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A DIFFERENTIABLE STRUCTURE FOR A BUNDLE OF C^*-ALGEBRAS ASSOCIATED WITH A DYNAMICAL SYSTEM

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Let (M, G) be a differentiable dynamical system, and σ be a transverse action for (M, G). We have a differentiable bundle (B, π, M, C) of C^*-algebras with respect to a flat family $\mathcal{F}_\sigma$ of local coordinate systems and we have a flat connection ∇ in B. If G is connected, the bundle B is a disjoint union of $\rho_x(C_r^*(\mathcal{G}))$ $(x \in M)$, where $\mathcal{G}$ is the groupoid associated with (M, G) and ρ_x is the regular representation of $C_r^*(\mathcal{G})$. We show that, for $f \in C_c^\infty(\mathcal{G})$, a cross section $cs(f) : x \mapsto \rho_x(f)$ is differentiable with respect to the norm topology, and calculate a covariant derivative $\nabla(cs(f))$. Though B is homeomorphic to the trivial bundle, the differentiable structure for B is not trivial in general. Let B^σ be a subbundle of B generated by elements f with the property $\nabla(cs(f)) = 0$. We show the triviality of the differentiable structure for B^σ induced from that for B when $C_r^*(\mathcal{G})$ is simple. We have a bundle $RM(B)$ of right multiplier algebras and it contains B as a subbundle. Let (M, G) be a Kronecker dynamical system and σ be a flow whose slope is rational. In this case, we have a subbundle D of $RM(B)$ whose fibers are $*$- isomorphic to $C(\mathbb{T})$. The flat connection ∇^r in D is not trivial and the bundle B decomposes into the trivial bundle B^σ and the non-trivial bundle D. Moreover, for a σ-invariant closed connected submanifold N of M with $\dim N = 1$, we show that $C_r^*(\mathcal{G}|N)$ is $*$-isomorphic to $C_r^*(D_x, \Phi_x)$, where Φ_x is the holonomy group of ∇^r with reference point x. If G is not connected, we also have sufficiently many differentiable cross sections of B and calculate their covariant derivatives.

0. Introduction. In the theory of C^*-algebras, one sometimes study a stable C^*-algebra $A \otimes \mathcal{K}$ instead of studying a given C^*-algebra A itself, where $\mathcal{K}$ is the algebra of all compact operators on the infinite dimensional separable Hilbert space. There are many other algebras D such that $D \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. Moreover, stable algebras do not have any identity elements. Therefore, given a stable C^*-algebra C, we want to find C^*- algebras A with the property $A \otimes \mathcal{K} \cong C$, especially unital ones with the property. We do not know any general answer to the question, but there is a method to construct such algebras A for foliation C^*- algebras. Let $(V, \mathcal{F})$ be a foliation and $C^*(V, \mathcal{F})$ be the foliation C^*-algebra introduced by A. Connes ([**1**], [**3**]). It follows from [**10**] that $C^*(V, \mathcal{F})$ is $*$-isomorphic to $C_r^*(\mathcal{G}|N) \otimes \mathcal{K}$, where $\mathcal{G}$ is the holonomy groupoid of $(V, \mathcal{F})$, where N is a complete transverse submanifold and where the groupoid $\mathcal{G}|N$ is the reduction of $\mathcal{G}$ by N. Suppose that V is compact. If we have $\dim N = \operatorname{codim} \mathcal{F}$, then the C^*-algebra $C_r^*(\mathcal{G}|N)$ is unital. To give an example, if $(V, \mathcal{F})$ is a Kronecker foliation, then the C^*- algebra $C_r^*(\mathcal{G}|N)$ is the irrational rotation algebra A_θ for an appropriate N. This example plays an important role in the theory of non-commutative differential geometry by A. Connes. We refer the reader to the works of A. Connes [**2**], [**3**], that of A. Connes and M.A. Rieffel [**4**] and that of M.A. Rieffel [**20**]. M.A. Rieffel also studied the example in [**17**], [**18**] from the viewpoint of Morita equivalence. The author studied another example of $C_r^*(\mathcal{G}|N)$ in [**12**], [**13**].

From these considerations, we begin to study C^*-algebras of reductions of differentiable dynamical systems. Let (M, G) be a differentiable dynamical system. We denote by $\mathcal{G}$ the topological groupoid $G \times M$ and denote by $C_r^*(\mathcal{G})$ the reduced C^*-algebra associated with $\mathcal{G}$. We have a regular representation ρ_x of $C_r^*(\mathcal{G})$ on a Hilbert space $\mathcal{H}_x$ for every $x \in M$. For the moment we assume that G is connected and that $C_r^*(\mathcal{G})$ is simple. We set $B_x = \rho_x(C_r^*(\mathcal{G}))$ and denote by B the disjoint union of C^*-algebras B_x $(x \in M)$. We may consider elements a of $C_r^*(\mathcal{G})$ to be cross sections $cs(a) : x \mapsto \rho_x(a)$ of the bundle B on M. Continuous fields of C^*- algebras have been studied by many authors. We refer the reader to the book of J. Dixmier [**5**], those of J.M.G. Fell and R.S. Doran [**8**], [**9**], the work of B.D. Evans [**6**] and that of M.A. Rieffel [**19**]. Since we study C^*-algebras associated with differentiable dynamical systems, it is natural to consider differentiable structure for fields of C^*-algebras. In the previous paper [**14**], the author introduced the notion of differentiable bundles of C^*-algebras and that of connections in them. A. Connes first introduced the notion of connections into the theory of C^*-algebras in [**2**]. He defined the notion in the setting of projective modules. On the other hand, our definition of connections is in the setting of bundles of C^*-algebras and it is a literal translation of that in the setting of vector bundles, except that our connections are compatible with $*$-algebraic structures possessed by fibers.

In this paper, we introduce a notion of a transverse action σ for (M, G) and we construct a family $\mathcal{F}_\sigma$ of local coordinate systems for B from local charts of (M, G) compatible with σ. Then $\mathcal{F}_\sigma$ defines a differentiable structure for B. Next, we prove that the above cross section $cs(f)$ is differentiable with respect to the norm topology for every $f \in C_c^\infty(\mathcal{G})$. We define a flat connection ∇ in B with respect to $\mathcal{F}_\sigma$. Though B is homeomorphic to the trivial bundle $M \times C_r^*(\mathcal{G})$, the differentiable structure for B is not trivial, that is, ∇ is not trivial. Let B^σ be the subbundle of B generated by elements f with the property $\nabla(cs(f)) = 0$. Then B^σ is trivial, that is, the restriction of ∇ to B^σ is trivial. We denote by $RM(B_x)$ the right multiplier algebra of B_x and denote by $RM(B)$ the disjoint union of Banach algebras $RM(B_x)$ $(x \in M)$. There exists a differentiable structure for $RM(B)$ such that B is a subbundle of $RM(B)$ and such that ∇ extends to a flat connection $\overline{\nabla^r}$ in $RM(B)$. In the case where (M, G) is a Kronecker dynamical system, we give a decomposition of B into a trivial part and a non-trivial part. There exists a subbundle D of $RM(B)$ such that every fiber D_x is $*$-isomorphic to the commutative C^*-algebra $C(\mathbb{T})$ and such that $B_x^\sigma D_x$ generates B_x. Let ∇^r be the restriction of $\overline{\nabla^r}$ to D and let Φ_x be the holonomy group of ∇^r with reference point x. Note that Φ_x is a subgroup of the group $\mathrm{Aut}(D_x)$ of all $*$-automorphisms of D_x. Let N be a σ-invariant closed connected submanifold of M with $\dim N = 1$. Then we show that the C^*-algebra $C_r^*(\mathcal{G}|N)$ is $*$- isomorphic to the reduced crossed product $C_r^*(D_x, \Phi_x)$ of D_x by Φ_x. This result means that B decomposes into the trivial bundle B^σ and the non-trivial bundle D and that D corresponds to the reduction of (M, G) by N. This situation was studied by M.A. Rieffel in [**17**], [**18**] from the viewpoint of projective modules. Our result describes the same situation from the viewpoint of vector bundles.

When G is not connected, we also define a differentiable bundle

B associated with a transverse action for (M, G) and define a flat connection ∇ in B. But, in this case, B_x is larger than $\rho_x(C_r^*(\mathcal{G}))$ and cross sections $cs(f)$ may not be differentiable. We define a cross section $cs_m(f)$ of B for $f \in C_c^\infty(\mathcal{G})$ and every connected component m of G, and we show that the cross sections $cs_m(f)$ are differentiable. The $*$-algebra $\mathcal{D}_x$ generated by elements of the form $cs_m(f)_x$ is dense in B_x with respect to the strong operator topology. The above results are valid even if G is discrete.

To find a transverse action for a given dynamical system (M, G), it may be useful to consider the universal covering space $\tilde{M}$ of M. Suppose that the action of G on M lifts to an action of G on $\tilde{M}$. (If G is simply connected, this assumption is satisfied.) If there exists a transverse action for $(\tilde{M}, G)$ and if it is compatible with the covering map, then we have a transverse action for (M, G). But we do not know any interesting examples of transverse actions for dynamical systems (M, G) such that the connected components G_e of G are not abelian, and it is difficult to find such examples. This is the problem for further investigation.

1. Preliminaries. (a) *Commutative dynamical systems.* Let (M, G) be a topological transformation group. We assume that a topological space M and a topological group G are second countable, Hausdorff and locally compact. We denote by $\mathcal{G}$ a topological groupoid $G \times M$ with the following operations; $s(g, x) = (e, x)$, $r(g, x) = (e, gx)$, $(g', gx)(g, x) = (g'g, x)$, $(g, x)^{-1} = (g^{-1}, gx)$ for $x \in M$ and g, $g' \in G$, where e is the unit of G. We set $\mathcal{G}_x = \{(g, x) \in \mathcal{G}; g \in G\}$ for $x \in M$. Let μ be a right Haar measure on G and Δ be the modular function of G. We define a right Haar system $\{\nu_x; x \in M\}$ on $\mathcal{G}$ by $\nu_x = \mu \times \delta_x$. Let $C_c(\mathcal{G})$ be the $*$-algebra of continuous functions with compact supports, where the product and the involution are defined as follows:

$$
\begin{aligned}
(f_1 * f_2)(g, x) &= \int_G f_1(g'^{-1}, g'gx) f_2(g'g, x)\, d\mu(g'), \\
f^*(g, x) &= \overline{f(g^{-1}, gx)}
\end{aligned}
$$

for f, f_1, f_2, $\in C_c(\mathcal{G})$ and $(g, x) \in \mathcal{G}$. We denote by $\mathcal{H}_x$ the Hilbert space $L^2(\mathcal{G}_x, \nu_x)$ for $x \in M$. We define the regular representation

ρ_x of $C_c(\mathcal{G})$ on $\mathcal{H}_x$ by

$$(\rho_x(f)\xi)(g,x) = \int_G f(gg'^{-1}, g'x)\xi(g',x)\, d\mu(g')$$

for $f \in C_c(\mathcal{G})$, $\xi \in \mathcal{H}_x$ and $(g,x) \in \mathcal{G}_x$. We define the reduced norm $||f||$ by $||f|| = \sup_{x\in M} ||\rho_x(f)||$. We denote by $C_r^*(\mathcal{G})$ the completion of $C_c(\mathcal{G})$ by the reduced norm. The representation ρ_x extends to a representation of $C_r^*(\mathcal{G})$, which we denote again by ρ_x . For details of groupoids and their C^*-algebras, we refer the reader to [**1**], [**3**] and [**16**].

LEMMA 1.1. *Let f be an element of $C_c(\mathcal{G})$ and D be a compact set in G such that* supp $f \subset D \times M$. *Then the following inequality holds: $||\rho_x(f)|| \leq I_D||f||_\infty$, where $||f||_\infty$ is the supremum norm of f and $I_D = \int_D \Delta^{1/2}(g)\, d\mu(g)$.*

Proof. Let χ_D be the characteristic function of D. For $\xi,\ \eta \in \mathcal{H}_x$, we have

$$\begin{aligned}
&\int_G |f(g'^{-1}, g'gx)\xi(g'g,x)\eta(g,x)|\, d\mu(g)\\
&\leq \Big(\int_G |f(g'^{-1}, g'gx)||\xi(g'g,x)|^2\, d\mu(g)\Big)^{1/2}\\
&\quad\cdot \Big(\int_G |f(g'^{-1}, g'gx)||\eta(g,x)|^2\, d\mu(g)\Big)^{1/2}\\
&\leq ||f||_\infty \chi_D(g'^{-1})||\eta||\Delta^{1/2}(g')||\xi||.
\end{aligned}$$

Then we have $|(\rho_x(f)\xi|\eta)| \leq I_D||f||_\infty||\eta||||\xi||$. □

We introduce a $*$-algebra of functions on $G \times G$. Let $\tilde{\mathcal{C}}$ be the set of bounded continuous functions K on $G\times G$ with the following property; there exists a compact set D in G such that supp $K \subset G \times D$. The set D may vary when K varies. Then $\tilde{\mathcal{C}}$ is a $*$-algebra with the following product and involution;

$$\begin{aligned}
(K_1 * K_2)(g,g') &= \int_G K_1(g, g''^{-1})K_2(g''g, g''g')\, d\mu(g''),\\
K^*(g,g') &= \overline{K(g'^{-1}g, g'^{-1})}
\end{aligned}$$

for K, K_1, $K_2 \in \tilde{\mathcal{C}}$ and $(g,g') \in G\times G$. We denote by $\mathcal{H}$ the Hilbert space $L^2(G,\mu)$. We define a $*$-representation ρ of $\tilde{\mathcal{C}}$ on $\mathcal{H}$ by

$$(\rho(K)\xi)(g) = \int_G K(g, g'^{-1})\xi(g'g)\, d\mu(g')$$

for $K \in \tilde{\mathcal{C}}$, $\xi \in \mathcal{H}$ and $g \in G$. We can prove the following lemma by a similar computation to that in the proof of Lemma 1.1.

LEMMA 1.2. *Let K be an element of $\tilde{\mathcal{C}}$ and D be a compact set in G such that* supp $K \subset G \times D$. *Then the following inequality holds:* $||\rho(K)|| \leq I_D ||K||_\infty$.

(b) *Differentiable bundles of C^*-algebras.* With a few modifications on the definitions in [**14**, §1], we summarize the necessary facts. Let $e_1, \ldots, e_n$ be the standart basis of $\mathbb{R}^n$ and $x_1, \ldots, x_n$ be the canonical coordinate functions of $\mathbb{R}^n$. Let Ω be an open subset of $\mathbb{R}^n$ and f be a map of Ω into a Banach space C. If there exists $\lim_{h \to 0} h^{-1}(f(x + he_i) - f(x))$ with respect to the norm in C, then we denote the limit by $(\partial f / \partial x_i)(x)$. We say that f is differentiable of class $(C^\infty)'$ on Ω if the partial derivatives $\partial^\alpha f / \partial x^\alpha$ exist and are continuous on Ω for all multi-indices α.

DEFINITION 1.3. (c.f. [**14**, Definition 1.1]). Let M be a finite dimensional real manifold of class C^∞ and $\mathcal{A}$ be the complete atlas defining the structure of M. A map f of M into a Banach space C is said to be of class C^∞ if $f \circ \varphi^{-1}$ is of class $(C^\infty)'$ on $\varphi(U)$ for every $(U, \varphi) \in \mathcal{A}$.

We assume that a real manifold M is second countable, Hausdorff and of class C^∞. Let B be a topological space, C be a C^*-algebra and π be a continuous map of B onto M. We set $B_x = \pi^{-1}(x)$ for $x \in M$ and suppose that B_x is a C^*-algebra. (It is easy to rewrite the rest of this section for Banach algebras C and B_x. We leave it to the reader.) Let $\{U_i\}$ be an open covering of M indexed by a set I and ψ_i be a homeomorphism of $\pi^{-1}(U_i)$ onto $U_i \times C$ such that $p_1 \circ \psi_i(b) = \pi(b)$ for $b \in \pi^{-1}(U_i)$, where $p_1 : U_i \times C \to U_i$ is the projection. For $x \in U_i$, we define a map $\psi_{i,x}$ of B_x into C by $\psi_{i,x}(b) = p_2 \circ \psi_i(b)$ for $b \in B_x$, where $p_2 : U_i \times C \to C$ is the projection. We denote by $\mathcal{F}$ the set of pairs (U_i, ψ_i) $(i \in I)$.

DEFINITION 1.4. (c.f. [**14**, Definition 1.2]). A quartet (B, π, M, C) is called a differentiable bundle of C^*-algebras with respect to $\mathcal{F}$ if $\mathcal{F}$ satisfies the following conditions:

(i) For every $i \in I$ and $x \in U_i$, $\psi_{i,x}$ is a $*$- isomorphism between C^*-algebras.

(ii) For i, $j \in J$ with $U_i \cap U_j \neq \varnothing$ and for a map f of $U_i \cap U_j$ into

C, define the map $f_{i,j}$ of $U_i \cap U_j$ into C by $f_{i,j}(x) = \psi_{i,x} \circ \psi_{j,x}^{-1} \circ f(x)$. If f is of class C^∞, then $f_{i,j}$ is of class C^∞.

Let $\mathcal{F}$ be a family satisfying the above condition (i). We say that $\mathcal{F}$ is a flat family of C^*-coordinate systems if it satisfies the following conditions:

(iii) For every $i, j \in I$ with $U_i \cap U_j \neq \varnothing$ and for every connected component U of $U_i \cap U_j$, there exists a $*$- automorphism α of the C^*-algebra C such that $\alpha = \psi_{i,x} \circ \psi_{j,x}^{-1}$ for all $x \in U$.

Let ξ be a map of an open set U of M into $\pi^{-1}(U)$ such that $\pi(\xi_x) = x$ for $x \in U$. For $i \in I$ with $U_i \cap U \neq \varnothing$, define the map $\tilde{\xi}_i$ of $U_i \cap U$ into C by $\tilde{\xi}_i(x) = \psi_{i,x}(\xi_x)$. We say that ξ is a differentiable cross section on U if $\tilde{\xi}_i$ is of class C^∞ for every $i \in I$ with $U_i \cap U \neq \varnothing$. We denote by $\Gamma(B)$ the $*$-algebra of all differentiable cross sections on M. Let TM be the tangent bundle on M, $\Gamma(TM)$ be the space of C^∞ vector fields on M and T^*M be the cotangent bundle on M. We denote by $T^*M \otimes B$ the tensor product of T^*M and B as real vector bundles. Let ξ be a cross section of $T^*M \otimes B$. If $x_1, \dots, x_n$ is a local coordinate system in M, then we have $\xi_x = \sum (dx_k)_x \otimes b_x^k$ with $b_x^k \in B_x$. We say that ξ is differentiable if the cross sections $x \mapsto b_x^k$ are differentiable. Let $\Gamma(T^*M \otimes B)$ be the two-sided $\Gamma(B)$-module of differentiable cross sections of $T^*M \otimes B$. We define the involution on $\Gamma(T^*M \otimes B)$ by $\xi_x^* = \sum (dx_k)_x \otimes (b_x^k)^*$. We denote by $C^\infty(M, \mathbb{R})$ the space of real-valued C^∞ functions on M.

DEFINITION 1.5. (c.f. [**14**, Definition 1.3]). Let (B, π, M, C) be a differentiable bundle of C^*-algebras and $\mathcal{D}$ be a $*$-subalgebra of $\Gamma(B)$ such that $f\xi \in \mathcal{D}$ for $f \in C^\infty(M; \mathbb{R})$ and $\xi \in \mathcal{D}$. A linear map ∇ of $\mathcal{D}$ into $\Gamma(T^*M \otimes B)$ is called a connection in B with domain $\mathcal{D}$ if it satisfies the following conditions: (i) $\nabla(f\xi) = df \otimes \xi + f\nabla\xi$, (ii) $\nabla(\xi\eta) = (\nabla\xi)\eta + \xi(\nabla\eta)$, (iii) $(\nabla\xi)(X) \in \mathcal{D}$, (iv) $\nabla(\xi^*) = (\nabla\xi)^*$ for $\xi, \eta \in \mathcal{D}$, $f \in C^\infty(M; \mathbb{R})$ and $X \in \Gamma(TM)$.

Suppose that the family $\mathcal{F}$ is flat. Let ∇ be a connection in B with domain $\Gamma(B)$ and $(V, x_1, \dots, x_n)$ be a local coordinate system in M. For $\xi \in \Gamma(B)$ and $i \in I$, we set $\tilde{\xi}_i(x) = \psi_{i,x}(\xi_x)$. We say that ∇ is a flat connection if we have

$$\psi_{i,x}((\nabla\xi)(X)_x) = \sum_{k=1}^{n} a_k(x) \frac{\partial \tilde{\xi}_i}{\partial x_k}(x) \quad (x \in V \cap U_i),$$

for $X \in \Gamma(TM)$ with $X_x = \sum a_k(x)(\partial/\partial x_k)_x$ (c.f. [**14**, Definition1.6], [**11**, Chapter II, §9]). Then the following lemma is obvious.

LEMMA 1.6. *If* (B, π, M, C) *is a differentiable bundle of* C^*-*algebras with respect to a flat family* $\mathcal{F}$, *then there exists a unique flat connection in* B.

2. Transverse actions and bundles of C^*-algebras. Let M be an n-dimensional real manifold of class C^∞ and G be a p-dimensional real Lie group of class C^∞. In the following sections, we assume that M and G are second countable and Hausdorff and that $0 < n < \infty$ and $0 \leq p < \infty$. If $p = 0$, then G is a countable discrete group. Moreover we assume that M is connected. Suppose that (M, G) is a differentiable dynamical system, that is, (M, G) is a transformation group and the map $(g, x) \mapsto gx$ of $G \times M$ into M is of class C^∞. Let G_e be the connected component of the unit e in G. We denote by $\mathcal{N}$ the countable discrete group G/G_e and denote by G_m the connected component of G corresponding to $m \in \mathcal{N}$. We take notations from §1, and also use the following notations; $\mathcal{G}_m = G_m \times M$, $\mathcal{G}_{m,x} = \mathcal{G}_m \cap \mathcal{G}_x$, $\mathcal{H}^m = L_2(G_m, \mu|G_m)$, $\mathcal{H}^m_x = L^2(\mathcal{G}_{m,x}, \nu_x|\mathcal{G}_{m,x})$, for $m \in \mathcal{N}$ and $x \in M$. Let $P^m_x \in \mathcal{B}(\mathcal{H}_x)$ be the projection on $\mathcal{H}^m_x$ and $P^m \in \mathcal{B}(\mathcal{H})$ be the projection on $\mathcal{H}^m$. We denote by $\mathcal{N}(\mathcal{G})$ the set of families $\zeta = (f_m)_{m\in\mathcal{N}}$ with the following properties; (1) $f_m \in C_c(\mathcal{G})$ $(m \in \mathcal{N})$, (2) $\sup_{m\in\mathcal{N}} ||f_m||_\infty < +\infty$, (3) there exists a compact set D in G such that $\operatorname{supp} f_m \subset D \times M$ for all $m \in \mathcal{N}$. We set $||\zeta|| = \sup_m ||f_m||_\infty$.

LEMMA 2.1. *For* $\zeta = (f_m)_{m\in\mathcal{N}} \in \mathcal{N}(\mathcal{G})$, *the sum* $\tilde{\rho}_x(\zeta) = \sum_{m\in\mathcal{N}} \rho_x(f_m)P^m_x$ *converges with respect to the strong operator topology in* $\mathcal{B}(\mathcal{H}_x)$, *and the following inequality holds:* $||\tilde{\rho}_x(\zeta)|| \leq J_D||\zeta||$, *where* D *is any compact set in* G *such that* $\operatorname{supp} f_m \subset D \times M (m \in \mathcal{N})$, *and* J_D *is a constant depending only on* D.

Proof. We set $D_m = D \cap G_m$. There exist elements $m(1), \dots, m(k)$ of $\mathcal{N}$ such that D is the disjoint union of non-empty sets $D_{m(1)}, \dots, D_{m(k)}$. Then we have $\rho_x(f_m)P^m_x = \sum_{l\in A(m)} P^l_x\rho_x(f_m)P^m_x$, where $A(m) = \{m(i)\ m; i = 1, \dots, k\}$. If we have $(P^l_x\rho_x(f_m)P^m_x\xi)(g, x) \neq 0$, then there exists $g' \in G_m$ such that $gg'^{-1} \in D_{lm^{-1}}$. This implies that we have $lm^{-1} = m(i)$ for some i with $1 \leq i \leq k$. We set

$B(l) = \{m(i)^{-1}l \in \mathcal{N}; i = 1, \dots, k\}$. We have

$$\left\| \sum_{m\in\mathcal{N}} \rho_x(f_m)P_x^m\xi \right\|^2 \leq k \sum_{l\in\mathcal{N}} \sum_{m\in B(l)} ||P_x^l\rho_x(f_m)P_x^m\xi||^2.$$

Note that $m \in B(l)$ if and only if $l \in A(m)$. Thus we have

$$\sum_{l\in\mathcal{N}} \sum_{m\in B(l)} ||P_x^l\rho_x(f_m)P_x^m\xi||^2 \leq I_D^2||\zeta||^2 \sum_{m\in\mathcal{N}} ||P_x^m\xi||^2.$$

□

Let B_x be the C^*-subalgebra of $\mathcal{B}(\mathcal{H}_x)$ generated by $\{\tilde{\rho}_x(\zeta); \zeta \in \mathcal{N}(\mathcal{G})\}$. Since we have $\tilde{\rho}_x(\zeta) = \rho_x(f)$ for $\zeta = (f_m)$ with $f_m = f$ for all $m \in \mathcal{N}$, B_x contains $\rho_x(C_r^*(\mathcal{G}))$. If G is connected, then we have $B_x = \rho_x(C_r^*(\mathcal{G}))$. If G is not connected, then B_x may not be separable. For $x \in M$ and $f \in C_c(\mathcal{G})$, we define $K_x^f \in \tilde{\mathcal{C}}$ by $K_x^f(g, g') = f(g', g'^{-1}gx)$. For $m \in \mathcal{N}$, we define $\chi_m \in C^\infty(G \times G)$ as follows; $\chi_m(g, g') = 1$ if $g'^{-1}g \in G_m$ and $\chi_m(g, g') = 0$ otherwise. For $\zeta = (f_m) \in \mathcal{N}(\mathcal{G})$, we define $K_x^\zeta \in \tilde{\mathcal{C}}$ by $K_x^\zeta = \sum_{m\in\mathcal{N}} K_x^{f_m}\chi_m$. We denote by C_x the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\{\rho(K_x^\zeta); \zeta \in \mathcal{N}(\mathcal{G})\}$. We define an isometry T of $\mathcal{H}_x$ onto $\mathcal{H}$ by $(T\eta)(g) = \eta(g, x)$ for $\eta \in \mathcal{H}_x$. We set $\tilde{\psi}_x(a) = TaT^*$ for $a \in B_x$. For $g \in G_e$ and $a \in C_x$, we set $\Psi(x, g)(a) = R_g a R_g^*$, where R is the right regular representation of G on $\mathcal{H}$. Then we have:

LEMMA 2.2. *For $x \in M$, there exists a unique spatial isomorphism $\tilde{\psi}_x$ of B_x onto C_x such that $\tilde{\psi}_x(\tilde{\rho}_x(\zeta)) = \rho(K_x^\zeta)$ for $\zeta \in \mathcal{N}(\mathcal{G})$.*

LEMMA 2.3. *For $x \in M$ and $g \in G_e$, there exists a unique spatial isomorphism $\Psi(x, g)$ of C_x onto C_{gx} such that $\Psi(x, g)(\rho(K_x^\zeta)) = \rho(K_{gx}^\zeta)$ for $\zeta \in \mathcal{N}(\mathcal{G})$.*

We denote by $\mathrm{Diff}_G(M)$ the group of diffeomorphisms of M which commute with the action of the connected component G_e on M. For $\alpha \in \mathrm{Diff}_G(M)$ and $m \in \mathcal{N}$, there exists a diffeomorphism α_m such that $g\alpha(x) = \alpha_m(gx)$ for all $g \in G_m$ and $x \in M$. If G is discrete, then we have $\mathrm{Diff}_G(M) = \mathrm{Diff}(M)$, the group of all diffeomorphisms on G. For $\alpha \in \mathrm{Diff}_G(M)$ and $\zeta = (f_m) \in \mathcal{N}(\mathcal{G})$, we define $\bar{\alpha}(\zeta) \in \mathcal{N}(\mathcal{G})$ by $\bar{\alpha}(\zeta) = (\tilde{\alpha}_m(f_m))$, where $\tilde{\alpha}_m(f_m)(g, x) = f_m(g, \alpha_m^{-1}(x))$. For $\zeta \in \mathcal{N}(\mathcal{G})$, we have $K_x^{\bar{\alpha}^{-1}(\zeta)} = K_{\alpha(x)}^\zeta$. Thus we have:

LEMMA 2.4. *For $\alpha \in \mathrm{Diff}_G(M)$ and $x \in M$, $C_x = C_{\alpha(x)}$.*

Remember that dim $M = n$ and dim $G = p$. We assume that $n \geq p$. Let $\sigma : \mathbb{R}^{n-p} \to \mathrm{Diff}_G(M)$ be a differentiable action, that is, σ is a homomorphism and the map $(x,t) \mapsto \sigma_t(x)$ is of class C^∞.

DEFINITION 2.5. Let U be a connected open set in M. Suppose that there exists a C^∞ diffeomorphism φ of U onto $S \times T$, where S is an open set in G_e with $e \in S$ and T is an open set in $\mathbb{R}^{n-p}$ with $0 \in T$. Then the pair (U, φ) is called a local chart of (M,G) compatible with σ if it satisfies the following conditions;
(i) $\varphi^{-1}(g,t) = g\varphi^{-1}(e,t)$,
(ii) $\varphi^{-1}(g,t) = \sigma_t(\varphi^{-1}(g,0))$ for all $(g,t) \in S \times T$.

Let (U,φ) be a local chart compatible with σ as above. We set $x_0 = \varphi^{-1}(e,0)$. For $x \in U$ with $\varphi(x) = (g,t)$, we have $g^{-1}x = \sigma_t(x_0)$. It follows from Lemmas 2.2, 2.3 and 2.4 that the map $\Psi(x, g^{-1}) \circ \tilde{\psi}_x$ is a spatial isomorphism of B_x onto C_{x_0} for $x \in U$ with $\varphi(x) = (g,t)$. We set $\psi_x = \Psi(x,g^{-1}) \circ \tilde{\psi}_x$. Then we have the following:

PROPOSITION 2.6. *Let (U_1, φ_1) and (U_2, φ_2) be local chatrs compatible with σ and U be a connected component of $U_1 \cap U_2$. If $\psi_{i,x}$ is the $*$-isomorphism of B_x onto C_{x_i} as above with respect to (U_i, φ_i) with $x_i = \varphi_i^{-1}(e,0)$ $(i = 1,2)$, then there exists a $*$-isomorphism α of C_{x_1} onto C_{x_2} such that $\alpha = \psi_{2,x} \circ \psi_{1,x}^{-1}$ for all $x \in U$.*

Proof. For $i = 1,2$, we set $\varphi_i(U_i) = S_i \times T_i$ as in Definition 2.5. We fix $x \in U$ and suppose that $\varphi_i(x) = (g_i, t_i)$ $(i = 1,2)$. Let x' be an element of U such that $\varphi_i(x') = (g_i', t_i')$ $(i = 1,2)$. We set $g_0 = g_1' g_1^{-1}$ and $t_0 = t_1 - t_1'$. Let U_0 be a sufficiently small neighborhood of x in U. For $x' \in U_0$, we have $g_0 x = \sigma_{t_0}(x')$, $\varphi_2(g_0 x) = (g_0 g_2, t_2)$ and $\varphi_2(\sigma_{t_0}(x')) = (g_2', t_0 + t_2')$. Since we have $(g_0 g_2, t_2) = (g_2', t_0 + t_2')$, we have $g_2^{-1} g_1 = g_2'^{-1} g_1'$. Since we have $\psi_{2,x} \circ \psi_{1,x}^{-1} = \Psi(x_1, g_2^{-1} g_1)$ and $\psi_{2,x'} \circ \psi_{1,x'}^{-1} = \Psi(x_1, g_2'^{-1} g_1')$, we have $\psi_{2,x} \circ \psi_{1,x}^{-1} = \psi_{2,x'} \circ \psi_{1,x'}^{-1}$. Since U_0 is a neighborhood of x, this completes the proof of Proposition 2.6. ∎

We denote by B the disjoint union of C^*-algebras $\{B_x; x \in M\}$ and denote by π the map of B onto M defined by $\pi(a) = x$ for $a \in B_x$. Let $\{(U_i, \varphi_i)\}$ be the set of all local charts of (M,G) compatible with σ indexed by a set I and let $\psi_{i,x}$ be the $*$-isomorphism of B_x onto C_{x_i} constructed as above from (U_i, φ_i) with $x_i = \varphi_i^{-1}(e, 0)$. We define a map ψ_i of $\pi^{-1}(U_i)$ onto $U_i \times C_{x_i}$ by $\psi_i(a) = (x, \psi_{i,x}(a))$ for $a \in B_x$. Let $\mathcal{F}_\sigma$ be the set of pairs (U_i, ψ_i) $(i \in I)$ constructed as above.

DEFINITION 2.7. A differentiable action σ is called a transverse action for (M, G) if the family $\{U_i; i \in I\}$ is an open covering of M.

In the following we assume that σ is a transverse action for (M, G). It follows from Proposition 2.6 that there exists a unique topology on B such that π is continuous and ψ_i is a homeomorphism for all $i \in I$. Since M is connected, the C^*-algebras C_x are mutually $*$ -isomorphic. Therefore, for a fixed $\tilde{x} \in M$, we set $C = C_{\tilde{x}}$ and fix a $*$-isomorphism between C and C_{x_i} for every $i \in I$, and then we identify C_{x_i} with C by this isomorphism. Thus we consider $\psi_{i,x}$ to be a $*$-isomorphism of B_x onto C and ψ_i to be a homeomorphism of $\pi^{-1}(U_i)$ onto $U_i \times C$. By virtue of Proposition 2.6, we have the following theorem:

THEOREM 2.8. *Suppose that σ is a transverse action for (M, G). Then the quartet (B, π, M, C) constructed above is a differentiable bundle of C^*-algebras with respect to the flat family $\mathcal{F}_\sigma$ of C^*-coordinate systems.*

3. Differentiable cross sections. For $f \in C_c^\infty(\mathcal{G})$ and $m \in \mathcal{N}$, we define an element $[f]_m = (f_m)$ of $\mathcal{N}(\mathcal{G})$ by $f_m = f$ and $f_k = 0$ if $k \neq m$, and define the cross section $cs_m(f)$ of B by $cs_m(f)_x = \tilde{\rho}_x([f]_m)$ $(x \in M)$, that is, $cs_m(f)_x = \rho_x(f)P_x^m$. If G is connected, we set $cs(f) = cs_e(f)$, where $\mathcal{N} = \{e\}$, and we have $cs(f)_x = \rho_x(f)$. Let $\sigma^m : \mathbb{R}^{n-p} \to \mathrm{Diff}\ (M)$ be a differentiable action such that $\sigma^m = g \circ \sigma \circ g^{-1}$ for every $g \in G_m$. We prepare a lemma for proving the differentiability of $cs_m(f)$.

LEMMA 3.1. *For $F \in C_c^\infty(\mathbb{R}^{n-p} \times \mathcal{G})$ and $t \in \mathbb{R}^{n-p}$, define an element F_t of $C_c^\infty(\mathcal{G})$ by $F_t(g, x) = F(t, g, x)$. Let t_0 be an element of $\mathbb{R}^{n-p}$.* (i) *The supremum norm $||F_t - F_{t_0}||_\infty$ converges to 0 as $t \to t_0$.* (ii) *Let J be an open interval in $\mathbb{R}$ containing 0, and let $t(\cdot)$ be a C^2 map of J into $\mathbb{R}^{n-p}$ with $t(0) = t_0$. Define an element f of $C_c^\infty(\mathcal{G})$ by $f(g, x) = \sum_{i=1}^{n-p} (\partial F / \partial t_i)(t_0, g, x)(dt_i/dh)(0)$, where*

$t(h) = (t_1(h), \dots, t_{n-p}(h))$. Then $||h^{-1}(F_{t(h)} - F_{t_0}) - f||_\infty$ converges to 0 as $h \to 0$.

The proof is elementary, and we omit it.

THEOREM 3.2. *The cross section $cs_m(f)$ is differentiable, that is, $cs_m(f) \in \Gamma(B)$, for every $f \in C_c^\infty(\mathcal{G})$ and $m \in \mathcal{N}$.*

Proof. We fix $i \in I$, that is, we fix (U_i, ψ_i) in $\mathcal{F}_\sigma$ and a local chart (U_i, φ_i) compatible with σ. Recall that φ_i is a diffeomorphism of U_i onto $S \times T$, where S and T are open sets of G_e and $\mathbb{R}^{n-p}$ respectively. Let (U_0, φ_0) be a local coordinate system of M such that $U_i \cap U_0 \neq \varnothing$. We set $U = U_i \cap U_0$ and $V = \varphi_0(U)$. We define C^∞ map $x(v)$ of V into U by $x(v) = \varphi_0^{-1}(v)$ and define C^∞ maps $g(v)$ of V into S and $t(v)$ of V into T by $\varphi_i(x(v)) = (g(v), t(v))$. We set $\xi = cs_m(f)$ and define maps $\tilde{\xi}_i$ of U_i into C and η of V into C by $\tilde{\xi}_i(x) = \psi_{i,x}(\xi_x)$ and $\eta = \tilde{\xi}_i \circ \varphi_0^{-1}$ respectively. It follows from Lemmas 2.2 and 2.3 that we have $\eta(v) = \rho(K^{[f]_m}_{g(v)^{-1}x(v)})$. We have $g(v)^{-1}x(v) = \sigma_{t(v)}(x_i)$, where $x_i = \varphi_i^{-1}(e, 0)$. We define an element F of $C^\infty(\mathbb{R}^{n-p} \times \mathcal{G})$ by $F(t, g, x) = f(g, \sigma_t^m(x))$. We have

$$\left|\left|K^f_{\sigma_{t(v)}(x_i)}\chi_m - K^f_{\sigma_{t(u)}(x_i)}\chi_m\right|\right|_\infty \leq ||F_{t(v)} - F_{t(u)}||_\infty, \text{ for } u,\ v \in V.$$

Let E be a compact set in G such that $\operatorname{supp} f \subset E \times M$. It follows from Lemma 1.2 that we have $||\eta(v) - \eta(u)|| \leq I_E||F_{t(v)} - F_{t(u)}||_\infty$. By virtue of Lemma 3.1 we know that η is continuous on V.

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$ and $v_1, \dots, v_n$ be coordinate functions of $\mathbb{R}^n$ associated with $e_1, \dots, e_n$. We fix an element u of V. For a fixed $k = 1, \dots, n$, let $\delta > 0$ be such that $u + he_k \in V$ for $|h| < \delta$. We denote by J the interval $\{h; |h| < \delta\}$ in $\mathbb{R}$. We define a C^∞ map τ of J into T by $\tau(h) = t(u + he_k)$. For $j = 1, \dots, n-p$, we define an element f_j^m of $C_c^\infty(\mathcal{G})$ by $f_j^m(g, x) = (\partial/\partial t_j)(f(g, \sigma_t^m(x)))|_{t=0}$. We set $t(v) = (t_1(v), \dots, t_{n-p}(v))$ and $\tau(h) = (\tau_1(h), \dots, \tau_{n-p}(h))$. We define an element a of $C_c^\infty(\mathcal{G})$ by $a(g, x) = \sum_{j=1}^{n-p}(\partial F/\partial t_j)(\tau(0), g, x)(d\tau_j/dh)(0)$. It follows from Lemma 3.1 that $h^{-1}(F_{\tau(h)} - F_{\tau(0)})$ converges to a as $h \to 0$. Let $\tilde{K}_h$

be a function on $G \times G$ such that $\tilde{K}_h(g, g')$ is equal to

$$h^{-1}\left\{K^f_{\sigma_{\tau(h)}(x_i)}\chi_m - K^f_{\sigma_{\tau(0)}(x_i)}\chi_m\right\}(g, g') \\ - \sum_{j=1}^{n-p}\left(K^{f^m_j}_{\sigma_{\tau(0)}(x_i)}\chi_m\right)(g, g')\frac{\partial t_j}{\partial v_k}(u).$$

We have $||\tilde{K}_h||_\infty \leq ||h^{-1}(F_{\tau(h)} - F_{\tau(0)}) - a||_\infty$. We set $\xi^j = cs_m(f^m_j)$ and define maps $\tilde{\xi}^j_i$ of U_i onto C and η^j of V into C by $\tilde{\xi}^j_i(x) = \psi_{i,x}(\xi^j_x)$ and $\eta^j = \tilde{\xi}^j_i \circ \varphi_0^{-1}$ respectively. It follows from Lemma 1.2 that we have

$$\left\|h^{-1}(\eta(u + he_k) - \eta(u)) - \sum_{j=1}^{n-p}\eta^j(u)\frac{\partial t_j}{\partial v_k}(u)\right\| \\ \leq I_E||h^{-1}(F_{\tau(h)} - F_{\tau(0)}) - a||_\infty.$$

Therefore we have $(\partial\eta/\partial v_k)(u) = \sum_{j=1}^{n-p}\eta^j(u)(\partial t_j/\partial v_k)(u)$. As we have seen in the first half of this proof, η^j is continuous on V. Therefore η is of class $(C^1)'$ in the sense of §1. Similarly η^j is of class $(C^1)'$ for $j = 1, \dots, n-p$. Therefore we know that η is of class $(C^\infty)'$ and that $\tilde{\xi}_i$ is of class C^∞ in the sense of Definition 1.3. This completes the proof of Theorem 3.2. □

Recall that $\Gamma(B)$ is not only a $*$-algebra but also a $C^\infty(M)$ - module. We denote by $\mathcal{D}$ the $*$-subalgebra of $\Gamma(B)$ generated by elements of the form $\omega \cdot cs_m(f)$ with $f \in C^\infty_c(\mathcal{G})$, $m \in \mathcal{N}$ and $\omega \in C^\infty(M)$. Then $\mathcal{D}$ is also a $C^\infty(M)$-submodule of $\Gamma(B)$. For $x \in M$, we set $\mathcal{D}_x = \{\xi_x \in B_x; \xi \in \mathcal{D}\}$. Note that $\mathcal{D}_x$ is the $*$-subalgebra of B_x generated by elements of the form $\rho_x(f)P^m_x$ with $f \in C^\infty_c(\mathcal{G})$ and $m \in \mathcal{N}$. If $\mathcal{N}$ is finite, then $\mathcal{D}_x$ is dense in the norm topology of B_x for every $x \in M$. If $\mathcal{N}$ is infinite, then $\mathcal{D}_x$ may not be dense in the norm topology, but it is dense in the strong operator topology of B_x by Lemma 2.1.

4. Flat connections. It follows from Lemma 1.6 that there exists a unique flat connection ∇ in B. In this section we calculate $\nabla(cs_m(f))$ explicitly.

LEMMA 4.1. *For $j = 1, \dots, n-p$, there exists an element w^j of $\Gamma(T^*M)$ such that $w^j_x(X_x) = X_x(t_j \circ p_2 \circ \varphi)$ $(X \in \Gamma(TM),\ x \in U)$*

for every local chart (U,φ) of (M,G) compatible with σ, where p_2 is the projection of $G_e \times \mathbb{R}^{n-p}$ onto $\mathbb{R}^{n-p}$ and t_j is the j-th coordinate function of $\mathbb{R}^{n-p}$.

Proof. Let $\{\omega_k; k = 1,2,\dots\}$ be a partition of unity on M subordinate to the cover $\{U_i; i \in I\}$. Let $i(k)$ be an element of I such that $\operatorname{supp}\omega_k \subset U_{i(k)}$. We define w^j by $w^j = \sum_{k=1}^{\infty}\omega_k d(t_j \circ p_2 \circ \varphi_{i(k)})$. □

THEOREM 4.2. *The flat connection ∇ in B satisfies the following equation;*

$$\nabla(cs_m(f)) = \sum_{j=1}^{n-p} w^j \otimes cs_m(f_j^m) \quad (f \in C_c^\infty(\mathcal{G})\ m \in \mathcal{N}),$$

where $f_j^m(g,x) = (\partial/\partial t_j)(f(g,\sigma_t^m(x)))|_{t=0}$. In particular, a cross section $(\nabla\xi)(X)$ is an element of $\mathcal{D}$ for every $\xi \in \mathcal{D}$ and $X \in \Gamma(TM)$.

Proof. Let $\{\omega_k\}$ be the partition of unity as in the proof of Lemma 4.1 and $i(k)$ be an element of I such that $\operatorname{supp}\omega_k \subset U_{i(k)}$. Let (V,ψ) be a local coordinate system of M and $x_1,\dots,x_n$ be coordinate functions associated with (V,ψ). We set $\xi = cs_m(f)$ and $\xi^j = cs_m(f_j^m)$, we set $\tilde{\xi}_{i(k)}(x) = \psi_{i(k),x}(\xi_x)$ and $\tilde{\xi}^j_{i(k)} = \psi_{i(k),x}(\xi^j_x)$, and then we set $\eta = \tilde{\xi}_{i(k)} \circ \psi^{-1}$ and $\eta^j = \tilde{\xi}^j_{i(k)} \circ \psi^{-1}$. We set $\tilde{t}_j^{i(k)} = t_j \circ p_2 \circ \varphi_{i(k)}$. It follows from the proof of Theorem 3.2 that we have $(\partial\eta/\partial v_l) = \sum_{j=1}^{n-p}\eta^j(\partial\tilde{t}_j^{i(k)} \circ \psi^{-1}/\partial v_l)$. Since we have $\sum_{k=1}^{\infty}(\partial\omega_k/\partial x_l) = 0$, we have

$$\sum_{k=1}^{\infty}\psi^{-1}_{i(k),x}\left(\frac{\partial(\omega_k\tilde{\xi}_{i(k)})}{\partial x_l}(x)\right) = \sum_{k=1}^{\infty}\sum_{j=1}^{n-p}\omega_k(x)\xi^j_x\frac{\partial\tilde{t}_j^{i(k)}}{\partial x_l}(x).$$

Let X be an element of $\Gamma(TM)$. It follows from Lemma 4.1 that we have $(\nabla\xi)(X)_x = \sum_{j=1}^{n-p} w^j_x(X_x)\xi^j_x$. This completes the proof of Theorem 4.2. □

In the rest of this section, we assume that G is connected and that $C_r^*(\mathcal{G})$ is simple. The following proposition shows that the bundle B is topologically trivial, but the differentiable structure for B is not trivial as we shall see in the next section.

PROPOSITION 4.3. *Suppose that G is connected and that $C_r^*(\mathcal{G})$ is simple. Then the bundle B is isomorphic to the product bundle $M \times C_r^*(\mathcal{G})$ as topological vector bundles.*

Proof. We set $A = C_r^*(\mathcal{G})$. Since G is connected, we have $\tilde{\rho}_x = \rho_x$. Since A is simple, ρ_x is a $*$- isomorphism of A onto B_x. For $i \in I$, we define a $*$-isomorphism $\Theta_{i,x}$ of A onto C by $\Theta_{i,x} = \psi_{i,x} \circ \rho_x$, where $(U_i, \psi_i) \in \mathcal{F}_\sigma$. For $a \in A$, we define a map η_a of U_i into C by $\eta_a(x) = \Theta_{i,x}(a)$. For $f \in C_c^\infty(\mathcal{G})$, it follows from the proof of Theorem 3.2 that η_f is continuous. Since $\Theta_{i,x}$ is isometry, the map $(x, a) \mapsto \eta_a(x)$ is continuous on $U_i \times A$. For $c \in C$, we define a map $\bar{\eta}_c$ of U_i into A by $\bar{\eta}_c(x) = \Theta_{i,x}^{-1}(c)$. The map $(x, c) \mapsto \bar{\eta}_c(x)$ is continuous on $U_i \times C$. We define a map Θ_i of $U_i \times A$ onto $U_i \times C$ by $\Theta_i(x, a) = (x, \eta_a(x))$. Then we have $\Theta_i^{-1}(x, c) = (x, \bar{\eta}_c(x))$. Therefore Θ_i is a homeomorphism. We define a map Θ of $M \times A$ onto B by $\Theta(x, a) = \rho_x(a)$. Then we have $\psi_i \circ \Theta = \Theta_i$ for every $i \in I$. Since the topology of B is determined by $\{\psi_i\}$, Θ is a homeomorphism. $\square$

We denote by $C_c^\infty(\mathcal{G})^\sigma$ the $*$-algebra of all elements f of $C_c^\infty(\mathcal{G})$ with the property that $\nabla(cs(f)) = 0$. It follows from Theorem 4.2 that f is an element of $C_c^\infty(\mathcal{G})^\sigma$ if and only if we have $f(g, \sigma_t(x)) = f(g, x)$ for all $t \in \mathbb{R}^{n-p}$ and $(g, x) \in \mathcal{G}$. Let $C_r^*(\mathcal{G})^\sigma$ be the C^*-subalgebra of $C_r^*(\mathcal{G})$ generated by $C_c^\infty(\mathcal{G})^\sigma$. We set $B_x^\sigma = \rho_x(C_r^*(\mathcal{G})^\sigma)$. We set $B^\sigma = \cup_{x \in M} B_x^\sigma$ and $\pi^\sigma = \pi | B^\sigma$, the restriction of π to B^σ. For $(U_i, \psi_i) \in \mathcal{F}_\sigma$, we set $\psi_i^\sigma = \psi_i | (\pi^\sigma)^{-1}(U_i)$ and $\psi_{i,x}^\sigma = \psi_{i,x} | B_x^\sigma$ $(x \in U_i)$. We denote by $\mathcal{F}_\sigma^\sigma$ the set of (U_i, ψ_i^σ) $(i \in I)$. Let C_x^σ be the C^*-subalgebra of C_x generated by elements $\rho(K_x^f)$ $(f \in C_c^\infty(\mathcal{G})^\sigma)$. Then $\psi_{i,x}^\sigma$ is a $*$-isomorphism of B_x^σ onto $C_{x_i}^\sigma$. Let $\tilde{x}$ be the element chosen in §2 so that we can identify C_{x_i} with $C = C_{\tilde{x}}$. We set $C^\sigma = C_{\tilde{x}}^\sigma$. Then we may identify the subalgebra $C_{x_i}^\sigma$ of C_{x_i} with the subalgebra C^σ of C. Thus we consider $\psi_{i,x}^\sigma$ to be a $*$-isomorphism of B_x^σ onto C^σ and ψ_i^σ to be a homeomorphism of $(\pi^\sigma)^{-1}(U_i)$ onto $U_i \times C^\sigma$. We denote by Θ^σ the restriction of Θ to $M \times C_r^*(\mathcal{G})^\sigma$, where Θ is the homeomorphism defined in the proof of Proposition 4.3. Then we have the following:

PROPOSITION 4.4. *Suppose that G is connected and $C_r^*(\mathcal{G})$ is simple. The quartet $(B^\sigma,\ \pi^\sigma,\ M,\ C^\sigma)$ is a differentiable bundle of C^*-algebras with respect to the family $\mathcal{F}_\sigma^\sigma$. Moreover the differentiable structure for B^σ is trivial in the following sense: There exists* *a homeomorphism Θ^σ of $M \times C_r^*(\mathcal{G})^\sigma$ onto B^σ with the following property; for every $(U_i, \psi_i^\sigma) \in \mathcal{F}_\sigma^\sigma$, there exists a $*$-isomorphism α_i of $C_r^*(\mathcal{G})^\sigma$ onto C^σ such that $\psi_i^\sigma \circ \Theta_i^\sigma = id_i \times \alpha_i$, where Θ_i^σ is the restriction of Θ^σ to $U_i \times C_r^*(\mathcal{G})^\sigma$ and id_i is the identity map of U_i onto itself.*

We denote by $RM(A)$ the Banach algebra of all right multipliers of a C^*-algebra A on a Hilbert space ([**15**, 3.12]). Let $RM(B)$ be the disjoint union of Banach algebras $RM(B_x)$ $(x \in M)$ and $\bar{\pi}$ be the map of $RM(B)$ onto M defined by $\bar{\pi}(a) = x$ for $a \in RM(B_x)$. Let (U_i, ψ_i) be an element of $\mathcal{F}_\sigma$. It follows from Lemmas 2.2 and 2.3 that $\psi_{i,x}$ is spatial for every $x \in U_i$. Therefore we can extend $\psi_{i,x}$ to an isomorphism $\bar{\psi}_{i,x}$ of $RM(B_x)$ onto $RM(C_{x_i})$. We define a map $\bar{\psi}_i$ of $\bar{\pi}^{-1}(U_i)$ onto $U_i \times RM(C_{x_i})$ by $\bar{\psi}_i(a) = (x, \bar{\psi}_{i,x}(a))$ for $a \in RM(B_x)$. We denote by $\overline{\mathcal{F}}_\sigma$ the set of $(U_i, \bar{\psi}_i)$ $(i \in I)$. Moreover we may identify $RM(C_{x_i})$ with $RM(C)$. Thus we consider $\bar{\psi}_{i,x}$ to be an isomorphism of $RM(B_x)$ onto $RM(C)$ and $\bar{\psi}_i$ to be a homeomorphism of $\bar{\pi}^{-1}(U_i)$ onto $U_i \times RM(C)$. Then the quartet $(RM(B),\ \bar{\pi},\ M,\ RM(C))$ is a differentiable bundle of Banach algebras with respect to the flat family $\overline{\mathcal{F}}_\sigma$ of Banach coordinate systems. It follows from Lemma 1.6 that there exists a unique flat connection $\bar{\nabla}$ in $RM(B)$.

5. Examples. (a) *Kronecker dynamical systems and irrational rotation algebras.* Let M be the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. For $\mu \in \mathbb{R} \cup \{\infty\}$, we define an action F^μ of $\mathbb{R}$ on M by $F_t^\mu(x_1, x_2) = (x_1 + t, x_2 + \mu t)$ if $\mu \in \mathbb{R}$ and by $F_t^\infty(x_1, x_2) = (x_1, x_2 + t)$ $((x_1, x_2) \in M, t \in \mathbb{R})$. Let G be the real line $\mathbb{R}$ and θ be an irrational number. We define an action of G on M by $t \cdot x = F_t^\theta(x)$ for $t \in G$ and $x \in M$. For $\mu \in \mathbb{Q} \cup \{\infty\}$, we define an action σ of $\mathbb{R}$ on M by $\sigma = F^\mu$. For $x_0 = (x_1^0, x_2^0) \in M$ and $\varepsilon > 0$, we set $S = T = \{t \in \mathbb{R}; |t| < \varepsilon\}$. We define a map φ_0 of $S \times T$ into M by $\varphi_0(t_1, t_2) = t_1 \cdot \sigma_{t_2}(x_0)$. We set $U = \varphi_0(S \times T)$. If ε is small enough, then φ_0 is a diffeomorphism onto U. In this case, we set $\varphi = \varphi_0^{-1}$ and (U, φ) is a local chart of (M, G) compatible with σ. Therefore σ is a transverse action for (M, G). It follows from Theorem 2.8 that there exists the differentiable bundle $(B,\ \pi,\ M,\ C)$ of C^*-algebras with respect to the flat family $\mathcal{F}_\sigma$. Let ∇ be the flat connection in B (Lemma 1.6). For $f \in C_c^\infty(\mathcal{G})$, it follows from Theorem 4.2 that we have, $\nabla(cs(f)) = (adx_1 + bdx_2) \otimes cs(f_1)$, where $a = -\theta/(\mu-\theta)$, $b = 1/(\mu-\theta)$ and $f_1 = \partial f/\partial x_1 + \mu(\partial f/\partial x_2)$, if $\mu \in \mathbb{Q}$ and we have $\nabla(cs(f)) = (-\theta dx_1 + dx_2) \otimes cs(\partial f/\partial x_2)$ if $\mu = \infty$.

First, we suppose that $\mu = \infty$. For $u \in C(\mathbb{T})$, we define an operator $rm(u)_x$ on $\mathcal{H}_x$ by $(rm(u)_x\zeta)(t,x) = u(x_2 + \theta t)\zeta(t,x)$ for $x = (x_1, x_2) \in M$, $\zeta \in \mathcal{H}_x$ and $t \in G$. For $f \in C_c(\mathcal{G})$, we have $\rho_x(f) rm(u)_x = \rho_x(f \cdot u)$, where $(f \cdot u)(t, x_1, x_2) = f(t, x_1, x_2)u(x_2)$. Therefore $rm(u)_x$ is an element of $RM(B_x)$. We denote by D_x the set of elements $rm(u)_x$ $(u \in C(\mathbb{T}))$. Then D_x is a C^*-subalgebra of $\mathcal{B}(\mathcal{H}_x)$ and D_x is $*$- isomorphic to $C(\mathbb{T})$. Note that f is an element of $C_c^\infty(\mathcal{G})^\sigma$ if and only if there exists an element $\tilde{f}$ of $C_c^\infty(\mathbb{R} \times \mathbb{T})$ such that $f(t, x_1, x_2) = \tilde{f}(t, x_1)$. Therefore $B_x^\sigma D_x$ generates B_x. Let D be the disjoint union of D_x $(x \in M)$, π^r be the restriction of $\bar{\pi}$ to D and ψ_i^r be the restriction of $\bar{\psi}_i$ to $(\pi^r)^{-1}(U_i)$ for $(U_i, \bar{\psi}_i) \in \overline{\mathcal{F}}_\sigma$. We denote by $\mathcal{F}_\sigma^r$ the set of (U_i, ψ_i^r) $(i \in I)$. Then the quartet $(D,\ \pi^r,\ M,\ C(\mathbb{T}))$ is a differentiable bundle of C^*-algebras with respect to the flat family $\mathcal{F}_\sigma^r$ of C^*-coordinate systems. We denote by ∇^r the unique flat connection in D (Lemma 1.6). Let $(U, \bar{\psi})$ be an element of $\overline{\mathcal{F}}_\sigma$ constructed from the above local chart (U, φ). We denote by ψ^r the restriction of $\bar{\psi}$ to $(\pi^r)^{-1}(U)$ and denote by ψ_x^r the restriction of $\bar{\psi}_x$ to D_x for $x \in U$. For $x = (x_1, x_2) \in U$, we have $(\psi_x^r(rm(u)_x)\zeta)(s) = u(-\theta(x_1 - x_1^0) + x_2 + \theta s)\zeta(s)$ for $u \in C(\mathbb{T})$, $\zeta \in \mathcal{H}$ and $s \in \mathbb{R}$. Let (x_1, x_2, x_3) be a natural coordinate system of $U \times \mathbb{T}$ as a subset of $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. We denote by $C_b^\infty(U \times \mathbb{T})$ the set of all C^∞ functions f on $U \times \mathbb{T}$ with the property that partial derivatives $\partial^\alpha f/\partial \tilde{x}^\alpha$ are bounded for every multi-index α and every natural coordinate system $\tilde{x}$. For $v \in C(U \times \mathbb{T})$, we define a map $rm(v)$ of U into D by $rm(v)_x = rm(v_x)_x$ for $x \in U$, where v_x is an element of $C(\mathbb{T})$ defined by $v_x(x_3) = v(x, x_3)$. As in the proof of Theorem 3.2, we can show that $rm(v)$ is a differentiable cross section of D on U for $v \in C_b^\infty(U \times \mathbb{T})$, and we have $\nabla^r(rm(v)) = dx_1 \otimes rm(v_1) + dx_2 \otimes rm(v_2)$, where $v_1 = \partial v/\partial x_1 - \theta(\partial v/\partial x_3)$ and $v_2 = \partial v/\partial x_2 + \partial v/\partial x_3$. Moreover we have $\nabla^r(rm(v)) = 0$ if and only if there exists an element u of $C^\infty(\mathbb{T})$ such that $v(x_1, x_2, x_3) = u(\theta(x_1 - x_1^0) - (x_2 - x_2^0) + x_3)$.

Let $[a, b]$ be a closed interval in $\mathbb{R}$, and $\gamma : [a, b] \to M$ be a smooth curve, that is, γ extends to be a C^∞ map of $(a - \varepsilon, b + \varepsilon)$ into M for some $\varepsilon > 0$, which we denote again by γ. We shall say that a map ξ of $[a,b]$ into D is a smooth curve in D if ξ extends to be a map of $(a-\varepsilon, b+\varepsilon)$ into D, which we denote again by ξ, such that $\pi^r(\xi(t)) = \gamma(t)$ and the map $t \mapsto \psi^r_{i,\gamma(t)}(\xi(t))$ is of class C^∞ for every $i \in I$. Next suppose that γ is a piecewise smooth curve. By definition there exists a partition $a = a_0 < a_1 < \cdots < a_k = b$ such that $\gamma|[a_j, a_{j+1}]$ is smooth for every j ([**21**, Definition 1.41]). We shall say that a map ξ of $[a,b]$ into D is a piecewise smooth curve in D if $\xi|[a_j, a_{j+1}]$ is smooth for every j. For a piecewise smooth curve ξ in D, we define $\nabla^r(\xi)(\dot\gamma(t)) \in D_{\gamma(t)}$ by $\nabla^r(\xi)(\dot\gamma(t)) = (\psi^r_{i,\gamma(t)})^{-1}((d/dt)(\psi^r_{i,\gamma(t)}(\xi(t))))$ (c.f. [**14**, §1]). A horizontal curve ξ in D is a piecewise smooth curve in D such that $\nabla^r(\xi)(\dot\gamma(t)) = 0$ for every $t \in [a,b]$ (c.f. [**11**, Chapter II, §3]). Then we have the following:

LEMMA 5.1. *Let $\gamma : [a,b] \to M$ be a piecewise smooth curve with $\gamma(a) = \gamma(b) = x$. For every $A \in D_x$, there exists a unique horizontal curve ξ_A in D such that $\xi_A(a) = A$. For $u \in C(\mathbb{T})$, define an element $h(u)$ of $C(\mathbb{T})$ by $\xi_A(b) = rm(h(u))_x$, where $A = rm(u)_x$. Then there exists an integer k such that $h(u)(s) = u(s+k\theta)$ $(s \in \mathbb{T})$ for every $u \in C(\mathbb{T})$.*

Proof. We fix $t_0 \in [a,b]$. Let (U, ψ^r) and (U, φ) be as above with $x_0 = \gamma(t_0)$. Let V be a connected neighborhood of t_0 such that $\gamma(t) \in U$ for every $t \in V$. Then we have $\xi_A(t) = (\psi^r_{\gamma(t)})^{-1} \circ \psi^r_{\gamma(t_0)}(\xi_A(t_0))$ for $t \in V$. This implies the existence and the uniqueness of ξ_A. Let u_t be an element of $C(\mathbb{T})$ such that $\xi_A(t) = rm(u_t)_{\gamma(t)}$. We set $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and $x_j(t_1, t_2) = \gamma_j(t_1) - \gamma_j(t_2)$ for $j = 1, 2$. Then we have $u_t(-\theta x_1(t, t_0) + \gamma_2(t) + \theta s) = u_{t_0}(\gamma_2(t_0) + \theta s)$ $(s \in \mathbb{T})$. Thus we have $h(u)(s) = u(s + k\theta)$ for an integer k. □

By virtue of Lemma 5.1, one can define a $*$-automorphism $\hat h_\gamma$ of D_x by $\hat h_\gamma(A) = \xi_A(b)$. This automorphism is called the parallel displacement along the curve γ. We denote by $C(x)$ the set of piecewise smooth curves starting and ending at x. The holonomy group Φ_x of ∇^r with reference point x is the group of all automorphisms $\hat h_\gamma$ $(\gamma \in C(x))$ (c.f. [**11**, Chapter II, §4]). We define an action α of $\mathbb{Z}$ on $C(\mathbb{T})$ by $\alpha_k(u)(t) = u(t + k\theta)$ for $u \in C(\mathbb{T})$, $k \in \mathbb{Z}$ and $t \in \mathbb{T}$. It follows from Lemma 5.1 that (D_x, Φ_x) is isomorphic to $(C(\mathbb{T}), \alpha)$. Therefore the reduced crossed product $C^*_r(D_x, \Phi_x)$ is $*$-isomorphic to the irrational rotation algebra A_θ. On the other hand, let N be a

σ-invariant closed connected submanifold with dim $N = 1$. Then N is of the form $\{x_1\} \times \mathbb{T}$ for some $x_1 \in \mathbb{T}$, and $C_r^*(\mathcal{G}|N)$ is $*$-isomorphic to A_θ. Therefore $C_r^*(\mathcal{G}|N)$ is $*$-isomorphic to $C_r^*(D_x, \Phi_x)$.

Next, we suppose that μ is rational, say $\mu = p/q$ for relatively prime integers p and q. There exist integers a and b such that $pb - qa = 1$. We define a diffeomorphism S of M as follows; $S(x_1, x_2) = (px_1 - qx_2, -ax_1 + bx_2)$ for $(x_1, x_2) \in M$. We set $\nu = (-a + b\theta)/(p - q\theta)$ and define actions $\tilde{F}$ and $\tilde{\sigma}$ by $\tilde{F}_t = S \circ F_t^\theta \circ S^{-1}$ and $\tilde{\sigma}_t = S \circ \sigma_t \circ S^{-1}$. Then we have $\tilde{F}_t = F^\nu_{(p-q\theta)t}$ and $\tilde{\sigma}_t = F^\infty_{t/q}$. Since the system $(M,\ F^\theta,\ \sigma)$ is conjugate to $(M,\ \tilde{F},\ \tilde{\sigma})$ by S, we have a similar result to that obtained above. Note that $C_r^*(\mathcal{G}|N)$ is $*$-isomorphic to A_ν for every σ-invariant closed connected submanifold N with dim $N = 1$. We can summarize the conclusion just obtained as follows:

THEOREM 5.2. *Let σ be a transverse action for (M, G) defined by $\sigma = F^\mu$ for $\mu \in \mathbb{Q} \cup \{\infty\}$ and let $(B,\ \pi,\ M,\ C)$ be a differentiable bundle of C^*-algebras with respect to $\mathcal{F}_\sigma$. Then there exists a subbundle $(D,\ \pi^r,\ M,\ C(\mathbb{T}))$ of $(RM(B),\ \bar{\pi},\ M,\ RM(C))$ with the following properties;* (i) *$B_x^\sigma D_x$ generates B_x for every $x \in M$,* (ii) *$C_r^*(\mathcal{G}|N)$ is $*$-isomorphic to $C_r^*(D_x, \Phi_x)$ for every $x \in M$, where N is a σ-invariant closed connected submanifold of M with dim $N = 1$ and Φ_x is the holonomy group of the flat connection ∇^r in D.*

(b) *An action of a semi-direct product group.* Let S be an element of $\mathrm{SL}(2, \mathbb{Z})$, λ be an eigenvalue of S and $(1, \theta)$ be an eigenvector of S with respect to λ. We suppose that θ is real and irrational. Let G be a semi-direct product group of $\mathbb{Z}$ and $\mathbb{R}$ defined by $(n, t)(m, s) = (n + M, \lambda^{-m}t + s)$ for $(n, t),\ (m, s) \in \mathbb{Z} \times \mathbb{R}$. We may identify the group $\mathcal{N}$ with $\mathbb{Z}$ and a connected component G_m is the set of elements of the form (m, t) $(t \in \mathbb{R})$ for $m \in \mathbb{Z}$. Let M be the torus $\mathbb{T}^2$. Since we have $SF_t^\theta = F_{\lambda t}^\theta S$, we can define an action of G on M by $(n, t) \cdot x = S^n F_t^\theta(x)$ for $(n, t) \in G$ and $x \in M$. Let ν be the other eigenvalue of S and $(1, \mu)$ be an eigenvector of S with respect to ν. We set $\sigma_t = F_t^\mu$ for $t \in \mathbb{R}$. As in (a), σ is a transverse action for (M, G). Let $(B,\ \pi,\ M,\ C)$ be a differentiable bundle of C^*- algebras with respect to the family $\mathcal{F}_\sigma$ and let ∇ be the flat connection in B. It follows from Theorem 4.2 that we have

$\nabla(cs_m(f)) = \nu^m(adx_1 + bdx_2) \otimes cs_m(f_1)$, where $a = -\theta/(\mu - \theta)$, $b = 1/(\mu - \theta)$ and $f_1 = \partial f/\partial x_1 + \mu(\partial f/\partial x_2)$ for $m \in \mathbb{Z}$ and $f \in C_c^\infty(\mathcal{G})$. We denote by N the submanifold $\{0\} \times \mathbb{T}$ of M and denote by $B(S, \lambda)$ the reduction C^*-algebra $C_r^*(\mathcal{G}|N)$. This algebra was studied in [**12**, **13**, **14**]. It follows from [**7**] and [**13**] that it is a simple algebra. We do not have any results concerning relations between the bundle and the algebra $B(S, \lambda)$. This is the problem for further investigation.

(c) *Actions of discrete groups.* Let (M, G) be a differentiable dynamical system with G discrete and let $\sigma : \mathbb{R}^n \to \mathrm{Diff}\,(M)$ be a differentiable action. Suppose that the differential of the map $t \mapsto \sigma_t(x)$ at 0 is an isomorphism for every $x \in M$. Then, for every $x_0 \in M$, there exist a neighborhood U of x_0 and a neighborhood T of 0 in $\mathbb{R}^n$ such that the map $\varphi_0 : t \mapsto \sigma_t(x_0)$ is a diffeomorphism of T onto U. We set $\varphi = \varphi_0^{-1}$. Then (U, φ) is a local chart compatible with σ and σ is a transverse action for (M, G). Let $(B,\ \pi,\ M,\ C)$ be a differentiable bundle of C^*-algebras with respect to $\mathcal{F}_\sigma$ and ∇ be the flat connection. It follows from Theorem 4.2 that we have, for $g \in G$ and $f \in C_c^\infty(\mathcal{G})$, $\nabla(cs_g(f)) = \sum_{k=1}^n d\varphi^k \otimes cs_g(f_k^g)$, where $\varphi = (\varphi^1, \ldots, \varphi^n)$ and $f_k^g(g', x) = (\partial/\partial t_k) f(g', g\sigma_t(g^{-1}x))|_{t=0}$.

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