psi=\theta_h$ for some h, $L_{\mathfrak{b}}(2(k+2s)+1-n,\psi^{-2})E(z,s;k,\psi,\mathfrak{b})$ is finite unless $\psi^2=1$, in which case it has a simple pole and

(7.6)

$$\operatorname{Res}_{s=1/2}\zeta_{\mathfrak{b}}(4s)E\left(z,s;\frac{n-1}{2},\psi,\mathfrak{b}\right)$$

$$=\pi^{d(\frac{n}{2}+1)}|N(\det T)|^{-\frac{1}{2}}N\mathfrak{d}^{-\frac{n}{2}}2^{d\frac{n-1}{2}-2}\Gamma\left(\frac{n}{2}\right)^{-d}\operatorname{Res}_{\sigma=1}\zeta_{\mathfrak{b}}(\sigma)$$

$$\cdot T[y]^{-\frac{d}{2}}\Bigg\{2^{-d}L_{\mathfrak{b}}\left(2-\frac{n+1}{2},\psi^{-1}\right)$$

$$\sum_{\substack{h\in L':T[h]=0,\\ T^v(h_v,\varepsilon_v)>0,v\in\boldsymbol{a}}}\prod_{v\nmid\mathfrak{b}:\nu_v(h>0)}G_{h,v}\left(\psi(\mathfrak{p}_v)q_v^{-\frac{n+1}{2}}\right)\boldsymbol{e}\left(\sum_{v\in\boldsymbol{a}}T^v(z_v,h_v)\right)\Bigg\}.$$

In (7.5) and (7.6), multiplying the residue by $T[y]^{sd}$ gives a holomorphic function.

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