PACIFIC JOURNAL OF MATHEMATICS
Vol. 169, No. 1, 1995

VOLUME ESTIMATES FOR LOG-CONCAVE DENSITIES WITH APPLICATION TO ITERATED CONVOLUTIONS

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A connection between volume estimates for a log-concave, symmetric density of a probability measure on $\mathbb{R}^n$ and its maximal value is established. As an application we prove for an absolute constant c_0

$$\underbrace{f * \cdots * f}_{m \text{ times}}(0) \leq \left(\frac{c_0}{\sqrt{m}}\right)^n f(0).$$

0. Introduction. Log-concave densities appear naturally in the theory of convex sets. Besides the normal distributions a lot of information is known about the cube $Q_n = [-1/2, 1/2]^n$. In particular, a modified form of Sudakow's inequality was proved by Carl and Pajor, see [**CP**].

THEOREM 1. *There is an absolute constant c_1, such that for every operator $u : \ell_2^n \to Y$ with* $\mathrm{rg}(u) \leq m$ *and all $k \in \mathrm{N}$*

$$\sqrt{k} \max\{d_k(u), e_k(u)\} \leq c_1(\ln(1 + m/k))^{1/2} \int_{Q_n} \|u(x)\|_Y \, dx.$$

Here d_k, e_k denotes the k-th Kolmogorov, entropy numbers, respectively.It is wellknown fact that the logarithmic factor can not be removed.In this paper we are interested in generalizations of Sudakows estimate for an arbitrary log-concave density which is closely related to upper bounds for the maximal value of a log-concave densities.This observation, based on ideas of Hensley and Ball, is contained in the following key

LEMMA 2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a log-concave, symmetric density of a probability measure on $\mathbb{R}^n$. There is an absolute constant c_0 such that for all $1 \leq p \leq 2n$*

$$\frac{1}{c_0} \leq f(0)^{1/n} \inf_{B \text{ convex body}} \left(\int_{\mathbb{R}}^n \|x\|_B^p f(x) \, dx\right)^{1/p} \mathrm{vol}(B)^{1/n} \leq c_0.$$

A more elaborated version for entropy estimates can be found in Chapter I. As an application Vaaler's and Ball's theorem about estimates of sections of the cube

$$1 \leq \mathrm{vol}_{n-k}(Q_n \cap H)^{1/k} \leq \sqrt{2},$$

H a k-codimensional subspace in $\mathbb{R}^n$ can be deduced from Theorem 1 and Lemma 2 (with a worse constant). But this abstract approach also applies for arbitrary completely symmetric convex bodies. For this the definition of the constant of isotropy is needed for a probability measure μ on $\mathbb{R}^n$.

$$L_\mu := \left((n!)^{-1} \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \mid \det(x_1, \dots, x_n) \mid^2 d\mu(x_1) \cdots d\mu(x_n) \right)^{1/2n}.$$

A probability measure μ is in isotropic position if it's covariance matrix is a multiple of the identity for all θ in the euclidean sphere one has

$$\int_{\mathbb{R}^n} \mid \langle x, \theta \rangle \mid^2 d\mu(x) = L_\mu^2.$$

The usual Lebesque measure is denoted by λ_n. A symmetric, convex body K is said to be in isotropic position if the measure $\chi_K \lambda_n$ is in isotropic position. Its constant of isotropy is denoted by L_K. It was observed by Hensley that the volume of hyperplanes of a convex body in isotropic position are merely constant. The same is true for arbitrary sections of a completely symmetric convex body.

THEOREM 3. *Let $K \subset \mathbb{R}^n$ be a symmetric, convex body in isotropic position and symmetric with respect to all hyperplanes $\langle x, e_j \rangle = 0$ $(j = 1, \dots, n)$. There is an absolute constant c_0 such that for all k-codimensional subspace $H \subset \mathbb{R}^n$ one has*

$$\frac{1}{c_0} \leq \mathrm{vol}_{n-k}(K \cap H)^{1/k} \leq c_0.$$

This result was obtained by Meyer and Pajor for the unit ball's of ℓ_p^n, whereas for $k = 1$ and arbitrary K was observed by Milman and Pajor. In Chapter II a symmetrization technique is used to prove an extremal property of the normal distribution under symmetric densities.

THEOREM 4. *Let f be a symmetric, bounded density of a probability measure$\mu = f\lambda_n$ on $\mathbb{R}^n$. Then there exists a matrix M with $|\det M| = 1$, such that for all $\alpha \in \mathbb{R}^m$ with $\|\alpha\|_2 = 1$, for all K-convex Banach space Y and all operator $u : \ell_2^n \to Y$ we have*

$$\left(\int_\Omega \left\|\sum_1^n uM(e_k)g_k\right\|_Y^2 d\mathbb{P}\right)^{1/2}$$
$$\leq c_0 K(Y)\|f\|_\infty^{1/n} \left(\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\left\|\sum_{j=1}^m \alpha_j u(x_j)\right\|_Y^2 dx_1\cdots dx_m\right)^{1/2}.$$

In particular, if f is the characteristic function of a convex body with volume 1 this implies for all k-codimensional subspace H

$$\lambda_{n-k}(M(K)\cap H)^{1/k} \leq c_0(1+\ln k).$$

Here $(g_k)_1^n$ denotes a sequence of independent, normalized gaussian variablies and $K(Y)$ the K-convexity constant of Y. The general formulation with an additional sequence α is motivated by the following definition of a symmetric norm induced by a symmetric probability measure μ on $\mathbb{R}^n$ and an arbitrary convex, symmetric body $B \subset \mathbb{R}^n$.

$$[[\alpha]]_B := \left(\int_{\mathbb{R}}^n\cdots\int_{\mathbb{R}}^n\left\|\sum_{j=1}^m \alpha_j x_j\right\| d\mu(x_1)\cdots d\mu(x_m)\right)^{1/2}; \quad \alpha\in\mathbb{R}^m.$$

Milman conjectured that this norm is comparable with the euclidean norm, provided some reasonable condition on μ and B are satisfied. We will prove a lower estimate.

THEOREM 5. *Let $\mu = f\lambda_n$ be a symmetric measure on $\mathbb{R}^n$ then we have for all symmetric, convex bodies $B \subset \mathbb{R}^n$ and all $m \in \mathbb{N}^m, \alpha \in \mathbb{R}^m$*

(i) $\sqrt{m} \leq c_0[[\underbrace{1,\dots,1}_{m\text{ times}}]]_B \operatorname{vol}(B)^{1/n}\|f\|_\infty^{1/n}$.

(ii)

$$\|\alpha\|_2 \leq c_0\min\{1+\ln L_\mu\|f\|_\infty^{1/2}, (1+\ln m)^{1/2}\}$$
$$\times \|f\|_\infty^{1/n}[[\alpha]]_B\operatorname{vol}(B)^{1/n}.$$

(iii) *If f is in addition log-concave one has for $m \geq n$*

$$L_\mu \sqrt{m} \leq c_0 [[\underbrace{1, \dots, 1}_{m\,\text{times}}]]_B \operatorname{vol}(B)^{1/n}, \quad \text{for}\, m \geq n.$$

Theorem 5 improves results from **[BMMP]** where (i) was only established for $m \geq n$ and for (iii) additional cotype conditions were needed. The proof of Theorem 5 is based on volume estimates for the norm $[[\alpha]]_B$. Via the key Lemma 2 the result can be reformulate in terms of iterated convolutions as folows.

COROLLARY 6. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a symmetric, log-concave density of a probability measure on $\mathbb{R}^n$. Then we have for all $m \in \mathbb{N}$*

(i) $\underbrace{f * \cdots * f}_{m\,\text{times}}(0) \leq \left(\frac{c_0}{\sqrt{m}}\right)^n f(0).$

(ii) $\underbrace{f * \cdots * f}_{m\,\text{times}}(0) \leq \left(\frac{c_0}{L_\mu \sqrt{m}}\right)^n$ *for all $m \geq n$*

(iii) *If $K \subset \mathbb{R}^n$ is a symmetric, convex body with* $\operatorname{vol}(K) = 1$ *then we have for all $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}^m$ with euclidean norm 1*

$$\left(\chi_{\alpha_1 K} * \cdots * \chi_{\alpha_m K}(0)\right)^{1/n} \leq c_0 (1 + \ln L_K) \prod_{j=1}^m \mid \alpha_j \mid .$$

Most of the results are contaned in the author's PhD Thesis.

Preliminaries. In what follows $c, c_0, c_1, \dots$ denote various absolute constants. A *convex body* $K \subset \mathbb{R}^n$ is a convex, compact, symmetric set with 0 as an interior point. Its gauge functional is denoted by $\|\ \ \|_K := \inf\{t > 0 \mid x \in tK\}$. For a subset $A \subset \mathbb{R}^n$ the outer k-dimensional Hausdorff measure is defined by

$$\lambda_k(A) := \lim_{\delta \to 0} \inf \left\{ v_k \left(\sum_{j=1}^{\infty} 2^{-k} \operatorname{diam}(B_j)^k \right) \mid A \subset \bigcup_{j=1}^{\infty} B_j \right.$$
$$\left. \text{and } \operatorname{diam}(B_j) \leq \delta \right\},$$

where $\operatorname{diam}(B) := \sup\{\|x - y\|_2 \mid x, y \in B\}$ is the diameter of B, $\|\ \ \|_2$ denotes the usual euclidean norm in $\mathbb{R}^n$, whose unit ball B_2^n

has volme v_n. A set B is called λ_k*–measurable* if for all sets $A \subset \mathbb{R}^n$ the equality

$$\lambda_k(B) = \lambda_k(B \cap A) + \lambda_k(B \backslash A)$$

holds. Obviously λ_k defines a translationinvariant measure on $\mathbb{R}^n$ which is also invariant under orthogomal transformations. For $1 \le k \le n$ and $k \in \mathbb{N}$ the measure λ_k induces the usual Lebesgue measure on all k–dimensional affine subspaces with the normalization

$$\lambda_k\{x \in \mathbb{R}^n \mid \|x\|_2 \le 1 \,\text{and}\, x_i = 0 \,\text{for}\, i > k\} = v_k = \pi^{k/2}\Gamma(1+k/2)^{-1},$$

where Γ denotes the Gamma function. For more precise information see [**FED**].

A *measure* μ on $\mathbb{R}^n$ is called *log-concave* if for all compact sets A, B and all $0 \le \lambda \le 1$ the following inequality is satisfied.

$$\mu(\lambda A + (1-\lambda)B) \ge \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

Here $\lambda A + (1-\lambda)B$ denotes the Minkowski sum of two sets. A positive *function* $f : \mathbb{R}^n \to \mathbb{R}$ is called *log-concave*, if the inequality

$$f(\lambda x + (1-\lambda)y) \ge f(x)^\lambda f(y)^{1-\lambda}$$

holds for all $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1$ (with the convention $0^\lambda = 0$). As usual, a positive function f is a *density* for a measure μ if

$$\mu(A) = \int_A f(x)\, d\lambda_n(x)$$

is valid for all λ_n–measurable sets $A \subset \mathbb{R}^n$. In this situation we simply write $\mu = f\lambda_n$. The connection between log-concave measures and log-concave densities was discovered independently by Borell and Prekopka, see [**BOL**], [**PRE**].

(1) Let μ be a measure with density f, then μ is log-concave if and only if there is a log-concave function $\tilde{f}$, such that $\mu = \tilde{f}\lambda_n$ almost everywhere.

For two integrable functions $f, g : \mathbb{R}^n \to \mathbb{R}$ the *convolution* is defined by

$$f * g(y) := \int_{\mathbb{R}^n} f(y-x)g(x)\, d\lambda_n(x).$$

Log-concave measure have important stability properties, which were discovered by Borell, see [**BOL**, Theorem 4.3; Theorem 4.4], and [**DHK**].

(2) Let $\mu = f\lambda_n$ be a log-concave probability measure on $\mathbb{R}^n$, $1 \leq m \leq n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ a linear operator, then the image measure

$$T^{-1}(\mu)(A) := \mu(T^{-1}(A)) \qquad A\ \lambda_m - \text{measurable}$$

is again log-concave and admits a log-concave density.

(3) The convolution of two integrable positive log-concave functions is again log-concave.

Now we will state some results from the so called theory of Banach spaces. For standard Banach space notations and informations about s-numbers we refer to the monographs of Pietsch, [**PI1**] and [**PI2**]. The presented exposition follows closely Pisier's book about the volume of convex bodies, [**PIS**]. Let K_1, K_2 be two subsets of a Banach space Y, we denote by $N(K_1, K_2)$ the smallest natural number N such that K_1 can be covered by translates of K_2, i.e. there are elements $(y_i)_1^N \subset Y$ with

$$K_1 \subset \bigcup_1^N y_i + K_2.$$

The unit ball of a Banach space X isdenoted by B_X. For an operator $T : X \to Y$ between two Banachspaces X, Y the n-th *Kolmogorov, entropy* and *volume number* are defined by

$$\begin{aligned} d_n(T) &:= \inf\{\|Q_E T\| \mid E \subset Y \text{ with } \dim E < n\}, \\ e_n(T) &:= \inf\{\varepsilon > 0 \mid N(T(B_X), \varepsilon B_Y) \leq 2^{n-1}\}, \\ v_n(T) &:= \inf\Big\{ \big(\lambda_n(Q_E T(B_X))/\lambda_n(B_{Y/E})\big)^{1/\kappa n} \mid \\ &\qquad\qquad E \subset Y \text{ with codim } E = n\Big\} \end{aligned}$$

where $Q_E : Y \to Y/E$ denotes the usual quotient mapping and the volume ratio is defined via an isomorphism between $\mathbb{K}^n$ and Y/E. Here κ is 1 in the real case and 2 in the complex case. The volume numbers were studied by Dudley, Milman and Pisier and introduced in its final form by Mascioni, see [**MA**]. By the surjectivity of the entropy numbers one can immediately deduce the following inequality

$$v_n(T) \leq 2e_n(T).$$

Accoding to a theorem of Carl, see [**CP**], entropy numbers and Kolmogorov numbers can be compared as follows.

(4) For $0 < \alpha, \beta < \infty$ there exists a constant $c(\alpha, \beta)$ such that for all $n \in \mathbb{N}$ and all operator $T : X \to Y$

$$\sup_{k\in\mathbb{N}} k^{\alpha}(\ln(1+n/k))^{-\beta} \max(e_k(T), e_k(T^*)) \\ \leq c(\alpha,\beta) \sup_{k\in\mathbb{N}} k^{\alpha}(\ln(1+n/k))^{-\beta} d_k(T).$$

For an operator $u : \ell_2^n \to Y$ the *ℓ–norm* of u is defined by

$$\ell(u) := \left\| \sum_1^n u(e_k) g_k \right\|_{L_2(\Omega;Y)},$$

where$(g_k)_1^n$ are independent, normalized gaussian variables. By trace duality the so called conjugate *ℓ^*–norm* of an operator $v : Y \to \ell_2^n$ is defined by

$$\ell^*(v) := \sup\{ |\operatorname{tr}(vu)| \mid \ell(u) \leq 1\}.$$

A Banach space Y is said to be *K–convex* if there is a constant $c \geq 0$ such that for all $n \in \mathbb{N}$ and $u : \ell_2^n \to Y$

$$\ell(u) \leq c\,\ell^*(u^*).$$

The best possible constant c is denotes by $K(Y)$. In general the K-convexity constant of n–dimensional Banach Y space is relatively small, namely $K(Y) \leq c(1 + \ln n)$. This can be deduced from the following interpolation result due to Pisier.

(5) Let Y be a Banach space and H a Hilbert space, such that (Y, H) is an interpolation couple. Then the interpolation space $[Y, H]_\theta$ is K-convex and the K–convexity constant satisfies

$$K([Y, H]_\theta) \leq c\theta^{-1}.$$

Here we use the complex interpolation method, which can be applied for Banach spaces over the reals after an appropriate complexification. We will only formaly use the above theorem and exactly in the same situation as in [**PIS**, Chapter 7]. Therefore we refer to this book for precise definitions and a proof.

At the end of this preliminaries we want to sketch the prove of a sort of converse inequality to (4).

(6) Let $n, m \in \mathbb{N} Y$ an m-dimensional Banach space and $u : \ell_2^n \to Y$ then we have for all $1 \leq k \leq n$

(i) $\sqrt{k} d_k(u) \leq c(1 + \ln m/k)^2 \sup_{j \geq k/16} \sqrt{j} v_j(u)$.

(ii) $d_k(u) \leq c\sqrt{m/k}(1 + \ln m/k)^2 v_{[k/16]}(u)$.

Sketch of the proof. The first step is based on the iteration procedure developed in [**PIS**, Chapter 9]. Since a similar result was proved in [**PAT**] we omit a proof. For every K-convex Banach space Y and $u : \ell_2^n \to Y$ one has

(i') $\sqrt{k} d_k(u) \leq c_1 K(Y) \sum_{j > k/6} j^{-1/2} v_j(u)$.

(ii') $\sqrt{k} d_k(u) \leq c_1 K^2(Y) \sqrt{n} v_{[k/6]}(u)$.

In fact (ii') can be deduced from (i') by K-convexity arguments and Alexandrov-Fenchels inequality, for similar arguments see [**PAT**]. Since the proof of (i) and (ii) is very similar and we will only prove (i). Let Y be a m-dimensional Banach space, by Pisier's existence proof of Milman's Ellipsoid, see [**PIS**], there is an isomorphism $w : \ell_2^m \to Y$ with

$$\sup_{k \in \mathbb{N}} k d_k(w^{-1}) \leq c_0 m \quad \text{and} \quad \sup_{k \in \mathbb{N}} k e_k(w) \leq c_0 m.$$

We will first assume Y complex. W.l.o.g. we can even assume that $Y = (\mathbb{C}^n, \| \ \|)$ and w is the formal identity. We consider the interpolation space $Y_\theta := [Y, \ell_2^n]_\theta$ and the operator $\iota_\theta : Y \to Y_\theta$. Then we have for $j = 1, \ldots, n$, see [**PI1**] or [**PIS**],

$$\begin{aligned} d_j(\iota_\theta) &\leq (d_j(\iota_0))^\theta = (d_j(w^{-1}))^\theta \leq c_0 (m/j)^\theta \quad \text{and} \\ e_j(\iota_\theta^{-1}) &\leq 2(e_j(\iota_0^{-1}))^\theta = 2(e_j(w))^\theta \leq c_0 (m/j)^\theta. \end{aligned}$$

Now we apply (i') for the operator $u_\theta := u\iota_\theta^{-1} : Y_\theta \to \ell_2^n$. Using the multiplicativity of the volume numbers and (5) we deduce for $\bar{k} := 3/4k$

$$\begin{aligned}\sqrt{k}d_{\bar{k}}(u_\theta) &\le \sqrt{4/3}c_1K(Y_\theta)\sum_{j>k/8} j^{-1/2}v_j(u_\theta)\\ &\le \sqrt{4/3}c_1c\theta^{-1}\sum_{j>k/8} j^{-1}v_j(\iota_\theta^{-1})\sup_{j>k/8}\sqrt{j}v_j(u)\\ &\le \sqrt{4/3}2c_0c_1c\theta^{-1}m^\theta\sum_{j=k/8}^{m} j^{-(1+\theta)}\sup_{j>k/8}\sqrt{j}v_j(u)\\ &\le cc_0c_1c_2\theta^{-2}m^\theta k^{-\theta}\sup_{j>k/8}\sqrt{j}v_j(u).\end{aligned}$$

The multiplicativity of the Kolmogorov numbers now implies

$$\begin{aligned}\sqrt{k}d_k(u) &\le \sqrt{k}d_{\bar{k}}(u_\theta)d_{[k/4]}(\iota_\theta)\\ &\le cc_0c_1c_2\theta^{-2}(m/k)^{2\theta}\sup_{j>k/8}\sqrt{j}v_j(u).\end{aligned}$$

Now we choose $\theta^{-1} := (1+\ln m/k)$ to obtain the result for some new constant c_3. In the real case we use a complexification and have to spend another factor $1/2$. $\square$

1. Log-concave densities and entropy estimates. In the following $f : \mathbb{R}^n \to \mathbb{R}$ will be a positive density of a probability measure $\mu = f\lambda_n$ on $\mathbb{R}^n$. The essential supremum of f is defined by

$$\|f\|_\infty := \inf_{\Omega\subset\mathbb{R}^n,\lambda_n(\Omega)=0}\ \sup_{x\in\mathbb{R}^n\setminus\Omega} f(x).$$

The following lemma is essentially contained in [**MIPA**], since it is basic for our results we sketch a proof.

LEMMA 1.1. *For all convex bodies $B\subset\mathbb{R}^n$ an all $0<p<\infty$ the following inequality holds*

$$(n/(n+p))^{1/p} \le \left(\int_{\mathbb{R}^n}\|x\|_B^p f(x)\quad d\lambda_n(x)\right)^{1/p}\|f\|_\infty^{1/n}\lambda_n(B)^{1/n}.$$

Proof. We can assume $A := \|f\|_\infty < \infty$. Then we define $g(x) := A^{-1}f(x)$. By [**MIPA**, Lemma 2.1] the function

$$F(p) := \left(((n+p)/n\lambda_n(B))\int_{\mathbb{R}^n}\|x\|_B^p g(x)\,d\lambda_n(x)\right)^{1/(p+n)}$$

is increasing. The inequality $F(p)^{1+n/p} \geq F(0)^{1+n/p}$ implies with $\int f(x)\, d\lambda_n(x) = 1$

$$\left(\int_{\mathbb{R}^n} \|x\|_B^p f(x)\, d\lambda_n(x)\right)^{1/p} \geq \left(\frac{n\lambda_n(B)A}{n+p}\right)^{1/p} (\lambda_n(B)A)^{-1/n-1/p}$$
$$\geq \left(\frac{n}{n+p}\right)^{1/p} (\lambda_n(B)A)^{-1/n}.$$

□

Before we discuss applications of Lemma 1.1 we want to prove a reverse inequality for log-concave densities.

PROPOSITION 1.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a log-concave, symmetric density of a probability measure $\mu = f\lambda_n$. For every $0 < p < \infty$ there exists a convex body B_p with*

$$\left(\int_{\mathbb{R}^n} \|x\|_{B_p}^p f(x)\, d\lambda_n(x)\right)^{1/p} \|f\|_\infty^{1/n} \lambda_n(B_p)^{1/n} \leq c_0(1 + p/n).$$

Proof. By [**BA**, Theorem 5.5] the function

$$\|x\|_r := \left(\int_0^\infty f(tx)t^r\, dt\right)^{-1/(r+1)} x \neq 0; \quad \|0\|_r := 0$$

defines a norm for all $0 < r < \infty$ with unit ball B_r, say. For $0 < p < \infty$ we set $r := p + n - 1$. For $x \neq 0$ we deduce from [**BA**, Theorem 5.3]

$$\left(\int_0^\infty f(tx)t^r\, d(t)\right)^n \leq$$
$$f(0)^{n-1-r}\Gamma(r+1)^n\Gamma(n)^{-(r+1)} \left(\int_0^\infty f(tx)t^{n-1}\, dt\right)^{r+1}.$$

Using polar coordinates this implies (S^{n-1} is the unit sphere in $\mathbb{R}^n$)

$$
\begin{aligned}
\lambda_n(B_r) &= v_n \lambda_{n-1}(S^{n-1})^{-1} \int_{S^{n-1}} \|x\|_r^{-n} \, d\lambda_{n-1}(x) \\
&= n^{-1} \int_{S^{n-1}} \left(\int_0^\infty f(tx) t^r \, dt \right)^{n/r+1} d\lambda_{n-1}(x) \\
&\leq n^{-1} f(0)^{-p/r+1} \Gamma(r+1)^{n/r+1} \Gamma(n)^{-1} \\
&\quad \times \int_{S^{n-1}} \int_0^\infty f(tx) t^{n-1} \, dt \, d\lambda_{n-1}(x) \\
&= f(0)^{-p/r+1} \Gamma(r+1)^{n/r+1} \Gamma(n+1)^{-1} \int_{\mathbb{R}^{n-1}} f(x) \, d\lambda_n(x).
\end{aligned}
$$

Since $\mu = f\lambda_n$ is a probability measure we obtain

$$
\begin{aligned}
&\lambda_n(B_r)^{1/n} \left(\int_{\mathbb{R}^n} \|x\|_{B_r}^p f(x) \, d\lambda_n(x) \right)^{1/p} \\
&\quad = \lambda_n(B_r)^{1/n} \left(\int_{S^{n-1}} \|x\|_r^p \int_0^\infty f(tx) t^{p+n-1} \, dt \, d\lambda_{n-1}(x) \right)^{1/p} \\
&\quad = \lambda_n(B_r)^{1/n} \Bigg(n v_n \lambda_{n-1}(S^{n-1})^{-1} \\
&\qquad\qquad \times \int_{S^{n-1}} \|x\|_r^{p-(r+1)} \, d\lambda_{n-1}(x) \Bigg)^{1/p} \\
&\quad = n^{1/p} \lambda_n(B_r)^{(n+p)/np} \\
&\quad \leq n^{1/p} f(0)^{-1/n} \Gamma(r+1)^{1/p} \Gamma(n+1)^{-1/n-1/p} \\
&\quad = f(0)^{-1/n} \Gamma(n+p)^{1/p} \Gamma(n)^{-1/p} (n!)^{-1/n}.
\end{aligned}
$$

Elementary compulations show

$$
\Gamma(p+n) \leq c_0 (p+n)^p \Gamma(n)
$$

for some absolute constant c_0, which implies

$$
\Gamma(n+p)^{1/p} \Gamma(n)^{-1/p} (n!)^{-1/n} \leq e c_0 (1 + p/n).
$$

Since a symmetric log-concave density admits its maximum in 0 the result is proved. □

As a consequence of Lemma 1.1 and proposition 1.2 we immediately get the key Lemma 2 from the introduction. In order to obtain satisfactory entropy estimates for a probability measure $\mu = f\lambda_n$ it is useful to consider the density of an orthogonal projection P_H onto a k-dimensional subspace H of $\mathbb{R}^n$ with orthogonal space $H^\perp$. Fubini's theorem guaranties that $f_H : H \to \mathbb{R}$

$$f_H(y) := \int_{y+H^\perp} f(x)\, d\lambda_{n-k}(x),$$

is a λ_k-measurable density for the image measure

$$\mu_H := P_H^{-1}(\mu); \quad \mu_H = f_H \lambda_k.$$

REMARK 1.3. *Let μ be a probability measure in isotropic position, the for all k-dimensional subspaces $H \subset \mathbb{R}^n$ one has*

$$1 \leq \sqrt{4\pi e} L_\mu \|f_H\|_\infty^{1/k}.$$

In particular, for $H = \mathbb{R}^n$ this means

$$1 \leq \sqrt{4\pi e} L_\mu \|f\|_\infty^{1/n}.$$

Proof. We apply Lemma 1.1 for f_H and the euclidean unit ball $B_H = B_2^n \cap H$. Since the image measure μ_H is also in isotropic position we obtain the assertion

$$(k/k+2)^{1/2} \leq L_\mu \sqrt{k} \|f_H\|_\infty^{1/k} \lambda_k(B_H)^{1/k} \leq \sqrt{2\pi e} L_\mu \|f_H\|_\infty^{1/k}.$$

□

For the following we define for $1 \leq m \leq n$

$$S_m := \sup_{1 \leq k \leq m} \sup_{H \subset \mathbb{R}^n, \dim H = k} \|f_H\|_\infty^{1/k}.$$

The next theorem establishes the equivalence between upper estimates for S_m and entropy estimates for processes induced by the probability measure $\mu = f\lambda_n$.

THEOREM 1.4. *Let $\mu = f\lambda_n$ be a probability measure on $\mathbb{R}^n$, Y a Banach space and $u : \ell_2^n \to Y$ an operator with $rg(u) \leq m$. Then we have for all $k \in \mathbb{N}$*

(i) $\sqrt{k} v_k(u) \leq e S_m \exp\left(\int_{R^n} \ln \|u(x)\| f(x)\, d\lambda_n(x)\right).$

(ii)

$$\sqrt{k}\max\{d_k(u), e_k(u), e_k(u*)\} \leq \\ c_0(\ln(1+m/k))^2\, S_m \exp\left(\int_{\mathbb{R}^n} \ln\, \|u(x)\|\, f(x)\, d\lambda_n(x)\right).$$

Vice versa, if f is in addition a log-concave density and c a constant such that for all $1 \leq k \leq m$, all Banach space Y and all operator $u : \ell_2^n \to Y$ with $rg(u) \leq k$ the inequality

$$\sqrt{k}v_k(u) \leq c\left(\int_{\mathbb{R}^n} \|u(x)\|_Y^k f(x)\, d\lambda_n(x)\right)^{1/k}$$

is satisfied, then we have $S_m \leq c_0 c$.

Proof. For the first part we have only to establish (i), because (ii) follows immediately from (i) using (4) and (6). By the definition of the volume numbers we can assume $u(\ell_2^n) = Y$ and Y is of dimension k. Therefore there exists a k-dimensional subspace H and an isomorphism $\widehat{u} : H \to Y$, such that $u = \widehat{u}P_H$. Clealy, we define $B := (\widehat{u})^{-1}(B_Y) \subset H$ and apply Lemma 1.1 to deduce for all $0 < p < \infty$

$$\begin{aligned}
\sqrt{k}(\lambda_k(B_H)/\lambda_k(B))^{1/k} &\leq \sqrt{2\pi e}\lambda_k(B)^{-1/k} \\
&\leq ((k+p)/k)^{1/p}\left(\int_H \|x\|_B^p f_H(x)\, d\lambda_k(x)\right)^{1/p} \|f_H\|_\infty^{1/k} \\
&\leq ((k+p)/k)^{1/p}\left(\int_{\mathbb{R}^n} \|P_H x\|_B^p f(x)\, d\lambda_n(x)\right)^{1/p} S_m \\
&\leq ((k+p)/k)^{1/p}\left(\int_{\mathbb{R}^n} \|u(x)\|_Y^p f(x)\, d\lambda_n(x)\right)^{1/p} S_m,
\end{aligned}$$

where B_H is the unit ball induced by euclidean norm on H. Sending p to 0 implies (i). For the second part let f be a log-concave density and $H \subset \mathbb{R}^n$ a k-dimensional subspace. By (2) f_H is again log-concave. Proposition 1.2 implies the existence of a convex body $B \subset H$ with

$$\left(\int_H \|x\|_B^k f_H(x)\, d\lambda_k(x)\right)^{1/k} \|f_H\|_\infty^{1/k} \lambda_k(B)^{1/k} \leq c_0.$$

Now we define $Y := (H, \| \ \|_B)$ and $u := P_H : \ell_2^n \to Y$ which is of rank k. Our assumption implies

$$\begin{aligned}\lambda_k(B)^{-1/k} &\leq \sqrt{k}(\lambda_k(B_H)/\lambda_k(B))^{1/k} \\ &\leq c\left(\int_{\mathbb{R}^n} \|u(x)\|_Y^k f(x)\, d\lambda_n(x)\right)^{1/k} \\ &= c\left(\int_{R^n} \|P_H(x)\|_B^k f_H(x)\, d\lambda_k(x)\right)^{1/k} \\ &\leq cc_0 \|f_H\|_\infty^{-1/k} \lambda_k(B)^{-1/k}.\end{aligned}$$

Taking the supremum over all $1 \leq k \leq m$ yields the assertion. □

Proof of Theorem 3. Let $K \subset \mathbb{R}^n$ in isotropic position and symmetric with respect to all hyperplanes $\langle x, e_j \rangle = 0$. In this situation the sequence of coordinate functionals $x_k : (K, \lambda_n) \to \mathbb{R}; x \mapsto \langle x, e_k \rangle$ has the same distribution as the sequence $(\varepsilon_k x_k)_1^n$, where $(\varepsilon_k)_1^n$ is a sequence of independent Bernoulli variables on $(\mathbb{D}, \nu)$. By Borell's lemma, see [**MS**, Appendix III.4] and a well-known symmetry argument, see [**PIS2**, proposition 3.2], we obtain for every operator $u : \ell_2^n \to Y$

$$L_K \int_{\mathbb{D}_n} \left\| \sum_1^n u(e_k)\varepsilon_k \right\|_Y d\nu \leq c_0 \int_K \|u(x)\|_Y \, dx.$$

With Theorem 1 we obtain the upper estimate from the second part of Theorem 1.4

$$L_K \lambda_{n-k}(K \cap H)^{1/k} \leq c_1,$$

for al k-codimensional subspaces $H \subset \mathbb{R}^n$. Since

$$\lambda_{n-k}(K \cap (y + H))$$

attains its maximum in 0 by Brunn-Minkowski's inequality, see [**MS**, Appendix III], the lower estimate follows immediately from Remark 1.3.

2. Symmetrization. In the following let $f : \mathbb{R}^n \to \mathbb{R}$ be a symmetric, bounded density of a probability measure $\mu = f\lambda_n$. For $k = 1, \ldots, n-1$ we consider the function $f_k : \mathbb{R}^k \to \mathbb{R}$ defined by

$$f_k(x_1, \ldots, x_k) := \int_{\mathbb{R}^{n-k}} f(x_1, \ldots, x_k, t_{k+1}, \ldots, t_n)\, dt_{k+1} \cdots dt_n.$$

We set $f_n = f$. In addition to the conditional expectaton f_k we need the following midpoint functions $m_k : \mathbb{R}^{k-1} \to \mathbb{R}$ $(k = 2, \dots, n)$

$$m_k(x_1, \dots, x_{k-1}) := \inf\{t \in \mathbb{R} \mid \int_{-\infty}^{t} f_k(x_1, \dots, x_{k-1}, s)\, ds = 1/2 f_{k-1}(x_1, \dots, x_{k-1})\}.$$

Clealy, f_k and m_k are measurable functions. For completeness we set $m_1 := 0$. The following lemma enables us to construct Bernoulli variables on $(\mathbb{R}^n, \mu)$.

LEMMA 2.1. *Let f be a symmetric density of a probability measure μ on $\mathbb{R}^n$, then the sequence $(\varepsilon_k)_1^n$ defined on $(\mathbb{R}^n, f\lambda_n)$ by*

$$\varepsilon_k(x) := \operatorname{sign}(x_k - m_k(x_1, \dots, x_{k-1}))$$

is a sequence of independent Bernoulli variablies.

Proof. By induction on k one can easily prove that for all $\delta \in \mathbb{D}_k$ one has

$$\mu(\varepsilon_1 = \delta_1, \dots, \varepsilon_k = \delta_k) = 2^{-k}.$$

In particular, we obtain that for $k = n$ and $A \subset \mathbb{D}_n$ we have

$$\mu((\varepsilon_1, \dots, \varepsilon_n) \in A) = 2^{-n} \operatorname{card}(A).$$

□

The next theorem ensures that the Bernulli sequence constracted above is strongly correlated to the coordinate functionals.

THEOREM 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a symmetric, bounded density of a probability measure on $\mathbb{R}^n$ then there exists a Bernoulli sequence $(\varepsilon_k)_1^n$ on $(\mathbb{R}^n, f\lambda_n)$ such that the matrix $A = (a_{ik})_{i,k=1}^{n}$*

$$a_{ik} := \int_{\mathbb{R}^n} \langle x, e_j \rangle \varepsilon_k(x) f(x)\, d\lambda_n(x)$$

is a lower triangle matrix and

$$1/4e \leq \mid \det\ A \mid^{1/n} \|f\|_\infty^{1/n}.$$

Proof. The definition of the midpoint functions is choosen such that for all $x_1, \dots, x_{k-1}$

$$\int_{-\infty}^{\infty} \operatorname{sign}(t - m_k(x_1, \dots, x_{k-1})) f_k(x_1, \dots, x_{k-1}, t)\, dt = 0.$$

For $i < k$ this immediately implies $a_{ik} = 0$ whereas for $i = k > 1$ we get

$$\begin{aligned} a_{kk} &= \int_{\mathbb{R}^{k-1}} \int_{-\infty}^{\infty} \operatorname{sign}(t - m_k(x_1, \dots, x_{k-1}))t \\ &\qquad \times f_k(x_1, \dots, x_{k-1}, t)\, dt\, dx_1, \dots, dx_{k-1} \\ &= \int_{\mathbb{R}^n} \mid x_k - m_k(x_1, \dots, x_{k-1}) \mid f(x)\, d\lambda_n(x) \\ &\quad + \int_{\mathbb{R}^{k-1}} m_k(x_1, \dots, x_{k-1}) \int_{-\infty}^{\infty} \operatorname{sign}(t - m_k(x_1, \dots, x_{k-1})) \\ &\qquad \times f_k(x-1, \dots, x_{k-1}, t)\, dt\, dx_1, \dots, dx_{k-1}. \end{aligned}$$

Therefore we have for all $k = 1, \dots, n$

$$a_{kk} = \int_{\mathbb{R}^n} \mid x_k - m_k(x_1, \dots, x_{k-1}) \mid f(x)\, d\lambda_n(x).$$

We define the volume preserving map $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T(x) := x - \sum_1^n m_k(x_1, \dots, x_{k-1}) e_k.$$

For the density $\tilde{f}$ of the image measure $\mu_T := T^{-1}(\mu)$ we obtain

$$\tilde{f}(x) := f(x + \sum_1^n m_k(x_1, \dots, x_{k-1}) e_k).$$

By Lemma 1.1 we deduce for $\tau = (a_{kk})_1^n$ and

$$B_1^n := \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_1^n \mid x_k \mid \leq 1 \right\}$$

$$
\begin{aligned}
n/n+1 &\le \left(\int_{\mathbb{R}^n} \|x\|_{D_\tau(B_1^n)}\tilde{f}(x)\, d\lambda_n(x)\right) \|\tilde{f}\|_\infty^{1/n}\lambda_n(D_\tau(B_1^n))^{1/n} \\
&\le \int_{\mathbb{R}^n} \sum_1^n a_{kk}^{-1}|x_k|\tilde{f}(x)\, d\lambda_n(x)\ \|\tilde{f}\|_\infty^{1/n} \left(\prod_1^n a_{kk}\right)^{1/n} 2(n!)^{-1/n} \\
&\le \int_{\mathbb{R}^n} \sum_1^n a_{kk}^{-1}|x_k - m_k(x_1, \dots , x_{k-1})| f(x)\, d\lambda_n(x) \\
&\qquad \times \|f\|_\infty^{1/n}\, |\det A|^{1/n} 2en^{-1} \\
&= 2e\, \|f\|_\infty^{1/n}\, |\det\ A|^{1/n}.
\end{aligned}
$$

$\square$

Proof of Theorem 4. Let f be a symmetric, bounded density of a probability measure $\mu = f\lambda_n$ and A be the matrix from Theorem 2.2. The assertion will be proved for $M := |\det\ A|A^{-1}$. First we want to show that for all operator $w : \ell_2^{nm} \to Y$ and

$$B := \sum_{j=1}^m \sum_{i=1}^n \sum_{k=1}^n a_{ik} e_{(j-1)n+i} \otimes e_{(j-1)n+k}$$

we have

$$
\begin{aligned}
(*) \quad & \ell^*((wB)^*) \le \sqrt{\pi/2} \\
& \times \left(\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \|w(x_1, \dots , x_m)\|_Y^2\, d\mu(x_1), \dots , d\mu(x_m)\right)^{1/2}.
\end{aligned}
$$

For $u : \ell_2^n \to Y$ and α with $\|\alpha\|_2 = 1$ we apply (*) to

$$w := \sum_{j=1}^m \sum_{k=1}^n \alpha_j e_{(j-1)n+k} \otimes u(e_k)$$

the definition of the K-convexity implies

$$
\begin{aligned}
\ell(uM) &\le 4e\sqrt{\pi/2}K(Y)\, \|f\|_\infty^{1/n} \\
& \times \left(\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left\|\sum_{j=1}^m \alpha_j u(x_j)\right\|_Y^2\, d\mu(x_1) \cdots\, d\mu(x_m)\right)^{1/2}
\end{aligned}
$$

If f is the characteristic function of a convex body the density $\chi_{M^{-1}(K)}$ is log-concave. In order to apply Theorem 1.4 we have to establish entropy estimates for this density and orthorgonal projections of rank k. Since the K convexity constant of space with dimension k is less than $c_0\ (1+\ln k)$ the entropy estimates follows from the transformation formula and entropy estimates of the ℓ-norm proved by Pajor/Tomczak, see [**PIS**]. For simplicity let us now assume $m=1$. Let $v:\ell_2^n\to Y^*$ with $\ell(v)\le 1$. Then we deduce from the comparison between Bernoulli and gaussian variables, see [**PIS2**],

$$\begin{aligned}
|tr((uA)^*v)| &= \left|\sum_1^n\sum_1^n a_{ik}\langle u(e_i),v(e_k)\rangle\right| \\
&= \left|\sum_1^n\sum_1^n \left(\int_{\mathbb{R}^n}\langle x,e_i\rangle\varepsilon_k(x)\ d\mu(x)\right)\langle u(e_i),v(e_k)\rangle\right| \\
&= \left|\int_{\mathbb{R}^n}\langle u(x),\sum_1^n\varepsilon_k v(e_k)\rangle\ d\mu(x)\right| \\
&\le \left(\int_{\mathbb{R}^n}\|u(x)\|_Y^2\ d\mu(x)\right)^{1/2}\left(\int_{\mathbb{R}^n}\left\|\sum_1^n\varepsilon_k v(e_k)\right\|_Y^2 *d\mu\right)^{1/2} \\
&\le \sqrt{\pi/2}\left(\int_{\mathbb{R}^n}\|u(x)\|_Y^2\ d\mu(x)\right)^{1/2}.
\end{aligned}$$

□

For the following we want to assume that f is a symmetric density of a probability measure $\mu=f\lambda_n$ in isotropic position and $\|f\|_\infty=1$. The next lemma studies the singular numbers of the matrix A defined by Theorem 2.2.

LEMMA 2.3. *For $1\le k\le n$ we have $d_k(A^{-1})\le c_1^{n/k}L_\mu^{(n-k)/k}$.*

Proof. By Theorem 4 and the isotropic position we have $\ell(A)\le c_0L_\mu n^{1/2}$. From (4) we deduce $e_n(A)\le c_2L_\mu$ and we set $\varepsilon:=(c_2L_K)^{-1}$. By [**GKS**] we can compare the singular numbers of A^{-1} with covering numbers of A^{-1}. Applying this and a well known maximality argument, see [**PAT**, Lemma 3.2] we obtain

$$\begin{aligned}
d_k(A^{-1}) &\le N(A^{-1}(B_2^n), \varepsilon B_2^n)^{1/k}\varepsilon \\
&\le 3^{n/k}\left(\lambda_n(A^{-1}(B_2^n) + \varepsilon B_2^n)/\lambda_n(\varepsilon B_2^n)\right)^{1/k}\varepsilon \\
&\le 3^{n/k}N(\varepsilon B_2^n, A^{-1}(B_2^n))^{1/k}\left(\lambda_n(2A^{-1}(B_2^n))/\lambda_n(\varepsilon B_2^n)\right)^{1/k}\varepsilon \\
&\le 3^{n/k}2^{n/k}(2^n|\det\ A^{-1}|\varepsilon^{-n})^{1/k}\varepsilon \le \left(36\sqrt{2\pi e}\right)^{n/k}\varepsilon^{(k-n)/k}.
\end{aligned}$$

□

Now we can prove a simular estimate as in Theorem 4 with the only difference that the L_μ constant instead of the K-convexity constant is involved.

PROPOSITION 2.4. *Let f a symmetric density of a probability measure $\mu = f\lambda_n$ in isotropic position with $\|f\|_\infty = 1$, A the matrix from Theorem 2.2, $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}^m$ with $\|\alpha\|_2 = 1$. Then for all Banach space Y, all operator $u : \ell_2^n \to Y$ and $1 \le k \le n$ we have*

$$\sqrt{k}d_k(uA) \le c_0(1+\ln L_\mu)\, n/k\,(1+\ln n/k)^2$$
$$\times\left(\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\left\|\sum_{j=1}^m \alpha_j u(x_j)\right\|_Y^2 d\mu(x_1)\cdots d\mu(x_m)\right)^{1/2}.$$

Proof. Let $1 \le k \le n$, by Lemma 2.3 we can find a subspace $H \subset \mathbb{R}^n$ with $\operatorname{codim} H < k$ and $\|P_H A^{-1}\| \le c_1^{n/k} L_\mu^{(n-k)/k}$. For simplicity let us assume $m = 1$. We denote by S the smallest constant c such that for all $u : \ell_2^n \to Y$ the inequality

$$\sqrt{j}d_j(uAP_H) \le c(1+\ln(rg(u)/j))\left(\int_K \|u(x)\|_Y^2\, d\lambda_n(x)\right)^{1/2}$$

holds. Now let

$$u : \ell_2^n \to Y \text{ with } rg(u) = m \le n \text{ and } \left(\int \|u(x)\|_Y^2\, d\mu(x)\right)^{1/2} = 1.$$

By the definition of S there is a subspace $F \subset Y$ with $\dim\ F < j$ and

$$\sqrt{j}\|Q_F uAP_H\| \le S(1+\ln m/j).$$

We define $H_1 \subset H$ as the orthogonal complement of $\ker(Q_F u A P_H)$ in H and $H_2 := A(H_1)$. Furthermore, we consider the convex body $B := A(Q_F u A P_H)^{-1}(B_{Y/F}) \cap H_2$ contained in H_2. By E_0, E_1 we denote an appropriate complexification of $(H_2, \| \ \|_B), (H_2, \| \ \|_2)$. Clealy, (E_0, E_1) is an interpolation couple and E_1 is a Hilbert space. Since μ is in isotropic position we obtain for $E_\theta := [E_0, E_1]_\theta$

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} \|P_{H_2}(x)\|_{E_\theta}^2 \, d\mu(x)\right)^{1/2} \\
&\le \left(\int_{\mathbb{R}^n} \|P_{H_2}(x)\|_{E_0}^2 \, d\mu(x)\right)^{(1-\theta)/2} \left(\int_{\mathbb{R}^n} \|P_{H_2}(x)\|_{E_1}^2 \, d\mu(x)\right)^{\theta/2} \\
&\le \sqrt{2}(L_\mu \sqrt{m})^\theta .
\end{aligned}$$

Now we denote $u_\theta := P_{H_2} : \ell_2^n(\mathbb{C}) \to E_\theta$ and $\iota_\theta : E_\theta \to E_0$. By the choice of F and H we have the following estimate for the norm of ι_θ

$$\begin{aligned}
\|\iota_\theta\|^{1/\theta} &\le \|\iota_1 : E_1 \to E_0\| \le \sqrt{2}\|Q_F u P_{H_2}\| \le \sqrt{2}\|Q_F u A P_H A^{-1}\| \\
&\le 2^{1/2}\|Q_F u A P_H\| \|P_H A^{-1}\| \le 2^{1/2} j^{-1/2}(m/j) S c_1^{n/k} L_\mu^{(n-k)/k}.
\end{aligned}$$

With Pajor/Tomczak's inequality, see [**PIS**], Theorem 4 and (5) we obtain for $u_0 = P_{H_2} : \ell_2^n \to E_0$

$$\begin{aligned}
\sqrt{j} d_j(u_0 A) &\le c_0 \|\iota_\theta\| \, \ell(u_\theta) \\
&\le c_0^2 \|\iota_\theta\| K(E_\theta) \left(\int_K \|P_{H_2}(x)\|_{E_\theta}^2 \, d\lambda_n(x)\right)^{1/2} \\
&\le c_0^3 \theta^{-1} \left(2^{1/2}(m/j)^{3/2} S (c_1 L_\mu)^{n/k}\right)^\theta .
\end{aligned}$$

Now we choose $\theta^{-1} := 1 + \ln\left(2^{1/2}(m/j)^{3/2} S (c_1 L_\mu)^{n/k}\right)$. Passing from the complex linear to the real linear operator and using the additivity of the Kolmogorov numbers, see [**PI2**] we obtain

$$\begin{aligned}
\sqrt{j} d_{3j-2}(u A P_H) &\le \sqrt{j} d_{2j-1}(Q_F u A P_H) \le \sqrt{j} d_j(u_0 A) \\
&\le c_3 (1 + \ln m/j)(1 + \ln(n/k L_\mu S)).
\end{aligned}$$

Passing from j to $3j-2$ we have proved

$$S \leq c_3\sqrt{3}(1+\ln 2)(n/k)\left(1+\ln\left(\frac{n}{k}L_\mu S\right)\right),$$

which is only possible if

$$S \leq c_4(n/k)(1+\ln n/k)(1+\ln L_\mu).$$

For an arbitrary operator $u : \ell_2^n \to Y$ we apply this estimate for $j = k$ and obtain

$$\begin{aligned}\sqrt{k}d_{2k-1}(uA) &\leq \sqrt{k}d_k(uAP_H)\\ &\leq c_4(1+\ln L_\mu)(n/k)(1+\ln n/k)^2\\ &\qquad \times\left(\int_{\mathbb{R}^n}\|u(x)\|_Y^2\,d\mu(x)\right)^{1/2}.\end{aligned}$$

□

The proof of Proposition 2.3 even shows that the random variable AP_H admits good entropy estimates in the sense of Theorem 1.4. In the whole argument the symmetry of f is not really used. It would be sufficent to assume that the hyperplane $x_1 = 0$ divides the measure space into equal parts.

3. Convolution and symmetric norms. In the following $f : \mathbb{R}^n \to \mathbb{R}$ will be a symmetric, bounded density of a probability measure $\mu = f\mu$. For $\alpha \in \mathbb{R}^m$ we denote by f_α the density of the vector valued random variable

$$Z_\alpha : (\mathbb{R}^{nm}, \bigotimes_{j=1}^{m}\mu) \to \mathbb{R}^n \ ; \ (x_1, x_2, \dots, x_m) \mapsto \sum_{j=1}^{m}\alpha_j x_j.$$

If f is log-concave also f_α, see (3). In this case the key Lemma 2 implies the equivalence between volume estimates for the norm $[[\alpha]]_B$ and upper estimates of f_α. We start with an easy lemma.

Lemma 3.1. *Let $B \subset \mathbb{R}^n$ be a convex body. For the volume of*

$$B_2^m(B) := \left\{(x_1, x_2, \dots, x_m) \in \mathbb{R}^{nm} \;\middle|\; \sum_{j=1}^{m}\|x_j\|_B^2 \leq 1\right\}.$$

we have $\lambda_{nm}(B_2^m(B)) = \Gamma(1+nm/2)^{-1}\Gamma(1+n/2)^m\lambda_n(B)^m$.

Proof. We use the formula

$$\lambda_n(K) = \Gamma(1+n/2)^{-1}\int_{\mathbb{R}^n} \exp(-\|x\|_K^2)\, d\lambda_n(x),$$

which was observed by Meyer and Pajor, see also [**BA2**, Lemma 7]. Hence we get

$$\begin{aligned}
\lambda_{nm}(B_2^m(B)) &= \Gamma(1+nm/2)^{-1} \\
&\quad \times \int_{\mathbb{R}^{nm}} \exp\left(-\sum_{j=1}^m \|x_j\|_B^2\right)\, d\lambda_n(x_1),\dots, d\lambda_n(x_m) \\
&= \Gamma(1+nm/2)^{-1}\left(\int_{\mathbb{R}^n}\exp(-\|x\|_B^2)\, d\lambda_n(x)\right)^m \\
&= \Gamma(1+nm/2)^{-1}\Gamma(1+n/2)^m\lambda_n(B)^m.
\end{aligned}$$

□

Proof of Theorem (5i), (6i). We will show that for all convex body $B \subset \mathbb{R}^n$ with $\lambda_n(B) = 1$ and for $\alpha = (1,\dots,1) \in \mathbb{R}^m$ we have

$$\sqrt{m} \le 16[[\alpha]]_B\ \|f\|_\infty^{1/n}.$$

For this we define the orthogonal $m\times m$ matrix $M = (m_{jk})_{j,k=1}^m$ by

$$m_{jk} := 2(2m+1)^{-1/2}\cos(2\pi jk/(2m+1)).$$

Since we have $|m_{jk}| \le \sqrt{2/m}$ the unconditionality of the norm $[[\ ]]$ implies

$$\sum_{k=1}^m [[(m_{jk})_{j=1}^m]]_B^2 \le 2[[\alpha]]_B^2.$$

Therefore we deduce from Lemma 1.1 and Lemma 3.1

$$
\begin{aligned}
\sqrt{2}[[\alpha]]_B &\geq \left(\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \sum_{k=1}^{m} \left\| \sum_{j=1}^{m} m_{jk} x_j \right\|_B^2 \, d\mu(x_1) \cdots d\mu(x_m) \right)^{1/2} \\
&= \Big(\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \| M \otimes Id_{\mathbb{R}^n}(x_1, \ldots, x_m) \|_{B_2^m(B)}^2 \\
&\qquad \times d\mu(x_1) \cdots d\mu(x_m) \Big)^{1/2} \\
&\geq 3^{-1/2} \lambda_{nm} (M \otimes Id_{\mathbb{R}^n}(B_2^m(B))^{-1/nm} \, \| f \cdots f \|_\infty^{-1/nm} \\
&= 3^{-1/2} \lambda_{nm} (B_2^m(B))^{-1/nm} \, \| f \|_\infty^{-1/n} \\
&= 3^{-1/2} \Gamma(1 + mn/2)^{1/nm} \Gamma(1 + n/2)^{-1/n} \, \| f \|_\infty^{-1/n} \\
&= 1/4e \, \pi^{1/2} 3^{-1/2} \sqrt{m} \, \| f \|_\infty^{-1/n}.
\end{aligned}
$$

□

Proof of Theorem (5iii), (6i). Let us denote by g the m-fold convolution of f, which is again log-concave and symmetric. By the key Lemma 2 it is sufficient to prove an upper estimate for g. We set $p = 2n$. By proposition 1.2 there is a convex body $B \subset \mathbb{R}^n$ with $\lambda_n(B) = 1$ and

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left\| \sum_{j=1}^{m} x_j \right\|_B^p \, d\mu(x_1) \cdots d\mu(x_m) \right)^{1/p} \\
&\qquad\qquad = \left(\int_{\mathbb{R}^n} \| x \|_B^p g(x) \, d\lambda_n(x) \right)^{1/p} \leq c_1 (g(0))^{-1/n}.
\end{aligned}
$$

We define the n-dimensional Banach space $Y = (\mathbb{R}^n, \| \ \|_B)$. Now let us fix $x_1, \ldots, x_m$ and consider the operator $u : \ell_2^m \to Y$ defined by

$$
u := \sum_{j=1}^{m} e_j \otimes x_j.
$$

Using Theorem 1 we obtain

$$
\sqrt{n} v_n(u) \leq c_0 \left(\int_{\mathbb{D}_m} \left\| \sum_{j=1}^{m} x_j \varepsilon_j \right\|_Y^p \, d\nu(\varepsilon) \right)^{1/p}.
$$

On the other hand the definition of the volume number and Cauchy/Binet's determinant formula tell us

$$\begin{aligned}\sqrt{n}v_n(u) &= \sqrt{n}\,(\lambda_n(u(B_2^m))/(B))^{1/n} = \sqrt{n}v_n^{1/n}|\det\ u^*u|^{1/2n}\\ &\geq |\det\ u^*u|^{1/2n} = \left(\sum_{1\leq i_1<\cdots<i_n\leq n} |\det(x_{i_1},\dots,x_{i_n})|^2\right)^{1/2n}.\end{aligned}$$

Now we integrate over these two inequalities and obtain

$$\begin{aligned}L_\mu^{2n}n!\binom{m}{n} &= \sum_{1\leq i_1<\cdots<i_n\leq n}\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}|\det(x_{i_1},\dots,x_{i_n})|^2\\ &\qquad\qquad \times d\mu(x_{i_1})\cdots d\mu(x_{i_n})\\ &= \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\sum_{1\leq i_1<\cdots<i_n\leq n}|\det(x_{i_1},\dots,x_{i_n})|^2\\ &\qquad\qquad \times d\mu(x_1)\cdots d\mu(x_m)\\ &\leq c_0^{2n}\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\int_{\mathbb{D}_m}\left\|\sum_{j=1}^m x_j\varepsilon_j\right\|_B^p d\nu(\varepsilon)d\mu(x_1)\dots d\mu(x_m)\\ &= c_0^{2n}\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\left\|\sum_{j=1}^m x_j\right\|_B^p d\mu(x_1)\dots d\mu(x_m).\end{aligned}$$

Where the last equality follows from the symmetry and independence of the sequence $(x_j)_{j=1}^m$. Finally we obtain

$$\sqrt{m}L_\mu g(0)^{1/n} \leq \sqrt{2e}c_0c_1.$$

□

It is easy to see that the above estimate is sharp, when μ is in isotropic position. As a corollary to Theorem (5iii) we want to state entropy estimates for the m-fold convolution, showing how fast iterated convolution leads to normal distributed variables.

COROLLARY 3.2. *Let f be a symmetric, log-concave density of a probability $\mu = f\lambda_n$ in isotropic position. Then for all operator*

$u : \ell_2^n \to Y$ with $rg(u) \le m$ and $1 \le k \le rg(u)$ we have

$$L_\mu \sqrt{k} d_k(u) \le c_0 (1 + \ln rg(u)/k)^2 \times \left(\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left\| \frac{1}{\sqrt{m}} \sum_{j=1}^m u(x_j) \right\|_Y^2 d\mu(x_1) \cdots d\mu(x_m) \right)^{1/2}.$$

Proof. Let us denote by g the m-fold convolution of f. By Theorem 1.4 we have to show that for all $1 \le k \le m$ and all subspace H with dim $H = k$ we have

$$L_\mu g_H(0)^{1/k} \le c_0 m^{-1/2}.$$

Here g_H is the density of the random variable

$$P_H Z : (\mathbb{R}^{nm}, \bigotimes_{j=1}^m \mu) \to H;$$
$$(x_1, x_2, \ldots, x_m) \mapsto P_H \left(\sum_{j=1}^m x_j \right) = \sum_{j=1}^m P_H(x_j).$$

Therefore g_H is the m-fold convolution of the density f_H of $\mu_H = (P_H)^{-1}(\mu)$ which is also in isotropic position and has the same constant of isotropy. An application of Theorem (5iii) yields the assertion. □

Proof of Theorem (5ii), (6iii) W.l.o.g. we can assume that $\mu = f\lambda_n$ is in isotropic position. Then we set $\tau := \|f\|_\infty^{-1/n}$ and $g(x) := \tau^n g(\tau x)$. The constant of isotropy of the probability measure $\nu = g\lambda_n$ satisfies $L_\nu = \tau^{-1} L_\mu$. For $\alpha \in \mathbb{R}^m$ with norm 1 and a convex body $B \subset \mathbb{R}^n$ we define $Y := (\mathbb{R}^n, \| \ \|_B)$ and take for $u : \ell_2^n \to Y$ the formal identity. Now we can apply proposition 2.3 to deduce with (4)

$$\sqrt{n} e_n(uA) \le c_0 (1 + \ln L_\nu) [[\alpha]]_{\nu,B} = c_0 (1 + \ln L_\mu \|f\|_\infty^{1/n}) \|f\|_\infty^{1/n} [[\alpha]]_B.$$

By Theorem 2.1 we have

$$\begin{aligned}(\lambda_n(B))^{-1/n} &\le |\det A^{-1}|^{1/n} \sqrt{n} (\lambda_n(B_2^n)/\lambda_n(A(B)))^{1/n} \\ &\le 3(8\pi e)^{1/2} 2 e_n(uA) \\ &\le 6(8\pi e)^{1/2} c_0 (1 + \ln L_\mu \|f\|_\infty^{1/n}) \|f\|_\infty^{1/n} [[\alpha]]_B.\end{aligned}$$

Now we turn to the logarithmic estimate in m. By symmetry and unconditionality of the norm $[[\ ]]_B$ we can assume that α is a non increasing, positive sequence. For fixed $1 \leq k \leq m$ we deduce as in Theorem (5i) with the unconditionality of $[[\ ]]_B$, see [**LTI**, Proposition 1.c.7],

$$\alpha_k\sqrt{k} \leq 16\alpha_k[[(\underbrace{1,\dots,1,0,\dots,0}_{k-\text{times}})]]_B\ \|f\|_\infty^{1/n}\lambda_n(B)^{1/n}$$
$$\leq 16[[\alpha]]_B\ \|f\|_\infty^{1/n}\lambda_n(B)^{1/n}.$$

Summing up over all k yields

$$\|\alpha\|_2 \leq 16(1+\ln m)^{1/2}\ [[\alpha]]_B\ \|f\|_\infty^{1/n}\lambda_n(B)^{1/n}.$$

If f is the characteristic function of a convex body with volume 1 we consider the random variable Z_α from the beginning of this chapter. Since f is log-concave the same is true for f_α. From the key Lemma 2 we deduce

$$f_\alpha(0)^{1/n} \leq c_0 \min\{1+\ln L_K, (1+\ln m)^{1/2}\}.$$

(6iii) follows after rewriting f_α in terms of iterated convolutions. □

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Received August 10, 1992 and in revised form May 26, 1993.

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