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WEIGHTED HADAMARD PRODUCTS OF HOLOMORPHIC FUCTIONS IN THE BALL

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Weighted Hadamard products of holomorphic functions in the unit ball B of $\mathbb{C}^n$ are studied, and are used to establish multiplier theorems for spaces of such functions on B. An interesting feature of such a product of two holomorphic functions f and g on B is that it is holomorphic on the unit polydisk U^n. Moreover, if, in addition, f belongs to the Hardy space $H^1(B)$ and g belongs to the Bloch space $\mathcal{B}(B)$, then the non-weighted Hadamard product of f and g belongs to BMOA(U^n), the space of holomorphic functions in U^n with bounded mean oscillation on the tours $(\partial U)^n$. Refinements of this result, as well as new charaterizations of spaces of multipliers of holomorphic functions in B, are also established.

1. Introduction. Hadamard products, their properties and related coefficient multipliers problems for spaces of holomorphic functions on the unit disk, are well-known and they have been studied by many authors (see, for example, [**5**] and the references therein). In the higher dimensional extension of such a study [**7**, **8**] one encounters with several natural, and quite interesting, questions concerning multi-index coefficient multipliers problems and the properties of weighted Haramard products of holomorphic functions of several complex variables. In this paper we shall address these questions in their higher dimensional setting by obtaining new charaterizations, some of which were unexpected, of spaces of multipliers of holomorphic functions in the ball, and in so doing we also extend and refine previously established results.

Before describing these characterizations and their background we need to set up some basic notation which shall also be used throughout the entire paper. By $H(\Omega)$ we denote the space of all holomorphic functions on a domain Ω in $\mathbb{C}^n$. For $z = (z_1, \cdots, z_n) \in$

$\mathbb{C}^n$, $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$, we let

$$\begin{aligned} \overline{z} &= (\overline{z}_1, \cdots, \overline{z}_n), & z^\alpha &= z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \\ |\alpha| &= \alpha_1 + \cdots + \alpha_n, & \alpha! &= \alpha_1! \cdots \alpha_n!, \\ \partial^\alpha &= \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, & \partial_j &= \partial/\partial z_j \quad (1 \leq j \leq n), \\ \nabla &= (\partial_1, \cdots, \partial_n), & \mathcal{R} &= \sum_{j=1}^n z_j \partial_j, \quad \mathcal{D} = 1 + \mathcal{R}, \end{aligned}$$

and

$$|z|_\infty = \max_{1 \leq j \leq n} |z_j|, \quad \|z\| = \{|z_1|^2 + \cdots + |z_n|^2\}^{1/2}.$$

Moreover, if also, $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{C}^n$ we then let

$$z \cdot \xi = (z_1 \xi_1, \cdots, z_n \xi_n), \quad \langle z, \xi \rangle = z_1 \overline{\xi}_1 + \cdots z_n \overline{\xi}_n,$$

and thus $\|z\|^2 = \langle z, z \rangle$, and $|z \cdot \xi| \leq |\xi|_\infty \|z\|$. With this notation, $B = B_n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ is the unit ball in $\mathbb{C}^n$, $U = B_1$ is the unit disk in $\mathbb{C}$, and thus $U^n = \{z \in \mathbb{C}^n : |z|_\infty < 1\}$ is the unit polydisk in $\mathbb{C}^n$. We also let $S = S_n = \partial B$, and $T = S_1$.

Let Ω be a complete Rheinhardt domain in $\mathbb{C}^n$, i.e. $z \in \Omega$ implies $z \cdot \xi \in \Omega$ for every $\xi \in \overline{U}^n$, and let $f \in H(\Omega)$. Then there exists a unique power series, respresenting f, i.e.

$$f(z) = \sum_\alpha a_\alpha z^\alpha, \quad (z \in \Omega)$$

with normal convergence in Ω, and with

$$a_\alpha = a_\alpha(f) = \{\partial^\alpha f(0)\}/\alpha! \qquad (\alpha \in \mathbb{Z}_+^n).$$

It follows that the space $H(\Omega)$ and, in particular, the spaces $H(B)$ and $H(U^n)$ may be regarded as spaces of multi-index sequences $\{a_\alpha\}$, $\alpha \in \mathbb{Z}_+^n$. For $q \geq 0$, we consider the multi-index sequence $\{\omega_\alpha(q)\}$ of positive numbers (weights), defined by

$$\omega_\alpha(q) = \frac{\alpha! \Gamma(n+q)}{\Gamma(n+q+|\alpha|)} \qquad (\alpha \in \mathbb{Z}_+^n).$$

Let X and Y be two vector spaces of multi-index sequences. A multi-index sequence $\{\lambda_\alpha\}$ is said to be a *multiplier* from X to Y if

$\{\lambda_\alpha a_\alpha\} \in Y$ whenever $\{a_\alpha\} \in X$. The set of all multipliers from X to Y is denoted by (X, Y). A multi-index sequence $\{\lambda_\alpha\}$ is said to be a *$q-$multiplier*, $q \geq 0$, from X to Y if $\{\lambda_\alpha \omega_\alpha(q) a_\alpha\} \in Y$ whenever $\{a_\alpha\} \in X$. The set of all *$q-$multipliers* from X to Y is denoted by $(X, Y)_q$. In general, however, $(X, Y)_q$ is not equal to $(Y, X)_q$, but it is so only when $n = 1$ and $q = 0$, in which case $\omega_\alpha(q) = 1$. The question of finding multipliers in (X, Y) when X and Y are subspaces of $H(B)$ has been considered and studied by several authors (see, for example, [**7**, pp. 118, 416], [**8**] and the references therein), and this is so, especially when $n = 1$, where more complete answers can be found. Indeed, recently, Mateljevic and Pavlovic [**5**] have shown that for $n = 1$, $\mathcal{B}(U) = (H^1(U), \mathrm{BMOA}(U))_0$, where $\mathcal{B}(B)$ and $H^1(B)$ are the familar Bloch and Hardy spaces, respectively, of functions in $H(B)$, and $\mathrm{BMOA}(U^n)$ is the space of functions in $H(U^n)$ with bounded mean oscillation on the torus T^n. The question of extending this result to $n > 1$ was treated by Shi [**8**] who was able to only establish that $\mathcal{B}(B) = (H^1(B), Y)_0$ with $Y = \mathcal{B}(U^n) \cap \{\cap_{0<p<\infty} H^p(B)\}$, where $\mathcal{B}(U^n)$ is the Bloch space of functions in $H(U^n)$. In this paper we shall address this higher dimensional question and thereby bridging the gap between the striking result of Mateljevic and Pavlovic when $n = 1$ and Shi's result for $n \geq 1$. In particular, it will be shown, amongst other things, that, in fact, $\mathcal{B}(B) = (H^1(B), \mathrm{BMOA}(U^n))_0$ (see Theorem 5.7) for every $n \geq 1$.

In light of the above question, we found it natural and quite interesting to study the problem of *$q-$multipliers* in its higher dimensional setting $n \geq 1$ and for any $q \geq 0$. In so doing we were also able to provide extensions and refinements of previous results. A key role in this study is played by the so-called '*weighted Hadamard products*' with weights $\omega_\alpha(q), \alpha \in \mathbb{Z}_+^n$, of functions in $H(B)$. The *q-Hadamard products* $(f * g)_q$ of two fuctions $f \in H(\Omega_1)$ and $g \in H(\Omega_2)$, where Ω_1 and Ω_2 are circular neighborhoods of $0 \in \mathbb{C}^n$, is defined as

$$(f * g)_q(z) = \sum_\alpha \omega_\alpha(q) a_\alpha(f) a_\alpha(g) z^\alpha$$

and thus $(f * g)_q = (g * f)_q$. And interesting feature of the product $(f * g)_q$ with f and g in $H(B)$ is that it is lying not only in $H(B)$ but also in $H(U^n)$, i.e. $H(B) \subset (H(B), H(U^n))_q$ (see Proposition 3.2). The reverse inclusion is also true, i.e. $H(B) = (H(B), H(U^n))_q$.

Indeed, if $g \in H(\Omega)$ where Ω is a circular neighborhood of $0 \in \mathbf{C}^n$, and if $(f * g)_q \in H(U^n)$ for every $f \in H(B)$, then we specialize to $f(z) = (1 - \langle z, \xi\rangle)^{-(n+q)}, z \in B$, where $\xi \in \overline{B}$. This implies that the power series $\sum_\alpha a_\alpha(g)\overline{\xi}^\alpha z^\alpha$ has, for every $\xi \in \overline{B}$, a normal convergence in $z \in U^n$. In particular, $g \in H(B)$ as asserted. When $n = 1$ and $q = 0$, the $q-$Hadamard product reduces to the classical Hadamard product of two holomorphic functions on the unit disk, which was also used in the recent work of Mateljevic and Pavlovic [**5**]. The higher dimensional product has been also studied by Shi [**8**] when $q = 0$. In this paper we study these products in the more general setting of $n \geq 1$ and $q \geq 0$, and in so doing also extend and refine the results of Shi as well as those of Mateljevic and Pavlovic.

The paper is organized as follows. In Sections 2 and 3, we introduce some relevant spaces of holomorphic functions on U^n and on B, and establish several preliminary results which will be needed in the remaining parts of the paper. In particular, we discuss the spaces $A^p_q = A^p_q(B), q \geq 0, 0 < p \leq \infty$ and the crucial generalized mean Lipschitz space $\mathcal{L}(U^n)$. In Section 4, we use duality arguments to identify the spaces $(A^p_q, H^\infty(U^n))_q$ (see Theorems 4.1, 4.2, 4.3 and 4.4). A significant refinement of these identifications, when $p = 1$, occurs in Section 5 which contains the main results of this paper (Theorems 5.1, 5.2, 5.6 and 5.7). In particular, we show that $\mathcal{B}(B) = (A^1_q, \mathcal{L}(U^n))_q = (A^1_q, \mathrm{BMOA}(U^n))_q$ which implies the result of Mateljevic and Pavlovic [**5**], as a special case, when $n = 1$ and $q = 0$.

For two complex-valued functions f and g on a nonvoid set Λ, we use the notation $f \approx g$ on Λ to mean that there exists a positive constant c so that $c^{-1}|g(\lambda)| \leq |f(\lambda)| \leq c|g(\lambda)|$ for every $\lambda \in \Lambda$.

2. Prerequisites and preliminaries. To deal efficiently with Hadamard products, it is convenient to introduce some further notation and recall some function theoretic concepts. We let $dv = dv^{(n)}$ denote the usual volume Lebesgue measure on $\mathbf{C}^n$ and we set $dA = \pi^{-1} dv^{(1)}$. For $u \in \mathbf{C}^n$, we let $D_u = \langle \nabla, \overline{u}\rangle$ and thus

$$D_u = \sum_{j=1}^n u_j \partial_j, \quad \overline{D}_u = \sum_{j=1}^n \overline{u}_j \overline{\partial}_j, \quad \mathcal{R} = D_z.$$

By $H(M_1, M_2)$ we denote the class of all holomorphic mappings from a complex manifold M_1 into another complex manifold M_2. Let Ω

be a bounded domain in $\mathbb{C}^n$, and let $u \in \mathbb{C}^n$. The *Kobayashi-Royden* metric $F_K^\Omega(z,u)$ for Ω at $z \in \Omega$ in the direction of u is defined by

$$F_K^\Omega(z,u) = \inf\{|\alpha| : \alpha \in \mathbb{C} \setminus \{0\}, \qquad \alpha f'(0) = u,\ f \in H(U,\Omega),\ f(0) = z\}.$$

In particular, F_K^Ω is holomorphic decreasing, i.e. if $\phi \in H(\Omega, \Omega_1)$ where Ω_1 is a domain in $\mathbb{C}^m$, then for any $z \in \Omega$ and any $u \in \mathbb{C}^n$,

$$F_K^{\Omega_1}(\phi(z), \phi_*(u)) \leq F_K^\Omega(z,u)$$

where $\phi_*(u) = (D_u\phi)(z) \in \mathbb{C}^m$. Let $f \in H(\Omega)$ and $z \in \Omega$. We define

$$(Q_\Omega f)(z) = \max\{(Q_\Omega f)(z,u) : u \in \mathbb{C}^n \setminus \{0\}\},$$

where

$$(Q_\Omega f)(z,u) = \frac{|(D_u f)(z)|}{F_K^\Omega(z,u)}, \quad u \in \mathbb{C}^n \setminus \{0\}.$$

The *Bloch-norm* $\|f\|_{\mathcal{B}(\Omega)}$ of $f \in H(\Omega)$ is defined by

$$\|f\|_{\mathcal{B}(\Omega)} = \sup\{(Q_\Omega f)(z) : z \in \Omega\}.$$

The *Bloch-space* $\mathcal{B}(\Omega)$ of Ω is defined as $\mathcal{B}(\Omega) = \{f \in H(\Omega) : \|f\|_{\mathcal{B}(\Omega)} < \infty\}$. In particular, $(\mathcal{B}(\Omega), \|\cdot\|_{\mathcal{B}(\Omega)})$ is a Banach space, provided constant functions are identified with zero. The *small Bloch space* $\mathcal{B}_0(\Omega) = \{f \in \mathcal{B}(\Omega) : \lim_{z\to\partial\Omega}(Q_\Omega f)(z) = 0\}$ is a closed subspace of $\mathcal{B}(\Omega)$.

PROPOSITION 2.1. *For $\phi \in H(\Omega, \Omega_1)$ and $f \in H(\Omega_1)$, $f \circ \phi \in H(\Omega)$ with $(Q_\Omega f \circ \phi)(z) \leq (Q_{\Omega_1} f)(\phi(z))$ for each $z \in \Omega$, and*

$$\|f \circ \phi\|_{\mathcal{B}(\Omega)} \leq \|f\|_{\mathcal{B}(\Omega_1)}.$$

In particular, $f \circ \phi \in \mathcal{B}(\Omega)$ whenever $f \in \mathcal{B}(\Omega_1)$.

Proof. Let $z \in \Omega$ and $u \in \mathbb{C}^n \setminus \{0\}$. Then

$$(Q_\Omega f \circ \phi)(z,u) = \frac{|D_u f \circ \phi)(z)|}{F_K^\Omega(z,u)} = \frac{|(D_{\phi_*(u)} f)(\phi(z))|}{F_K^\Omega(z,u)}.$$

If $\phi_*(u) = 0$, then $(Q_\Omega f \circ \phi)(z,u) = 0 \leq (Q_{\Omega_1} f)(\phi(z))$. If $\phi_*(u) \neq 0$, then

$$(Q_\Omega f \circ \phi)(z,u) \leq \frac{|(D_{\phi_*(u)} f)(\phi(z))|}{F_K^{\Omega_1}(\phi(z), \phi_*(u))} \leq (Q_{\Omega_1} f)(\phi(z)).$$

Thus

$$(Q_\Omega f \circ \phi)(z) \leq (Q_{\Omega_1} f)(\phi(z))$$

as desired. Moreover,

$$\begin{aligned}\|f \circ \phi\|_{\mathcal{B}(\Omega)} &= \sup\{(Q_\Omega f \circ \phi)(z) : z \in \Omega\} \\ &\leq \sup\{(Q_{\Omega_1} f)(\phi(z)) : z \in \Omega\} \\ &\leq \sup\{(Q_{\Omega_1} f)(w) : w \in \Omega_1\} \\ &= \|f\|_{\mathcal{B}(\Omega_1)},\end{aligned}$$

and the proof is complete. $\square$

Let Ω be a complete Rheinhardt domain in $\mathbb{C}^n$, and let f be a function on Ω. We define $f^*(z) = \overline{f(\overline{z}))}$ and $f_\xi(z) = f(\xi \cdot z)$ for any $\xi \in \overline{U}^n$. If λ is a scalar with $|\lambda| \leq 1$, we also write f_λ for $f_{\lambda \mathbf{1}}$ where $\mathbf{1} = (1, \cdots, 1) \in T^n$. Clearly, f^* and f_ξ, $\xi \in \overline{U}^n$, are functions on Ω such that f^*, $f_\xi \in H(\Omega)$ if also $f \in H(\Omega)$. It is also clear that $f_\xi \in H(\overline{\Omega})$ for each $\xi \in U^n$ whenever $f \in H(\Omega)$. Moreover, we have:

PROPOSITION 2.2. *Let Ω be a bounded complete Rheinhardt domain in $\mathbb{C}^n$, and let $f \in \mathcal{B}(\Omega)$. Then, f^* and f_ξ, $\xi \in \overline{U}^n$, are in $\mathcal{B}(\Omega)$ with $\|f^*\|_{\mathcal{B}(\Omega)} = \|f\|_{\mathcal{B}(\Omega)}$ and $\|f_\xi\|_{\mathcal{B}(\Omega)} \leq \|f\|_{\mathcal{B}(\Omega)}$.*

Proof. The assertion concerning f^* is obvious from the definition of $\|\cdot\|_{\mathcal{B}(\Omega)}$. The assertion concerning f_ξ follows from Proposition 2.1 by taking $\phi \in H(\Omega, \Omega)$ as $\phi(z) = \xi \cdot z$, $z \in \Omega$, and the proof is complete. $\square$

PROPOSITION 2.3. *Let $f \in \mathcal{B}(B)$ and $\xi \in \overline{B}$. Then $f_\xi \in \mathcal{B}(U^n)$ with $\|f_\xi\|_{\mathcal{B}(U^n)} \leq \|f\|_{\mathcal{B}(B)}$.*

Proof. Let $\phi(z) = \xi \cdot z$, $z \in U^n$. Then $\phi \in H(U^n, B)$. By Proposition 2.1, $f_\xi = f \circ \phi \in \mathcal{B}(U^n)$ with $\|f_\xi\|_{\mathcal{B}(U^n)} = \|f \circ \phi\|_{\mathcal{B}(U^n)} \leq \|f\|_{\mathcal{B}(B)}$, concluding the proof. $\square$

Let $u \in \mathbb{C}^n$. A simple, and well-known computation, shows that

$$F_K^{U^n}(z, u) = \max_{1 \leq j \leq n} \frac{|u_j|}{1 - |z_j|^2}, \quad z \in U^n,$$

and

$$F_K^B(z, u) = \frac{\{(1 - \|z\|^2)\|u\|^2 + |\langle z, u\rangle|^2\}^{1/2}}{1 - \|z\|^2}, \quad z \in B.$$

It follows that for $f \in H(U^n)$,

$$\frac{1}{n}(Q_{U^n} f)(z) \le \max_{1\le j\le n}\{(1-|z_j|^2)|(\partial_j f)(z)|\} \\ \le (Q_{U^n} f)(z), \quad z \in U^n,$$

and so

$$\frac{1}{n}\|f\|_{\mathcal{B}(U^n)} \le \sup_{z\in U^n}\max_{1\le j\le n}\{(1-|z_j|^2)|(\partial_j f)(z)|\} \le \|f\|_{\mathcal{B}(U^n)}.$$

In particular, defining

$$\|f\|_\beta = \sup_{z\in U^n}\sum_{j=1}^{n}(1-|z_j|^2)|f(z)+nz_j\partial_j f(z)|$$

we deduce that

$$\|f\|_\beta \approx |f(0)| + \|f\|_{\mathcal{B}(U^n)}.$$

Accordingly, we may express the Bloch space $\mathcal{B}(U^n)$ as the space $\{f \in H(U^n) : \|f\|_\beta < \infty\}$, in which case $\mathcal{B}(U^n)$ is a Banach space with the norm $\|\cdot\|_\beta$. Moreover, $\mathcal{B}_0(U^n)$ is the $\mathcal{B}(U^n)$-closure of the holomorphic polynomials in $z \in \mathbb{C}^n$.

In a similar fashion (see [**2**]) one also shows that for $f \in H(B)$,

$$\|f\|_{\mathcal{B}(B)} \approx \sup_{z\in B}(1-\|z\|^2)\|(\nabla f)(z)\| \approx \sup_{z\in B}(1-\|z\|^2)|(\mathcal{R}f)(z)|,$$

and thus $\mathcal{B}(B) = \{f \in H(B) : \|f\|_{\mathcal{B}} < \infty\}$ where

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in B}(1-\|z\|^2)\|(\nabla f)(z)\|.$$

In this case $\mathcal{B}(B)$ is a Banach space with the norm $\|\cdot\|_{\mathcal{B}}$, and $\mathcal{B}_0(B)$ is the $\mathcal{B}(B)$-closure of the holomorphic polynomials in $z \in \mathbb{C}^n$.

We let $\delta : Z \to Z_+$ be defined by $\delta(m) = \max(m,0), m \in Z$. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in Z_+^n$, we define $\delta(\alpha) = (\delta(\alpha_1), \cdots, \delta(\alpha_n)) \in Z_+^n$ and $\pi(\alpha) = \prod_{j=1}^{n}\{\alpha_j : \alpha_j \ne 0\}$. In particular, $\pi(0) = 0$ and $\pi(\alpha) = \alpha_1 \cdots \alpha_n$ if $\alpha_j \ne 0$ for all $1 \le j \le n$. We also let $\pi_\alpha(z) = z^\alpha, z \in \mathbb{C}^n$. Let $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, and let $1 \le j \le n$. We let $z_{(j)} = (z_1, \cdots, z_{j-1}) \in \mathbb{C}^{j-1}$ if $j > 1$ and $z^{(j)} = (z_{j+1}, \cdots, z_n) \in \mathbb{C}^{n-j}$ if $j < n$. In particular, $(z_{(j)}, z^{(j)}) \in \mathbb{C}^{n-1}$ if $n > 1$, and if f is a function defined in a $\mathbb{C}^n$-neighborhood $\mathcal{N}(z)$ of z then $f(z_{(j)}, \cdot, z^{(j)})$

is a function defined on $\mathcal{N}_j^n(z) = \{\lambda \in \mathbb{C} : (z_{(j)}, \lambda, z^{(j)}) \in \mathcal{N}^n(z)\}$. Of course, $\mathcal{N}^n(z) = \mathcal{N}_j^n(z)$, $f = f(z_{(j)}, \cdot, z^{(j)}), j = 1$, when $n = 1$.

Let Ω be a complete Rheinhardt domain in $\mathbb{C}^n$, and let $f \in H(\Omega)$ with

$$f(z) = \sum_\alpha a_\alpha z^\alpha \quad (z \in \Omega),$$

where $a_\alpha = a_\alpha(f), \alpha \in Z_+^n$. We define

$$(D^n f)(z) = \sum_\alpha \pi(\alpha) a_\alpha z^{\delta(\alpha-1)}$$

and, for $s \in \mathbb{C}$,

$$(\mathcal{D}^s f)(z) = \sum_\alpha (1 + |\alpha|)^s a_\alpha z^\alpha.$$

Note that D^n is a differential operator of order n with $D^1 = d/dz$ and that $\mathcal{D}^m = (1+\mathcal{R})^m$ for any $m \in Z$. Moreover, if $\sigma = \operatorname{Re}(s) > 0$, then

$$(\mathcal{D}^{-s} f)(z) = \frac{1}{\Gamma(\sigma)} \int_0^\infty f(e^{-t} z) e^{-t} t^{s-1} dt.$$

By $H^\infty(U^n)$ we denote the Hardy space $H^\infty(U^n) = \{f \in H(U^n) : \|f\|_\infty < \infty\}$, where $\|f\|_\infty = \sup\{|f(z)| : z \in U^n\}$. This is, of course, a Banach space with the norm $\|\cdot\|_\infty$. Another Banach space that we shall consider is the space $\mathrm{BMOA}(U^n) = \{f \in H(U^n) : \|f\|_* < \infty\}$, the space of functions in $H(U^n)$ with *bounded mean oscillation* on the torus T^n (see [4, p. 238] when $n = 1$) normed by $\|f\|_* = (|f(0)|^2 + \|f\|^2_{\mathrm{BMOA}})^{1/2}$, $f \in H(U^n)$, where

$$\|f\|^2_{\mathrm{BMOA}} = \sup_{\xi \in U^n} \int_{U^n} |(D^n f)(z)|^2 \left(\prod_{j=1}^n \frac{(1 - |z_j|^2)(1 - |\xi_j|^2)}{|1 - z_j \bar{\xi}_j|^2} \right) \cdot \frac{dv(z)}{\pi^n}.$$

The closure in $\mathrm{BMOA}(U^n)$ of the holomorphic polynomials in $z \in \mathbb{C}^n$ gives rise to the space of functions in $H(U^n)$ with vanishing mean oscillation on T^n and is denoted by $\mathrm{VMOA}(U^n)$.

To proceed, we introduce two new sectional subspaces of $\mathrm{BMOA}(U^n)$ and $\mathrm{VMOA}(U^n)$. For $f \in H(U^n)$, we define

$$\|f\|^2_{SB} = |f(0)|^2 + \sum_{j=1}^n \sup_{(z_{(j)}, z^{(j)}) \in U^{n-1}} \sup_{\tau \in U} \int_U |\partial_j f(z)|^2 \frac{(1 - |z_j|^2)(1 - |\tau|^2)}{|1 - \bar{\tau} z_j|^2} dA(z_j)$$

and we set $SB(U^n) = \{f \in H(U^n) : \|f\|_{SB} < \infty\}$. The closure in $SB(U^n)$ of the holomorphic polynomials in $z \in \mathbb{C}^n$ is denoted by $SB_0(U^n)$. It is clear that $SB(U^n)$ and $SB_0(U^n)$ are closed subspaces of BMOA(U^n) and VMOA(U^n), respectively. Moreover, $SB(U^n) =$ BMOA(U^n) and $SB_0(U^n) =$ VMOA(U^n) when $n = 1$. Next, for a continuous function f on U^n, $0 \le r < 1$, and $(z_{(j)}, z^{(j)}) \in U^{n-1}$, $1 \le j \le n$, we define

$$M_\infty\left(r, (z_{(j)}, z^{(j)}) : f\right) = \sup\left\{|f(z_{(j)}, r\lambda, z^{(j)})| : \lambda \in T\right\}.$$

We now consider a *generalized mean Lipschitz space* $\mathcal{L}(U^n) = \{f \in H(U^n) : \|f\|_{\mathcal{L}} < \infty\}$ where

$$\|f\|_{\mathcal{L}}^2 = |f(0)|^2 + \sum_{j=1}^{n} \sup_{(z_{(j)}, z^{(j)}) \in U^{n-1}} \int_0^1 M_\infty^2(r, (z_{(j)}, z^{(j)}) : \partial_j f)(1 - r^2)2r dr.$$

Equipped with the norm $\|\cdot\|_{\mathcal{L}}$, $\mathcal{L}(U^n)$ becomes a Banach space. Moreover, using the subharmonicity of $|\partial_j f|, f \in H(U^n)$, on U^n and the Lebesgue dominated convergence theorem, one shows easily that the holomorphic polynomials in $z \in \mathbb{C}^n$ are dense in $\mathcal{L}(U^n)$. Also, as is well-known, $H^\infty(U^n) \subset SB(U^n) \subset$ BMOA$(U^n) \subset \mathcal{B}(U^n)$ and $SB_0(U^n) \subset$ VMOA$(U^n) \subset \mathcal{B}_0(U^n)$, with the inclusions being continuous. We now prove:

PROPOSITION 2.4. *The space $\mathcal{L}(U^n)$ is continuously contained in $SB_0(U^n)$, with $\|f\|_{SB} \le \|f\|_{\mathcal{L}}$ for each $f \in H(U^n)$.*

Proof. Let $f \in H(U^n)$, $z \in U^n, \tau \in U$ and $1 \le j \le n$. Then, using polar coordinates

$$\begin{aligned}
&\int_U |\partial_j f(z)|^2 \frac{(1 - |z_j|^2)}{|1 - \overline{\tau} z_j|^2} dA(z_j) \\
&= \frac{1}{\pi} \int_0^1 \left(\int_T \frac{\left|\partial_j f\left(z_{(j)}, r\lambda, z^{(j)}\right)\right|^2}{|1 - r\lambda\overline{\tau}|^2} |d\lambda| \right) (1 - r^2) r dr \\
&\le \frac{1}{\pi} \int_0^1 M_\infty^2\left(r, \left(z_{(j)}, z^{(j)}\right) : \partial_j f\right) \left(\int_T |1 - r\lambda\overline{\tau}|^{-2} |d\lambda| \right) (1 - r^2) r dr \\
&= \int_0^1 M_\infty^2\left(r, (z_{(j)}, z^{(j)}\right) : \partial_j f)(1 - r^2|\tau|^2)^{-1}(1 - r^2)2r dr,
\end{aligned}$$

and thus

$$\int_U |\partial_j f(z)|^2 \frac{(1-|z_j|^2)(1-|\tau|^2)}{|1-z_j\overline{\tau}|^2} dA(z_j)$$
$$\leq \int_0^1 M_\infty^2 \left(r, \left(z_{(j)}, z^{(j)}\right) : \partial_j f\right) (1-r^2) 2r dr.$$

This implies that $\|f\|_{SB} \leq \|f\|_{\mathcal{L}}$ and hence $\mathcal{L}(U^n) \subset SB(U^n)$. Since $SB_0(U^n)$ is the closure in $SB(U^n)$ of the holomorphic polynomials in $z \in \mathbb{C}^n$ and since these polynomials are dense in $\mathcal{L}(U^n)$, the desired result follows and the proof is complete. □

3. Some prerequisites on the unit ball. Let $d\sigma$ be the normalized surface measure on $S = \partial B$ and let, for $q > 0, dv_q$ stand for the probability measure on $\overline{B}$, defined by

$$dv_q(z) = \frac{1}{\pi^n} \frac{\Gamma(n+q)}{\Gamma(q)} (1 - \|z\|^2)^{q-1} dv(z) \qquad (z \in \overline{B}).$$

As $q \to 0^+$, dv_q tends, in the weak* limit sense, to $d\sigma$ which is also denoted by dv_0. For $0 < p \leq \infty$ and $q \geq 0$, we let L_q^p be the quasi-Banach space $L^p(dv_q)$ with the quasi-norm $\|\cdot\|_{p,q}$, defined by

$$\|f\|_{p,q} = \left\{\int_B |f|^p dv_q\right\}^{1/p}.$$

For two dv_q-measurable functions f and g such that $fg \in L_q^1$, we define

$$\langle f, g\rangle_q = \int f\overline{g} dv_q.$$

Clearly, $\langle\ ,\ \rangle_q$ also serves as the inner product of L_q^2. When $q > 0$, the space $A_q^p = L_q^p \cap H(B)$ is a closed subspace of L_q^p. The limiting space A_0^p is identified in the usual way as the Hardy space $H^p = H^p(B)$ of functions in $H(B)$. In particular, for $f \in H(B)$,

$$\|f\|_{p,0} = \sup_{0<r<1} M_p(r, f)$$

with

$$M_p(r, f) = \|f_r\|_{p,0} = \left\{\int_S |f(rz)|^p dv_0(z)\right\}^{1/p},$$

and $A_0^p = H^p = \{f \in H(B) : \|f\|_{p,0} < \infty\}$ may be identified in usual way as a closed subspace of L_0^p. Note also the identity

$$\|f\|_{p,q} = \left\{\int_0^1 M_p^p(\sqrt{r}, f)d\nu_q(r)\right\}^{1/p}$$
$$= \left\{\int_0^1 \|f_{\sqrt{r}}\|_{p,0}^p d\nu_q(r)\right\}^{1/p} \qquad (f \in H(B))$$

where $d\nu_q(r) = (r^{n-1}(1-r)^{q-1}/B(n,q))dr$ is, for $q > 0$, a probability measure on $(0,1)$. Here $B(n,q) = \Gamma(n)\Gamma(q)/\Gamma(n+q)$ is the usual beta function of n and $q > 0$. The above identity is also correct when $q = 0$; in fact, for $f \in H(B)$, $\lim_{q\to 0^+} \|f\|_{p,q} = \|f\|_{p,0}$.

In this paper, unless stated otherwise, we assume that $q \geq 0$ is fixed. If $f, g \in H(B)$, then $\langle f_r, g_r\rangle_q$ exists for every $0 \leq r < 1$, and we define the *q-pairing* $(f,g)_q$ as $(f,g)_q = \lim_{r\to 1^-}\langle f_r, g_r\rangle_q$, whenever the limit exists. Obviously, $(f,g)_q = \langle f, g\rangle_q$ whenever $fg \in A_q^1$. We let P_q denote the orthogonal projection of L_q^2 onto A_q^2. The latter is a functional Hilbert space on B with the reproducing kernel K_q, given by

$$K_q(z,\xi) = (1 - \langle z, \xi\rangle)^{-(n+q)} \qquad (z, \xi \in B).$$

In particular, for any $z \in B$,

$$(P_q f)(z) = \langle f, K_q(\cdot, z)\rangle_q \qquad (f \in L_q^2)$$

or

$$(P_q f)(z) = \int f(\xi)K_q(z,\xi)dv_q(\xi).$$

As is well-known (see, for example, [**2**]), P_q extends to a continuous projection of L_q^p onto A_q^p, $1 < p < \infty$, with norm $m_q(p)$ satisfying $m_q(2) = 1$ and $m_q(p) = m_q(p')$ where $p' = p/(p-1)$. In particular, the dual, with respect to the q-pairing, of A_q^p is isomorphic to $A_q^{p'}$. We also consider the space $\Gamma_q = \Gamma_q(B)$ of functions f in A_q^2 such that

$$\|f\|_{*,q} = \sup\left\{|\langle f, g\rangle_q| : g \in A_q^2 \text{ with } \|g\|_{1,q} \leq 1\right\}$$

is finite. Evidently, Γ_q is a Banach space of functions in $H(B)$ with the norm $\|\cdot\|_{*,q}$, and it serves as the dual of A_q^1 with respect to the q-pairing. In particular, $A_q^\infty \subset \Gamma_q \subset A_q^p$ for every $0 < p < \infty$, and $f \in \Gamma_q$ if and only if $f \in H(B)$ and there exists a constant $c_f \geq 0$ so that $|(f,g)_q| \leq c_f \|g\|_{1,q}$ for every $g \in H(\overline{B})$. Moreover, one shows easily that $\Gamma_q = P_q(L_q^\infty)$, where for $f \in P_q(L_q^\infty)$, $\|f\|_{*,q} = \min\left\{\|\tilde{f}\|_{\infty,q} : \tilde{f} \in L_q^\infty, P_q(\tilde{f}) = f\right\}$, and thus P_q is a continuous projection of L_q^∞ onto Γ_q with norm 1. As is well-known (see, for example, [**3**]), Γ_0 may also be identified with the space $\mathrm{BMOA}(B)$, the space of functions in $H(B)$ with *(non-isotropic) bounded mean oscilation* on S. Moreover, for $q > 0$, the space Γ_q is independent of q, i.e. $\Gamma_{q_1} = \Gamma_{q_2}$, with equivalent norms, for any $q_1, q_2 > 0$ (see [**2**]). In fact, $\Gamma_q, q > 0$, may be identified with the previously mentioned Bloch space $\mathcal{B}(B)$, and one can show (see [**2**]) that for any $f \in H(B)$

$$\|f\|_{\mathcal{B}} \approx \sup_{z \in B} (1 - \|z\|^2)^s |\mathcal{D}^s f(z)|$$

for each $s > 0$.

We shall also consider the familar *Lipschitz space* $\Lambda_s = \Lambda_s(B)$, $s > 0$, of functions $f \in H(B)$ so that there exists an integer $m > s$ with

$$|||f|||_{m,s} = \sup_{z \in B_n} (1 - \|z\|^2)^{m-s} |\mathcal{D}^m f(z)| < \infty.$$

Evidently, Λ_s is a Banach space of functions in $H(B)$ with norm $||| \cdot |||_{m,s}$ which is independent of the integer $m > s$, provided constant functions are identified with zero. Moreover, using the methods of proofs found in [**2**] one can show that the dual of A_q^p, $0 < p < 1$, $q \geq 0$, with respect to the q-pairing, is isomorphic to $\Lambda_{(n+q)(1/p-1)}$. In particular, for every $q \geq 0$, any of the equivalent norms $||| \cdot |||_{m,s}$ of Λ_s is equivalent to the the norm $\| \cdot \|_{\Lambda_s}$ defined by

$$\|f\|_{\Lambda_s} = \sup\left\{ |\langle f, g\rangle_q| : g \in H(\overline{B}),\ \|g\|_{\frac{n+q}{n+q+s}, q} \leq 1 \right\}.$$

By $C(\overline{B})$ we denote the Banach-algebra of all continuous functions on $\overline{B}$ with the norm $\| \cdot \|_\infty$, and by $A(B)$ we denote the classical ball-algebra $C(\overline{B}) \cap H(B)$. Naturally, $A(B)$ may be identified as a closed subspace of $C(\overline{B})$ as well as of $A_q^\infty = H^\infty$. Moreover, $\Lambda_s(B) \subset A(B)$ for each $s > 0$. We also let $\mathcal{M} = \mathcal{M}(\overline{B})$ be the Banach space of all finite complex measures μ on $\overline{B}$ with the total variation norm $\|\mu\|_\dagger = \int |d\mu|$. For $\mu \in \mathcal{M}$, we define

$$(k_q \mu)(z) = \int K_q(z, \xi) d\mu(\xi) \qquad (z \in B),$$

and we let $\Omega_q = \Omega_q(B) = \{k_q(\mu) : \mu \in \mathcal{M}\}$. It follows that $\Omega_q \subset H(B)$ and that k_q is a linear operator of $\mathcal{M}$ onto Ω_q. We equip Ω_q with the norm

$$\|f\|_{\dagger,q} = \inf\{\|\mu\|_\dagger : \mu \in \mathcal{M}, k_q(\mu) = f\} \qquad (f \in \Omega_q),$$

and thus Ω_q is a Banach space. In particular, k_q is a continuous linear operator of $\mathcal{M}$ onto Ω_q whose norm, since $k_q(dv_q) = 1$, is 1.

Before proceeding with some preliminary results, we make a simple remark concerning $\mu \in \mathcal{M}$. Recall that for any measurable mapping ϕ of $\overline{B}$ into $\overline{B}$, the induced measure $\mu_{(\phi)} = \mu \circ \phi^{-1}$ is the unique measure in $\mathcal{M}$ such that

$$\int (f \circ \phi) d\mu = \int f d\mu_{(\phi)}$$

for all $f \in C(\overline{B})$, and thus, $|\mu(\overline{B})| \leq \|\mu_{(\phi)}\|_\dagger \leq \|\mu\|_\dagger$. In particular, $\|\mu_{(\phi)}\|_\dagger = \|\mu\|_\dagger$ if μ is also non-negative. For $\xi \in \overline{U}^n$, $\mu_{(\xi)}$ denotes the induced measure $\mu_{(\phi)}$ where $\phi(z) = z \cdot \overline{\xi}$, $z \in \overline{B}$. One then verify easily that

$$[k_q(\mu)]_\xi = k_q(\mu_{(\xi)})$$

for every $q \geq 0$.

PROPOSITION 3.1. *Let $f \in H(B)$, $q \geq 0, s > 0$, and $\xi \in \overline{U}^n$. Then*

(i) $\|f_\xi\|_{p,q} \leq \|f\|_{p,q}$ *for any* $0 < p \leq \infty$*;*

(ii) $\|f_\xi\|_{*,q} \leq \|f\|_{*,q}$*;*

(iii) $\|f_\xi\|_{\Lambda_s} \leq \|f\|_{\Lambda_s}$*;*

(iv) *If also $f \in \Omega_q$, then $f_\xi \in \Omega_q$ with $\|f_\xi\|_{\dagger,q} \leq \|f\|_{\dagger,q}$;*

(v) *If also $f \in A^p_q$, $0 < p < \infty$, then*

$$\lim_{\xi \to 1} \|f_\xi - f\|_{p,q} = 0.$$

Proof. To prove (i), we observe that the case $p = \infty$ is trivial, and so we assume that $0 < p < \infty$. Let $\xi = (r_1 e^{i\theta_1}, \cdots, r_n e^{i\theta_n}), \theta_j \in R, r_j \geq 0$ with $r = \max\{r_1, \cdots, r_n\} \leq 1$. It follows from the subharmornicity of $|f|^p$ on B that

$$\begin{aligned}
\|f_\xi\|_{p,q}^p &= \int |f(\xi \cdot z)|^p dv_q(z) \\
&= \int |f(r_1 z_1 e^{i\theta_1}, \cdots, r_n z_n e^{i\theta_n})|^p dv_q(z) \\
&= \int |f(r_1 z_1, \cdots, r_n z_n)|^p dv_q(z) \\
&= \int_0^1 (\int_S |f(\sqrt{\eta} r_1 z_1, \cdots, \sqrt{\eta} r_n z_n)|^p dv_0(z)) d\nu_q(\eta) \\
&\leq \int_0^1 (\int_S |f(\sqrt{\eta} z)|^p dv_0(z)) d\nu_q(\eta) \\
&= \int_0^1 M_p^p(\sqrt{\eta}, f) d\nu_q(\eta) \\
&= \|f\|_{p,q}^p,
\end{aligned}$$

which is the desired result. To prove (ii), we may, of course, assume that $f \in \Gamma_q$. In particular, $f \in A_q^1$ and by (i) for $p = 1$, also, $f_\xi \in A_q^1$. For $g \in H(\overline{B})$, $\langle f_\xi, g \rangle_q = \langle f, g_\xi \rangle_q$, and thus

$$\|f_\xi\|_{*,q} = \sup \left\{ |\langle f, g_\xi \rangle_q| : g \in H(\overline{B}), \|g\|_{1,q} \leq 1 \right\}.$$

But, by definition, and (i) with $p = 1$,

$$|\langle f, g_\xi \rangle_q| \leq \|f\|_{*,q} \|g_\xi\|_{1,q} \leq \|f\|_{*,q} \|g\|_{1,q}.$$

It follows that $\|f_\xi\|_{*,q} \leq \|f\|_{*,q}$ as desired. To prove (iii), we may assume that $f \in \Lambda_s$. In particular $f \in A(B)$ and so $\langle f_\xi, g \rangle_0 = \langle f, g_\xi \rangle_0$ for any $g \in H(\overline{B})$. It follows that

$$\|f_\xi\|_{\Lambda_s} = \sup \left\{ |\langle f, g_\xi \rangle_0| : g \in H(\overline{B}), \|g\|_{n/(n+s),0} \leq 1 \right\},$$

and hence, by definition, and (i) with $p = n/(n+s), q = 0$,

$$|\langle f, g_\xi \rangle_0| \leq \|f\|_{\Lambda_s} \|g_\xi\|_{n/(n+s),0}.$$

This means that $\|f_\xi\|_{\Lambda_s} \leq \|f\|_{\Lambda_s}$ and (iii) is proved.

To prove (iv), we assume that $f \in \Omega_q$. It follows that $f = k_q(\mu)$ for some $\mu \in \mathcal{M}$, and thus, by the preceeding remark, $f_\xi =$

$[k_q(\mu)]_\xi = k_q(\mu_{(\xi)}) \in \Omega_q$. Moreover, again by the preceeding remark, we have

$$\begin{aligned}
\|f\|_{\dagger,q} &= \inf\{\|\mu\|_\dagger : \mu \in \mathcal{M}, f = k_q(\mu)\} \\
&\geq \inf\{\|\mu\|_\dagger : \mu \in \mathcal{M}, f_\xi = [k_q(\mu)]_\xi\} \\
&= \inf\left\{\|\mu\|_\dagger : \mu \in \mathcal{M}, f_\xi = k_q(\mu_{(\xi)})\right\} \\
&\geq \inf\left\{\|\mu_{(\xi)}\|_\dagger : \mu \in \mathcal{M}, f_\xi = k_q(\mu_{(\xi)})\right\} \\
&\geq \inf\{\|\nu\|_\dagger : \nu \in \mathcal{M}, f_\xi = k_q(\nu)\} \\
&= \|f_\xi\|_{\dagger,q},
\end{aligned}$$

and the proof of (iv) is complete.

Finally, we prove (v). If $f \in A^p_q$, $0 < p < \infty$, and $\xi \in \overline{U}^n$ then, as $\xi \to \mathbf{1}$, $f_\xi \to f$ almost everywhere on $\overline{B}$ (pointwise on B and, when $q = 0$, also almost everywhere on S), and so, by Fatou's lemma, $\|f\|_{p,q} \leq \underline{\lim}_{\xi\to\mathbf{1}}\|f_\xi\|_{p,q}$. But, by (i), $\overline{\lim}_{\xi\to\mathbf{1}}\|f_\xi\|_{p,q} \leq \|f\|_{p,q}$ and thus $\lim_{\xi\to\mathbf{1}}\|f_\xi\|_{p,q} = \|f\|_{p,q}$. It follows from a well-known stronger version of the Lebesgue dominated convergence theorem that $\lim_{\xi\to\mathbf{1}}\|f_\xi - f\|_{p,q} = 0$, and the proof is complete. □

PROPOSITION 3.2. *For $f, g \in H(B)$ and any $0 \leq r < 1$, we have*

$$(f * g)_q(rz) = \langle f_r, g^*_{\bar{z}}\rangle_q = \int f(r\xi)g(z\cdot\overline{\xi})dv_q(\xi)$$

for every $z \in U^n$. In particular, $(f * g)_q \in H(U^n)$.

Proof. Since $0 \leq r < 1$ and $z \in U^n$, f_r and $g^*_{\bar{z}}$ are in $H(\overline{B})$, and hence the above integral is absolutely convergent. In particular,

$$f(r\xi)g(z\cdot\overline{\xi}) = \sum_{\alpha,\beta} a_\alpha(f)a_\beta(g)r^{|\alpha|}\xi^\alpha\overline{\xi}^\beta z^\beta,$$

and term by term integration with respect to dv_q is allowed. Thus, using the orthogonality of the monomials π_α, $\alpha \in \mathbb{Z}^n_+$, and the fact that

$$\omega_\alpha(q) = \langle\pi_\alpha, \pi_\alpha\rangle_q = \int \xi^\alpha\overline{\xi}^\alpha dv_q(\xi)$$

we conclude that

$$\int f(r\xi)g(z\cdot\overline{\xi})dv_q(\xi) = \sum_\alpha a_\alpha(f)a_\alpha(g)(rz)^\alpha = (f * g)_q(rz),$$

and the proof is complete. □

PROPOSITION 3.3. *Let $1 \leq p, s, t \leq \infty$ with $1 + t^{-1} = p^{-1} + s^{-1}$, and let $f, g \in H(B)$. Then $(f * g)_q \in H(U^n)$ with $\|(f * g)_q\|_{t,q} \leq \|f\|_{p,q}\|g\|_{s,q}$. Moreover, if also $t = \infty$, i.e. if $s = p' = p/(p-1)$, then $\|(f * g)_q\|_\infty \leq \|f\|_{p,q}\|g\|_{p',q}$. In particular, $(f * g)_q \in H^\infty(U^n)$ if $f \in A_q^p$ and $g \in A_q^{p'}$.*

Proof. That $(f * g)$ is in $H(U^n)$ is a special case of Proposition 3.2. To prove the inequality, we let $0 \leq r < 1$ and $z \in U^n$. We also let $a = t, b^{-1} = p^{-1} - t^{-1}$ and $c^{-1} = s^{-1} - t^{-1}$. Then $1 \leq a, b, c \leq \infty$ with $a^{-1} + b^{-1} + c^{-1} = 1$. By Proposition 3.2 and Hölder's inequality

$$\begin{aligned} |(f * g)_q(rz)| &\leq \int |f(r\xi)||g(z \cdot \bar{\xi})| dv_q(\xi) \\ &= \int |f(r\xi)|^{p(1/a+1/b)} |g(z \cdot \bar{\xi})|^{s(1/a+1/c)} dv_q(\xi) \\ &\leq \left(\int |g(z \cdot \bar{\xi})|^s |f(r\xi)|^p dv_q(\xi) \right)^{1/a} \\ &\quad \cdot \left(\int |f(r\xi)|^p dv_q(\xi) \right)^{1/b} \left(\int |g(z \cdot \bar{\xi})|^s dv_q(\xi) \right)^{1/c}. \end{aligned}$$

If $t < \infty$, then by Proposition 3.1(i),

$$|(f * g)_q(rz)|^t \leq \|f\|_{p,q}^{pa/b} \|g\|_{s,q}^{sa/c} \left(\int |g(z \cdot \bar{\xi})|^s |f(r\xi)|^p dv_q(\xi) \right).$$

It follows from Fubini's theorem and Proposition 3.1 that

$$\|(f * g)_{q,r}\|_{t,q}^t \leq \|f\|_{p,q}^t \|g\|_{s,q}^t,$$

and since this is true for any $0 \leq r < 1$, the desired result follows when $t < \infty$.

When $t = \infty$, we find, using Proposition 3.1(i), that $|(f * g)_q(rz)| \leq \|f\|_{p,q}\|g\|_{p',q}$ and so $\|(f * g)_q\|_\infty \leq \|f\|_{p,q}\|g\|_{p',q}$. This concludes the proof. □

LEMMA 3.4. *Let $g \in H(B)$ and define $T_g(f) = (f * g)_q$. Then T_g is a linear operator of $H(B)$ into $H(U^n)$. Let $0 < p \leq \infty$ and let $Y = Y(U^n)$ be a functional quasi-Banach space of functions in*

$H(U^n)$, and assume that $T_g(A_q^p) \subset Y$. Then T_g is a continuous linear operator of A_q^p into Y, i.e. there exists a non-negative constant $c_g = c_g(A_q^p, Y)$ so that $\|(f * g)_q\|_Y \le c_g \|f\|_{p,q}$ for every $f \in A_q^p$.

Proof. The fact that T_g is a linear operator from $H(B)$ into $H(U^n)$ is a trivial consequence of Proposition 3.3. To prove the remaining part of the lemma, we assume that T_g is a linear operator of A_q^p into $Y = Y(U^n)$ and we shall show that T_g is closed. To this end we assume that $f_k \to f$ in A_q^p and $h_k = T_g(f_k) \to h$ in Y, and hence we must show that $T_g(f) = h$. Fix $z \in U^n$. By assumption, $h_k(z) \to h(z)$. Similarly, by Proposition 3.3, $|((f - f_k) * g)_q(z)| \le \|f_k - f\|_{p,q}\|g\|_{p',q} \to 0$, $p' = p/(p-1)$. It follows that

$$
\begin{aligned}
&|\{T_g(f)\}(z) - h(z)| \\
&\qquad\qquad \le |((f - f_k) * g)_q(z)| + |h_k(z) - h(z)| \to 0,
\end{aligned}
$$

and hence $T_g(f) = h$, and the proof is complete. □

To proceed, we introduce two special holomorphic functions on the unit disk U. For $a \in \mathbb{C}$ and $m \in \mathbb{Z}_+$, $(a)_m$ stands for 1 if $m = 0$ and $a(a+1)\cdots(a+m-1)$ if $m > 0$. Let $a, b, c \in \mathbb{C}, \lambda \in U$ and define

$$F(a, b; c : \lambda) = \sum_{m+0}^{\infty} \frac{(a)_m (b)_m}{m!(c)_m} \lambda^m,$$

and

$$G_{a,b}(\lambda) = \sum_{m=0}^{\infty} (m+1)^{-b} \frac{(a)_m}{m!} \lambda^m.$$

As is well-known the hypergeometic function $F(a, b; c : \cdot)$ satisfies the *Gauss formula* $F(a, b : c : \lambda) = (1-\lambda)^{c-a-b)} F(c-a, c-b; c : \lambda)$. Moreover, if $\mathrm{Re}(c) > \max(0, \mathrm{Re}(a), \mathrm{Re}(c))$ then we have the *Gauss theorem*, namely $F(a, b; c : 1) = \Gamma(c)\Gamma(c-a-b)/\Gamma(c-a)\Gamma(c-b)$.

For $a \in R, q \ge 0$ and $z \in B$, we define

$$I_{a,q}(z) = \int |1 - \langle z, \xi \rangle|^{-a} dv_q(\xi).$$

It follows that

$$I_{a,q}(z) = F(a/2, a/2; n + q : \|z\|^2),$$

and thus $I_{a,q}(z) = (1 - \|z\|^2)^{n+q-a} I_{2(n+q)-a,q}(z)$. Moreover, for $z \in B, I_{a,q}(z) \geq I_{a,q}(0) = 1$ and , $I_{a,q}(z) \leq 2^{-a}$ if $a \leq 0$, and $I_{a,q}(z) \leq I_{a,q}(1) = \Gamma(n+q)\Gamma(n+q-a)/\{\Gamma(n+q-a/2)\}^2$ if $n+q > a > 0$. These arguments establish the first two parts of the next proposition. The proof of the third part may be found in [**2**].

PROPOSITION 3.5. *Let $a, b \in R$ and $q \geq 0$.*

(i) *If $a < n+q$, then $I_{a,q}(z) \approx 1$, $z \in B$, with $1 \leq I_{a,q}(z) \leq 2^{-a}$ if $a \leq 0$ and*

$$1 \leq I_{a,q}(z) \leq \frac{\Gamma(n+q)\Gamma(n+q-a)}{\{\Gamma(n+q-a/2)\}^2}$$

when $a > 0$;

(ii) *If $a > n+q$, then $I_{a,q}(z) \approx (1 - \|z\|^2)^{-(a-n-q)}, z \in B$, with*

$$1 \leq (1 - |z|^2)^{a-n-q} I_{a,q}(z) \leq \frac{\Gamma(n+q)\Gamma(a-n-q)}{\{\Gamma(a/2)\}^2};$$

(iii) *If $a > b$, then $G_{a,b}(\lambda) = (1-\lambda)^{-(a-b)}F(\lambda)$, $\lambda \in U$, where F is in the Lipschitz class $\Lambda_{a-b}(U)$ with $F(0) = 1$.*

PROPOSITION 3.6. *Let $f \in H(B)$, $0 < p < \infty$, $q \geq 0$, and $\xi \in B$. Then:*

(i) *$|f(\xi)| \leq \int |f(z)|^p |K_q(z,\xi)|^2 K_q(\xi,\xi)^{-1} dv_q(z)$ with equality if and only if f is constant on B;*

(ii) *$|f(\xi)| \leq \{K_q(\xi,\xi)\}^{1/p} \|f\|_{p,q}$ with equality if and only if $f = \lambda \{K_q(\cdot,\xi)\}^{2/p}$ for some constant $\lambda \in \mathbb{C}$.*

Proof. We first prove (i) when $\xi = 0$. In this case, since $|f|^p$ is subharmonic on B, we have

$$|f(0)|^p \leq \int_S |f(\sqrt{r}\eta)|^p dv_0(\eta) = \|f_{\sqrt{r}}\|_{p,0}^p$$

for any $0 \leq r < 1$, with equality if and only if f is constant on B. This gives the desired result when $q = 0$. When $q > 0$, we integrate both sides of the above inequality with respect to the probability measure $d\nu_q(r)$, $0 \leq r < 1$. This gives

$$|f(0)|^p \leq \int |f(\eta)|^p dv_q(\eta)$$

with equality if and only if f is constant on B. This establishes (i) when $\xi = 0$. For any other $\xi \in B$, we replace f by $f \circ \varphi$ where φ is a holomorphic automorphism of B with $\varphi(0) = \xi$, and thus

$$|f(\xi)|^p \leq \int |f(\varphi(\eta))|^p dv_q(\eta)$$

with equality if and only if f is constant on B. This proves (i) by observing that

$$K_q(\varphi(0), \varphi(0)) dv_q(\eta) = |K_q(\varphi(\eta), \varphi(0))|^2 dv_q(\varphi(\eta))$$

for each $\eta \in \overline{B}$.

Finally, (ii) follows from (i) by replacing f with $f\{K_q(\cdot, \xi)\}^{-2/p}$, and the proof is complete. □

We shall also need the following identity for $H^p = H^p(B)$ functions. Its proof appears in [**1**].

PROPOSITION 3.7. *Let* $0 < p < \infty$ *and* $f \in H(B)$. *Then*

$$\|f\|_{p,0}^p - |f(0)|^p = \frac{p^2}{2n} \int_B |\mathcal{R} f(z)|^2 |f(z)|^{p-2} \|z\|^{-2n} \log \frac{1}{\|z\|} dv_1(z).$$

This proposition leads to the following lemma:

LEMMA 3.8. *Let* $0 < p \leq 2$ *and* $f \in H(B)$. *Then*

$$p^2 \int_0^1 \frac{1}{r}(1-r) M_p^2(r, \mathcal{R} f) dr \leq \|f - f(0)\|_{p,0}^2.$$

Proof. Let $g = f - f(0)$ and use Proposition 3.7. This gives

$$\|g\|_{p,0}^p = p^2 \int_0^1 \frac{1}{r} \log \frac{1}{r} \left(\int_S |\frac{\mathcal{R} g(r\xi)}{g(r\xi)}|^2 |g(r\xi)|^p d\sigma(\xi) \right) dr.$$

By Hölder's inequality, with $2/p \geq 1$, and Proposition 3.1(i)

$$\begin{aligned}
&\int_S |\mathcal{R}g(r\xi)|^p d\sigma(\xi) \\
&= \int_S \left|\frac{\mathcal{R}g(r\xi)}{g(r\xi)}\right|^p |g(r\xi)|^p d\sigma(\xi) \\
&\leq \left\{\int_S |\frac{\mathcal{R}g(r\xi)}{g(r\xi)}|^2 |g(r\xi)|^p d\sigma(\xi)\right\}^{p/2} \|g_r\|_{p,0}^{p(2-p)/2} \\
&\leq \left\{\|g\|_{p,0}^{2-p} \int_S |\frac{\mathcal{R}g(r\xi)}{g(r\xi)}|^2 |g(r\xi)|^p d\sigma(\xi)\right\}^{p/2}.
\end{aligned}$$

It follows that

$$\|g\|_{p,0}^2 \geq p^2 \int_0^1 \frac{1}{r} \log \frac{1}{r} M_p^2(r, \mathcal{R}g)dr \geq p^2 \int_0^1 \frac{1}{r}(1-r)M_p^2(r, \mathcal{R}g)dr,$$

and the proof is complete. □

In the next section we shall show, by duality methods, that $(A_q^p, H^\infty(U^n))_q = \Lambda_{(n+q)(1/p-1)}$, $0 < p < 1$; $(A_q^p, H^\infty(U^n))_q = A_q^{p/(p-1)}$, $1 < p < \infty$; $(A_q^\infty, H^\infty(U^n))_q = \Omega_q$ and $(A_q^1, H^\infty(U^n))_q = \Gamma_q$. The latter result means that $(A_0^1, H^\infty(U^n))_0 = \mathrm{BMOA}(B)$ and $(A_q^1, H^\infty(U^n))_q = \mathcal{B}(B)$ for every $q > 0$. A significant refinement, and an extension in the case $q > 0$, is provided in Section 5. There it will be shown that, in fact, for any $q \geq 0$

$$\begin{aligned}
\mathcal{B}(B) &= \left(A_q^1, \mathcal{L}(U^n)\right)_q = \left(A_q^1, \mathrm{VMOA}(U^n)\right)_q \\
&= \left(A_q^1, \mathrm{BMOA}(U^n)\right)_q = \left(A_q^1, \mathcal{B}_0(U^n)\right)_q = \left(A_q^1, \mathcal{B}(U^n)\right)_q.
\end{aligned}$$

In particular,

$$\mathcal{B}(U) = (H^1(U), \mathcal{B}(U))_0 = (H^1(U), \mathcal{B}_0(U))_0 = (H^1(U), \mathrm{VMOA}(U))_0$$

when $n = 1$ and $q = 0$, which is the result of Mateljevic and Pavlovic [**5**], mentioned in the introduction.

We conclude this section by establishing the following inclusion relationships.

PROPOSITION 3.9. *Let $s > q$ and $1 \leq p < (n+s)/(n+q)$. Then $A_q^1 \subset \Omega_q \subset A_s^p$, and the inclusions are continuous.*

Proof. To prove $A_q^1 \subset \Omega_q$, we assume $f \in A_q^1$ and $z \in B$. Since $g(z) = \langle g, K_q(\cdot, z)\rangle_q$ for every $g \in A_q^2$, a simple density argument shows that also $f(z) = \langle f, K_q(\cdot, z)\rangle_q$. In particular, $f = k_q(\mu)$ where $d\mu = f dv_q$, and thus $f \in \Omega$ with $\|f\|_{\dagger,q} \le \|f\|_{1,q}$, which is the desired result. To prove the inclusion $\Omega_q \subset A^p_s$, we assume $\mu \in \mathcal{M}$ and use Hölder's inequality and Fubini's theorem to obtain

$$\begin{aligned}\|k_q(\mu)\|^p_{p,s} &= \int |(k_q(\mu)(z)|^p dv_s(z)\\ &\le \|\mu\|^{p-1}_\dagger \int \left\{\int |K_q(z,\xi)|^p dv_s(z)\right\} |d\mu(\xi)|\\ &= \|\mu\|^{p-1}_\dagger \int I_{(n+q)p,s}(\xi)|d\mu(\xi)|.\end{aligned}$$

Since $n + s > (n+q)p$, Proposition 3.5(i) shows that

$$I_{(n+q)p,s}(\xi) \le c^p,$$

where $c = c(n,p,q,s) = \left\{\frac{\Gamma(n+s)\Gamma(n+s-(n+q)p)}{\Gamma^2(n+s-(n+q)p/2)}\right\}^{1/p}$, for every $\xi \in \overline{B}$. It follows that $\|k_q(\mu)\|_{p,s} \le c\|\mu\|_\dagger$, and the proof is complete. □

Proposition 3.10. *Let* $0 < p \le \infty$, $0 < s \le \min(p,1)$, $0 \le r < 1$ *and* $f, g \in H(B)$. *Then*

$$\|(f * g)_{q,r}\|_{p,q} \le (1-r)^{-(n+q)(1-s)/s}\|f\|_{p,q}\|g\|_{s,q}.$$

In particular, $A^1_q \subset (A^p_q, A^p_q)_q$ *whenever* $1 \le p \le \infty$.

Proof. Let $h = (f * g)_q$ and $\xi \in U^n$. By Proposition 3.2,

$$h(r^2\xi) = \int f(r\xi \cdot z) g(r\overline{z}) dv_q(z),$$

and thus

$$|h(r^2\xi)| \le \|F_r\|_{1,q}$$

where

$$F(z) = f(z\cdot\xi)\overline{g(\overline{z})}, \quad z \in B,$$

i.e. $F = f_\xi g^*$. Since $F \in H(B)$ we have, using Proposition 3.6(ii),

$$\|F_r\|_\infty \le (1-r^2)^{-(n+q)/s}\|F\|_{s,q},$$

and so by Proposition 3.1(i), since $0 < s \le 1$,

$$\begin{aligned}\|F_r\|_{1,q} &= \int |F_r|^{1-s}|F_r|^s dv_q\\ &\le \|F_r\|^{1-s}_\infty \|F_r\|^s_{s,q}\\ &\le \left(1-r^2\right)^{-(n+q)(1-s)/s}\|F\|_{s,q}.\end{aligned}$$

It follows that

$$(1-r)^{(n+q)(1-s)}|h(r\xi)|^s \\ \leq \|F\|_{s,q}^s = \int |g(\overline{z})|^s |f(z\cdot\xi)|^s dv_q(z),$$

and thus by Minkowski's inequality, with $p/s \geq 1$, and Proposition 3.1(i),

$$(1-r)^{(n+q)(1-s)} \left(\int |h(r\xi)|^p dv_q(\xi)\right)^{s/p} \\ \leq \int |g(\overline{z})|^s \left(\int |f(z\cdot\xi)|^p dv_q(\xi)\right)^{s/p} dv_q(z) \\ \leq \|f\|_{p,q}^s \|g\|_{s,q}^s,$$

concluding the proof. $\square$

A special case of the last proposition, namely when $n = 1$ and $q = 0$, appears also, as the main result, in Pavlovic [**6**] with a different proof. Moreover, the particular inclusion $A_q^1 \subset (A_q^p, A_q^p)_q$ for $1 \leq p \leq \infty$, may also be deduced from Proposition 3.3 with $s = 1$, and thus $t = p$. For $n > 1$, this inclusion is not quite sharp. Indeed, by Propositoin 3.3 with $s = 1$ and $t = p = \infty$, $A_q^1 \subset (A_q^\infty, H^\infty(U^n))_q \subset (A_q^\infty, A_q^\infty)_q$.

4. Multipliers of A_q^p into $H^\infty(U^n)$. In this section we identify $(A_q^p, H^\infty(U^n))_q$ for $0 < p \leq \infty$. Some parts of these identifications appear, with different proofs, also in Shi [**8**] when $1 < p < \infty$ and $q = 0$.

THEOREM 4.1. *Let $1 < p < \infty$ and $g \in H(B)$. Then $g \in A_q^{p'}, p' = p/(p-1)$, if and only if $(f * g)_q \in H^\infty(U^n)$ for every $f \in A_q^p$. Equivalently, $(A_q^p, H^\infty(U^n))_q = A_q^{p'}$.*

Proof. The inclusion $A_q^{p'} \subset (A_q^p, H^\infty(U^n))_q$ is a special case of Proposition 3.3. To prove the converse we assume that $g \in H(B)$ and that $(f*g)_q \in H^\infty(U^n)$ for every $f \in A_q^p$, and invoke Lemma 3.4. It follows that there exists a constant $c_g \geq 0$ so that $\|(f*g)_q\|_\infty \leq c_g \|f\|_{p,q}$ for every $f \in A_q^p$. Let $0 \leq r < 1$ and $f \in A_q^p$. By Proposition 3.2, $(f*g)_q(r1) = \langle f, g_r^* \rangle_q$ and so

$$|\langle f, g_r^* \rangle_q| = |(f*g)_q(r\mathbf{1})| \leq \|(f*g)_q\|_\infty \leq c_g \|f\|_{p,q}.$$

It follows by duality that $||g_r||_{p',q} = ||g_r^*||_{p,q} \leq c_g$ for every $0 \leq r < 1$. In particular, $g \in A_q^{p'}$ with $||g||_{p',q} \leq c_g$, and the proof is complete. $\square$

The cases $p = 1$ and $p = \infty$ of the above theorem are contained in the following two results:

THEOREM 4.2. *Let $g \in H(B)$. Then $g \in \Gamma_q$ if and only if $(f*g)_q \in H^\infty(U^n)$ for every $f \in A_q^1$. Equivalently, $(A_q^1, H^\infty(U^n))_q = \Gamma_q$. In particular, $(H^1, H^\infty(U^n))_0 = \mathrm{BMOA}(B)$ and $(A_q^1, H^\infty(U^n))_q = \mathcal{B}(B)$ for every $q > 0$.*

Proof. To prove the inclusion $\Gamma_q \subset (A_q^1, H^\infty(U^n))_q$, we assume that $g \in \Gamma_q$ and $f \in A_q^1$. Let $z \in U^n$. By Proposition 3.2, $(f * g)_q(z) = \langle g, f_{\bar{z}}^* \rangle_q$, and since Γ_q is the dual of A_q^1 with respect to the $q-$pairing, we conclude, using Proposition 3.1(i), that

$$\begin{aligned}|(f * g)_q(z)| &\leq ||g||_{*,q}||f_{\bar{z}}^*||_{1,q} \\ &\leq ||g||_{*,q}||f^*||_{1,q} = ||g||_{*,q}||f||_{1,q}.\end{aligned}$$

It follows that $(f * g)_q \in H^\infty(U^n)$ with $||(f * g)_q||_\infty \leq ||f||_{1,q}||g||_{*,q}$, as desired.

To prove the converse, we assume that $g \in H(B)$ and that $(f * g)_q \in H^\infty(U^n)$ for every $f \in A_q^1$, and apply Lemma 3.4. Thus, there exists a constant $c_g \geq 0$ so that $||(f * g)_q||_\infty \leq c_g||f||_{1,q}$ for every $f \in A_q^1$. Let $0 \leq r < 1$ and $f \in A_q^1$. It follows from Proposition 3.2 that

$$|\langle f, g_r^* \rangle_q| = |(f * g)_q(r\mathbf{1})| \leq ||(f * g)_q||_\infty \leq c_g||f||_{1,q}.$$

Let $s, t \in (0, 1)$ with $\max(r, s) < t$. A direct computation gives

$$\langle f, g_r^* - g_s^* \rangle_q = \langle f_{rt^{-1}} - f_{st^{-1}}, g_t^* \rangle_q$$

and so

$$|\langle f, g_r^* - g_s^* \rangle_q| \leq c_g||f_{rt^{-1}} - f_{st^{-1}}||_{1,q}.$$

It follows from Proposition 3.1(v) that the $q-$pairing $(f, g^*)_q \doteq \lim_{r\to 1^-}\langle f, g_r^* \rangle_q$ exists, and thus $|(f, g^*)_q| \leq c_g||f||_{1,q}$ for every $f \in A_q^1$. This implies that g^*, and therefore also g, is in the dual $(A_q^1)^*$ of A_q^1 with respect to the $q-$pairing. Since $\Gamma_q = (A_q^1)^*$, we find that $g \in \Gamma_q$, and the proof is complete. $\square$

THEOREM 4.3. *Let $g \in H(B)$. Then $g \in \Omega_q$ if and only if $(f * g)_q \in H^\infty(U^n)$ for every $f \in A_q^\infty (= H^\infty)$. Equivalently, $\left(A_q^\infty, H^\infty(U^n)\right)_q = \Omega_q$.*

Proof. To prove the inclusion $\Omega_q \subset (A_q^\infty, H^\infty(U^n))_q$, we assume that $g \in \Omega_q$ and $f \in A_q^\infty$. Let $z \in U^n$. By Proposition 3.2,

$$(f * g)_q(z) = \int f(\xi) g(z \cdot \overline{\xi}) dv_q(\xi),$$

and since $g \in \Omega_q$, there exists a measure $\mu \in \mathcal{M}$ such that $g = k_q(\mu)$. It follows from Fubini's theorem and from the reproducing property of K_q that

$$\begin{aligned}(f * g)_q(z) &= \int \left[\int f(\xi) K_q(z \cdot \overline{\xi}, \eta) dv_q(\xi)\right] d\mu(\eta) \\ &= \int \left[\int f(\xi) K_q(z \cdot \overline{\eta}, \xi) dv_q(\xi)\right] d\mu(\eta) \\ &= \int f(z \cdot \overline{\eta}) d\mu(\eta),\end{aligned}$$

and so $|(f*g)_q(z)| \leq \|f\|_{\infty,q} \|\mu\|_\dagger$, or $\|(f*g)_q\|_\infty \leq \|f\|_{\infty,q} \|\mu\|_\dagger$. This means that $\|(f * g)_q\|_\infty \leq \|f\|_{\infty,q} \|g\|_{\dagger,q}$, and the desired inclusion follows. To prove the converse, we assume that $g \in H(B)$ and that $(f * g)_q \in H^\infty(U^n)$ for every $f \in A_q^\infty$, and apply Lemma 3.4. In particular, there exists a constant $c_g \geq 0$ so that $\|(f * g)_q\|_\infty \leq c_g \|f\|_{\infty,q}$ for every $f \in A_q^\infty$. Let $0 \leq r < 1$ and $f \in A_q^\infty$, and define

$$H_r(f) = \int f(\xi) g(r\overline{\xi}) dv_q(\xi).$$

Since, by Proposition 3.2, $H_r(f) = (f * g)_q(r\mathbf{1})$ we deduce that $|H_r(f)| \leq \|(f * g)_q\|_\infty \leq c_g \|f\|_{\infty,q}$. It follows that H_r is a continuous linear functional on A_q^∞ with norm $\|H_r\| \leq c_g$. In particular, H_r is a continuous linear functional on the ball-algebra $A(B)$ with norm $\|H_r\| \leq c_g$. Since $A(B)$ is a subspace of $C(\overline{B})$, H_r has a norm-preserving extension $\tilde{H}_r$ to $C(\overline{B})$. It follows from the Riesz representation theorem that there exists a unique measure $\mu_r \in \mathcal{M}$ with $\|\mu_r\|_\dagger = \|\tilde{H}_r\| = \|H_r\| \leq c_g$ so that

$$\tilde{H}_r(f) = \int f(\overline{\xi}) d\mu_r(\xi)$$

for every $f \in C(\overline{B})$. Since $\|\mu_r\|_\dagger \leq c_g$ for every $0 \leq r < 1$, we deduce that there exists a sequence $\{r_k\} \subset (0,1)$ with $r_k \to 1$ so that μ_{r_k} converges in $\mathcal{M}$ to a measure $\mu \in \mathcal{M}$. It follows that $\tilde{H}_{r_k}$ converges to $\tilde{H}$, where

$$\tilde{H}(f) = \int f(\overline{\xi})d\mu(\xi)$$

for every $f \in C(\overline{B})$. In particular, $\lim_{k\to\infty} H_{r_k}(f) = \tilde{H}(f)$ for every $f \in A(B)$, and thus upon taking $f = \pi_\alpha, \alpha \in \mathbb{Z}_+^n$, we obtain

$$\lim_{k\to\infty} H_{r_k}(\pi_\alpha) = \int \overline{\xi}^\alpha d\mu(\xi).$$

Since, however, $H_{r_k}(\pi_\alpha) = a_\alpha(g) r_k^{|\alpha|}\omega_\alpha(q)$ we deduce that

$$a_\alpha(g) = \frac{1}{\omega_\alpha(q)} \int \overline{\xi}^\alpha d\mu(\xi) \qquad (\alpha \in \mathbb{Z}_+^n),$$

and thus

$$g(z) = \sum_\alpha a_\alpha(g) z^\alpha = \int \left(\sum_\alpha \frac{1}{\omega_\alpha(q)} z^\alpha \overline{\xi}^\alpha \right) d\mu(\xi) \qquad (z \in B).$$

It follows, since

$$K_q(z,\xi) = \sum_\alpha \frac{1}{\omega_\alpha(q)} z^\alpha \overline{\xi}^\alpha,$$

that $g(z) = \int K_q(z,\xi)d\mu(\xi), z \in B$, or that $g = k_q(\mu)$ with $\mu \in \mathcal{M}$, i.e. $g \in \Omega_q$. This concludes the proof. □

Finally, we now deal with the case $0 < p < 1$.

THEOREM 4.4. *Let $0 < p < 1$ and $g \in H(B)$. Then $g \in \Lambda_{(n+q)(1/p-1)}$ if and only if $(f * g)_q \in H^\infty(U^n)$ for every $f \in A_q^p$. Equivalently, $(A_q^p, H^\infty(U^n))_q = \Lambda_{(n+q)(1/p-1)}$.*

Proof. Let $s = (n+q)(1/p-1)$. Assume that $g \in \Lambda_s, f \in A_q^p$ and $z \in U^n$. By Proposition 3.2, $(f * g)_q(z) = \langle g, f_{\overline{z}}^* \rangle_q$, and since Λ_s is the dual of A_q^p with respect to the q-pairing, we deduce, using Proposition 3.1(i), that

$$|(f * g)_q(z)| \leq \|g\|_{\Lambda_s} \|f_{\overline{z}}^*\|_{p,q} \leq \|g\|_{\Lambda_s} \|f^*\|_{p,q} = \|g\|_{\Lambda_s} \|f\|_{p,q}.$$

It follows that $\Lambda_s \subset (A^p_q, H^\infty(U^n))_q$. To prove the converse, we assume that $g \in H(B)$ and that $(f*g)_q \in H^\infty(U^n)$ for every $f \in A^p_q$. Invoking Lemma 3.4, we infer the existence of a constant $c_g \geq 0$ so that $\|(f*g)_q\|_\infty \leq c_g\|f\|_{p,q}$ for every $f \in A^p_q$. Let $0 \leq r < 1$, $0 < t_1, t_2 < 1$ and $f \in A^p_q$. Then

$$|\langle f_{t_1}, g^*_r\rangle_q| = |(f*g)(rt_1\mathbf{1})| \leq c_g\|f\|_{p,q},$$

and by Proposition 3.1(v)

$$|\langle f_{t_1}, g^*_r\rangle_q - \langle f_{t_2}, g^*_r\rangle_q| \leq \|g_r\|_{\Lambda,s}\|f_{t_1} - f_{t_2}\|_{p,q} \to 0,$$

as $t_1 \to t_2$. Consequently, the q–pairing $(f, g^*_r)_q$ exists with

$$|(f, g^*_r)_q| \leq c_g\|f\|_{p,q}$$

for every $f \in A^p_q$. It follows from duality that $g^*_r \in \Lambda_s$ and $\|g^*_r\|_{\Lambda_s} \leq c_g$ for every $0 \leq r < 1$. Equivalently, $g_r \in \Lambda_s$ and $\|g_r\|_{\Lambda_s} \leq c_g$ for every $0 \leq r < 1$. In particular, using Proposition 3.1(iii), $g \in \Lambda_s$ and $\|g\|_{\Lambda,s} \leq c_g$, and the proof is complete. □

5. Multipliers of A^1_q into $\mathcal{L}(U^n)$. In view of Theorem 4.2, we have $\mathcal{B}(B) = (A^1_q, H^\infty(U^n))_q \subset \left(A^1_q, \mathrm{BMOA}(U^n)\right)_q \subset \left(A^1_q, \mathcal{B}(U^n)\right)_q$, provided $q > 0$. The main purpose of this section is to show that $\left(A^1_q, \mathcal{L}(U^n)\right)_q = \mathcal{B}(B)$ for any $q \geq 0$. This will be accomplished by showing that $\left(A^1_q, \mathcal{B}(U^n)\right)_q \subset \mathcal{B}(B)$ and that $\mathcal{B}(B) \subset \left(A^1_q, \mathcal{L}(U^n)\right)_q$ (see Theorems 5.1 and 5.6 below) for every $q \geq 0$.

We first prove the following, and rather crucial, theorem:

THEOREM 5.1. *For any $q \geq 0$, we have $\left(A^1_q, \mathcal{B}(U^n)\right)_q \subset \mathcal{B}(B)$.*

Proof. Let $g \in H(B)$ and assume that $(f*g)_q \in \mathcal{B}(U^n)$ for every $f \in A^1_q$. Since, for example, $\mathcal{B}(U^n)$ is continuously contained in $A^2(U^n) = L^2(U^n) \cap H(U^n)$, $\mathcal{B}(U^n)$ is also a functional Banach space of functions in $H(U^n)$. We can therefore invoke Lemma 3.4 to infer the existence of a constant c_g so that

$$\|(f*g)_q\|_\beta \leq nc_g\|f\|_{1,q}$$

for every $f \in A_q^1$. Equivalently, for $h = (f * g)_q$ with $f \in A_q^1$ we have

$$\sup_{z \in U^n} \sum_{j=1}^{n} \left(1 - |z_j|^2\right) |h(z) + n z_j \partial_j h(z)| \leq n c_g \|f\|_{1,q}.$$

In particular, for $0 \leq r < 1$,

$$\left(1 - r^2\right) \sum_{j=1}^{n} |h(r\mathbf{1}) + n r \partial_j h(r\mathbf{1})| \leq n c_g \|f\|_{1,q},$$

and thus

$$\left(1 - r^2\right) |\mathcal{D} h(r\mathbf{1})| \leq c_g \|f\|_{1,q}.$$

Fix $b > 0$ and $\xi \in S$ and let $f = G_{n+q,-b}(\langle \cdot, r\overline{\xi} \rangle)$. It follows that for $\eta \in U^n$

$$\begin{aligned} h(\eta) = (f * g)(\eta) &= \int g(z \cdot \eta) G_{n+q,-b}\left(\langle \overline{z}, r\overline{\xi} \rangle\right) dv_q(z) \\ &= \sum_{\alpha} (|\alpha| + 1)^b a_\alpha(g) (r\xi\eta)^\alpha, \end{aligned}$$

and thus

$$h(\eta) = (f * g)(\eta) = \mathcal{D}^b g_{r\xi}(\eta).$$

In particular,

$$\mathcal{D} h(r\mathbf{1}) = \mathcal{D}^{b+1} g(r^2 \xi).$$

Moreover, using Proposition 3.5(iii), we find that $G_{n+q,-b}(\lambda) = (1 - \lambda)^{-(n+q+b)} F(\lambda), \lambda \in U$, with $F \in \Lambda_{n+q+b}(U)$. Let $c_0 = \|F\|_\infty$. It follows, again from Proposition 3.5(ii), that

$$\|f\|_{1,q} \leq c_0 I_{n+q+b,q}(r\overline{\xi}) \leq c(1 - r^2)^{-b}$$

with $c = c_0 \Gamma(n+q)\Gamma(b) / \left\{\Gamma((n+q+b)/2)\right\}^2$. Accordingly,

$$(1 - r^2)^{b+1} |\mathcal{D}^{b+1} g(r^2 \xi)| \leq c c_g$$

for any $\xi \in S$ and $0 \leq r < 1$. This implies $\|g\|_{\mathcal{B}} < \infty$, or $g \in \mathcal{B}(B)$, and the proof is complete. $\square$

We also establish the following crucial estimate:

THEOREM 5.2. *Let $f, g \in H(B)$, $h = (f * g)_0$ and $1 \leq k \leq n$. Then*

$$\int_0^1 M_\infty^2 \left(r, \left(z_{(k)}, z^{(k)}\right) : \mathcal{D}_k(\partial_k h)\right) (1 - r^2)^3 dr \\ \leq 4^{2-\delta_{n1}} \|f - f(0)\|_{1,0}^2 \|g - g(0)\|_{\mathcal{B}(B)}^2$$

for all $(z_{(k)}, z^{(k)}) \in U^{n-1}$, where $\mathcal{D}_k = 1 + z_k \partial_k$. Here $\delta_{n1} = 0$ if $n > 1$ and $\delta_{11} = 1$.

Proof. Without loss of generality, we may assume that $f \in H(\overline{B})$. We fix $z \in U^n$, and we let $\lambda \in \overline{U}^n$. By Proposition 3.2, $h \in H(U^n)$ and

$$h(\lambda \cdot z) = \int_S f(\lambda \cdot \xi) g(z \cdot \overline{\xi}) d\sigma(\xi) \\ = \int_S f(\lambda \cdot \xi) g_{\overline{\xi}}(z) d\sigma(\xi).$$

It follows that

$$\lambda_k \partial_k h(\lambda \cdot z) = \int_S f(\lambda \cdot \xi) \partial_k g_{\overline{\xi}}(z) d\sigma(\xi),$$

and so, by differentiating with respect to λ_k,

$$\mathcal{D}_k \partial_k h(\lambda \cdot z) = \int_S \xi_k \partial_k f(\lambda \cdot \xi) \partial_k g_{\overline{\xi}}(z) d\sigma(\xi).$$

Letting $F = f - f(0)$, $G = g - g(0)$ and $H_k = \mathcal{D}_k \partial_k h$, we arrive at

$$H_k(\lambda \cdot z) = \int_S \xi_k \partial_k F(\lambda \cdot \xi) \partial_k G_{\overline{\xi}}(z) d\sigma(\xi).$$

It follows from Proposition 2.3 that

$$|H_k(\lambda \cdot z)| \leq \int_S \left|\xi_k\right| \left|\partial_k F(\lambda \cdot \xi)\right| \left|\partial_k G_{\overline{\xi}}(z)\right| d\sigma(\xi) \\ \leq \left(1 - |z_k|^2\right)^{-1} \int_S |\xi_k| |\partial F(\lambda \cdot \xi)| \|G_{\overline{\xi}}\|_{\mathcal{B}(U^n)} d\sigma(\xi) \\ \leq \left(1 - |z_k|^2\right)^{-1} \|G\|_{\mathcal{B}(B)} \int_S |\xi_k| |\partial_k F(\lambda \cdot \xi)| \, d\sigma(\xi),$$

and thus

$$|H_k(\lambda \cdot z)|(1 - r^2) \leq \|G\|_{\mathcal{B}(B)} \int_S |\xi_k| \, |\partial_k F(\lambda \cdot \xi)| \, d\sigma(\xi),$$

we now specify $\lambda \in \overline{U}^n$ by taking $\lambda_k = z_k$ and $\lambda_j = 1$ for $j \neq k, 1 \leq j \leq n$. This gives

$$(1-r^2)\left|H_k\left(z_{(k)}, z_k^2, z^{(k)}\right)\right| \\ \leq \|G\|_{\mathcal{B}(B)} \int_S |\xi_k| \left|\partial_k F(\xi_{(k)}, z_k\xi_k, \xi^{(k)})\right| d\sigma(\xi)$$

and thus

$$(1-r^2)M_\infty\left(r^2, (z_{(k)}, z^{(k)}) : H_k\right) \\ \leq \frac{1}{r}\|G\|_{\mathcal{B}(B)} \int_S \left|\mathcal{R}_1 F_{(\xi')}\left(r\frac{\xi_k}{|\xi_k|}\right)\right| d\sigma(\xi),$$

where for any $\xi \in S$, $F_{(\xi')}$ is a function in $H(\overline{U})$ defined by

$$F_{(\xi')}(w) = F\left(\xi_{(k)}, |\xi_k|w, \xi^{(k)}\right), \qquad w \in \overline{U},$$

and $\mathcal{R}_1$ is the one-dimensional radial-derivative $\mathcal{R}$, given by $\mathcal{R}_1 = w\frac{d}{dw}$. We now use the fact that any $\xi \in S$ can be represented as $\xi = (\xi, \xi')$ where $\xi' = (\xi_{(k)}, \xi^{(k)}) \in B_{n-1}$ and $|\xi_k|^2 = 1 - \|\xi'\|^2 = 1 - \|\xi_{(k)}\|^2 - \|\xi^{(k)}\|^2$. In particular, for $\xi \in S$, $F_{(\xi')} = F$ when $n = 1$. It follows that

$$\int_0^1 M_\infty^2\left(r^2, (z_{(k)}, z^{(k)}\right) : H_k)(1-r^2)^3 r dr \\ \leq \|G\|_{\mathcal{B}(B)}^2 \int_0^1 \frac{(1-r^2)}{r}\left\{\int_S \left|\mathcal{R}_1 F_{(\xi')}\left(r\frac{\xi_k}{|\xi_k|}\right)\right| d\sigma(\xi)\right\}^2 dr \\ \leq 2\|G\|_{\mathcal{B}(B)}^2 \int_0^1 \frac{1-r}{r}\left\{\int_S \left|\mathcal{R}_1 F_{(\xi')}\left(r\frac{\xi_k}{|\xi_k|}\right)\right| d\sigma(\xi)\right\}^2 dr \\ = 2\|G\|_{\mathcal{B}(B)}^2 \int_0^1 \frac{1-r}{r}\left\{\int_{B_{n-1}}\left(\int_0^{2\pi} |\mathcal{R}_1 F_{(\xi')}(re^{i\theta})| \frac{d\theta}{2\pi}\right) dv_1^{(n-1)}(\xi)\right\}^2 dr \\ = 2\|G\|_{\mathcal{B}(B)}^2 \int_0^1 \frac{1-r}{r}\left(\int_{B_{n-1}} M_1\left(r, \mathcal{R}_1 F_{(\xi')}\right) dv_1^{(n-1)}(\xi')\right)^2 dr,$$

and thus, using Minkowski's inequality and Lemma 3.8 with $n = 1$ and $p = 1$,

$$\begin{aligned}
&\int_0^1 M_\infty^2\left(r^2, (z_{(k)}, z^{(k)}) : H_k\right)(1-r^2)^3 r\,dr \\
&\le 2\|G\|^2_{\mathcal{B}(B)} \left\{ \int_{B_{n-1}} \left[\int_0^1 \frac{1}{r}(1-r) M_1^2(r, \mathcal{R}_1 F_{(\xi')})dr \right]^{1/2} dv_1^{(n-1)}(\xi') \right\}^2 \\
&\le 2\|G\|^2_{\mathcal{B}(B)} \left\{ \int_{B_{n-1}} \|F_{(\xi')} - F_{(\xi')}(0)\|_{1,0} dv_1^{(n-1)}(\xi') \right\}^2 .
\end{aligned}$$

For $n = 1$, the last integral reduces to $\|F - F(0)\|_{1,0} = \|F\|_{1,0}$. On the other hand, when $n > 1$, Proposition 3.6(ii) shows that for any $\xi \in S$, $\|F_{(\xi')}(0)\|_{1,0} = |F_{(\xi')}(0)| \le \|F_{(\xi')}\|_{1,0}$ and so

$$\begin{aligned}
&\int_{B_{n-1}} \|F_{(\xi')} - F_{(\xi')}(0)\|_{1,0} dv_1^{(n-1)}(\xi') \\
&\quad \le 2 \int_{B_{n-1}} \|F_{(\xi')}\|_{1,0} dv_1^{(n-1)}(\xi') \\
&\quad = 2 \int_{B_{n-1}} \left(\int_0^{2\pi} \left| F(\xi_{(k)}, |\xi_k| e^{i\theta}, \xi^{(k)}) \right| \frac{d\theta}{2\pi} \right) dv_1^{(n-1)}(\xi') \\
&\quad = 2 \int_S |F(\xi)| d\sigma(\xi) \\
&\quad = 2\|F\|_{1,0}.
\end{aligned}$$

It follows that

$$\int_0^1 M_\infty^2(r^2, (z_{(k)}, z^{(k)}) : H_k)(1-r^2)^3 r\,dr \le 2 \cdot 4^{1-\delta_{n1}} \|F\|^2_{1,0} \|G\|^2_{\mathcal{B}(B)}$$

and hence

$$\begin{aligned}
&\int_0^1 M_\infty^2\left(r, \left(z_{(k)}, z^{(k)}\right) : H_k\right)(1-r)^3 dr \\
&\quad = 2 \int_0^1 M_\infty^2(r^2, (z_{(k)}, z^{(k)}) : H_k)(1-r^2)^3 r\,dr \\
&\quad \le 4^{2-\delta_{n1}} \|F\|^2_{1,0} \|G\|^2_{\mathcal{B}(B)}.
\end{aligned}$$

This completes the proof. □

At this stage we wish to show that $\mathcal{B}(B) \subset (A_0^1, \mathcal{L}(U^n))_0$, i.e. to show that the estimate on $h = (f * g)_0$ in the last theorem will eventually lead to h being in $\mathcal{L}(U^n)$. To accomplish this we require a Hardy-Littlewood type estimate for functions in $H(U)$. Toward this end we first establish the following simple lemma. For $0 < x, y \leq \infty$, we let $m(x) = \min(x, 1)$ and $\epsilon(x, y) = [m(m(y)/x)]^{-1} - 1$.

LEMMA 5.3. *Let $f \in H(U)$, $0 < p \leq \infty$, $a > 0$ and $0 \leq \eta < r < 1$. Then*

$$M_p(r, f) \leq M_p(r, \mathcal{D}f)$$

and

$$r^a M_p^a(r, f) - 2^{\epsilon(a,p)} \eta^a M_p^a(\eta, f) \leq 2^{\epsilon(a,p)} (r - \eta)^a M_p^a(r, \mathcal{D}f).$$

Proof. Let $z \in U$ and $0 < s < 1$. Since $\mathcal{D}f(rz) = d(sf(sz))/ds$, we deduce that

$$rf(rz) - \eta f(\eta z) = \int_\eta^r \mathcal{D}f(sz)ds,$$

and thus

$$|rf(rz) - \eta f(\eta z)| \leq (r - \eta) \sup_{\eta < s < r} |\mathcal{D}f(sz)|.$$

If $p \leq 1$, then it follows from the Hardy-Littlewood maximal theorem that

$$r^p M_p^p(r, f) - \eta^p M_p^p(\eta, f) \leq (r - \eta)^p M_p^p(r, \mathcal{D}f).$$

If, on the other hand, $p \geq 1$ then Minkowski's inequality applied to the above identity gives

$$rM_p(r, f) - \eta M_p(\eta, f) \leq (r - \eta) M_p(r, \mathcal{D}f).$$

It follows that for any $0 < p \leq \infty$

$$r^{m(p)} M_p^{m(p)}(r, f) - \eta^{m(p)} M_p^{m(p)}(\eta, f) \\ \leq (r - \eta)^{m(p)} M_p^{(p)}(r, \mathcal{D}f).$$

The first inequality of the lemma follows from this inequality by letting $\eta = 0$. The second inequality follows by raising the inequality to the power $a/m(p)$, and the proof is complete. □

PROPOSITION 5.4. *Let $f \in H(U)$, $0 < p \leq \infty, a, q > 0$ and let*

$$K_0(f) = \int_0^1 M_p^a(r, \mathcal{D}f)(1-r)^{a+q-1}dr$$

and

$$K(f) = \int_0^1 M_p^a(r, f)r^a(1-r)^{q-1}dr.$$

Then $K(f) \leq cK_0(f)$, where $c = c(a,q,p)$ is constant given by

$$c(a,q,p) = \frac{2(a+q)}{q}\left\{2^{1/q}2^{\epsilon(a,p)/q} + 4 \cdot 2^{3(a+q)\epsilon(a,p)/q}\right\}.$$

Proof. Let $\delta = 2^{-[1+\epsilon(a,p)]/q}$, and define $r_m = 1-\delta^m$, $m = 0, 1, \cdots$. Then

$$\begin{aligned} K(f) &= \sum_{m=0}^{\infty} \int_{r_m}^{r_{m+1}} M_p^a(r,f)r^a(1-r)^{q-1}dr \\ &\leq \sum_{m=0}^{\infty} M_p^a(r_{m+1}, f)r_{m+1}^a \int_{r_m}^{r_{m+1}} (1-r)^{q-1}dr \\ &= \frac{1}{q}(1-\delta^q)\sum_{m=0}^{\infty} M_p^a(r_{m+1}, f)r_{m+1}^a\delta^{qm} \\ &= \frac{1}{q}(1-\delta^q)(J_1 + J_1 + J_3), \end{aligned}$$

where

$$J_1 = M_p^a(r_1, f)r_1^a, \quad J_2 = 2^{\epsilon(a,p)}\sum_{m=1}^{\infty} M_p^a(r_{m-1}, f)r_{m-1}^a\delta^{qm},$$

and

$$J_3 = \sum_{m=1}^{\infty}\left\{M_p^a(r_{m+1}, f)r_{m+1}^a - 2^{\epsilon(a,p)}M_p^q(r_{m-1}, f)r_{m-1}^a\right\}\delta^{qm}.$$

By Lemma 5.3,

$$\begin{aligned} J_1 &\leq M_p^a(r_1, f) \leq M_p^a(r_1, \mathcal{D}f) \\ &= \frac{a+q}{\delta}\left(\int_{r_1}^1 (1-r)^{a+q-1}dr\right) M_p^a(r_1, \mathcal{D}f) \\ &\leq \frac{a+q}{\delta}\int_{r_1}^1 M_p^a(r, \mathcal{D}f)(1-r)^{a+q-1}dr \\ &\leq \frac{a+q}{\delta}\int_0^1 M_p^a(r, \mathcal{D}f)(1-r)^{a+q-1}dr, \end{aligned}$$

and so

$$J_1 \leq \frac{a+q}{\delta} K_0(f).$$

Also,

$$\begin{aligned}
J_2 &= 2^{\epsilon(a,p)}\delta^q \sum_{m=0}^{\infty} M_p^a(r_m, f) r_m^q \delta^{qm} \\
&\leq 2^{\epsilon(a,p)}\delta^q \sum_{m=0}^{\infty} \left(\int_{r_m}^{r_{m+1}} M_p^a(r,f) r^a (1-r)^{q-1} dr \right) \frac{q\delta^{qm}}{(1-\delta^q)\delta^{qm}} \\
&= 2^{\epsilon(a,p)} \frac{q\delta^q}{1-\delta^q} \int_0^1 M_p^a(r,f) r^a (1-r)^{q-1} dr,
\end{aligned}$$

and thus

$$J_2 \leq 2^{\epsilon(a,p)} \frac{q\delta^q}{1-\delta^q} K(f).$$

Similarly, again by Lemma 5.3,

$$\begin{aligned}
J_3 &\leq 2^{\epsilon(a,p)} \sum_{m=1}^{\infty} (r_{m+1} - r_m)^a M_p^a(r_{m+1}, \mathcal{D}f) \delta^{qm} \\
&= 2^{\epsilon(a,p)} \delta^{-a} (1-\delta^2)^a \sum_{m=1}^{\infty} M_p^a(r_{m+1}, \mathcal{D}f) \delta^{(a+q)m} \\
&\leq 2^{\epsilon(a,p)} \delta^{-a} (1-\delta^2) \frac{(a+q)\delta^{-a-q}}{1-\delta^{a+q}} \sum_{m=1}^{\infty} \int_{r_m}^{r_{m+1}} M_p^a(r, \mathcal{D}f)(1-r)^{a+q-1} dr \\
&\leq 2^{\epsilon(a,p)} (a+q) \delta^{-(2a+q)} \frac{(1-\delta^2)^a}{1-\delta^{a+q}} \int_0^1 M_p^a(r, \mathcal{D}f)(1-r)^{a+q-1} dr,
\end{aligned}$$

and so

$$J_3 \leq 2^{\epsilon(a,p)} (a+q)^{-(2a+q)} \frac{(1-\delta^2)^a}{1-\delta^{a+q}} K_0(f).$$

It follows that

$$\begin{aligned}
&\left(1 - 2^{\epsilon(a,p)}\delta^q\right) K(f) \\
&\qquad \leq \frac{1}{q}(1-\delta^q)(J_1 + J_2) \\
&\qquad \leq \frac{(a+q)(1-\delta^q)}{q\delta} \left(1 + 2^{\epsilon(a,p)} \frac{\delta(1-\delta^2)^a}{\delta^{2a+q}(1-\delta^{a+q})}\right) K_0(f),
\end{aligned}$$

and so

$$K(f) \leq 2\frac{a+q}{q} \left\{ 2^{1/q} 2^{\epsilon(a,p)/q} + 4 \cdot 2^{3a/q} 2^{3(a+q)\epsilon(a,p)/q} \right\} K_0(f),$$

as desired. □

In the next two results, we let

$$c_n = 3 \cdot 2^{3-\delta_{n1}} \sqrt{2n(1+2^{10})}.$$

LEMMA 5.5. *Let $f \in A_0^1 (= H^1)$, $g \in \mathcal{B}(B)$ and $h = (f * g)_0$. Then $h \in \mathcal{L}(U^n)$ with*

$$\|h - h(0)\|_{\mathcal{L}} \leq c_n \|f - f(0)\|_{1,0} \|g - g(0)\|_{\mathcal{B}(B)}.$$

In particular, $\mathcal{B}(B) \subset (A_0^1, \mathcal{L}(U^n))_0$.

Proof. Fix an integer k, $1 \leq k \leq n$, and use Proposition 5.4, with $p = \infty$ and $a = q = 2$, for $\partial_k h(z_{(k)}, \cdot, z^{(k)}) \in H(U)$, $(z_{(k)}, z^{(k)}) \in U^{n-1}$. Note that by Proposition 3.2, $h \in H(U^n)$. It follows, using a change of variable, monotoneity and Theorem 5.2, that

$$\begin{aligned}
&\int_0^1 M_\infty^2 \left(r, \left(z_{(k)}, z^{(k)}\right) : \partial_k h\right) (1 - r^2) 2r dr \\
&\quad = 3 \int_0^1 M_\infty^2 \left(r^{3/2}, \left(z_{(k)}, z^{(k)}\right) : \partial_k h\right) (1 - r^3) r^2 dr \\
&\quad \leq 9 \int_0^1 M_\infty^2 \left(r, \left(z_{(k)}, z^{(k)}\right) : \partial_k h\right) r^2 (1 - r) dr \\
&\quad \leq 9c(2,2,\infty) \int_0^1 M_\infty^2 \left(r, \left(z_{(k)}, z^{(k)}\right) : \mathcal{D}_k(\partial_k h)\right) (1 - r)^3 dr \\
&\quad \leq 9c(2,2,\infty) 4^{2-\delta_{n1}} \|f - f(0)\|_{1,0}^2 \|g - g(0)\|_{\mathcal{B}(B)}^2 .
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|h - h(0)\|_{\mathcal{L}}^2 \\
&\quad = \sum_{k=1}^n \sup_{(z_{(k)}, z^{(k)}) \in U^{n-1}} \int_0^1 M_\infty^2 \left(r, \left(z_{(k)}, z^{(k)}\right) : \partial_k h\right) (1 - r^2) 2r dr \\
&\quad \leq n 3^2 2^{7-2\delta_{n1}} (1 + 2^{10}) \|f - f(0)\|_{1,0}^2 \|g - g(0)\|_{\mathcal{B}(B)}^2 ,
\end{aligned}$$

and the proof is complete. □

THEOREM 5.6. *Let $q \geq 0, f \in A_q^1, g \in \mathcal{B}(B)$ and $h = (f * g)_q$. Then $h \in \mathcal{L}(U^n)$ with*

$$\|h - h(0)\|_{\mathcal{L}} \leq c_n \|f - f(0)\|_{1,q} \|g - g(0)\|_{\mathcal{B}(B)}.$$

In particular, $\mathcal{B}(B) \subset (A^1_q, \mathcal{L}(U^n))_q$.

Proof. By Lemma 5.5 we may assume that $q > 0$. For $0 < r < 1$ and $z \in \overline{U}^n$ we have

$$(f_{r^2} * g_{r^2})_0(z) = \int_S f_{r^2}(\xi) g_{r^2}(z \cdot \overline{\xi}) d\sigma(\xi)$$

and thus

$$h = \int_0^1 (f_r * g_r)_0 d\nu_q(r)$$

where $d\nu_q$ is the previously defined probability measure on $(0,1)$. It follows from Lemma 5.5, the triangle inequality, and Proposition 2.2 that

$$\begin{aligned} \|h - h(0)\|_{\mathcal{L}} &\leq \int_0^1 \|(f_r * g_r)_0\|_{\mathcal{L}} d\nu_q(r) \\ &\leq c_n \int_0^1 \|f_r - f(0)\|_{1,0} \|g_r - g(0)\|_{\mathcal{B}(B)} d\nu_q(r) \\ &\leq c_n \|g - g(0)\|_{\mathcal{B}} \int_0^1 \|f_r - f(0)\|_{1,0} d\nu_q(r). \end{aligned}$$

The desired result now follows by observing that $\|f_r - f(0)\|_{1,0} \leq \|f_{\sqrt{r}} - f(0)\|_{1,0}$, and that

$$\|f - f(0)\|_{1,q} = \int_0^1 \|f_{\sqrt{r}} - f(0)\|_{1,0} d\nu_q(r).$$

□

Finally, we prove the following result which was alluded in the previous sections.

THEOREM 5.7. *For any $q \geq 0$, we have*

$$\begin{aligned} \mathcal{B}(B) &= \left(A^1_q(B), \mathcal{L}(U^n)\right)_q = \left(A^1_q(B), \mathrm{VMOA}(U^n)\right)_q \\ &= \left(A^1_q(B), \mathrm{BMOA}(U^n)\right)_q = \left(A^1_q(B), \mathcal{B}_0(U^n)\right)_q \\ &= \left(A^1_q(B), \mathcal{B}(U^n)\right)_q. \end{aligned}$$

Proof. Using Proposition 2.4, Theorem 5.1 and Theorem 5.6, we have

$$\begin{aligned} \mathcal{B}(B) &\subset \left(A^1_q, \mathcal{L}(U^n)\right)_q \subset \left(A^1_q, \mathrm{VMOA}(U^n)\right)_q \\ &\subset \left(A^1_q, \mathrm{BMOA}(U^n)\right)_q \subset \left(A^1_q, \mathcal{B}(U^n)\right)_q \subset \mathcal{B}(B) \end{aligned}$$

and

$$\begin{aligned}\mathcal{B}(B) &\subset \left(A_q^1, \mathcal{L}(U^n)\right)_q \subset \left(A_q^1, \mathrm{VMOA}(U^n)\right)_q \\ &\subset \left(A_q^1, \mathcal{B}_0(U^n)\right)_q \subset \left(A_q^1, \mathcal{B}(U^n)\right)_q \subset \mathcal{B}(B),\end{aligned}$$

concluding the proof. □

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