PACIFIC JOURNAL OF MATHEMATICS
Vol. 169, No. 1, 1995

A SPLITTING CRITERION FOR RANK 2 VECTOR BUNDLES ON $\mathbf{P}^n$

E. BALLICO

This is an addendum to a recent paper of V. Ancona, T. Peternell and J. Wisniewski. Here we prove (using heavily their paper) two criteria for the splitting of rank 2 algebraic vector bundles (one on $\mathbf{P}^n$ and one on certain algebraic complete manifolds).

More precisely, the aim here is to show why the proofs of [**1**, Th. 10.5], and [**1**, Th. 10.13], give the following two theorems.

THEOREM 1. *Let E be a rank 2 algebraic vector bundle on $\mathbf{P}^n$ which satisfies the assumptions of* [**1**, Th. 10.5]. *Then E splits.*

THEOREM 2. *Let E be a rank 2 algebraic vector bundle on a projective manifold X with (X, E) satisfying the assumption of* [**1**, Th. 10.13]. *Then E splits.*

The assumptions on X in Theorem 2 are very restrictive (e.g. X is a Fano manifold with $\mathrm{Pic}(X) \cong \mathbf{Z}$). We only remark that the assumptions of Theorem 1 are satisfied if there is a two dimentional projective family, S, of lines in $\mathbf{P}^n$ such that the splitting type of $E|L$ is the same for all $L \in S$.

Proof of Theorem 1. By the statement of [**1**, Th. 10.13], E numerically splits, i.e. it has the same Chern classes of a direct sum of 2 line bundles, i.e. there are integers a_1, a_2 with $a_1 \leq a_2$ such that $c_1(E) = a_1 + a_2$ and $c_2(E) = a_1 a_2$. The key remark is that the proof of [**1**, Th. 10.5], shows the existence of a line L such that $E|L \cong \mathbf{O}_L(a_1) \oplus \mathbf{O}_L(a_2)$. Since $4c_2(E) - c_1(E)^2 \leq 0$, E is not stable. Hence there is an integer $t \geq (a_1 + a_2)/2$ such that $H^0(\mathbf{P}^n, E(-t)) \neq 0$; take as t the minimal one; the corresponding section s of $E(-t)$ will vanish on a codimension 2 subscheme, Z, with $\deg(Z) = c_2(E(-t))$. Since $c_2(E(-x)) < 0$ if $a_1 < x < a_2$, we have $t \geq a_2$. If $t = a_2$ we obtain $Z = \varnothing$; hence E splits. Hence we may assume $t > a_2$. This implies that $\mathbf{s}|D = 0$ for every line D such that $E|D \cong \mathbf{O}_D(b_1) \oplus \mathbf{O}_D(b_2)$ with $a_1 \leq b_1 \leq b_2 \leq a_2$; in particular by semicontinuity this is true for a general line of $\mathbf{P}^n$. Hence $\mathbf{s} = 0$, contradiction. □

The proof of Theorem 2 is simply the remark (following [**1**], Remark 10.12) that, having Theorem 1 instead of the statement of [**1**, Th. 10.5], we obtain the stronger assertion of Theorem 2 instead of the numerical splitting asserted by [**1**, Th. 10.13].

The proof of Theorem 1 (i.e. of the small part of [**1**] needed) works in positive characteristic. The same remark applies to Theorem 2 if we assume $\mathrm{Pic}(X) \cong \mathbf{Z}$ instead of making the assumptions on X which by [**2**] imply in characteristic 0 that $\mathrm{Pic}(X) \cong \mathbf{Z}$.

We think that [**1**, Remark 10.12], (on the extension of [**1**, §10], to other manifolds) is potentially very interesting and we hope that some reader will be able to use it.

References

[1] V. Ancona, T. Peternell and J. Wisniewski, *Fano bundles and splitting theorems on projective spaces and quadrics*, Pacific J. Math., (to appear).

[2] J. Wisniewski, *On a conjecture of Mukai*, Manuscripta Math., **68** (1990), 135-141.

Received August 12, 1992.

University of Trento
38050 Provo (TN)
Italy
E-mail address: ballico@itncisca.bitnet
ballico@science.unitn.it