PACIFIC JOURNAL OF MATHEMATICS
Vol. 169, No. 2, 1995

NILPOTENT CHARACTERS

GABRIEL NAVARRO

In this note we study modular characters of finite p-solvable groups which are induced from p-nilpotent subgroups and its π-version.

1. Introduction. There is at least one reason to study such characters. In [**2**], for any block B of a finite group G, Alperin and Broué found a successful and natural Sylow B-theory which synthesized local group theory with several results on blocks by Brauer. This approach led to the Broué-Puig idea of nilpotent blocks. From the local representation point of view, therefore, the nilpotent blocks are the most natural blocks.

It is well known that theorems on p-blocks, in general, become far more accessible when we restrict our attention to the p-solvable groups. Sometimes, as it happens with the cyclic defect theory, they almost become trivial. This is not the case with the nilpotent blocks. Puig described the block algebra of a nilpotent block of a p-solvable group in [**11**].

Here we focus ourselves with the characters of the block. If φ is a modular character lying in a p-block B of a finite p-solvable group, we show that B is nilpotent if and only if φ is induced from a p-nilpotent subgroup. With this approach and applying Isaacs π-theory we are able to introduce nilpotent π-blocks (π-blocks have been studied by Robinson, Staszewski, Slattery and others) and to describe them satisfactorily: they only have a unique modular character φ (which is induced from a subgroup K with a normal Hall π-subgroup), and its $|\operatorname{Irr}(D)|$ ordinary characters are also induced from convenient characters of K (D is defect group of the block). Finally, we will find the Fong characters associated with φ (the characters α of a Hall π-subgroup with $\alpha^G = \Phi_\varphi$).

Of course, when the set of primes π is just the complement of a prime p, π-blocks are just the ordinary blocks.

2. Subpairs and nilpotent blocks. If B is a p-block of a finite group G, a B-subpair is a pair (P, b_P) where P is a p-subgroup of G and b_P is a block of $PC_G(P)$ inducing B (we are using Alperin's book notation [**1**]). If P is a defect group of B, then (P, b_P) is said to be a Sylow B-subpair. It is one of the main results in [**2**] to show that Sylow B-subpairs are G-conjugate and that each B-subpair is contained in one Sylow B-subpair (a natural but not obvious definition of containment is given in [**1**]). It is worth to mention that if the block B is the principal block, local block theory is just Sylow theory.

Inspired by Frobenius Theorem, Broue and Puig defined nilpotent blocks: a block B is said to be nilpotent if whenever (P, b_P) is a B-subpair then $N_G(P, b_P)/C_G(P)$ is a p-group.

We begin with a Lemma. It is not in general true that if b^G is defined and nilpotent, then b is nilpotent (we will give some example below). However, in some special conditions more is true. (We recall that notation used in [**2**] and [**4**], is entirely equivalent to that in [**1**]: just apply V. 3.5 of [**5**]).

LEMMA 1. *Let B be a block of a p-solvable group G. Let $\theta \in \mathrm{Irr}(O_{p'}(G))$ be covered by B and let $b \in Bl(T)$ cover θ and induce B, where T is the inertia group of θ in G. Then B is nilpotent if and only if b is nilpotent.*

Proof. Suppose that B is nilpotent and let (P, b_P) be a b-subpair. We wish to show that $N_T(P, b_P)/C_T(P)$ is a p-group. Let us denote by $^* : \mathrm{Irr}_P(O) \to \mathrm{Irr}(C_O(P))$ the Glauberman Correspondence (see Chapter 13 of [**6**]), where $O = O_{p'}(G)$.

By applying, for instance, Lemma (4.4) of [**13**] to T, we have that if b_P covers ψ^*, where $\psi \in \mathrm{Irr}_P(O)$, then b covers ψ. Therefore, we have that b_P lies over θ^*. We observe that $N_T(P)$ is the inertia subgroup of θ^* in $N_G(P)$. This is because $N_G(P)$ acts on O fixing the P-invariant characters and commuting with the correspondence (see Theorem (13.1) (c) of [**6**]).

By Theorem (1.2.4) of [**4**], we know that $b_P^{PC_G(P)}$ is nilpotent; so let δ be the unique Brauer character in $b_P^{PC_G(P)}$. Since δ lies over θ^* and $PC_T(P)$ is the inertia group of θ^* in $PC_G(P)$, let $\mu \in \mathrm{IBr}(PC_T(P)|\theta^*)$ such that $\mu^{PC_G(P)} = \delta$. By Fong-Reynolds (Theorem V.2.5 of [**5**]), we know that μ is the only modular character in b_P. Therefore, if $x \in N_T(P, b_P)$ then $\mu^x = \mu, \delta^x = \delta$ and consequently $x \in N_G(P, b_P^{PC_G(P)})$. Then $N_T(P, b_P)/C_T(P)$ is isomorphic to a subgroup of $N_G(P, b_P^{PC_G(P)})/C_G(P)$, which is a p-group by hypothesis.

Now assume that b is nilpotent and let (P, b_P) be a B-subpair. We want to prove that $N_G(P, b_P)/C_G(P)$ is a p-group. Let $H = PC_G(P)$. We note that b_P^{HO} covers θ^x, for some $x \in G$. This can be seen, for instance, by taking an irreducible character of b_P^{HO} lying under some irreducible character of B (by Theorem B of [**3**]). Since P is contained in a defect group of b_P^{HO}, it follows that some O-conjugate of P, say P^o, stabilizes θ^x, by Fong-Reynolds. Therefore, P stabilizes θ_1 and b_P^{HO} covers θ_1, where $\theta_1 = \theta^{xo^{-1}}$. Let T_1 be the stabilizer of θ_1 in G. If we denote by $\theta_1^* \in \mathrm{Irr}(C_O(P))$ the Glauberman correspondent of θ_1 with respect to P, by an earlier argument we have that $N_{T_1}(P)$ is the stabilizer of θ_1^* in $N_G(P)$.

Now let $\gamma^* \in \mathrm{Irr}(C_O(P))$be covered by b_P. Then γ is covered by b_P^{HO}, and therefore $\gamma = \theta_1^c$, for some $c \in C_G(P)$. Thus $\theta_1^* = (\gamma^*)^{c^{-1}}$ is also covered by b_P. Since $PC_{T_1}(P)$ is the stabilizer in $PC_G(P)$ of θ_1^*, we find $e \in Bl(PC_{T_1}(P)|\theta^*)$ such that $e^{PC_G(P)} = b_P$. By an earlier argument, e^{T_1} lies over θ_1, and, since it induces B, it follows that e^{T_1} is a G-conjugate of b. Therefore, it is nilpotent. By Theorem (1.2) of [**4**], e is also nilpotent and thus it contains a unique modular character, say δ. By Fong-Reynolds, $\delta^{PC_G(P)}$ is the unique modular character in b_P.

Suppose now that $y \in N_G(P, b_P)$. Then y fixes P and $\delta^{PC_G(P)}$. By Clifford Theory, $(\theta_1^*)^y = (\theta_1^*)^c$, for some $c \in C_G(P)$. Thus $yc^{-1} \in N_{T_1}(P)$ and by the uniqueness in the Clifford Correspondence, $\delta^{yc^{-1}} = \delta$. Then $yc^{-1} \in N_{T_1}(P, e)$. Consequently, $N_G(P, b_P) \subseteq N_{T_1}(P, e)C_G(P)$. Thus $N_G(P, b_P)/C_G(P)$ is isomorphic to a subgroup of $N_{T_1}(P, e)/C_{T_1}(P)$, which is a p-group. □

LEMMA 2. *Let B be a nilpotent block of a p-solvable group G and let $\theta \in \mathrm{Irr}(O_{p'}(G))$ covered by B. If θ is G-invariant then G is p-nilpotent.*

Proof. We argue by induction on $|G|$. Write $O = O_{p'}(G)$.

By Fong Theory, (see, for instance, Theorem (2.1) of [**13**]), we know that the Sylow p-subgroups of G are the defect groups of B. Fix P a Sylow p-subgroup of G and let (P, b_P) be a Sylow B-

subpair. By Frobenius Theorem, it suffices to show that if Q is any p-subgroup of P then $N_G(P)/C_G(P)$ is a p-group. By Theorem (16.3) of [**1**], let $(Q, b_Q) \leq (P, b_P)$. Since b_Q is nilpotent, let δ be the unique Brauer character in b_Q. By earlier arguments in Lemma 1, if $\theta^* \in \mathrm{Irr}(C_O(Q))$ is the Q-Glauberman correspondent of $\theta \in \mathrm{Irr}_Q(O)$, then b_Q lies over θ^* and θ^* is $N_G(Q)$-invariant. By local group theory, it is well known that $C_O(Q) = O_{p'}(N_G(Q))$. If $QC_G(Q) < G$, by induction, we have that $QC_G(Q)/O_{p'}(N_G(Q))$ is a p-group. Therefore, by Green's Theorem (see, for instance, (3.1) of [**8**]), $\delta_{O_{p'}(N_G(Q))} = \theta^*$, and since δ is the only Brauer character lying over θ^*, we have that δ and θ^* determine one each other uniquely. Therefore, δ is $N_G(Q)$-invariant, and so it is b_Q. Thus, $N(Q, b_Q)/C_G(Q) = N_G(Q)/C_G(Q)$ is a p-group in any case, and Frobenius Theorem applies. □

3. π-characters. If G is a π-separable group, we denote by $I_\pi(G)$ the set of Isaacs π-characters of G. Of course, when $\pi = p', I_\pi(G)$ is just the set of Brauer characters of G. We refer the reader to [**7**] and [**8**], for definitions, notation and basic properties of the set $I_\pi(G)$. We recall that there exists a canonical subset of the irreducible characters of $G, B_\pi(G)$, such that restriction to π-elements gives a bijection from $B_\pi(G)$ onto $I_\pi(G)$ (Theorem (9.3) of [**7**]).

We certainly will use that any π-character is induced from a π-degree π-character (Huppert's Theorem, see (3.4) of [**8**]), and other fact proved recently in [**9**]. If $\varphi \in I_\pi(G)$ and $\varphi = \delta^G = \mu^G$, where $\delta \in I_\pi(K)$ and $\mu \in I_\pi(J)$ have π-degree, then the Hall π'-subgroups of K and J are G-conjugate: this invariant is the vertex of a π-character.

We say that $\varphi \in I_\pi(G)$ is *nilpotent* if $\varphi = \delta^G$, where $\delta \in I_\pi(K)$ with $K = O_{\pi\pi'}(K)$.

LEMMA 3. *Let G be a π-separable group and let $\varphi \in I_\pi(G)$ be nilpotent. If $\theta \in \mathrm{Irr}(O_\pi(G))$ is G-invariant and lies under φ then $G = O_{\pi\pi'}(G)$.*

Proof. Write $\varphi = \delta^G$, where $\delta \in I_\pi(K)$ with $K = O_{\pi\pi'}(K)$, and let $O = O_\pi(G)$. Since OK has a normal Hall π-subgroup, by replacing (K, δ) by (OK, δ^{OK}), we may assume that $O \subseteq K$. Now, by (3.4) of [**8**], let $\beta \in I_\pi(R)$ with π-degree be such that $\beta^K = \delta$. Since β^{OR} has also π-degree (because $|OR : R|$ is a π-number), we also may assume that δ has π-degree.

By comments above, observe that if P is a Hall π'-subgroup of K, then P is a vertex of φ.

Let $U = O_{\pi\pi'}(G)$. We claim that $\varphi_U = e\eta$, where $\eta \in I_\pi(U)$ and $\eta_O = \theta$. To see this, let $\chi \in B_\pi(G)$ be a lifting of φ (see Theorem (9.3) of [**7**]), and let $\psi \in B_\pi(U)$ be under χ ((7.5) of [**7**]). Then, by (6.3) and (6.5) of [**7**], $\psi_U = \theta$ and ψ is the only B_π-character lying over θ. Therefore, ψ is G-invariant and so it is $\psi^o = \eta \in I_\pi(U)$, its restriction to π-elements. This proves the claim.

Now, since ψ has π-degree, by (5.4) of [**7**], ψ is π-special and therefore, (U, ψ) is a subnormal π-factorable pair in the sense of [**7**]. Therefore, $(U, \psi) \leq (W, \alpha)$, where $(W, \alpha), \alpha$ a π-special character of W, is a nucleous of χ (definition (4.6) of [**7**]). Thus $\alpha^{o^G} = \varphi$, and by Theorem B of [**9**], it follows that P^x is a Hall π'-subgroup of W, for some $x \in G$. Then $P^x \cap U$ is a Hall π'-subgroup of U, and thus $U \subseteq OP \subseteq K$.

Now, since U/O and $O_\pi(K)/O$ are normal subgroups of K/O of coprime order it follows that $O_\pi(K)/O \subseteq C_{G/O}(U/O) \subseteq U/O$, by Lemma 1.2.3. Therefore, we conclude that $O_\pi(K) = O$. Let $V = O_{\pi\pi'\pi}(G)$. Since K/U and V/U have coprime orders it follows that $V \cap K = U$. Observe that $\delta_U = \eta$, by (3.1) of [**8**], and that δ^{KV} has π-degree. Therefore, $\eta^V = (\delta^{KV})_V \in I_\pi(V)$. Since ψ lifts η, necessarily $\psi^V \in \mathrm{Irr}(V)$. Since ψ is G-invariant, by problem (6.1) of [**6**], for instance, it follows that $U = V = G$, as wanted. □

LEMMA 4. *Let G be a π-separable group and let Y be a normal π-subgroup of G. Let $\varphi \in I_\pi(G)$, let $\theta \in \mathrm{Irr}(Y)$ under φ and let $\delta \in I_\pi(T|\theta)$ with $\delta^G = \varphi$, where T is the stabilizer of θ in G. Then φ is nilpotent if and only if δ is nilpotent.*

Proof. By the definition, certainly φ is nilpotent if δ is nilpotent. So assume that φ is nilpotent and write $\varphi = \psi^G$, where $\psi \in I_\pi(K)$, with K having a normal Hall π-subgroup. Since YK has also a normal Hall π-subgroup, we may replace K by YK and assume that K contains Y. Also, by replacing K by some G-conjugate, we may assume that ψ lies over θ. If $\alpha \in I_\pi(K \cap T|\theta)$ induces ψ, by uniqueness in the Clifford correspondence, (3.2) of [**8**], it follows that $\alpha^T = \delta$, and the proof of the Lemma is complete. □

Now we prove.

THEOREM 5. *Let B be a p-block of a p-solvable group and let $\varphi \in \mathrm{IBr}(B)$. Then B is nilpotent if and only if φ is nilpotent.*

Proof. We argue by induction on $|G|$. Let $\theta \in \mathrm{Irr}(O_{p'}(G))$ be under φ, let $\delta \in IBr(T|\theta)$ with $\delta^G = \varphi$, where T is the stabilizer of θ in G, and let $b \in Bl(T)$ be the block of δ. If $T = G$, by Lemma 2 and Lemma 3, we have that, in both cases, G is p-nilpotent and so every block and every character are nilpotent. If $T < G$, by induction and Lemma 1 and Lemma 4, we have that φ is nilpotent if and only if δ is nilpotent if and only if b is nilpotent if and only if B is nilpotent. □

4. π-Blocks. Brauer himself considered the idea of generalizing p-blocks to π-blocks, for a set of primes π. Later, Robinson and others introduced several definitions of π-blocks. We will follow the Isaacs-Slattery's approach which certainly coincides with Robinson's when the group is π-separable. We refer the reader to [**12**] and [**13**], for definition, notation and further comments on the subject.

THEOREM 6. *Let G be a π-separable group and let $\varphi \in I_\pi(G)$ be nilpotent. Let B be the π-block of φ. Then*
(a) *φ is the only modular character in B.*

(b) *If $\delta^G = \varphi$, where $\delta \in I_\pi(K)$ has π-degree and K has a normal Hall π-subgroup, then the map $\psi \to \psi^G$ from $\mathrm{Irr}(K|\delta_{O_\pi(K)}) \to \mathrm{Irr}(B)$ is a bijection.*

(c) *With the notation of* (b), *$(\delta_{O_\pi(K)})^G = \Phi_\varphi$. Thus, if H is a Hall π-subgroup of G containing $O_\pi(K)$, then $(\delta_{O_\pi(K)})^H \in \mathrm{Irr}(H)$ is a Fong character for φ.*

Proof. (a) Let $\theta \in \mathrm{Irr}(O)$ under φ, where $O = O_\pi(G)$. Let $\delta \in I_\pi(T|\theta)$ with $\delta^G = \varphi$, where T is the stabilizer of θ in G, and let b be the π-block of δ. If $T = G$, by Lemma 3, G has a normal Hall π-subgroup. Also by (2.8) of [**12**], we know that the modular characters in B are the π-characters over θ. By (6.3) of [**7**], it follows that φ is the only one. On the other hand, if $T < G$, by Lemma 3, induction and Theorem (2.10) of [**12**], the result follows.

(b) We argue by induction on $|G|$.

Since δ has π-degree, we have that $\alpha = \delta_{O_\pi(K)} \in \mathrm{Irr}(O_\pi(K))$.

Let $V = OO_\pi(K)$. Since $|OK : V|$ is a π'-number, we have that $\alpha^V = (\delta^{OK})_V \in \mathrm{Irr}(V)$. Since α is K-invariant, by (4.3) of [**7**], it follows that the map $\psi \to \psi^{OK}$ is a bijection from $\mathrm{Irr}(K|\alpha) \to \mathrm{Irr}(OK|\alpha^V)$. Now let $\theta \in \mathrm{Irr}(O)$ be under α^V and let $\epsilon \in I_\pi(T \cap OK|\theta)$ be such that $\epsilon^{OK} = \delta^{OK}$, where T is the stabilizer of θ in G. If $\mu = \epsilon^T$, observe that $\mu \in I_\pi(T|\theta)$ and $\mu^G = \varphi$. By Lemma 4, notice that μ is nilpotent. If $T = G$, by Lemma 3, we have that O is a Hall π-subgroup of G. Also, $\varphi_O = \theta$, which forces $OK = G$. In this case, $V = O, \alpha^V = \theta$ and we know that $\psi \to \psi^G$ is a bijection from $\mathrm{Irr}(K|\alpha) \to \mathrm{Irr}(G|\theta)$. Since $\mathrm{Irr}(B) = \mathrm{Irr}(G|\theta)$, by (2.8) of [**12**], in this case, we are done. So we may assume that $T < G$ and by induction we have that the map $\psi \to \psi^T$ is a bijection from $\mathrm{Irr}(T \cap OK|\epsilon_{T\cap V}) \to \mathrm{Irr}(b)$.Since $\epsilon_{T\cap V}$ is $T \cap OK$-invariant and induces α^V, by (4.3) of [**7**], it follows that the map $\psi \to \psi^{OK}$ is a bijection from $\mathrm{Irr}(T \cap OK|\epsilon_{T\cap V}) \to \mathrm{Irr}(OK|\alpha^V)$ (observe that $(T \cap OK)V = OK$,because they have coprime indices). By the above and Theorem (2.10) of [**12**], we have that the map $\psi \to \psi^G$ is a bijection from $\mathrm{Irr}(T \cap OK|\epsilon_{T\cap V}) \to \mathrm{Irr}(B)$ and therefore so it is the map $\psi \to \psi^G$ from $\mathrm{Irr}(OK|\alpha^V) \to \mathrm{Irr}(B)$. This proves (b).

(c) By Lemma (2.3) of [**10**], it suffices to show that $(\delta_{O_\pi(K)})^G = \Phi_\varphi$. If $\chi \in \mathrm{Irr}(B)$, by (b), we have that $\chi^o = (\chi(1)/\varphi(1))\,\varphi$. Then,

$$\begin{aligned}\Phi_\varphi &= \sum_{\chi\in\mathrm{Irr}(B)} (\chi(1)/\varphi(1))\,\chi = \sum_{\psi\in\mathrm{Irr}(K|\delta_{O_\pi(K)})} (\psi(1)/\delta(1))\,\psi^G \\ &= \left(\sum_{\psi\in\mathrm{Irr}(K|\delta_{O_\pi(K)})} (\psi(1)/\delta(1))\,\psi\right)^G \\ &= \left(\left(\delta_{O_\pi(K)}\right)^K\right)^G = \left(\delta_{O_\pi(K)}\right)^G.\end{aligned}$$

It is not difficult to show that all Fong characters associated with φ arise this way. □

We think it is worth to remark that if an irreducible character χ is induced from a p-nilpotent character the p-block of χ need not to be nilpotent. For instance, consider χ an irreducible character of degree 3 in the symmetric group on four letters and $p = 2$. The block of χ is the principal block which is not nilpotent (because G is not p-nilpotent). However, χ is induced from a Sylow 2-subgroup of G.

5. An example. We mentioned above that if a block b^G is defined and nilpotent, then b needs not to be nilpotent. More surprisingly, if a block nilpotent b covers a block e, e needs not to be nilpotent (this fact was communicated to the author by L. Puig, and we take this opportunity for thanking him). We give an easy

EXAMPLE 7. Let $D = \langle x, y\rangle$ be the dihedral group of order 8, with $C = \langle x\rangle$ of order 4 and $x^y = x^{-1}$ and let D act on $P = \langle z\rangle$ of order 3 by $z^y = z^{-1}$ and C acting trivially. Let $G = PD$ be the semidirect product and put $p = 3$. Let $\lambda \in \mathrm{Irr}(C)$ of order 4 and $\hat{\lambda} = \lambda \times 1_P \in \mathrm{Irr}(P \times C)$. Then $\chi = (\hat{\lambda})^G \in \mathrm{Irr}(G)$. Observe that, by (7.1) of [**7**], $\chi \in B_2(G)$ and thus, $\varphi = \chi^o \in \mathrm{IBr}(G)$. Observe that φ is nilpotent. Let $J = P\langle y\rangle$ and let $H = J \times Z \triangleleft G$, where $Z = \langle x^2\rangle$. Then $\chi_H = \mu_1 + \mu_2$, where $\mu_1 \in \mathrm{Irr}(H/P)$, and μ_i is linear. Then, μ_i, which is normal constituent of a nilpotent character φ, is not nilpotent (since H is not p-nilpotent). This shows that, in general, nilpotent characters do not lie over nilpotent characters. Also, $\mu_i^G = \varphi$, and hence the nonnilpotent block of μ_i induces the block of φ.

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Received December 10, 1992. Research partially supported by DGICYT.PB 90-0414-C03-01.

Universitat de Valencia
46100 Burjassot, Valencia
Spain
E-mail address: gabriel@vm.ci.uv.es