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CONTROLLING TIETZE-URYSOHN EXTENSIONS

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This paper explores some of the possibilities for controlling properties of continuous extensions of continuous functions. For example, particular attention is paid to the problem of preserving some desirable common property (e.g. pairwise disjointness, partition of unity, etc.) of a collection of functions, when the entire collection is extended simultaneously and continuously from a closed subset A of a normal topological space X to the whole space. The functions treated here are mainly real-valued, but in the last section, a procedure introduced by Dugundji is used to show how to preserve the equicontinuity of a collection of functions whose ranges lie in a locally convex metric linear space.

0. Introduction. If A is a closed subset of a normal topological space X, and if $f : A \to \mathbf{R}$ is continuous function, then by the extension theorems of Tietze [**T**] and Urysohn [**U**] (the combination of which we will refer to as [**TU**]), there exists a continuous extension $\hat{f}$ of f to all of X. Several generalizations of these results have been obtained, notably Katětov's result [**K**] for uniformly continuous functions on uniform spaces, and Dugundji's result [**D**] for continuous functions into linear topological spaces. In this article, we present some results which deal with the degree of control we have over the extensions of continuous functions; that is, how we may choose the sets where extended functions take on certain specified values, or satisfy some predetermined relationship.

In Section 1, we show that we may choose, with reasonable topological restrictions, any finite number of disjoint closed subsets of X as level sets for $\hat{f}$.

We show in Section 2 that if we have a continuous function f and a nonnegative (or nonpositive) continuous function g defined on a closed subset A of a compact metric space X, such that their product fg satisfied $fg = h|_A$ for some continuous function $h : X \to \mathbf{R}$, then under suitable conditions f and g may be extended to continuous functions $\hat{f}$ and $\hat{g}$ on all of X such that $\hat{f}\hat{g} = h$. At the end of this section we outline a much more general problem, some aspects of which are treated in later sections.

Section 3 contains two propositions on extending pairwise disjoint continuous functions. The second one employs a construction recently used by Mandelkern in [**M**].

In section 4, we show that a partition of unity defined on a closed subset of a normal space may be extended to the whole space.

In section 5 we show that an extension procedure defined by Dugundji in [**D**], when applied to functions from a closed subset A of a metric space (X, d) to a locally convex metric linear space (Y, ϱ), preserves the equicontinuity of any pointwise bounded equicontinuous collection of functions. A related result (Corollary 9) states that for any compact subset $\{f_\gamma\}$ of $C(A, Y)$, there exists a compact subset $\{\hat{f}_\gamma\}$ of $C(X, Y)$, such that $\hat{f}_\gamma|_A = f_\gamma$ for each γ.

1. Extending level sets. Let X be a normal topological space, let A be a closed subset of X, and let $f : A \to \mathbf{R}$ be continuous. By [**TU**], there exists a continuous extension $\hat{f}$ of f to all of X. Moreover, if $A_i := f^{-1}(\{a_i\})$ for arbitrary real numbers $a_1, \ldots, a_n$ in the range of f, then observe that in extending f we have also extended the level sets A_i of f to level sets $\hat{A}_i := \hat{f}^{-1}(\{a_i\})$ of $\hat{f}$. It is therefore reasonable to ask whether, starting with subsets $\hat{A}_i$ of X satisfying $\hat{A}_i \cap A = A_i$ for each i, we can find a continuous extension $\hat{f}$ of f such that $\hat{f}^{-1}(\{a_i\}) = \hat{A}_i$ for each i.

Clearly this implies that each $\hat{A}_i$ must be a closed G_δ-subset of X. Moreover, if we assume that $a_1 < \cdots < a_n$, and let

$$\hat{U}_0 := \hat{f}^{-1}((-\infty, a_1)), \hat{U}_1 := \hat{f}^{-1}((a_1, a_2)), \cdots, \hat{U}_{n-1} := \hat{f}^{-1}((a_{n-1}, a_n)), \hat{U}_n := \hat{f}^{-1}((a_n, \infty)),$$

then obviously the $\hat{U}_i$ are pairwise disjoint and $X \setminus (\bigcup_{i=1}^n \hat{A}_i) = \bigcup_{i=0}^n \hat{U}_i$. Notice that, generally speaking, $\hat{A}_i$ must separate $X \setminus \hat{A}_i$ into two disjoint open sets.

It turns out that if the sets $\hat{A}_i$ satisfy these minimal requirements, then the desired extension $\hat{f}$ exists (Theorem 2). The following theorem, used as an auxiliary result for Theorem 2, is in its own right a reasonable extension of [**TU**] in the case when f is a bounded function.

THEOREM 1. *Let X be a normal topological space, let A be a closed subset of X, and let $f : A \to [c,d]$ be a continuous function, such that the sets $C := f^{-1}(\{c\})$ and $D := f^{-1}(\{d\})$ are nonempty. If $\hat{C}$ and $\hat{D}$ are disjoint closed G_δ-sets in X such that $\hat{C} \cap A = C$ and $\hat{D} \cap A = D$, then there exists a continuous extension $\hat{f} : X \to [c,d]$ of f such that $\hat{C} = \hat{f}^{-1}(\{c\})$ and $\hat{D} = \hat{f}^{-1}(\{d\})$.*

Proof. Define the function $f_1 : A \cup \hat{C} \cup \hat{D} \to [c,d]$ by

$$f_1(x) := \begin{cases} f(x) & \text{if } x \in A \\ c & \text{if } x \in \hat{C} \\ d & \text{if } x \in \hat{D}. \end{cases}$$

Clearly f_1 is an extension of f to $A \cup \hat{C} \cup \hat{D}$ such that $f_1^{-1}(\{c\}) = \hat{C}$ and $f_1^{-1}(\{d\}) = \hat{D}$. The fact that f_1 is continuous follows from the "pasting lemma"[1] of elementary topology applied to the functions $f : A \to [c,d]$, $\mathbf{c} : \hat{C} \to \{c\}$, and $\mathbf{d} : \hat{D} \to \{d\}$. Since $A \cup \hat{C} \cup \hat{D}$ is closed, it follows from [**TU**] that there exists a continuous extension $\hat{f}_1 : X \to [c,d]$ of f_1. Now if $\hat{f}_1^{-1}(\{c\}) = \hat{C}$ and $\hat{f}_1^{-1}(\{d\}) = \hat{D}$, we can set $\hat{f} = \hat{f}_1$ and be done. Therefore let us assume that the sets $C_1 := \hat{f}_1^{-1}(\{c\}) \setminus \hat{C}$ and $D_1 := \hat{f}_1^{-1}(\{d\}) \setminus \hat{D}$ are nonempty (later we will handle the case when exactly one of C_1, D_1 is empty).

Our strategy is to define a continuous function φ on X such that $\hat{f} = \hat{f}_1 + \varphi$ satisfies the conclusion of the theorem. Clearly we must have $\varphi(A \cup \hat{C} \cup \hat{D}) = \{0\}$, $\varphi|_{C_1} > 0$, and $\varphi|_{D_1} < 0$ in order to make the proper adjustments to $\hat{f}_1$. Moreover, φ should not be too large in magnitude, or else we will create unwanted points x where $\hat{f}(x) \le c$ or $\hat{f}(x) \ge d$.

To begin, observe that by hypothesis, $\hat{C}$ is closed G_δ; hence we can write $\hat{C} = \bigcap_{n=1}^\infty G_n$ for a decreasing sequence $\{G_n\}$ of open sets. Moreover, by the normality of X, we may assume that $G_n^c \cap \overline{G_{n+1}} = \emptyset$ for each n, where G_n^c denotes the complement of G_n and $\overline{G_{n+1}}$ denotes the closure of G_{n+1}. It follows from Urysohn's lemma that for each n, there exists a continuous function $\gamma_n : \hat{C} \cup C_1 \to [0, 2^{-n}]$

[1] If g_1 and g_2 are continuous functions defined on closed sets C_1 and C_2, respectively, and if g_1 and g_2 agree on $C_1 \cap C_2$, then they can be used to define a new function piecewise on $C_1 \cup C_2$ (in the obvious way) which is continuous on $C_1 \cup C_2$.

such that

$$\gamma_n((\hat{C} \cup C_1) \cap G_n^c) = \{2^{-n}\} \quad \text{and} \quad \gamma_n((\hat{C} \cup C_1) \cap \overline{G_{n+1}}) = \{0\}.$$

We now define the function γ on $\hat{C} \cup C_1$ by

$$\gamma(x) := \sum_{n=1}^{\infty} \gamma_n(x).$$

Since for each n, γ_n is continuous and $|\gamma_n| \leq 2^{-n}$, the above series converges uniformly and thus $\gamma : \hat{C} \cup C_1 \to [0,1]$ is continuous. Clearly γ is zero on $\hat{C}$ and strictly positive on C_1.

In a similar fashion, we can define a continuous function $\delta : \hat{D} \cup D_1 \to [-1, 0]$ which is zero on $\hat{D}$ and strictly negative on D_1. It follows from the pasting lemma that the function $\alpha : A \cup \hat{C} \cup C_1 \cup \hat{D} \cup D_1 \to [-1, 1]$ given by

$$\alpha(x) := \begin{cases} \gamma(x) & \text{if } x \in \hat{C} \cup C_1 \\ \delta(x) & \text{if } x \in \hat{D} \cup D_1 \\ 0 & \text{otherwise} \end{cases}$$

is continuous. Our final step in the construction of φ is to use [**TU**] to extend α to a continuous function $\hat{\alpha} : X \to [-1, 1]$, and define

$$\varphi := \frac{\mathbf{c} - \hat{f}_1}{2} \vee \hat{\alpha} \wedge \frac{\mathbf{d} - \hat{f}_1}{2}.$$

Observe that if $t_1 \leq 0 \leq t_2$ for numbers t_1 and t_2, then $(t_1 \vee t) \wedge t_2 = t_1 \vee (t \wedge t_2)$ for any $t \in \mathbf{R}$, so no parentheses are needed in the above formula for φ. We claim that $\hat{f} := \hat{f}_1 + \varphi$ is the desired extension of f. It is clear that $\hat{f}$ is continuous and $\hat{f}|_A = f$, because $\varphi|_A = \mathbf{0}$. The following implications are also easily verified:

(1) $$x \in \hat{C} \Rightarrow \hat{\alpha}(x) = 0 \Rightarrow \varphi(x) = 0 \Rightarrow \hat{f}(x) = c$$

(2) $$x \in \hat{D} \Rightarrow \hat{\alpha}(x) = 0 \Rightarrow \varphi(x) = 0 \Rightarrow \hat{f}(x) = d$$

$$
\begin{aligned}
(3)\qquad x \in C_1 &\Rightarrow \hat{\alpha}(x) > 0 \\
&\Rightarrow \hat{f}(x) = c + \left(\frac{c-c}{2} \vee \hat{\alpha}(x) \wedge \frac{d-c}{2}\right) \\
&= c + \left(\hat{\alpha}(x) \wedge \frac{d-c}{2}\right) \\
&\Rightarrow c < \hat{f}(x) \le \frac{c+d}{2} < d \\
&\Rightarrow \hat{f}(x) \in (c, d)
\end{aligned}
$$

$$
\begin{aligned}
(4)\qquad x \in D_1 &\Rightarrow \hat{\alpha}(x) < 0 \\
&\Rightarrow \hat{f}(x) = d + \left(\frac{c-d}{2} \vee \hat{\alpha}(x) \wedge \frac{d-d}{2}\right) \\
&= d + \left(\frac{c-d}{2} \vee \hat{\alpha}(x)\right) \\
&\Rightarrow c < \frac{c+d}{2} \le \hat{f}(x) < d \\
&\Rightarrow \hat{f}(x) \in (c, d)
\end{aligned}
$$

$$
\begin{aligned}
(5)\quad x \in \hat{f}_1^{-1}\big((c,d)\big) \Rightarrow \hat{f}(x) &= \hat{f}_1(x) \\
&\quad + \left(\frac{c-\hat{f}_1(x)}{2} \vee \hat{\alpha}(x) \wedge \frac{d-\hat{f}_1(x)}{2}\right) \\
&\Rightarrow c < \frac{c+\hat{f}_1(x)}{2} \le \hat{f}(x) \le \frac{d+\hat{f}_1(x)}{2} < d \\
&\Rightarrow \hat{f}(x) \in (c, d).
\end{aligned}
$$

Thus $\hat{f}$ maps X into $[c, d]$, with $\hat{f}^{-1}(\{c\}) = \hat{C}$ and $\hat{f}_1(\{d\}) = \hat{D}$, so $\hat{f}$ is the desired extension of f.

To treat the case when C_1 (respectively, D_1) is empty, we can construct α so that α is zero on $A \cup \hat{C} \cup \hat{D}$ and strictly positive on C_1 (respectively, zero on $A \cup \hat{C} \cup \hat{D}$ and strictly negative on D_1); the rest of the proof is essentially the same. ☐

THEOREM 2. *Let X be a normal topological space, let A be a closed subset of X, and let $f : A \to \mathbf{R}$ be continuous. Let*

$a_1, a_2, \ldots, a_n$ belong to $f(A)$ with $a_1 < a_2 < \cdots < a_n$, let $A_i := f^{-1}(\{a_i\})$ for each i, and let

$$U_0 := f^{-1}\Big((-\infty, \alpha_1)\Big), U_1 := f^{-1}\Big((a_1, a_2)\Big), \cdots U_{n-1} := f^{-1}\Big((a_{n-1}, a_n)\Big), U_n := f^{-1}\Big((a_n, \infty)\Big).$$

If $\hat{A}_1, \ldots, \hat{A}_n$ are disjoint closed G_δ-sets in X such that $\hat{A}_i \cap A = A_i$ for each i and $X \setminus (\bigcup_{i=1}^n \hat{A}_i) = \bigcup_{i=0}^n \hat{U}_i$, where $\hat{U}_0, \ldots, \hat{U}_n$ are disjoint open sets satisfying $\hat{U}_i \cap A = U_i$ for each i, then there exists a continuous extension $\hat{f}$ of f to all of X such that $\hat{f}^{-1}(\{a_i\}) = \hat{A}_i$ for each i.

Proof. For $i = 1, \ldots, n-1$, let $f_i := f|_{A_i \cup U_i \cup A_{i+1}}$. By Theorem 1, there exist continuous extensions $\hat{f}_i : \hat{A}_i \cup \hat{U}_i \cup \hat{A}_{i+1} \to [a_i, a_{i+1}]$ of each function f_i, such that $\hat{f}_i^{-1}(\{a_i\}) = \hat{A}_i$ and $\hat{f}_i^{-1}(\{a_{i+1}\}) = \hat{A}_{i+1}$. Moreover, $f_0 := f|_{A_1 \cup U_0}$ and $f_n := f|_{A_n \cup U_n}$ can similarly be extended to continuous functions $\hat{f}_0 : \hat{A}_1 \cup \hat{U}_0 \to (-\infty, a_1]$ and $\hat{f}_n : \hat{A}_n \cup \hat{U}_n \to [a_n, \infty)$ such that $\hat{f}_0^{-1}(\{a_1\}) = \hat{A}_1$ and $\hat{f}_n^{-1}(\{a_n\}) = \hat{A}_n$. To see that this is true, consider (say) f_n. As in the proof of the Theorem 1, we can extend f_n to a continuous function $(\hat{f}_n)_1 : \hat{A}_n \cup \hat{U}_n \to [a_n, \infty)$ such that $\hat{A}_n \subseteq (\hat{f}_n)_1^{-1}(\{a_n\})$; if $(\hat{f}_n)_1^{-1}(\{a_n\}) \neq \hat{A}_n$, then we can remedy this by adding to $(\hat{f}_n)_1$ a continuous function $\gamma : \hat{A}_n \cup \hat{U}_n :\to [a_n, \infty)$ similar to that constructed in the proof of Theorem 1, which is zero on $\hat{A}_n$ and strictly positive on $\hat{U}_n$, so that $\hat{f}_n := (\hat{f}_n)_1 + \gamma$ is the desired extension of f_n. The extension of f_0 is handled similarly, using a negative analog of γ.

To complete the proof, we can simply define $\hat{f}$ piecewise on X in the obvious way (for example, let $\hat{f}|_{\hat{A}_0 \cap \hat{U}_0} = \hat{f}_0$), and note that $\hat{f}$ is continuous on X by the pasting lemma. □

REMARK. It is easy to see that continuity in the above theorems cannot be replaced by uniform continuity or bounded uniform continuity. For example, let $X := \mathbf{R}^2$, $A := \mathbf{R} \times \{0\}$, $f(x, y) := |x| \wedge 1$, $A_1 := f^{-1}(\{0\}) := \{(0,0)\}$, $\hat{A}_1 := A_1 \cup \{(x,y)|y = 1/x\}$. Then f is bounded and uniformly continuous on A, but f cannot be extended to a uniformly continuous function $\hat{f}$ on X such that $\hat{f}^{-1}(\{0\}) = A_1$.

2. Extending products. Let X be a normal topological space, let A be a closed subset of X, and let $f, g : A \to \mathbf{R}$ be continuous. By [**TU**], there exist continuous extensions $\hat{f}$ and $\hat{g}$ of f and g to all of X. Obviously, in extending f and g, we also create a continuous extension $\hat{f}\hat{g}$ of the product fg. It is therefore natural to ask whether, given a continuous function $h : X \to \mathbf{R}$ such that $fg = h|_A$, we can find continuous extensions $\hat{f}$ and $\hat{g}$ to all of X such that $\hat{f}\hat{g} = h$.

We begin with two examples which show that, in general, such extensions may fail to exist. These examples are illustrated in Figures 1 and 2 below. In each case, we take X to be a square in $\mathbf{R}^2$, with A being the closed left half of X, and it is easy to see that we can find continuous functions f, g, and h with the values indicated in the figures.

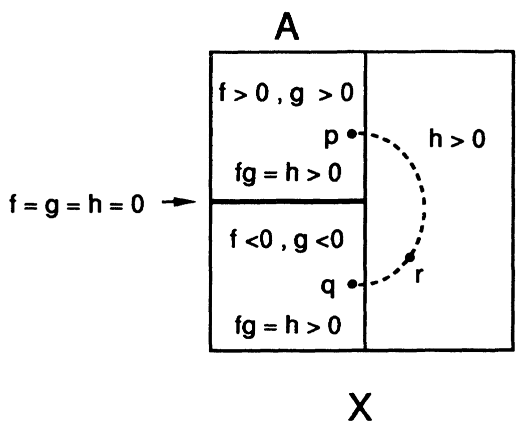

FIGURE 1.

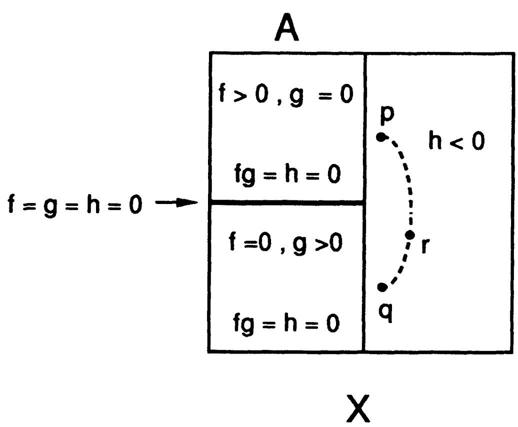

FIGURE 2.

Suppose that continuous extensions $\hat{f}$ and $\hat{g}$ of f and g existed in Figure 1. Since$\hat{f}(p) > 0$ and $\hat{f}(q) < 0$ (with p and q as indicated), there must be a point r outside A on the dotted curve at which $\hat{f}(r) = 0$. But then we cannot have $\hat{f}(r)\hat{g}(r) = h(r) > 0$.

For the second case, suppose that continuous extensions $\hat{f}$ and $\hat{g}$ of f and g existed in Figure 2, such that $\hat{f}\hat{g} = h$. We could then find points p and q as indicated, with $\hat{f}(p) > 0$ and $\hat{g}(q) > 0 \Rightarrow \hat{f}(q) < 0$ since $h(q) < 0$. Again there must be a point r outside A on the dotted curve such that $\hat{f}(r) = 0$, which contradicts $\hat{f}(r)\hat{g}(r) = h(r) < 0$.

In the first case, we had the simple relationship $g^{-1}(\{0\}) = f^{-1}(\{0\})$ between zero sets, but the mere fact that both f and g were allowed to take on both positive and negative values allowed us to preclude the possibility of suitable extensions. In the second case, both f and g were nonnegative, but by choosing their zero sets carefully, we were able to preclude the existence of the extensions again. Thus even when X is a compact metric space, the desired extensions are not guaranteed. However, our next theorem shows that the above cases are, roughly speaking, the worst possible; that is, by requiring that at least one of f, g be nonnegative (or nonpositive), and by demanding an inclusion relation on their zero sets, we obtain a sufficient condition for the existence of the desired extensions.

THEOREM 3. *Let (X, d) be a compact metric space, and let A be a closed subset of X. Suppose that $f, g : A \to \mathbf{R}$ and $h : X \to \mathbf{R}$ are continuous, with $g \geq 0$ (or $g \leq 0$), such that $fg = h|_A$. If $g^{-1}(\{0\}) \subseteq f^{-1}(\{0\})$, then there exist continuous extensions $\hat{f}$ of f and $\hat{g}$ of g to all of X such that $\hat{f}\hat{g} = h$.*

Proof. For definiteness we will consider the case $g \geq 0$. First note that if $g > 0$ (i.e. if g is strictly positive on A), then we can apply [**TU**] to extend g to a continuous function $\hat{g} > 0$ on all of x (by using $\hat{g} + d(x, A)$ if necessary). In the case, defining $\hat{f} := h/\hat{g}$ completes the proof.

Therefore let us assume that $g^{-1}(\{0\}) \neq \emptyset$. Defining $Z := h^{-1}(\{0\})$, we have by hypothesis that

$$Z \cap A = f^{-1}(\{0\}) \supseteq g^{-1}(\{0\}) \neq \emptyset.$$

Using [**TU**], let $\hat{g}_1 \geq 0$ be a continuous extension of g to all of X. Again, by adding the function $d(x, A)$ to $\hat{g}_1$ if necessary, we may assume that $\hat{g}_1 > 0$ on $X \setminus A$. The function $\hat{g}_1$ is our first approximation to a suitable extension of g; our first approximation to a suitable extension of f is the function $\hat{f}_1$ defined by

$$\hat{f}_1(x) := \begin{cases} f(x) & \text{if } x \in A \\ h(x)/\hat{g}_1(x) & \text{if } x \in X \setminus A. \end{cases}$$

Since $\hat{g}_1|_A = g$, we have $\hat{f}_1(x) = h(x)/\hat{g}_1(x)$ whenever $g(x) \neq 0$, so $\hat{f}_1$ is continuous at all points of $X \setminus g^{-1}(\{0\})$. Moreover, if a point $x_0 \in g^{-1}(\{0\})$ belongs to the interior of $A \cup Z$, then $\hat{f}_1$ is continuous at x_0, because $\hat{f}_1|_{A\cup Z}$ is continuous on $A \cup Z$ (this follows from the pasting lemma applied to the functions $f : A \to \mathbf{R}$ and $\mathbf{0} : A \to 0$). Hence the only type of point at which $\hat{f}_1$ might not be continuous is a point $x_0 \in g^{-1}(\{0\}) \cap \partial(A \cup Z)$, where $\partial(A \cup Z)$ denotes the boundary of $A \cup Z$.

From now on, we consider only those n large enough so that the sets

$$H_n := \{x \in X |\, |h(x)| \geq 4^{-(n+1)}\}$$

are nonempty. Since$X$ is compact, h and $\hat{g}_1$ are uniformly continuous, and since each H_n is compact, $\hat{g}_1$ is bounded away from zero on H_n. It follows that $h/\hat{g}_1$ is uniformly continuous on H_n. Moreover, we can find a decreasing sequence of positive real numbers δ_n which satisfy the following rules:

$$\delta_n < \inf\{d(x,y) | x \in H_n,\ y \in (H_{n+1})^c\} \tag{1}$$

$$\delta_n < \inf\{d(x,y) | x \in H_n,\ y \in Z\} \tag{2}$$

$$(x, y \in H_{n+1} \text{ and } d(x,y) < \delta_n) \Rightarrow |h(x)/\hat{g}_1(x) - h(y)/\hat{g}_1(y)| < 2^{-(n+1)}. \tag{3}$$

Next, for each n, we define the set A_n by

$$A_n := \{x \in X | d(x, A) \geq \delta_n/2\}.$$

Now $A \cup Z$ and $A_n \cap H_n$ are disjoint closed sets for each n, so we can find a function $\gamma_n : X \to [0, 2^{-n}]$ for each n which is zero on

$A \cup Z$ and equal to 2^{-n} on $A_n \cap H_n$. As in the proof of Theorem 1, the function $\gamma : X \to [0,1]$ given by

$$\gamma(x) := \sum_{n=1}^{\infty} \gamma_n(x)$$

is continuous, and $\gamma(A \cup Z) = \{0\}$. We claim that the function $\hat{f}$ and $\hat{g}$ given by

$$\hat{g} := \hat{g}_1 + \gamma \text{ and } \hat{f}(x) := \begin{cases} f(x) & \text{if } x \in A \\ h(x)/\hat{g}(x) & \text{if } x \in X \setminus A \end{cases}$$

are the desired extensions of f and g.

It is easy to check that: $\hat{g}$ is continuous on X, $\hat{g}|_A = g$, $\hat{f}|_A = f$ and $\hat{f}\hat{g} = h$. As in our previous remarks about $\hat{f}_1, \hat{f}$ is continuous everywhere, except possibly at points of $g^{-1}(\{0\}) \cap \partial(A \cup Z)$. Therefore let $x_0 \in g^{-1}(\{0\}) \cap \partial(A \cup Z)$, let $\epsilon > 0$, and choose $n \in \mathbf{N}$ such that $2^{-n} < \epsilon$. Since $\hat{f}_1|_{A \cup Z}$ is continuous, we may choose n large enough (δ_n small enough) so that

$$(4) \qquad d(x, x_0) < \delta_n \Rightarrow |\hat{f}_1(x)| = |\hat{f}_1(x) - \hat{f}_1(x_0)| < \epsilon/2$$

for all $x \in A \cup Z$. Let B_0 be a ball centered at x_0 with radius $< \delta_n/2$. Since $\gamma(A \cup Z) = \{0\}$, we have $\hat{f}|_{A \cup Z} = \hat{f}_1|_{A \cup Z}$, so by (4),

$$x \in B_0 \cap (A \cup Z) \Rightarrow |\hat{f}(x) - \hat{f}(x_0)| = |\hat{f}_1(x) - \hat{f}_1(x_0)| < \epsilon/2 < \epsilon.$$

Suppose now that $x \in B_0 \setminus (A \cup Z)$; then since $H_m \uparrow X \setminus Z$, there is a smallest m such that $x \in H_m$, and $m > n$ by (2), because $d(x, Z) < \delta_n$. We have $4^{-(m+1)} \leq |h(x)| < 4^{-m}$. Suppose that x also belongs to A_m, and hence to $A_m \cap H_m$. Then since $\hat{f}(x_0) = 0$,

$$\begin{aligned} |\hat{f}(x) - \hat{f}(x_0)| = |\hat{f}(x)| &= \frac{|h(x)|}{\hat{g}(x)} < \frac{|h(x)|}{\gamma_m(x)} < \frac{4^{-m}}{2^{-m}} \\ &= 2^{-m} < 2^{-n} < \epsilon. \end{aligned}$$

On the other hand, if $x \notin A_m$, then there exists $y \in A$ such that $d(x,y) < \delta_m/2$. Moreover, $\delta_m/2 < \inf\{d(w,z) | w \in H_m, z \in$

$(H_{m+1})^c\}$ by (1), so y must belong to H_{m+1}. Since also $x \in H_m \subset H_{m+1}$, we have from (3) that

$$(5)\quad \begin{aligned} |\hat{f}(x) - \hat{f}(x_0)| &= |\hat{f}(x)| = |h(x)/\hat{g}(x)| \le |h(x)/\hat{g}_1(x)| \\ &\le |h(x)/\hat{g}_1(x) - h(y)/\hat{g}_1(y)| + |h(y)/\hat{g}_1(y)| \\ &< 2^{-(m+1)} + |\hat{f}_1(y)| < \epsilon/2 + |\hat{f}_1(y)|. \end{aligned}$$

To estimate $|\hat{f}_1(y)|$, note that

$$d(y, x_0) \le d(y, x) + d(x, x_0) < \delta_m/2 + \delta_n/2 < \delta_n,$$

so (4) implies that $|\hat{f}_1(y)| < \epsilon/2$, and thus from (5),

$$|\hat{f}(x) - \hat{f}(x_0)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence we have shown that $x \in B_0 \Rightarrow |\hat{f}(x) - \hat{f}(x_0)| < \epsilon$, so $\hat{f}$ is continuous at x_0. It follows that $\hat{f}$ is continuous on X and that $\hat{f}$ and $\hat{g}$ are the desired extensions of f and g. □

COROLLARY 4. *Let $(X, d), A$ and h be as in Theorem 3. Let $f, g_1, \dots, g_n : A \to \mathbf{R}$ be continuous functions with $g_i \ge 0$ or $g_i \le 0$ and*

$$f^{-1}(\{0\}) \supseteq g_n^{-1}(\{0\}) \supseteq \cdots \supseteq g_1^{-1}(\{0\}),$$

such that $f g_1 \cdots g_n = h|_A$. Then there exists a continuous extension $\hat{f}$ of f and a continuous extension $\hat{g}_i$ of each g_i to all of X such that $\hat{f}\hat{g}_1 \cdots \hat{g}_n = h$.

Proof. The proof goes by induction on n. Suppose the theorem is true for any $n-1$ functions $g_1, \dots, g_{n-1}$ as described above (we know it is true for a single g by Theorem 3). Since each of the g_i is continuous and nonnegative (or nonpositive), their product is also. Hence by Theorem 3, there exists a continuous extension $\hat{f}$ of f and a continuous extension $\hat{G}$ of the product $g_1 \cdots g_n$ to all of X such that $\hat{f}\hat{G} = h$. Then $(g_1 \cdots g_{n-1})g_n = \hat{G}|_A$, and by hypothesis there exist continuous extensions $\hat{g}_i$ of each g_i to all of X such that $\hat{g}_1 \cdots \hat{g}_n = \hat{G}$. Hence $\hat{f}\hat{g}_1 \cdots \hat{g}_n = h$. □

REMARKS. The problems considered in this section are special cases of a much more general problem, which we will now formulate.

Let A be a closed subset of a normal topological space X, and let $P(u_1, \ldots, u_n)$ be a real-valued function of n real variables. Suppose we have n continuous functions $f_1, \ldots, f_n : A \to \mathbf{R}$ and a continuous function $h : X \to \mathbf{R}$ such that

$$(*) \qquad h(x) = P(f_1(x), \ldots, f_n(x)) \quad \text{for each } \; x \in A.$$

The question we wish to pose is: Do there exist continuous extensions $\hat{f}_1, \ldots, \hat{f}_n : X \to \mathbf{R}$ of the f_i such that $(*)$ holds for all $x \in X$? As we shall see, certain special cases of this problem have interesting interpretations. Let us note first that we have just explored the cases when $P(u_1, u_2) = u_1 u_2$ and $P(u_1, \ldots, u_n) = u_1 u_2 \cdots u_n$. Also, the problem is an easy one if $P(u_1, \ldots, u_n) = u_1 + \cdots + u_n$. We merely use [**TU**] to extend $f_2, \ldots, f_n$ to $\hat{f}_2, \ldots, \hat{f}_n$ and define $\hat{f}_1(x) := h(x) - \big(\hat{f}_2(x) + \cdots + \hat{f}_n(x)\big)$. The case when $P(u_1, u_2) = u_1/u_2$ is also easy if f_2 is of constant sign (say $f_2 > 0$) and A is a G_δ-set; we extend f_2 to $\hat{f}_2 > 0$ and define $\hat{f}_1(x) := h(x)\hat{f}_2(x)$. There are many other cases, however, for which the answers are not clear.

Some important special cases occur if we put restrictions on h and perhaps on the functions $f_1, \ldots, f_n$ and their extensions. For example, suppose we have $P(u_1, \ldots, u_n) = u_1 + \cdots + u_n$ and $h = \mathbf{1}$, and we demand that the functions $f_1, \ldots, f_n$ and their extensions be nonnegative. Then the above question is equivalent to asking: If a finite partition of unity is defined on A, can it be extended to a partition of unity on all of X? Of course there is usually a topological restriction on a partition of unity, namely that it be subordinate to some predetermined open covering, and thus we might want the extended collection $\{\hat{f}_1, \ldots, \hat{f}_n\}$ to be subordinate to some predetermined open covering of X. This problem is formulated more precisely and solved in Section 4.

Another special case of interest is the case when $P(u_1, \ldots, u_n) = \sum_{i \neq j} |u_i u_j|$. If we take $h = \mathbf{0}$, then the condition is the same as requiring that $|f_i| \wedge |f_j| = \mathbf{0}$ whenever $i \neq j$, which is equivalent to saying that the collection $\{f_1, \ldots, f_n\}$ of functions is "pairwise disjoint", a property which is often useful in analysis. Our goal is then to find a pairwise disjoint set of extensions $\{\hat{f}_1, \ldots, \hat{f}_n\}$. This problem is solved in the next section.

3. Extending pairwise disjoint collections. Recall that two continuous functions f_1 and f_2 are *disjoint* if $|f_1| \wedge |f_2| = \mathbf{0}$. An equivalent statement in terms of their supports F_1 and F_2 is that $\text{int}(F_1 \cap F_2) = \emptyset$. We have the following proposition for extending finite collections of pairwise disjoint continuous functions:

PROPOSITION 5. *Let A be a closed subset of a normal space X, and let the functions $f_1, \dots, f_n : A \to \mathbf{R}$ be continuous and pairwise disjoint. Then there exist pairwise disjoint continuous extensions $\hat{f}_1, \dots, \hat{f}_n$ of the respective f_i to all of X.*

Proof. It will suffice to consider the case when $f_1, \dots, f_n \geq 0$. For each $1 \leq k \leq n$ we can find a continuous extension $g_k : X \to [0, \infty)$ of f_k, by [**TU**]. For each such k, define $\hat{f}_k$ by

$$\hat{f}_k(x) := \left(g_k(x) - \bigvee_{i \neq k} g_i(x) \right) \vee 0.$$

Clearly each $\hat{f}_k$ is continuous. To show that $\hat{f}_k|_A = f_k$ for each k, let $a \in A$. From the above definition, we see that

$$f_k(a) = g_k(a) = 0 \Rightarrow \hat{f}_k(a) = 0.$$

On the other hand, if $f_k(a) > 0$, then by disjointness condition, $g_i(a) = f_i(a) = 0$ for all $i \neq j$, so $\hat{f}_k(a) = g_k(a) = f_k(a)$.

Finally, to show the pairwise disjointness of the $\hat{f}_i$, fix $1 \leq j,\ k \leq n$ with $j \neq k$. Then

$$\hat{f}_k(x) > 0 \Rightarrow g_j(x) < g_k(x) \Rightarrow \hat{f}_j(x) = 0.$$

□

An obvious question at this point would be whether we can extend an infinite collection of pairwise disjoint continuous functions. The answer is affirmative in the case of a countable collection, but the proof is rather technical and will be included in a later work. For the case of an arbitrary infinite collection we do not know the answer, but we do know that arbitrary collections can be extended in the case when X is a metric space. This is proved in the following proposition, which employs a construction used by Mandelkern in

[**M**], and which is similar to ones used by F. Riesz in [**Ke**], Tietze in [**T**], and Hausdorff in [**H**].

PROPOSITION 6. *Let A be a closed subset of a metric space (X, d), and let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a set of pairwise disjoint continuous functions from A to $\mathbf{R}$. Then there exists a set $\{\hat{f}_\gamma\}_{\gamma\in\Gamma}$ of pairwise disjoint continuous functions from X to $\mathbf{R}$, such that $\hat{f}_\gamma|_A = f_\gamma$ for each $\gamma \in \Gamma$.*

Proof. Again it will suffice to consider the case when $f_\gamma \geq 0$ for each $\gamma \in \Gamma$. For each γ, let $\varphi_\gamma : A \to [1,2)$ be the function $\varphi_\gamma(x) := \frac{2}{\pi}\arctan(f_\gamma(x)) + 1$, so that $\varphi_\gamma(x) = 1 \Leftrightarrow f_\gamma(x) = 0$. We are done if we can find a continuous extension $\hat{\varphi}_\gamma : X \to [1,2)$ of each φ_γ such that $\hat{\varphi}_\alpha \wedge \hat{\varphi}_\beta = \mathbf{1}$ whenever $\alpha \neq \beta$; for then we can define $\hat{f}_\gamma$ by $\hat{f}_\gamma(x) := \tan(\frac{\pi}{2}(\hat{\varphi}_\gamma(x) - 1))$, and have $|\hat{f}_\alpha| \wedge |\hat{f}_\beta| = \mathbf{0}$ whenever $\alpha \neq \beta$.

Folowing the construction used in [**M**], we define $\hat{\varphi}_\gamma$ for each γ by $\hat{\varphi}_\gamma(x) := \varphi_\gamma(x)$ if $x \in A$, and

$$\hat{\varphi}_\gamma(x) := \inf_{a\in A} \varphi_\gamma(a)\frac{d(x,a)}{d(x,A)} \quad \text{if } x \notin A.$$

It is clear that $\hat{\varphi}_\gamma : X \to [1,2]$. If $\hat{\varphi}_\gamma^{-1}(\{2\}) \neq \emptyset$, then we can remedy this by taking the infimum of $\hat{\varphi}_\gamma$ with a continuous function which is equal to 1 and 2, respectively, on the disjoint closed sets $\hat{\varphi}_\gamma^{-1}(\{2\})$ and A. Thus we may assume that $\hat{\varphi}_\gamma : X \to [1,2)$.

Now in [**M**] it is shown that if φ_γ is uniformly continuous, then so is $\hat{\varphi}_\gamma$. With only slight modifications, the same proof applies if we replace uniform continuity with continuity. Hence we only need show, as mentioned above, that $\hat{\varphi}_\alpha \wedge \hat{\varphi}_\beta = \mathbf{1}$ whenever $\alpha \neq \beta$.

Clearly this true at all points of A, so fix $x \in X \setminus A$, and choose two distinct $\alpha, \beta \in \Gamma$. For each γ let G_γ be the set $G_\gamma := \{a \in A | \varphi_\gamma(a) > 1\}$. Let us fix x and α. Then $d(x, G_\alpha) \geq d(x, A)$. If $d(x, G_\alpha) > d(x, A)$, then $d(x, A \setminus G_\alpha) = d(x, A)$, and since $\varphi_\alpha(a) = 1$ for each $a \in A \setminus G_\alpha$, we have

$$\hat{\varphi}_\alpha(x) \leq \inf_{a\in A\setminus G_\alpha} \varphi_\alpha(a)\frac{d(x,a)}{d(x,A)} = \inf_{a\in A\setminus G_\alpha} \frac{d(x,a)}{d(x,A)} = 1,$$

so $\hat{\varphi}_\alpha(x) = 1$. On the other hand, if $d(x, G_\alpha) = d(x, A)$, then using the fact that $\varphi_\beta(a) = 1$ for each $a \in G_\alpha$ (by the pairwise disjointness of f_α and f_β), we have

$$\hat{\varphi}_\beta(x) \le \inf_{a\in G_\alpha} \varphi_\beta(a)\frac{d(x,a)}{d(x,A)} = \inf_{a\in G_\alpha} \frac{d(x,a)}{d(x,A)} = 1,$$

so $\hat{\varphi}_\beta(x) = 1$. Thus $\hat{\varphi}_\alpha \wedge \hat{\varphi}_\beta = 1$, and the proposition is proved. □

4. Extending partitions of unity. The following theorem affirmatively answers a natural question on extending partitions of unity.

THEOREM 7. *Let A be a closed subset of a normal topological space X, and let $f_1, \dots, f_n : A \to [0,1]$ form a partition of unity subordinate to an open covering $\{U_i\}_{i=1}^n$ of A. If $\{\hat{U}_i\}_{i=1}^n$ is an open covering of X such that $\hat{U}_i \cap A = U_i$ for each i, then there exists a partition of unity $\{\hat{f}_i\}_{i=1}^n$ on X subordinate to $\{\hat{U}_i\}_{i=1}^n$ such that $\hat{f}_i|_A = f_i$ for each i.*

Proof. Let us put $F_i := \operatorname{supp}(f_i)$ for each i, and define $A_1 := F_1 \cup (\hat{U}_2 \cup \cdots \cup \hat{U}_n)^c$. Then $A_1 \subset \hat{U}_1$ and we can find an open set V_1 with $A_1 \subset V_1 \subset \overline{V_1} \subset \hat{U}_1$. Moreover, $\{V_1, \hat{U}_2, \dots, \hat{U}_n\}$ covers X. Inductively, given an open covering $\{V_1, \dots, V_{k-1}, \hat{U}_k, \dots, \hat{U}_n\}$ of X, let $A_k := F_k \cup (V_1 \cup \cdots \cup V_{k-1} \cup \hat{U}_{k+1} \cup \cdots \cup \hat{U}_n)^c$, and choose an open set V_k with $A_k \subset V_k \subset \overline{V_k} \subset \hat{U}_k$. At the nth step of the process, we obtain an open covering $\{V_i\}_{i=1}^n$ of X with $F_i \subset V_i \subset \overline{V_i} \subset \hat{U}_i$ for each i. Similarly, starting with $\{V_i\}_{i=1}^n$, we can find an open covering $\{W_i\}_{i=1}^n$ of X such that $F_i \subset W_i \subset \overline{W_i} \subset V_i$ for each i.

Next we extend each f_i to a continuous function $g_i : X \to [0,1]$, and we choose a continuous function $\gamma_i : X \to [0,1]$ for each i such that $\gamma_i(\overline{W_i}) = \{1\}$ ad $\gamma_i(V_i^c) = \{0\}$. Then it is easy to check that $F_i \subset \operatorname{supp}(g_i\gamma_i) \subset \hat{U}_i$ and $g_i\gamma_i|_A = f_i$. If we had $\sum g_i\gamma_i > 0$, we could define $\hat{f}_k := g_k\gamma_k/(\sum g_i\gamma_i)$ and be done. However, since this may not be the case, we make the following adjustment.

Let $B := (\sum g_i\gamma_i)^{-1}(\{1\})$. Then B is a closed G_δ-set containing A and we can find, as in previous proofs, a nonnegative continuous function β on X which is zero on B and strictly positive on B^c. For each i, we now define

$$\hat{g}_i := g_i\gamma_i + \beta\gamma_i.$$

Since $\hat{g}_i$ is a multiple of γ_i, we have $\operatorname{supp}\hat{g}_i \subset \overline{V_i} \subset U_i$ for each i. To show that $\sum \hat{g}_i > 0$, observe that if $x \in B$ then $\beta(x) = 0$, so $\sum \hat{g}_i(x) = \sum g_i(x)\gamma_i(x) = 1$. On the other hand, if $x \notin B$, x still belongs to W_k for some k, and thus $\sum \hat{g}_i(x) \geq \hat{g}_k(x) > 0$, because $\beta(x)\gamma_k(x) = \beta(x) > 0$.

Finally, we can define, for $k = 1, \dots, n$,

$$\hat{f}_k := \frac{\hat{g}_k}{\sum \hat{g}_i}.$$

Clearly $\hat{f}_k$ is continuous and $\sum \hat{f}_i = \mathbf{1}$. Also, $\operatorname{supp}(\hat{f}_k) = \operatorname{supp}(\hat{g}_k) \subset \hat{U}_k$, and $\hat{f}_k$ is an extension of f_k, because $A \subset B$ implies

$$\hat{f}_k|_A = \frac{\hat{g}_k|_A}{\sum \hat{g}_i|_A} = \frac{g_k\gamma_k|_A}{\sum g_i\gamma_i|_A} = \frac{f_k}{\sum f_i} = f_k.$$

□

REMARK. Again the obvious question: Does Theorem 7 hold for an infinite partition of unity ?

5. Extending equicontinuous sets of functions. In this section we will generalize the range space to be a locally convex metric linear space (Y, ϱ) over the real or complex field. Recall that for a metrizable Y, we can always make the additional assumption that the metric ϱ is translation-invariant, i.e., $\varrho(\vec{y} + \vec{w}, \vec{z} + \vec{w}) = \varrho(\vec{y}, \vec{z})$ for all $\vec{y}, \vec{w}, \vec{z} \in Y$.

Let us recall that a collection $\{f_\gamma\}_{\gamma \in \Gamma}$ of functions from a metric space (X, d) to a metric space (Y, ϱ) is called *equicontinuous at a point* $x \in X$ if, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\varrho(f_\gamma(x'), f_\gamma(x)) < \epsilon$ for all $\gamma \in \Gamma$ whenever $d(x', x) < \delta$. The collection is *equicontinuous* if it is equicontinuous at every point of X. It is in keeping with the spirit of preceding theorems to ask: If A is a closed subset of a metric space (X, d), and if $\mathcal{F} := \{f_\gamma\}_{\gamma \in \Gamma}$ is an equicontinuous subset of $C(A, Y)$, does there exist an equicontinuous subset $\hat{\mathcal{F}} := \{\hat{f}_\gamma\}_{\gamma \in \Gamma}$ of $C(X, Y)$ such that $\hat{f}_\gamma|_A = f_\gamma$ for each $\gamma \in \Gamma$? A little thought reveals that some additional restriction is needed. For example, let $X := [0, 1]$, $Y := \mathbf{R}$, $A := \{0, 1\}$, and $\mathcal{F} := \{f_n\}_{n \in N}$, where for each n, f_n is the uniformly continuous function defined on A by $f_n(x) := nx$. Suppose by way of contradiction that $\{g_n\}_{n\in N} \subset C(X)$ is an equicontinuous set of extensions of the f_n. Then for each $x \in [0,1]$ we can find an interval $I_x := (x-\delta_x, x+\delta_x)$ such that for any n, $|g_n(y)-g_n(x)| < 1/2$ whenever $y \in I_x$. But $[0,1]$ is covered by $I_{x_1}, \dots, I_{x_k}$ for some $x_1, \dots, x_k$, and this implies that $n = g_n(1) \leq k$ for each n, which is clearly a contradiction.

Note that in the previous example, the set $\{f_n(1)\}_{n\in N}$ was unbounded. To get the right restriction, we recall that a subset S of a linear topological space Y is said to be *bounded* if for each neighborhood V of $\vec{0}$ there exists $m > 0$ such that $\alpha\vec{s} \in V$ for all $\vec{s} \in S$ whenever a scalar α satisfies $|\alpha| < m$. If we require that for each $a \in A$ the set $\{f_\gamma(a)\}_{\gamma\in\Gamma}$ be bounded in Y, then the desired extensions of the f_γ exist (Theorem 8).

Before presenting the theorem, we should point out that the proof makes use of an extension defined by Dugundji in [**D**]. Indeed, in the process of proving the equicontinuity of $\hat{\mathcal{F}}$ on ∂A, we will almost repeat the proof of Dugundji's extension theorem. We also need to state a preliminary result from [**D**].

Recall that an open covering $\mathfrak{U}$ of a topological space X is called *locally finite* if for each $x \in X$, there exists a neighborhood V of x, such that V meets only finitely many $U \in \mathfrak{U}$ (indeed, we may assume that V is subset of each such U). If $\mathfrak{U}$ and $\mathfrak{V}$ are two open coverings of X, then $\mathfrak{V}$ is *refinement* of $\mathfrak{U}$ if for each $V \in \mathfrak{V}$ there is a set $U \in \mathfrak{U}$ containing it. We need the following result from [**D**].

If (X,d) is a metric space, and if A is a closed subset of X, then there exists an open covering $\mathfrak{U}$ of $X \setminus A$ such that

(i) $\mathfrak{U}$ is locally finite,

(ii) any neighborhood of any $a \in \partial A$ contains infinitely many sets from $\mathfrak{U}$,

(iii) given any neighborhood W of $a \in A$, there exists a neighborhood W' of a with $W' \subset W$ such that the implication $U \cap W' \neq \emptyset \Rightarrow U \subset W$ holds for each $U \in \mathfrak{U}$.

Using the terminology of [**D**], we refer to such a covering as a *canonical covering.*

THEOREM 8. *Let (X,d) be a metric space, let (Y,ϱ) be a locally convex metric linear space, and let A be a closed subset of X. Suppose that $\mathcal{F} := \{f_\gamma\}_{\gamma\in\Gamma}$ is an equicontinuous subset of $C(A,Y)$,* *and suppose that for each $a \in A$, the set $\{f_\gamma(a)\}_{\gamma\in\Gamma}$ is bounded in Y. Then there exists an equicontinuous subset $\hat{\mathcal{F}} := \{\hat{f}_\gamma\}_{\gamma\in\Gamma}$ of $C(X,Y)$ such that $\hat{f}_\gamma|_A = f_\gamma$ for each $\gamma \in \Gamma$.*

Proof. Let $\mathfrak{U}$ be a canonical covering of $X \setminus A$, and define for each $U \in \mathfrak{U}$ a function $\lambda_U : X \setminus A \to [0,1]$ by

$$(**) \qquad \lambda_U(x) := \frac{d(x, X\setminus U)}{\sum_{U'\in\mathfrak{U}} d(x, X \setminus U')}.$$

Since $\mathfrak{U}$ is locally finite, there exists for each $x \in X \setminus A$ a neighborhood W_x of x such that W_x meets only (say) $U_1, \ldots, U_n \in \mathfrak{U}$, and $W_x \subset U_1 \cap \cdots \cap U_n$. Thus the sum in the denominator of $(**)$ is finite and positive, so λ_U is well-defined. Moreover, λ_U is clearly continuous on W_x (and hence at every point of $X \setminus A$), and for any such x, $\sum_{U\in\mathfrak{U}} \lambda_U(x) = \lambda_{U_1}(x) + \cdots + \lambda_{U_n}(x) = 1$.

Next, for each U choose $x_U \in U$. Then for each x_U choose $a_U \in A$ such that $d(x_U, a_U) < 2d(x_U, A)$ (we assume of course that $a_U = a_{U'}$ whenever $x_U = x_{U'}$). We define the extensions of the f_γ as follows: for each $\gamma \in \Gamma$, let

$$\hat{f}_\gamma(x) := \begin{cases} \sum_{U\in\mathfrak{U}} \lambda_U(x) f_\gamma(a_U) & \text{if } x \notin A \\ f_\gamma(x) & \text{if } x \in A. \end{cases}$$

Clearly $\hat{\mathcal{F}} := \{\hat{f}_\gamma\}_{\gamma\in\Gamma}$ is equicontinuous on $\text{int}(A)$, so let us first choose $x_0 \in \partial A$ and let $\epsilon > 0$. For each γ let us write $\vec{y}_\gamma := \hat{f}_\gamma(x_0) = f_\gamma(x_0)$. If $B_\varrho(\vec{y}_\gamma, \epsilon)$ denotes the open ball of radius ϵ centered at $\vec{y}_\gamma$, we can use the translation invariance of ϱ to find a convex neighborhood V of $\vec{0}$ in Y such that $V + \vec{y}_\gamma := \{\vec{y} + \vec{y}_\gamma | \vec{y} \in V\}$ satisfies $\vec{y}_\gamma \in V + \vec{y}_\gamma \subset B_\varrho(\vec{y}_\gamma, \epsilon)$ for each $\gamma \in \Gamma$. Since there is a small ball in V centered at $\vec{0}$, there is a ball of equal radius in each $V + \vec{y}_\gamma$ centered at $\vec{y}_\gamma$. Hence we may choose $\delta > 0$ such that for any γ, $\hat{f}_\gamma(x) = f_\gamma(x) \in V + \vec{y}_\gamma$ whenever $x \in B_d(x_0, \delta) \cap A$. Observe that if $x_U \in B_d(x_0, \delta/3)$,then

$$\begin{aligned} d(a_U, x_0) &\leq d(a_U, x_U) + d(x_U, x_0) < 2d(x_U, A) + d(x_U, x_0) \\ &\leq 2d(x_U, x_0) + d(x_U, x_0) < 2\delta/3 + \delta/3 = \delta. \end{aligned}$$

Hence for any $\gamma \in \Gamma$, $x_U \in B_d(x_0, \delta/3) \Rightarrow f_\gamma(a_U) \in V + \vec{y}_\gamma$.

Finally, using the fact that $\mathfrak{U}$ is canonical, choose a neighborhood W_0 of x_0, with $W_0 \subset B_d(x_0, \delta/3)$,such that $U \cap W_0 \neq \emptyset \Rightarrow U \subset B_d(x_0, \delta/3)$. Then $x \in W_0 \setminus A$ implies that x belongs only to some $U_1, \dots, U_n$ and $x_{U_1}, \dots, x_{U_n} \in B_d(x_0, \delta/3)$. Hence for any γ, $\hat{f}_\gamma(x)$ is a convex combination

$$\hat{f}_\gamma(x) = \lambda_{U_1}(x) f_\gamma(a_{U_1}) + \cdots + \lambda_{U_n}(x) f_\gamma(a_{U_n})$$

of vectors $f_\gamma(a_{U_i})$ belonging to the convex set $V + \vec{y}_\gamma$. Therefore $\hat{f}_\gamma(x) \in V + \vec{y}_\gamma$, which implies $\hat{f}_\gamma(x) \in B_\varrho(\vec{y}_\gamma, \epsilon)$. This proves that $\hat{\mathcal{F}}$ is equicontinuous at each $x \in \partial A$.

To finish the proof, let $x_0 \in X \setminus A$ and let $\epsilon > 0$. As before, let us write for each γ, $\vec{y}_\gamma := \hat{f}_\gamma(x_0) = \sum_{i=1}^n \lambda_{U_i}(x_0) f_\gamma(a_{U_i})$, where $U_1, \dots, U_n$ are precisely those $U \in \mathfrak{U}$ which contain x_0, and let $W_0 \subset U_1 \cap \cdots \cap U_n$ be a neighborhood of x_0 disjoint from every other U. Again let V be a convex neighborhood of $\vec{0}$ satisfying $\vec{y}_\gamma \in V + \vec{y}_\gamma \subset B_\varrho(\vec{y}_\gamma, \epsilon)$ for each $\gamma \in \Gamma$. With n fixed, it is known that there exists a neighborhood V' of $\vec{0}$ contained in V such that $\vec{v}_1 + \cdots + \vec{v}_n \in V$ for any n vectors $\vec{v}_1, \dots, \vec{v}_n \in V'$. Now since each set $\{f_\gamma(a_{U_i})\}_{\gamma \in \Gamma}$ is bounded, so is the union of these n sets. Hence there exists a number $m > 0$ such that for any γ, $\alpha f_\gamma(a_{U_1}), \dots, \alpha f_\gamma(a_{U_n}) \in V'$ for any α with $|\alpha| < m$. Then since $\lambda_1, \dots, \lambda_n$ are continuous, we can find a sufficiently small neighborhood $W_0' \subset W_0$ of x_0 such that whenever $x \in W_0'$, we have $(\lambda_{U_i}(x) - \lambda_{U_i}(x_0)) f_\gamma(a_{U_i}) \in V'$ for $1 \leq i \leq n$ and for every $\gamma \in \Gamma$. Hence the vector

$$\begin{aligned}\hat{f}_\gamma(x) - \vec{y}_\gamma = &(\lambda_{U_1}(x) - \lambda_{U_1}(x_0)) f_\gamma(a_{U_1}) \\ &+ \cdots + (\lambda_{U_n}(x) - \lambda_{U_n}(x_0)) f_\gamma(a_{U_n})\end{aligned}$$

belongs to V, so $\hat{f}_\gamma(x) \in V + \vec{y}_\gamma \Rightarrow \hat{f}_\gamma(x) \in B_\varrho(\vec{y}_\gamma, \epsilon)$ whenever $x \in W_0'$. Since the choice of W_0' was independent of γ, this proves that $\hat{\mathcal{F}}$ is equicontinuous at x_0. □

REMARK. It is clear that we cannot remove the requirement in the above theorem that A be closed. For example, the set $\{f_n(x) := x^n\}_{n \in N}$ of real-valued functions is equicontinuous on the interval $[0, 1)$, but not on $[0, 1]$.

The following corollary is a slight variation of Theorem 8, and in view of the Ascoli-Arzela theorem, it takes on a pleasing form.

COROLLARY 9. *Let (X, d) be a locally compact metric space, let (Y, ϱ) be a locally convex metric linear space, and let $A \subset X$ be closed. Let us consider the compact-open topology on $C(A, Y)$ and $C(X, Y)$, and suppose that $\mathcal{F} := \{f_\gamma\}_{\gamma\in\Gamma}$ is a compact subset of $C(A, Y)$ for some indexing set Γ. Then there exists a compact subset $\hat{\mathcal{F}} := \{\hat{f}_\gamma\}_{\gamma\in\Gamma}$ of $C(X, Y)$ such that $\hat{f}_\gamma|_A = f_\gamma$ for each $\gamma \in \Gamma$.*

Proof. We will show that the set $\hat{\mathcal{F}}$ as defined in Theorem 8 is the desired subset of $C(X, Y)$. We already know that $\hat{\mathcal{F}}$ is an equicontinuous set of extensions of the f_γ. By the Ascoli-Arzela theorem, we can show that $\hat{\mathcal{F}}$ has compact closure by proving that for each $x_0 \in X \setminus A$, the set $\{\hat{f}_\gamma(x_0)\}_{\gamma\in\Gamma}$ is compact in Y (clearly this true for any $x_0 \in A$). To prove this, let $x_0 \in X \setminus A$ and let $\{\hat{f}_k(x_0)\}_{k=1}^{\infty}$ be any sequence (we assume that $0, 1, 2, \ldots \in \Gamma$). Since $\mathcal{F}$ is compact, the sequence $\{f_k\}_{k=1}^{\infty}$ has a convergent subsequence $\{f_{k_m}\}_{m=1}^{\infty}$ such that $f_{k_m} \overset{m}{\to} f_0$ [2] for some $f_0 \in \mathcal{F}$. We will prove the compactness of $\{\hat{f}_\gamma(x_0)\}_{\gamma\in\Gamma}$ by reindexing the convergent subsequence as $\{f_m\}_{m=1}^{\infty}$ and showing that $\hat{f}_m(x_0) \to \hat{f}_0(x_0)$.

As in the proof of Theorem 7, let $U_1, \ldots, U_n$ be precisely those $U \in \mathfrak{U}$ which contain x_0 and let $a_{U_1}, \ldots, a_{U_n} \in A$ be the corresponding points in A. Since convergence in the compact-open topology on $C(A, Y)$ implies pointwise convergence on A, we have

$$f_m(a_{U_1}) \overset{m}{\to} f_0(a_{U_1}), \ldots, f_m(a_{U_n}) \overset{m}{\to} f_0(a_{U_n})$$

in Y. Since scalar multiplication is continuous, we also have

$$\lambda_{U_1}(x_0)f_m(a_{U_1}) \overset{m}{\to} \lambda_{U_1}(x_0)f_0(a_{U_1}), \ldots, \lambda_{U_n}(x_0)f_m(a_{U_n}) \overset{m}{\to} \lambda_{U_n}(x_0)f_0(a_{U_n}),$$

and since vector addition is also continuous,

$$\sum_{i=1}^{n} \lambda_{U_i}(x_0)f_m(a_{U_i}) \overset{m}{\to} \sum_{i=1}^{n} \lambda_{U_i}(x_0)f_0(a_{U_i}),$$

which is equivalent to $\hat{f}_m(x_0) \to \hat{f}_0(x_0)$.

We will finish the proof by showing that $\hat{\mathcal{F}}$ is in fact closed. Let $\{\hat{f}_\delta\}_{\delta\in\Delta}$ be any net in $\hat{\mathcal{F}}$ converging to a function in $C(X, Y)$, where

[2]For brevity, we will hereafter use notation of the form $P_i \overset{i}{\to} Q$ to indicate that $P_i \to Q$ as $i \to \infty$.

Δ is a directed set. Since $\mathcal{F}$ is closed, we have $f_\delta \to f_0$ for some $f_0 \in \mathcal{F}$. We are done if we show that $\hat{f}_\delta \to \hat{f}_0$, and this can be proved by showing that for any $x_0 \in X \setminus A$, we have $\hat{f}_\delta(x_0) \to \hat{f}_0(x_0)$ in Y, because the compact-open topology and the pointwise convergence topology coincide on $\hat{\mathcal{F}}$. Having fixed x_0, let $U_1, \ldots, U_n$ and $a_{U_1}, \ldots, a_{U_n} \in A$ be as described above. By an argument similar to that used above, we have the following three results:

$$f_\delta(a_{U_1}) \xrightarrow{\delta} f_0(a_{U_1}), \ldots, f_\delta(a_{U_n}) \xrightarrow{\delta} f_0(a_{U_n}),$$

$$\lambda_{U_1}(x_0) f_\delta(a_{U_1}) \xrightarrow{\delta} \lambda_{U_1}(x_0) f_0(a_{U_1}), \ldots, \lambda_{U_n}(x_0) f_\delta(a_{U_n}) \xrightarrow{\delta} \lambda_{U_n}(x_0) f_0(a_{U_n}),$$

$$\sum_{i=1}^{n} \lambda_{U_i}(x_0) f_\delta(a_{U_i}) \xrightarrow{\delta} \sum_{i=1}^{n} \lambda_{U_i}(x_0) f_0(a_{U_i}),$$

the last result being equivalent to $\hat{f}_\delta(x_0) \to \hat{f}_0(x_0)$. □

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