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GENERALIZED WAHL MAPS AND ADJOINT LINE BUNDLES ON A GENERAL CURVE

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For any two line bundles on a smooth curve, there are so called Wahl maps, that can be viewed as generalizations of the ordinary Gaussian. These maps govern various properties of the projective embeddings of C, like for example the first order deformations of the projective cone that smooth the vertex. In this paper we investigate these maps from the point of view of the intrinsic geometry of C, by applying an approach of Voisin for the case $L = N = K$.

1. Introduction. Consider a smooth projective curve C and two line bundles L and N on it. It is well known that there is a linear map, given by section multiplication

$$\mu : H^0(C, L) \otimes H^0(C, N) \longrightarrow H^0(C, L + N).$$

We define the *module of relations* of L and N, denoted $R(L, N)$, to be the kernel of μ. The Wahl map, or Gaussian map

$$\gamma_{L,N} : R(L, N) \longrightarrow H^0(K + L + N)$$

(where K denotes the canonical line bundle on C) is defined by making sense of the expression $\gamma_{L,N}(s, t) =: sdt - tds$. These maps have attracted increasing attention since Wahl's basic observation that they relate to the deformation theory of the projective cone over C ([**W88**]). In fact, if L is a very ample line bundle on C the cokernels of $\gamma_{K,L^{i-1}}$, for i positive, are dual to the first order deformations of the projective cone which smooth the vertex. From this it follows, for example, that if C is the hyperplane section of a (projective) $K3$ surface, then $\gamma_{K,K}$ is not surjective. This was proved from a deformation theoretic point of view by Wahl, and along different lines by Beauville and Merindol ([**BM87**]).

This circle of ideas has led to the question of the behavior of $\gamma_K = \gamma_{K,K}$ on a general curve. In fact, this map being onto implies that the general canonical curve is not the hyperplane section of a smooth surface. Ciliberto *et al* have proved, using degeneration methods, that for a curve with general moduli γ_K surjects ([**CHM**]). Mukai then observed ([**M87**]) that if C is a smooth curve lying on a $K3$ surface and such that the class of C generates $\mathrm{Pic}(C)$, then on C there are minimal pencils for which the adjoint line bundle is not projectively normal. Voisin has then generalized this observation into a new conceptual approach to the problem ([**V**]). Namely, she shows that if C is a Petri general curve for which γ_K is not onto, then on C there exist complete linear series of dimension one and minimal degree such that for all of these the adjoint line bundle is not linearly normal. In other words, if A is a minimal pencil and γ_K is not surjective the multiplication map $S^2H^0(C, K-A) \longrightarrow H^0(C, 2K-2A)$ can't be onto. Then she shows that this cannot happen on a general curve, thereby proving surjectivity of γ_K in this case. Her proof uses two very different arguments in the odd genus case and in the even genus one.

At the same time, there has been growing activity concerning the problem of the surjectivity of $\gamma_{L,N}$, for arbitrary line bundles L and N on C. This more general question has been explored, among others, by Bertram, Ein and Lazarsfeld ([**BEL89**]), and by Wahl ([**W88**],[**W89**],[**W90**],[**W**]). The first three authors have found conditions on the degree of L involving the Clifford index of the curve that guarantee surjectivity of $\gamma_{K,L}$. Wahl has found other conditions, and he formulated a conjecture to the effect that $\gamma_{K,L}$ is onto as soon as $\deg(L) \geq 2g+$ some suitable constant. More generally, he has posed the question of finding a geometric interpretation of the failure of $\gamma_{L,N}$ to surject, and of the resulting stratification of the Picard group of C in terms of the corank of $\gamma_{L,N}$. The object of this article is to show that even for these more general Wahl maps one can still interpret the failure of $\gamma_{L,N}$ to surject in terms of the existence of pencils of small degree for which suitable section multiplication maps are not surjective.

Before describing the results, let me recall that $W^r_d(C)$ denotes the subvariety of $\mathrm{Pic}^d(C)$ consisting of the line bundles L on C satisfying $\deg(L) = d$ and $h^0(L) \geq r+1$. Then the main theorem is

THEOREM 1.1. *Let C be a general curve of genus $g > 8$ and L be a line bundle on it. Then*

(i) *If $g = 2s$ and $\deg(L) > 3s$, or if $g = 2s+1$ and $\deg(L) > 3s+3$ and $\gamma_{K,L}$ is not onto, then section multiplication $H^0(K-A) \otimes H^0(L-A) \longrightarrow H^0(K+L-2A)$ is not surjective, for general $A \in W^1_{s+2}$.*

(ii) *Suppose that L is chosen generally and that $\deg(L) \geq 3s+10$ when $g = 2s$ or that $\deg(L) \geq 3s+6$ when $g = 2s+1$. Then the above multiplication is onto, for a general choice of such an A.*

From the theorem it immediately follows

COROLLARY 1.1. *Let C be as above.For a general line bundle L on C with the above lower bounds on the degree, $\gamma_{K,L}$ is onto.*

Furthermore with a little argument one also obtains

COROLLARY 1.2. *Let C be a general curve of genus $g > 8$ and L be an arbitrary line bundle on it, of degree $\geq 5s+12$ if $g = 2s$ or $\geq 5s+8$ when $g = 2s+1$. Then $\gamma_{K,L}$ is onto.*

REMARK 1.1. To obtain Corollary 1.2 from Corollary 1.3, one shows that if $\gamma_{K,L}$ is onto for a general line bundle of degree d, then it is onto for an arbitrary line bundle of degree $d+g+2$. In a private communication, Jan Stevens recently showed with a more sophisticated argument that this can be improved to $d+g$.

The same attack can be applied to Wahl maps of the kind $\gamma_{L,N}$, with L and N any two line bundles on C. In this direction I prove the following

THEOREM 1.2. *Let C be a general curve of genus $g > 8$ and L, N be two line bundles on it. Then*

(i) *assume that $\deg(L) \geq 3s+5$ if $g = 2s+1$ (resp., $\geq 3s+4$ if $g = 2s$), and that $\deg(N) \geq \deg(L) + g - 1$. Then if L and N are chosen general $\gamma_{L,N}$ is onto.*

(ii) *if L and N are arbitrary and $\deg(L) \geq 5s+7$ if $g = 2s+1$ (resp., $\geq 5s+5$ if $g = 2s$) and $\deg(N) \geq \deg(L) + g - 1$,then $\gamma_{L,N}$ is onto.*

Corollary 2 and Theorem 2(2) should be compared with the similar results obtained in [**BEL89**].

The paper is organized as follows. In the first part, Voisin's point of view is applied to the situation at hand. Specifically, in §1 we explain the relation between gaussian maps of type $\gamma_{K,L}$ and section multiplications $H^0(C, K-A) \otimes H^0(C, L-A) \longrightarrow H^0(C, K+L-2A)$, for A a pencil on C. In particular, it is shown that the proof of the first statement of Theorem A follows from the surjectivity of

$$\phi: \bigoplus_{A \in W^1_{s+2}(C)} H^0(K+L-2A) \otimes \wedge^2 H^0(A) \longrightarrow H^0(2K+L)$$

given by the composition of $id \otimes \gamma_{A,A}$ with section multiplication. The surjectivity of ϕ is dealt with in §2.

In §3 a degeneration argument is used to show that the above multiplications are surjective on the general curve, thereby obtaining a surjectivity statement for $\gamma_{K,L}$ under suitable conditions on L. The proof is given by an induction on the genus.

In §4 these results are extented to the case of the Gaussian maps $\gamma_{L,N}$ and in §5 an application to higher Wahl maps is given.

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2. A basic commutative diagram. As before, consider a smooth projective curve C and a line bundle L on it. If A is any other line bundle on C and $V \subset H^0(A)$ is a pencil of sections of A, then we have a commutative diagram

(1)
$$\begin{array}{ccc} H^0(L-A) \otimes H^0(K-A) \otimes \wedge^2 V & \xrightarrow{\alpha} & H^0(L+K-2A) \otimes \wedge^2(V) \\ \beta \downarrow & & \delta \downarrow \\ R(L,K) & \xrightarrow{\gamma_{L,K}} & H^0(2K+L) \end{array}$$

where α is section multiplication, and $\beta((s \otimes t) \otimes (u_1 \wedge u_2)) = (s \cdot u_1) \otimes (t \cdot u_2) - (s \cdot u_2) \otimes (t \cdot u_1)$, while δ is the composition of $id \otimes \gamma_A$ with section multiplication (observe that $\wedge^2 V \subset \wedge^2 H^0(A) \subset R(A,A)$).

Recall the following facts about the varieties W^r_d:

(i) W^r_d is a connected subvariety of $\mathrm{Pic}^d(C)$;

(ii) for any curve $C, \dim(W^r_d)$ is at least equal to the expected dimension expressed by the Brill-Noether number $\varrho(r,d,g) = g-(r+1)\cdot(g-d+r)$;

(iii) if C is Petri general, than $\dim(W^r_d) = \varrho(r,d,g)$ and W^r_d is smooth off W^{r+1}_d.

In particular it follows that, if C is general and $\varrho(r+1,d,g)<0$, then W^r_d is a smooth irreducible variety of the expected dimension.

Let us assume in what follows that C is a Petri general curve. Then $\dim(W^1_{s+2}) = 2$ when $g = 2s$ and $= 1$ when $g = 2s+1$; by the above in both cases W^1_{s+2} is smooth and irreducible, and nondegenerate in $\mathrm{Pic}^d(C)$. For any nonempty open subset $U \subset W^1_{s+2}$ let us define

$$R_U =: \bigoplus_{A\in U} H^0(L-A)\otimes H^0(K-A)\otimes W_A$$

$$S_U =: \bigoplus_{A\in U} H^0(L+K-2A)\otimes W_A$$

where $W_A = \wedge^2 H^0(C,A)$. We then have a commutative diagram

$$\begin{array}{ccc} R_U & \longrightarrow & S_U \\ \big\downarrow & & \phi\big\downarrow \\ R(L,K) & \xrightarrow{\gamma_{L,K}} & H^0(2K+L) \end{array} \tag{2}$$

and we clearly have:

REMARK 2.1. Assume that ϕ above is surjective. Then if $\gamma_{L,N}$ is not onto, not all multiplications $H^0(L-A)\otimes H^0(K-A)\longrightarrow H^0(L+K-2A)$, for $A\in W^1_{s+2}$, can be surjective.

We can say more:

PROPOSITION 2.1. *Let C be a Petri general curve of genus $g>8$ and L be a line bundle on it. Suppose that ϕ is surjective.*

(i) *Assume that* $\deg(L)>3s$ *if $g=2s$, or that* $\deg(L) > 3s \mp 3$ *if $g = 2s+1$, and that $L\neq K$ is general. Define $U =: \{A\in W^1_{s+2} | h^0(K+A-L)=0\}$. Then if $\gamma_{K,L}$ is not onto the multiplications $H^0(L-A)\otimes H^0(K-A)\longrightarrow H^0(L+K-2A)$ are never surjective, for any $A\in U$*

(ii) *If γ_K is not surjective, then $S^2H^0(K-A) \longrightarrow H^0(2K-2A)$ is not surjective, for any $A \in W^1_{s+2}$.*

REMARK 2.2. In (i), the condition on $\deg(L)$ implies that $\deg(K+A-L) < g$ and so, for a general choice of such an L, U will be a nonempty open subset of W^1_{s+2}; the complement of U consists of those points at which the dimension of $H^0(C, L-A)$ jumps up. In (2), $H^0(K-A)$ has constant rank on W^1_{s+2}. Also, (ii) is the content of ([**V**, Lemma 10]).

Proof. For (1), for a general choice $(A_1, \dots, A_n) \in U \times \cdots \times U$ where n is sufficiently large

$$\bigoplus_{i=1}^{n} H^0(K+L-2A_i) \longrightarrow H^0(2K+L)$$

is onto. On the other hand, since $H^0(2K+L)$ and $H^0(K+L-2A)$ have constant rank on U, if one of the above multiplications is onto for some $A \in U$ the same is true for the general point of U. So if the statement was false a general choice of $(A_1, \dots, A_n)$ would yield a composition of surjections

$$\bigoplus_{i=1}^{n} H^0(L-A_i) \otimes H^0(K-A_i) \to \bigoplus_{i=1}^{n} H^0(L+K-2A_i) \to H^0(2K+L)$$

and then $\gamma_{K,L}$ would be onto. For (2), use the same argument with $U = W^1_{s+2}$ □

3. Surjectivity of ϕ. As we have seen, we are led to the question of the surjectivity of

$$\phi : \bigoplus_{A \in U} H^0(K+L-2A) \longrightarrow H^0(2K+L). \tag{3}$$

We'll prove:

THEOREM 3.1. *Suppose that C is a Petri general curve of genus $g > 8$ and that L is a general line bundle on C, with* $\deg(L) >$ $2\deg(A)$, *and let $U \subset W^1_{2s+2}$ be open and nonempty. Then ϕ is onto.*

Proof. The following argument is an adaptation of Voisin's. We'll study the kernel of the dual map

$$\phi^* : H^1(T_C - L) \longrightarrow \bigoplus_{A \in U} H^1(2A-L) \tag{4}$$

given by $\phi^*(u) = \oplus_{A \in W^1_{s+2}} u \cdot R_A$, where $R_A \in H^0(2A+K)$ denotes the ramification divisor of the morphism $\phi_A : C \longrightarrow \mathbb{P}^1$ associated to the pencil A. Choose a double cover $\pi : \widetilde{C} \longrightarrow C$ ramified along a general $B \in |2L|$. We then have the basic isomorphism

$$\pi_* \mathcal{O}_{\tilde{C}} \simeq \mathcal{O}_C \oplus L^{-1} \tag{5}$$

$$K_{\tilde{C}} \simeq \pi^*(K_C \otimes L) \tag{6}$$

and so

$$H^1(\widetilde{C}, T_{\tilde{C}}) \simeq H^1(C, T_C - L) \oplus H^1(C, T_C - 2L)$$

Hence we can interpret $u \in H^1(C, T_C - L)$ as a first order deformation of $\widetilde{C}$. It is easily checked that the natural cup product map

$$H^1(C, T_C - L) \longrightarrow \mathrm{Hom}(H^0(C, K_C + L), H^1(C, \mathcal{O}_C)) \tag{7}$$

is a component of the period map of $\widetilde{C}$; it is furthermore still injective, because its dual is given by section multiplication $H^0(K_C + L) \otimes H^0(K_C) \longrightarrow H^0(2K_C + L)$ and this is surjective because $\deg(2K + L) > 4g + 2$ and by a theorem of Mark Green. So to prove the theorem it is sufficient to show that any $u \in \mathrm{Ker}(\phi^*)$ maps to zero under the morphism in equation 7.

Observe that, since L is non-torsion, the pull-back map $\pi^* : \mathrm{Pic}^{s+2}(C) \longrightarrow \mathrm{Pic}^{2s+4}(\widetilde{C})$ is injective. Hence $\pi^*(W^1_{s+2}(C)) \subset \mathrm{Pic}^{2s+4}(\widetilde{C})$ is a smooth subvariety, isomorphic to $W^1_{s+2}(C)$. □

LEMMA 3.1. *$\pi^*(W^1_{s+2})$ is an irreducible component of $W^1_{2s+4}(\widetilde{C})$.*

Proof. Clearly $\pi^*(W^1_{s+2}(C)) \subset W^1_{2s+4}(\widetilde{C})$. By Brill-Noether theory [**ACGH84**] the statement will follow if we show that, at the general point of $\pi^*(W^1_{s+2}(C))$ the Petri homomorphism

$$\mu_{\pi^*(A)} : H^0(\widetilde{C}, \pi^*(A)) \otimes H^0(\widetilde{C}, K_{\widetilde{C}} \otimes \pi^*(-A)) \longrightarrow H^0(\widetilde{C}, K_{\widetilde{C}}) \tag{8}$$

has corank equal to the dimension of $W^1_{s+2}(C)$. Assume first that A is globally generated. From (5) and (6) one has that (8) splits as the direct sum of

$$\mu_A : H^0(C, A) \otimes H^0(C, K - A) \longrightarrow H^0(K_C) \tag{9}$$

$$\nu_A : H^0(C, A) \otimes H^0(C, K + L - A) \longrightarrow H^0(C, K_C + L) \tag{10}$$

and (9) is the Petri homomorphism of A, which by the assumption on C has corank equal to $\dim(W^1_{s+2}(C))$, while the base point free pencil trick applied to A, together with the fact that $H^1(K+L-2A)=0$, show that (10) is a surjection. So the proof will be complete if we show that the general point $A \in W^1_{s+2}(C)$ is a globally spanned line bundle on C. Consider first the case $g=2s$. Then $\dim(W^1_{s+1}(C)) = 0$ and each $B \in W^1_{s+1}(C)$ is spanned, because $\varrho(1,d,2s) < 0 \quad \forall d < s+1$. Also, if $P \in C$ and $B \in W^1_{s+1}(C)$ then it is easily checked that $B+P \in W^1_{s+2}(C)$, and that the base point locus of $B+P$ is $\{P\}$. Hence we get a finite family of (disjoint) copies of C in $W^1_{s+2}(C)$, one for each element of W^1_{s+1}. If $A \in W^1_{s+2}(C)$ is not globally generated and P is a base point of A, then $B = A - P \in W^1_{s+1}(C)$, i.e. A lies on one of these curves. Hence it is clear that the lemma holds in this case. As to the case $g = 2s+1$, it is easy to see that all $A \in W^1_{s+2}(C)$ are globally generated. □

Let's now return to the proof of Theorem 3.1. Suppose $u \in \mathrm{Ker}(\phi^*)$, so that for all $A \in U$ we have $A \cdot R_A \in H^1(2A-L)$. Since $H^0(2A-L)=0, H^1(2A-L)$ has constant rank on $W^1_{s+2}(C)$, and so we actually have $u \cdot R_A = 0$ in $H^1(C, 2A-L), \forall A \in W^1_{s+2}(C)$. This has the following deformation theoretic interpretation. First of all, observe that the first order deformation of $\widetilde{C}$ induced by u induces a first order deformation of $\mathrm{Pic}^{2s+4}(\widetilde{C})$. Next we have:

CLAIM 3.1. *Suppose that $u \in \mathrm{Ker}(\phi^*)$, and that $A \in W^1_{s+2}(C)$. Then $\pi^*(A)$ deforms together with its sections along the first order deformation induced by u.*

Proof of the Claim. By Brill- Noether theory, a line bundle N on $\widetilde{C}$ deforms together with its sections along the first order deformation induced by $\xi \in H^1(\widetilde{C}, T_{\widetilde{C}})$ if and only if ξ annihilates the image of the Wahl map $\gamma_{N,K_{\widetilde{C}}-N}$ which maps the kernel of the Petri homomorphism of N to $H^0(\widetilde{C}, 2K_{\widetilde{C}})$. Suppose first that A is base point free. Applying the base point free pencil trick to A we get that

$$\mathrm{Ker}\{\mu_{\pi^*(A)}\} \simeq H^0(C, K+L-2A).$$

Now $\gamma_{L,K_{\widetilde{C}}-L}$ split as the direct sum of maps $\alpha_1 : H^0(C, K+L-2A) \longrightarrow H^0(C, 2K+2L)$ and $\alpha_2 : H^0(C, K+L-2A) \longrightarrow$

$H^0(C, 2K+L)$, where α_2 is given by multiplication with R_A. Therefore, since $u \cdot R_A = 0$ u annihilates the image of α_2. Because $H^1(K+L) = 0$, it also kills the image of α_1, and the Claim follows in this case. If $g = 2s$ and $A \in W^1_{s+2}$ is not spanned than A has exactly one base point P, and $A - P \in W^1_{s+1}$. Now we apply the base point free pencil trick to $A-P$, and this yields an exact sequence $0 \to H^0(C, K_C+L+P-2A) \to H^0(C, A) \otimes H^0(K+L-A) \to H^0(C, K+L-P)$ and since the latter space injects into $H^0(C, K+L)$ we obtain

$$\mathrm{Ker}\{\mu_{\pi^* A}\} \simeq H^0(C, K_C+L+P-2A) \subset H^0(C, K_C+L+2P-2A).$$

Now $H^0(C, K_C+2P-2A) \longrightarrow H^0(C, 2K_C+L)$ is given by cupping with R_A, and so the statement follows in this case also. □

This proves following:

COROLLARY 3.1. *The first order deformation of* $\mathrm{Pic}^{2s+4}(C)$ *associated to* $u \in \mathrm{Ker}(\phi^*)$ *contains a first order deformation of* $W^1_{s+2}(C)$.

Observe that given an inclusion of algebraic manifolds $Y \subset X$ and a first order deformation of X containing a first order deformation of Y there is a commutative diagram

$$\begin{array}{ccc} H^0(X, \Omega^1_X) & \xrightarrow{u_X} & H^1(X, \mathcal{O}_X) \\ \downarrow & & \downarrow \\ H^0(Y, \Omega^1_Y) & \xrightarrow{u_Y} & H^1(Y, \mathcal{O}_Y) \end{array}$$

where $u_X \in H^1(X, T_X)$ and $u_y \in H_1(Y, T_Y)$ are the extension classes of the two first order deformations.

On the other hand, we have the isomorphisms

$$H^0(X, \Omega^1_{\mathrm{Pic}^{2s+4}(\tilde{C})}) \simeq H^0(C, K+L) \oplus H^0(C, K)$$
$$H^1(\mathrm{Pic}^{2s+4}(\tilde{C}), \mathcal{O}_{\mathrm{Pic}^{2s+4}(\tilde{C})}) \simeq H^1(C, \mathcal{O}_C) \oplus H^1(C, -L)$$

and so we get the following commutative diagram

(11)

$$\begin{array}{ccc}
H^0(C, K+L) & \xrightarrow{u} & H^1(C, \mathcal{O}_C) \\
{\scriptstyle a}\downarrow & & {\scriptstyle d}\downarrow \\
H^0(\mathrm{Pic}^{2s+4}(\widetilde{C}), \Omega_{\mathrm{Pic}^{2s+4}(\widetilde{C})}) & \longrightarrow & H^1(\mathrm{Pic}^{2s+4}(\widetilde{C}), \mathcal{O}_{\mathrm{Pic}^{2s+4}(\widetilde{C})}) \\
{\scriptstyle b}\downarrow & & {\scriptstyle e}\downarrow \\
H^0(W^1_{s+2}(C), \Omega_{W^1_{s+2}(C)}) & \longrightarrow & H^1(W^1_{s+2}(C), \mathcal{O}_{W^1_{s+2}(C)})
\end{array}$$

For the proof of Theorem 3.1 we are then reduced to:

LEMMA 3.2. *In diagram* (11) *we have* $ba = 0$ *and* ed *is injective.*

Proof. b is the composition

$$H^0(K_{\widetilde{C}}) \longrightarrow H^0(K_C) \hookrightarrow H^0(\Omega^1_W).$$

Hence the first assertion follows, because the first map above is just projection along $H^0(K+L)$. Next observe that ed is the composition

$$H^1(\mathcal{O}_C) \hookrightarrow H^1(\mathcal{O}_{W^1_{s+2}(C)}) \simeq H^0(\Omega_{W^1_{s+2}(C)})$$

where the last map is injective by [**FL81**]. □

This completes the proof of the Theorem. □

REMARK 3.1. Since $H^0(C, K_C - A)$ has constant rank on W^1_{s+2}, we apply this argument to the case $L = K$. With respect to the proof in [**V**], dealing with W^1_{s+2} rather than W^1_{s+1} in the even genus case avoids the hypothesis $L = K$ and simplifies the argument. However, this is done at the numerical cost of dealing with pencils that are only next to minimal rather than minimal in the case of even genus. In other words, when the above theorem is applied to the particular case $L = K$ and $g = 2s$ we only get that if γ_K is not onto then $K_C - A$ is not projectively normal, for $A \in W^1_{s+2}$, rather than for $W^1_{s+1}(C)$. In spite of what I was erroneously claiming in a first draft of this paper this does not imply the stronger numerical statement that $K_C - A$ is not projectively normal, for $A \in W^1_{s+1}$.

4. Surjectivity of $\gamma_{K,L}$. Referring to diagram 2, we have shown that under appropriate conditions the map ϕ is onto. It follows as remarked in §2 that if $\gamma_{K,L}$ is not onto then no multiplication $H^0(C, K-A) \otimes H^0(C, L-A) \longrightarrow H^0(C, K+L-2A)$ is onto, for any $A \in U$. This is the statement of Proposition 2.1. We'll prove:

THEOREM 4.1. *Let $g \geq 4$ be either $2s$ or $2s+1$, and let (C, L, A) be a general choice of a smooth curve of genus g, and line bundles L and A on C, with $A \in W^1_{s+2}(C)$ and* $\deg(L) \geq 3s+10$ *if $g = 2s$ and* $\deg(L) \geq 3s+6$ *for $g = 2s+1$. Then $H^0(C, K+A-L) = 0$ and section multiplication $H^0(C, K-A) \otimes H^0(C, L-A) \longrightarrow H^0(C, K+L-2A)$ is surjective.*

Before proving the Theorem, let's remark that it implies the following

COROLLARY 4.1. *Let C be a general curve of genus $g > 8$, with $g = 2s$ or $g = 2s+1$. Then if L is a general line bundle on C, with* $\deg(L) \geq 3s+10$ *when $g = 2s$ and* $\deg(L) \geq 3s+6$ *when $g = 2s+1$ the Wahl map $\gamma_{K,L}$is onto.*

From this we may deduce a result already contained in [**BEL89**]:

COROLLARY 4.2. *Let C be a general curve of genus $g > 8$ as before, and let N be a line bundle on C, satisfying* $\deg(N) \geq 5s+12$ *when $g = 2s$ and* $\deg(N) \geq 5s+8$ *when $g = 2s+1$. Then $\gamma_{K,N}$ is onto.*

Proof. If B is a general line bundle on C of degree $g+2$, we may assume that B is spanned and that $\gamma_{K,N-B}$ is onto, by virtue of the previous corollary. Consider a pencil of section $V \subset H^0(C, B)$ which generates V. We have a commutative diagram

$$\begin{array}{ccc} R(K, N-B) \otimes V & \xrightarrow{\gamma_{K,N-B}} & H^0(C, 2K+N-B) \otimes V \\ \downarrow & & \downarrow{\scriptstyle \beta} \\ R(K, N) & \xrightarrow{\gamma_{K,N}} & H^0(C, 2K+N) \end{array} \tag{12}$$

By assumption, $\gamma_{K,N-B}$ is onto, and the base point free pencil trick shows that so is β. Hence $\gamma_{K,N}$ is also onto. $\square$

Proof of Theorem 4.1. Let us first consider the case $g = 2s + 1$. The statement of the Theorem in this case will follow from the following

PROPOSITION 4.1. *Let C be a general curve of genus $g > 3$, and let L be a general line bundle on C, with* $\deg(L) \geq 3s + 4$ *if* $g = 2s$ *and* $\deg(L) \geq 3s + 6$ *if* $g = 2s + 1$. *Then there exists* $A \in W^1_{d_{\min}}(C)$ *satisfying* $H^0(C, K + A - L) = 0$ *and such that* $H^0(C, K - A) \otimes H^0(C, L - A) \longrightarrow H^0(C, K + L - 2A)$ *is onto.*

Recall that $d_{\min} = s + 1$ when $g = 2s + 1$ and $d_{\min} = s + 2$ when $g = 2s+1$. The odd genus case of the Proposition is the same as the odd genus case of the Theorem. The even genus case and the odd genus case of the Proposition can be proved simultaneously with an induction argument.

Proof of Proposition 4.1. We proceed by induction on g. To begin with, if C is Petri general of genus $g \geq 4$ then by Riemann-Roch and Brill-Noether number calculations one easily checks that, $\forall A \in W^1_{d_{\min}}(C), K - A$ is spanned for $g \geq 4$ and birationally very ample for $g > 3$ (and very ample for $g \geq 10$). So let C be Petri general of genus 4, so that $\deg(A) = 3$ and $K - A$ is spanned. In fact, $K - A \in W^1_3(C)$ and by the base point free pencil trick the surjectivity of section multiplication follows if we have $H^1(C, L - K) \simeq H^0(C, 2K - L)^* = 0$. But under the given assumptions $\deg(2K - L) \leq 2$ and a general choice of L does the job. On the other hand $\deg(K + A - L) < 1$ and so $H^0(C, K + A - L) = 0$ can also be arranged. One deals similarly with the case $g = 5$.

So now suppose given a general curve C, which we'll also assume to be Petri general, of genus $g = 2s \geq 4$ and line bundles A and L on C with $A \in W^1_{s+1}(C)$ and $\deg(L) \geq 3s+4$, satisfying the conclusions of Proposition 4.1. We may assume without loss that $L - A$ is very ample, and we know that $K - A$ is spanned and birationally very ample. Therefore we have two nondegenerate morphisms $\varphi_{K-A} : C \longrightarrow \mathbb{P}^{s-1}$ and $\varphi_{L-A} : C \hookrightarrow \mathbb{P}^l$, and hence a product embedding $\varphi : C \hookrightarrow \mathbb{P}^{s-1} \times \mathbb{P}^l$. Choose points $P, Q \in C$ generally and let $l_1 \subset \mathbb{P}^{s-1}$ and $l_2 \subset \mathbb{P}^l$ meeting C nontangentially at P and Q and at no other point. Identifying l_1 and l_2 the product embedding gives a smooth rational curve $\triangle \subset \mathbb{P}^{s-1} \times \mathbb{P}^l$ meeting $\varphi(C)$ nontangentially at P and Q. Define $C' =: C \cup \triangle$; then C' is a nodal curve, of genus $g' = 2s + 1$. The proof of the following lemma will be given later: □

LEMMA 4.1. *C' can be smoothed in $\mathbb{P}^{s-1} \times \mathbb{P}^l$.*

Next let

$$A' =: K_{C'} \otimes \mathcal{O}_{\mathbb{P}^{s-1}}(-1)$$

Then $\deg(A') = \deg(A'|_C) + \deg(A'|_\triangle)$ and so

$$\deg(A') = 2g' - 2 - \deg(K_C) + \deg(A) - 1 = s + 2$$

and by Riemann-Roch

$$\begin{aligned} h^0(C', A') &= h^0(C', K_{C'} - A') + s + 2 + 1 - g' \\ &= h^0(C, K - A) + h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) + 2 - s = 2. \end{aligned}$$

Next define L' on C' by

$$L' - A' = \mathcal{O}_{\mathbb{P}^l}(1)|_{C'}$$

so that

$$\deg(L') = \deg(A') + \deg(L - A) + 1 = \deg(L) + 2.$$

Let's first check that $H^0(C', K_{C'} + A' - L') = 0$. This is equivalent to $h^0(C', L' - A') = \deg(L' - A') + 1 - g'$. But the right hand side is $h^0(C, L - A) + h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) - 2 = h^0(C, L - A)$, while the left hand side is $\deg(L - A) + 1 - g = h^0(L - A)$. So this step of the induction is reduced to the following:

LEMMA 4.2. *$H^0(C', L' - A') \otimes H^0(C', K' - A') \longrightarrow H^0(C', K' + L' - 2A')$ is onto.*

Let's postpone the proof of the above lemma and proceed to the second part of the induction. So assume given a triple (C, L, A) with C a Petri general curve of genus $g = 2s + 1$ and L, A line bundles on C satisfying $A \in W^1_{s+2}(C), \deg(L) \geq 3s + 6$, $H^0(C, K + A - L) = 0$ and such that the surjectivity statement of the proposition holds. As before we may assume that $L - A$ is very ample, and we consider the product embedding $\varphi_{K-A} \times \varphi_{L-A} : C \longrightarrow \mathbb{P}^{s-1} \times \mathbb{P}^l$; however, we now consider a copy $l_1 \subset \mathbb{P}^{s-1}$ of $\mathbb{P}^1$ embedding in degree two, and meeting $\varphi_{K-A}(C)$ nontangentially at $\varphi_{K-A}(P)$ and $\varphi_{K-A}(Q), P, Q \in C$, and at no other point. Also, let $l_2 \subset \mathbb{P}^l$ be a line meeting C nontangentially at $\varphi_{L-A}(P)$ and $\varphi_{L-A}(Q)$ and at no other point. Again, we identify these two copies of $\mathbb{P}^1$ and call $\triangle$ the image under the product embedding in $\mathbb{P}^{s-1} \times \mathbb{P}^l$. Let $\varphi =: \varphi_{K-A} \times \varphi_{L-A} : C \hookrightarrow \mathbb{P}^{s-1} \times \mathbb{P}^l$ and define $C' =: C \cup \triangle$. Then C' is a nodal curve of genus $g' = g + 1$, and on it we consider the line bundle $A' =: K_{C'} \otimes \mathcal{O}_{\mathbb{P}^{s-1}}(-1)|_{C'}$. By the same computation as above one checks that $\deg(A') = s + 2$ and $h^0(C', A') = 2$. Defining L' by $L' - A' =: \mathcal{O}_{\mathbb{P}^l}(-1)|_{C'}$ we have $\deg(L') = \deg(L) + 1$. Exactly as before one checks that $\dim |L' - A'|$ has the expected value, and so the induction step will be completed by providing a proof of the corresponding variants of Lemmas 4.1 and 4.2. Furthermore, observe that in passing from one odd genus to the next $\deg(L)$ increases by three, and this is exactly what it takes to keep the induction going. We now prove Lemmas 4.1 and 4.2.

Proof of Lemma 4.1. Let $Y = \mathbb{P}^{s-1} \times \mathbb{P}^l$. The smoothability of C' in Y will follow if we can show that $H^1(C', N_{C'/Y}) = 0$ and that $H^0(C', N_{C'/Y}) \longrightarrow T^1_{C'}$ is onto. In fact, since $C' \subset Y$ is a local complete intersection the first condition implies that the Hilbert scheme of Y is smooth at C', while the second says that there are embedded first order deformations of C' which smooth the nodes (cf. [**HH83**]).

Recall that φ_{K-A} is birationally very ample, while φ_{L-A} is very ample. On the other band, just by Petri generality we have $H^1(C, T_{\mathbb{P}}|_C) = 0$ in both cases. From this it is clear that

$$H^1(C, T_Y|_C) = 0.$$

Next recall the exact sequence

$$0 \to \mathcal{O}_{C'} \to \mathcal{O}_C \oplus \mathcal{O}_\triangle \to \mathcal{O}_{C\cap\triangle} \to 0$$

from which we obtain the sequence

$$0 \to T_Y|_{C'} \to T_Y|_C \oplus T_Y|_\triangle \to T_Y(P) \oplus T_Y(Q) \to 0$$

where P and Q are the intersection points of $C$and $\triangle$. Since T_Y is spanned, the latter sequence is exact on global sections, and so we get that $H^1(C', T_Y|_{C'}) = 0$. Now the exact sequence

$$0 \to T_{C'} \to T_Y|_{C'} \to N_{C'/Y} \to T^1_{C'} \to 0$$

can be chopped off in two short exact sequences, and from this we see that both of the above conditions are satisfied. □

Proof of Lemma 4.2. By assumption, on C we have the exact sequence

$$0 \to R(K-A, L-A) \to H^0(K-A) \otimes H^0(L-A) \\ \to H^0(K+L-2A) \to 0$$

where $R(K-A, L-A) \subset H^0(Y, \mathcal{O}_Y)$ is the linear series of the (1,1)-divisors containing $\varphi(C)$. We have a similar sequence on C':

$$0 \to R(K'-A, L'-A') \to H^0(L'-A') \otimes H^0(K'-A') \\ \to H^0(K'+L'-2A')$$

and one easily checks that $h^0(L-A) = h^0(L'-A'), h^0(K-A) = h^0(K'-A')$ and $h^0(K'+L'-2A') = h^0(K+L-2A)+1$. Hence to prove the statement it is sufficient to show that dim $R(K'-A', L'-A') <$ dim $R(K-A, L-A)$, i.e. that there exist (1,1) divisors in Y containing C but not $\triangle$.

Let us first consider the case $g = 2s$, so that $\triangle$ has bidegree (1,1). Observe that then $\triangle \subset Y$ depends on the identification of the lines l_1 and l_2. When we change this identification by an automorphism of $\mathbb{P}^1$, the image of $\triangle$ sweeps the surface

$$\langle \varphi_{K-A}(P), \varphi_{K-A}(Q) \rangle \times \langle \varphi_{L-A}(P), \varphi_{L-A}(Q) \rangle$$

where $\langle , \rangle$ denotes the line joining the given points in the appropiate spaces. The statement in this case follows from the following

CLAIM 4.1. *Let C be a smooth curve, and let $\varphi_1 : C \longrightarrow \mathbb{P}^m$ and $\varphi_2 : C \longrightarrow \mathbb{P}^n$ be nondegenerate morphisms. Define*

$$\widetilde{\mathrm{S}}\mathrm{ec}(C) =: \bigcup_{P,Q \in C} \langle \varphi_1(P), \varphi_1(Q) \rangle \times \langle \varphi_2(P), \varphi_2(Q) \rangle.$$

Then $\widetilde{\mathrm{S}}\mathrm{ec}(C)$ is not contained in any (1,1)-divisor $D \subset \mathbb{P}^m \times \mathbb{P}^n$.

Proof of the Claim. Provisionally let $Y = \mathbb{P}^m \times \mathbb{P}^n$ and for $P \in C$ fixed consider the projections $\pi_1 : \mathbb{P}^m \setminus \{\varphi_1(P)\} \longrightarrow \mathbb{P}^{m-1}$ and

$\pi_2 : \mathbb{P}^n \setminus \{\varphi_2(P)\} \longrightarrow \mathbb{P}^{n-1}$, where $\mathbb{P}^{m-1}$ and $\mathbb{P}^{n-1}$ are two fixed hyperplanes. We then have a product morphism

$$\pi_P : (\mathbb{P}^m \setminus \{\varphi_1(P)\}) \times (\mathbb{P}^n \setminus \{\varphi_2(Q)\}) \longrightarrow \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}.$$

Let $D' =: D \cap \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}, \widetilde{C} =: \pi_P(C)$. I claim that

$$D \supset \widetilde{\mathrm{S}}\,\mathrm{ec}(C) \Rightarrow D' \supset \widetilde{\mathrm{S}}\,\mathrm{ec}(\widetilde{C}). \tag{13}$$

Note that by induction this reduces the proof of the Lemma to the case where either $m = 1$ or $n = 1$, and then it is trivial.

In fact it is easily checked that $\pi_P(\widetilde{\mathrm{S}}\,\mathrm{ec}(C)) \subset \widetilde{\mathrm{S}}\,\mathrm{ec}(\widetilde{C})$ is a Zariski dense open subset, so that a general point in the latter variety can be written as $\pi_P(Q)$, where

$$Q = (Q_1, Q_2) \in \langle \varphi_1(A),\ \varphi_1(B)\rangle \times \langle \varphi_2(A),\ \varphi_2(B)\rangle$$

for suitable $A, B \in C$. Assume that in this situation

$$D \supset \langle \varphi_1(P), Q_1\rangle \times \langle \varphi_2(P), Q_2\rangle. \tag{14}$$

It then follows that

$$D' \supset [\langle \varphi_1(P), Q_1\rangle \times \langle \varphi_2(P), Q_2\rangle] \cap (\mathbb{P}^{m-1} \times \mathbb{P}^{n-1})$$

and this establishes (13).

It remains to prove (14). By assumption we have

$$\begin{aligned} D' \supset\ & [\langle \varphi_1(P), \varphi_1(A)\rangle \times \langle \varphi_2(P), \varphi_2(A)\rangle] \\ & \cup [\langle \varphi_1(P), \varphi_1(B)\rangle \times \langle \varphi_2(P), \varphi_2(B)\rangle] \\ & \times [\langle \varphi_1(A), \varphi_1(B)\rangle \times \langle \varphi_2(A), \varphi_2(B)\rangle]. \end{aligned}$$

In other words, we have

$$D \supset (l_1 \times l_2) \cup (r_1 \times r_2) \cup (s_1 \times s_2)$$

where $l_1, r_1, s_1 \subset \mathbb{P}^m$ and $l_2, r_2, s_2 \subset \mathbb{P}^n$ are triples of lines contained in the same plane Λ_1 and Λ_2 respectively. Intersecting D with $\Lambda_1 \times \Lambda_2$ we are thus reduce to proving that a (1,1)-divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ cannot contain such a union of lines, and this is a well known fact. A simpler version of the same argument deals with the case $g = 2s + 1$. □

Let's now come to the even genus case of Theorem 4.1. Recall that we want to prove the following statement: if (C, L, A) is a general choice of a smooth curve of genus $g = 2s$ and line bundles L and A on C, with $A \in W^1_{s+2}(C)$ and $\deg(L) \geq 3s + 10$, then $H^0(C, K + A - L) = 0$ and section multiplication $H^0(C, K - A) \otimes H^0(C, L - A) \longrightarrow H^0(C, K + L - 2A)$ is onto. Clearly the first conclusion is true, and the second follows from a slight modification of the previous degeneration argument, as follows. Let (C, L, A) be a general choice of a curve of odd genus $g = 2s + 1$ and line bundles L and A on it , satisfying both conclusions of Theorem 4.1. Such a choice exists by the argument above. Now apply the same construction, but take $\triangle$ to be of bidegree (1,1). C' has genus $g' = 2s + 1$, and if we define A' and L' in the same manner we obtain $\deg(A') = s + 3$ and $h^0(C', A') = 2$; everything else stays unchanged. □

One may also obtain the statement for genera divisible by 4 from the following covering argument. Let (C', L', A') be a general choice of a curve of genus $g' = 2k + 1$ and of line bundles L' and A' on it satisfying both conclusions of Theorem 3.1. Such a triple exists by the argument above. Pick a general point $P \in C'$; without loss of generality, we may assume that if $N =: L' + 3P$, then $H^0(C', N - A') \otimes H^0(C', K' - A') \longrightarrow H^0(C', N + K' - 2A')$ is also onto, where we have let $K' =: K_{C'}$. Also, we may assume C' to be Petri general, so that $d_{\min} = k + 2$.

Now consider the double cover $C \xrightarrow{\pi} C'$ ramified along $R = 6P$. Letting g be the genus of C, we have $g = 2(g' + 1)$ and so $d_{\min}(C) + 1 = g' + 3 = 2(k + 2)$. Now let $A =: \pi^*(A')$, so that $\deg(A) = 2\deg(A') = 2(k + 2)$ and $h^0(A) = 2$, and let $L =: \pi^*(N)$, so that $\deg(L) \geq 3(g' + 1) + 10$. Finally observe that the multiplication map $H(C, L - A) \otimes H^0(C, K - A) \longrightarrow H^0(C, L + K - 2A)$ splits as the direct sum of various section multiplications on C', two of which are $H^0(C', K' - A') \otimes H^0(C', L' - A') \longrightarrow H^0(C', K' + L' - 2A')$ and $H^0(C', K' - A') \otimes H^0(C', N' - A') \longrightarrow H^0(C', K' + N' - 2A')$. From this is easy to deduce the statement.

5. The Wahl maps $\gamma_{L,N}$. As before, consider a smooth projective curve C and let L, N be two line bundles on it. Suppose $A \in W^1_{s+2}$. Then $r(L - A), r(N - A) \geq 0$ when $\deg(L)$, $\deg(N) \geq 3s + 2$

in case $g = 2s$ and $\deg(L)$, $\deg(N) \geq 3s+3$ when $g = 2s+1$. Define

$$U =: \{A \in W^1_{s+2} | h^0(C, K + A - L) = h^0(K + A - L) = 0\}$$

and consider the variant of diagram (2), where we let $W_A =: \wedge^2 H^0(C, A)$:

(15)

$$\begin{array}{ccc} \bigoplus_{A\in U} H^0(L-A) \otimes H^0(N-A) \otimes W_A & \longrightarrow & \bigoplus_{A\in U} H^0(N+L-2A) \otimes W_A \\ \big\downarrow & & \big\downarrow \phi \\ R(L,N) & \xrightarrow{\gamma_{L,N}} & H^0(K+L+N) \end{array}$$

PROPOSITION 5.1. *Let C be a Petri general curve and suppose* $\deg(L)$, $\deg(N) \geq 3s+2$ *if* $g = 2s$ *and* $\deg(L)$, $\deg(N) \geq 3s+3$ *if* $g = 2s+1$. *Then ϕ from diagram* (4-2) *is onto.*

Proof. Write $N = K - R$, where $\deg(R) \leq s-4$ if $g = 2s$, $\deg(R) \leq s-3$ if $g = 2s+1$. We look at

$$\phi^* : H^1(C, T_C - (L-R)) \longrightarrow \bigoplus_{A\in U} H^1(C, 2A - (L-R)).$$

Consider the double cover $\pi : \widetilde{C} \to C$ ramified along the general element $B \in |2(L-R)|$. The argument used in the proof of Theorem 2.1 goes over verbatim. □

COROLLARY 5.1. *In the situation of the Proposition, if $\gamma_{L,N}$ is not onto then no multiplication $H^0(C, L-A) \otimes H^0(C, N-A) \longrightarrow H^0(C, L+N-2A)$ is onto, for any $A \in U$.*

We can now prove

THEOREM 5.1. *Let C be a general curve and L, N be two general line bundles on it. Assume that*

(i) $\deg(L) \geq 3s+4$ *if* $g = 2s$, *or that* $\deg(L) \geq 3s+5$ *if* $g = 2s+1$

(ii) $\deg(N) > \deg(L) + g - 2$.

Then $\gamma_{L,N}$ is onto. Furthermore, assume that L, N are any two line bundles on C such that (2) *holds and* $\deg(L) \geq 5s+7$ *if* $g = 2s$ *or* $\deg(L) \geq 5s+5$ *when* $g = 2s+1$. *Then $\gamma_{L,N}$ is onto.*

Proof. Let's start with the first statement. For a general choice of such an L and a general $A \in U$, $L - A$ is spanned. Let $V \subset$

$H^0(C, L-A)$ be a pencil spanning $L-A$, so that there is an exact sequence $0 \to A-L \to V \otimes \mathcal{O}_C \to L-A \to 0$. Twisting by $N-A$ we get the exact sequence

$$0 \to N-L \to V \otimes (N-A) \to L+N-2A \to 0$$

and this exact on global sections because by condition (2) and the assumed generality of L and N we may assume $H^1(C, N-L) = 0$. The first statement follows.

To prove the second part of the theorem, observe that for general line bundles B and D on C of degree $g+1$ we may assume that B and D are spanned and that $\gamma_{N-D,L-B}$ is onto, by the first statement of this theorem. We may also assume that $r(B) = r(D) = 1$, so that we have exact sequences $0 \to -B \to H^0(C, B) \to B \to 0$ and $0 \to -D \to H^0(C, D) \to D \to 0$. Letting $V =: H^0(C, B), W =: H^0(C, D)$ we obtain the commutative diagram

(16)
$$\begin{array}{ccc} R(L-B, N-C) \otimes V \otimes W & \xrightarrow{\gamma_{L-B,N-D}} & H^0(C, K+L-B+N-C) \otimes V \otimes W \\ \downarrow & & \downarrow \beta \\ R(L, N) & \xrightarrow{\gamma_{L,N}} & H^0(C, K+L+N) \end{array}$$

and using the above sequences it is easy to see that β is onto. The theorem follows. □

6. Higher Wahl maps. Consider as before a smooth projective curve C and line bundles L and N on it. The map $\gamma_{L,N}$ that we have been considering so far generalizes to a hierarchy of Wahl map $\gamma^i_{L,N}$ defined as follows (cf. [**W88**]). On the product $C \times C$ consider the line bundle $p_1^*(L) \otimes p_2^*(N)$, which we'll abbreviate to $L_1 \otimes N_2$, and let $\triangle \subset C \times C$ denote the diagonal. Then $H^0(C \times C, L_1 \otimes N_2) \simeq H^0(C, L) \otimes H^0(C, N)$ is filtered by the subspaces $R_l(L, N) = H^0(C \times C, L_1 \otimes N_2(-l\triangle))$, $l \geq 0$. For each l, we have an exact sequence on $C \times C$

$$0 \to L_1 \otimes N_2(-(l+1)\triangle) \to L_1 \otimes N_2(-l\triangle) \to L_1 \otimes N_2(-l\triangle)|_\triangle \to 0$$

and we simply define

$$(17) \qquad \gamma^l_{L,N} : R_l(L, N) \longrightarrow H^0(C, lK_C + L + N)$$

to be the induced map on global sections. Notice that $R_{l+1}(L,N) = \operatorname{Ker}\gamma^l_{L,N}$. In particular, $\gamma^0_{L,N}$ is just section multiplication, and $\gamma^1_{L,N}$ is the usual Wahl map. The approach used in the previous paragraphs to deal with the first Wahl map can be generalized to $\gamma^l_{L,N}$, as follows. As before, let's assume the genus of C is $g = 2s$ or $g = 2s+1$, and let's denote $W = W^1_{s+2}$. Let $U =: \{A_1, \cdots, A_l) \in W \times \cdots \times W | h^0(C, K + \sum_{i=1}^l A_i - L) = h^0(C, K + \sum_{i=1}^l A_i - N) = 0\}$, and define

$$V(L,N;A_1,\cdots,A_l) =: H^0\left(L - \sum_{i=1}^l A_i\right) \otimes H^0\left(N - \sum_{i=1}^l A_i\right) \otimes \wedge^2 H^0(A_1) \otimes \cdots$$

and

$$W(L,N;A_1,\cdots,A_l) =: H^0\left(L + N - 2\sum_{i=1}^l A_i\right) \otimes \wedge^2 H^0(A_1) \otimes \cdots \otimes \wedge^2 H^0(A_l)$$

and consider the following commutative diagram, which generalizes (15):

(18)

$$\begin{array}{ccc} \bigoplus_{(A_1,\cdots,A_l)\in U} V(L,N;A_1,\cdots,A_l) & \longrightarrow & \bigoplus_{(A_1,\cdots,A_l)\in U} W(L,N;A_1,\cdots,A_l) \\ f\downarrow & & \phi\downarrow \\ R_l(L,N) & \xrightarrow{\gamma^l_{L,N}} & H^0(C, lK + L + N) \end{array}$$

Assume that ϕ is onto. Then, by the usual argument, if $\gamma^l_{L,N}$ is not onto none of the multiplications

$$H^0\left(L - \sum_{i=1}^l A_i\right) \otimes H^0\left(N - \sum_{i=1}^l A_i\right) \longrightarrow H^0\left(L + N - 2\sum_{i=1}^l A_i\right)$$

can be surjective. Now we can proceed inductively to draw the same kind of conclusions as in the case of $\gamma_{L,N}$

THEOREM 6.1. *Assume that C is Petri general and that* $\deg(L) + \deg(N) > 2g - 2 + 2l(s+2)$. *Then*

$$\phi : \bigoplus_{(A_1,\cdots,A_l)\in U} H^0\left(L + N - 2\sum_{i=1}^n A_i\right) \longrightarrow H^0(lK + L + N)$$

is onto.

Proof. We have

$$\phi^* : H^1((l-1)T - L - N) \longrightarrow \bigoplus H^1\left(2\sum_{i=1}^{l} A_i + K - L - N\right)$$

given by $u \mapsto \otimes u \cdot R_{A_1} \cdots R_{A_l}$. If $u \in \text{Ker } \phi^*$, then $u \cdot R_{A_1} \cdots R_{A_l} = 0$, for every $(A_1, \cdots, A_l) \in W$.

Now $u \cdot R_{A_1} \cdots R_{A_{l-1}} \in H^1(T - (L + N - K - 2\sum_{i=1}^{l-1} A_i))$, and we have $\deg(L + N - K - \sum_{i=1}^{l-1} 2A_i) > 2 \deg(A)$. By Theorem 3.1, this implies $u \cdot R_{A_1} \cdots R_{A_{l-1}} = 0$ for all $(A_1, \cdots, A_{l-1}) \in U$, and now the statement follows by induction. □

COROLLARY 6.1. *In the situation of Theorem* 6.1, *if* $\gamma^l_{L,N}$ *fails to be surjective then*

$$H^0\left(C, L - \sum_{i=1}^{l} A_i\right) \otimes H^0\left(C, L - \sum_{i=1}^{l} A_i\right) \longrightarrow H^0\left(C, L + N - 2\sum_{i=1}^{l} A_i\right)$$

is never surjective, for any $(A_1, \ldots, A_l) \in U$.

THEOREM 6.2. *Assume that L and N are two general line bundles on C, with* $\deg(L) \geq g + 1 + l(s+2)$ *and* $\deg(N) > g - 2 + \deg(L)$. *Then* $\gamma^l_{L,N}$ *is onto. If L and N are arbitrary and* $\deg(L) > 2g + 2 + l(s+2)$ *and* $\deg(N) > g - 2 + \deg(L)$, *then* $\gamma^l_{L,N}$ *is onto*

Proof. Use the same argument as in the proof of Theorem 5.1, replacing $L - A$ by $L - \sum_{i=1}^{l} A_i$ and similarly for $N - A$. □

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Note added in proof: Since this paper was written, there has been further progress in the study of Wahl maps, due to J. Stevens, A. Lopez and G. Pareschi. Pareschi's work, in particular, is highly relevant to the present approach, and leads to some general results about the surjectivity of these maps.