PACIFIC JOURNAL OF MATHEMATICS
Vol. 169, No. 2, 1995

CONVERGENCE OF INFINITE EXPONENTIALS

GENNADY BACHMAN

In this paper we give two tests of convergence for an infinite exponential $a_1^{a_2^{a_3^{\cdot^{\cdot^{\cdot}}}}}$. We also show that these tests are essentially the best possible.

1. Introduction and Statement of Results. Given a sequence of positive real numbers a_n, $n = 1, 2, 3, \ldots$, we associate with it a sequence of partial exponentials E_n, $n = 1, 2, 3, \ldots$, defined by

$$E_n = a_1^{a_2^{\cdot^{\cdot^{\cdot^{a_n}}}}} . \tag{1.1}$$

We will call $\{a_n\}$ a sequence of exponents and the sequence $\{E_n\}$ an infinite exponential. As in the study of sums and products one would like to develop tests of convergence of an infinite exponential. Euler [**E**] was the first to give such a test. He showed that in the special case $a_1 = a_2 = a_3 = \cdots = a$, E_n is convergent if and only if $e^{-e} \leq a \leq e^{1/e}$. This result has been rediscovered by many authors. An extensive bibliography of papers containing this and related results may be found in the survey paper by Knoebel [**K**].

In the general case of non-constant exponents the best known results are due to Barrow [**B**]. He showed (although some of his arguments are rather sketchy) that $\{E_n\}$ is convergent for $e^{-e} \leq a_n \leq e^{1/e}$, $n \geq n_0$. He also considered the cases $a_n \geq e^{1/e}$ and $a_n \leq e^{-e}$. In the first case, writing $a_n = e^{1/e} + \epsilon_n$, with $\epsilon_n \geq 0$, he showed that $\{E_n\}$ is convergent if

$$\lim_{n\to\infty} \epsilon_n n^2 < \frac{e^{1/e}}{2e}, \tag{1.2}$$

and is divergent if

$$\lim_{n\to\infty} \epsilon_n n^2 > \frac{e^{1/e}}{2e}. \tag{1.3}$$

In the second case, writing $a_n = e^{-e} - \epsilon_n$, with $\epsilon_n \geq 0$, he obtained the conditions $\lim_{n\to\infty} \epsilon_n = 0$ and $\lim_{n\to\infty} n^q \epsilon_n = 0$ for some $q > 1$, as necessary and sufficient for the convergence of $\{E_n\}$ respectively.

Ramanujan made the following entry (without a proof) on page 30 of his third notebook (see [**R**], page 390, also posed as an unsolved problem at the 1991 West Coast Number Theory Conference, see Problem 91:06 in [**G**]): E_n is convergent when

$$1 + \log\log a_n \leq \frac{1}{2}\left\{\frac{1}{n^2} + \frac{1}{(n\log n)^2} + \frac{1}{(n\log n\log\log n)^2} + \cdots\right\}, \tag{1.4}$$

and is divergent when the left hand side is greater than the right hand side with any 1 replaced by $1+\epsilon$. This statement requires some clarification. What Ramanujan probably had in mind was a test of convergence of an infinite exponential of a sequence of exponents $a_n \geq 1$. A sufficient condition for the convergence was furnished by the inequality $1 + \log\log a_n \leq f(n)$, $n \geq n_0$ for an appropriate, possibly any, function $f(n)$ with an asymptotic expansion given by the right hand side of (1.4) as $n \to \infty$. An easy calculation shows that Barrow's satements (1.2), even in the much stronger form with $<$ replaced by $\leq$, and (1.3) are contained in Ramanujan's assertion with the right hand side of (1.4) truncated after the first term.

The main purpose of this paper is to give a proof of Ramanujan's test of convergence of an infinite exponential and to generalize it to the case of complex exponents a_n. In order that the exponentiation be unambiguous we assume that the sequence of complex numbers b_n, $n = 1, 2, 3, \ldots$ is given and set

$$a_n = e^{b_n}. \tag{1.5}$$

With this definition of the sequence $\{a_n\}$ (1.1) is well defined. The case of complex exponents has also been considered before. The best known results here are due to Shell [**S**], in the case of equal exponenents, and Thron [**T**], in the general case. We state here only the results of Thron, who showed that $\{E_n\}$ is convergent if $|b_n| \leq 1/e$, $n \geq n_0$. We first give the following test of convergence of an infinite exponential with complex exponents:

THEOREM 1. *Let* $\{a_n\}$ *and* $\{E_n\}$ *be defined by* (1.5) *and* (1.1) *respectively. Now set*

$$\hat{a}_n = e^{|b_n|} \quad [n \geq 1], \tag{1.6}$$

and define $\hat{E}_n$*,* $n = 1, 2, 3, \ldots$*, by (1.1) in terms of the sequence* $\{\hat{a}_n\}$*. Then if* $\hat{E}_n$ *converges, then so must* E_n*.*

The above test of convergence is of independent interest. In particular, Thron's result follows immediately from Barrow's results for real exponents a_n, $1 \leq a_n \leq e^{1/e}$, and Theorem 1.

To state our results concerning Ramanujan's test of convergence we introduce the following notation for the iterated logarithm. Setting $x_1 = e$ and

$$L_1(x) = L(x) = \log(x), \quad \text{for } x \geq e,$$

we define recursively x_k and $L_k(x)$, for $k \geq 2$, by $x_k = e^{x_{k-1}}$, and

$$L_k(x) = L_{k-1}(L(x)), \quad \text{for } x \geq x_k.$$

With this notation we have:

THEOREM 2. *Let* $\{E_n\}$ *be defined by* (1.5) *and* (1.1) *respectively. Then the infinite exponential converges if there exist positive integers* k_0 *and* n_0 *such that for all* $n \geq n_0$ *we have*

$$\begin{aligned} &1 + \log|\log a_n| = 1 + \log|b_n| \\ &\leq \frac{1}{2}\left\{\frac{1}{n^2} + \frac{1}{(nL_1(n))^2} + \frac{1}{(nL_1(n)L_2(n))^2} + \cdots \right. \\ &\qquad \left. + \frac{1}{(nL_1(n)L_2(n)\cdots L_{k_0}(n))^2}\right\}. \end{aligned} \tag{1.7}$$

To complement this result we prove:

THEOREM 3. *Let* E_n *be defined by* (1.1) *in terms of a sequence of real numbers* a_n *satisfying* $a_n > 1$ *and*

(1.8)

$$\begin{aligned} &1 + \log\log a_n \\ &\geq \frac{1}{2}\left\{\frac{1}{n^2} + \frac{1}{(nL_1(n))^2} + \cdots \right. \\ &\qquad \left. + \frac{1}{(nL_1(n)L_2(n)\cdots L_{k_0-1}(n))^2} + \frac{1+\epsilon}{(nL_1(n)L_2(n)\cdots L_{k_0}(n))^2}\right\} \end{aligned}$$

for $n \geq n_0$*, for some positive integers* k_0 *and* n_0*, and* $\epsilon > 0$*. Then the infinite exponential diverges.*

2. Preliminaries. In this section we prove three lemmas. The first of these reduces the principal case of our problem to an equivalent problem which is easier to handle. We will find it convenient to use the notation

$$[x_1, x_2, \dots, x_n] = x_1^{x_2^{\cdot^{\cdot^{x_n}}}} \qquad \text{and} \qquad [x_1, x_2, x_3, \dots]$$

to denote partial exponents and an infinite exponential respectively. We also set

$$(2.1)\ \ l_0(x) = \frac{1}{x} \ \text{ and } \ l_k(x) = \frac{1}{xL_1(x)L_2(x)\cdots L_k(x)} \quad [k \geq 1].$$

LEMMA 1. *Let a sequence of real numbers x_n, $n = 1, 2, 3, \dots$, satisfying $x_n > 1$ be given. Define a sequence X_n, $n = 1, 2, 3, \dots$ by*

$$x_n = \exp\left(\frac{1 + X_n}{e}\right). \tag{2.2}$$

Then $[x_1, x_2, x_3, \dots]$ converges if and only if there exists a sequence Y_n, $n = 1, 2, 3, \dots$, satisfying $Y_n \geq -1$ and such that the inequality

$$1 + Y_n \geq (1 + X_n)e^{Y_{n+1}} \tag{2.3}$$

holds.

Proof. Since $x_n > 1$ the sequence $[x_1, x_2, x_3, \dots]$ is monotonically increasing. Hence to show that it is convergent it suffices to show that it is bounded. But this follows immediately from (2.2) and (2.3) since

$$[x_1, x_2, \dots, x_n] \leq [x_1, x_2, \dots, x_n, e^{1+Y_{n+1}}] \leq e^{1+Y_1}.$$

In the opposite direction suppose that the infinite exponential $[x_1, x_2, x_3, \dots]$ is convergent. Since $x_n > 1$ then so must be an infinite exponential $[x_n, x_{n+1}, x_{n+2}, \dots]$ for any $n \geq 1$. Denoting a limit of $[x_n, x_{n+1}, x_{n+2}, \dots]$ by e^{1+Y_n}, we observe that $Y_n \geq -1$ and that the sequence $\{Y_n\}$ satisfies

$$e^{1+Y_n} = \left[x_n, e^{1+Y_{n+1}}\right] = e^{(1+X_n)e^{Y_{n+1}}}.$$

This gives (2.3) with equality and completes the proof of the lemma. $\square$

The next two lemmas are the main ingredients in the proofs of Theorems 2 and 3.

LEMMA 2. *Let T_n^k, C_n^k and X_n^k be defined by*

$$T_n^k = \sum_{j=0}^{k} l_j(n-1), \tag{2.4}$$

$$C_n^k = \frac{1}{2}\sum_{j=0}^{k} l_j^2(n), \tag{2.5}$$

and

$$1 + X_n^k = \left(1 + T_n^k\right) e^{-T_{n+1}^k}, \tag{2.6}$$

where $l_j(n)$ is given by (2.1), for any integers $k \geq 0$ and $n \geq 2$ for which the right hand sides of (2.4) and (2.5) are defined. Then there exists a sequence of integers $\{n_k\}$ such that for all $n \geq n_k$ we have

$$C_n^k < X_n^k < C_n^{k+1}. \tag{2.7}$$

Proof. Let an integer $k \geq 0$ be fixed. We write T_n and X_n to denote T_n^k and X_n^k respectively. By (2.4) and (2.1) we have $T_n = O_k(1/n) < 1$, for $n \geq n_k$ sufficiently large in terms of k. For such integers n we can expand the right hand side of (2.6) in a Taylor series to obtain

(2.8)
$$\begin{aligned} 1 + X_n &= (1+T_n)e^{-T_{n+1}} \\ &= (1+T_n)\left(1 - T_{n+1} + \frac{1}{2}(T_{n+1})^2 - \frac{1}{6}(T_{n+1})^3 + O_k\left(\frac{1}{n^4}\right)\right) \\ &= 1 + T_n - T_{n+1} + \frac{1}{2}(T_{n+1})^2 - T_nT_{n+1} + \frac{1}{2}T_n(T_{n+1})^2 \\ &\quad - \frac{1}{6}(T_{n+1})^3 + O_k\left(\frac{1}{n^4}\right). \end{aligned}$$

Now, by (2.4) and (2.1), expanding T_n about $n+1$ we get

$$T_n = T_{n+1} - T'_{n+1} + \frac{1}{2}T''_{n+1} - \frac{1}{6}T'''_{\xi}, \tag{2.9}$$

for some ξ with $n < \xi < n+1$, where

$$T'_{n+1} = \sum_{j=0}^{k} l'_j(n) = -\sum_{j=0}^{k} l_j(n) \sum_{i=0}^{j} l_i(n), \tag{2.10}$$

$$\begin{aligned} T''_{n+1} &= \sum_{j=0}^{k} l''_j(n) = -\left(\left(l_0^2(n)\right)' + \sum_{j=1}^{k}\sum_{i=0}^{j}(l_j(n)l_i(n))'\right) \\ &= \frac{2}{n^3} + O_k\left(\frac{1}{n^3 \log n}\right), \end{aligned} \tag{2.11}$$

and

$$T'''_{\xi} = \sum_{j=0}^{k} l'''_j(\xi - 1) = O_k\left(\frac{1}{n^4}\right). \tag{2.12}$$

Substituting (2.9)–(2.12) into (2.8) and simplifying the resulting expression we obtain

$$1 + X_n = 1 - T'_{n+1} - \frac{1}{2}(T_{n+1})^2 + \frac{1}{3n^3} + O_k\left(\frac{1}{n^3 \log n}\right).$$

Hence by (2.10), (2.4) and (2.5) we have

$$\begin{aligned} X_n &= -T'_{n+1} - \frac{1}{2}(T_{n+1})^2 + \frac{1}{3n^3} + O_k\left(\frac{1}{n^3 \log n}\right) \\ &= \frac{1}{2}\sum_{j=0}^{k} l_j^2(n) + \frac{1}{3n^3} + O_k\left(\frac{1}{n^3 \log n}\right) \\ &= C_n^k + \frac{1}{3n^3} + O_k\left(\frac{1}{n^3 \log n}\right). \end{aligned} \tag{2.13}$$

This, for $n \geq n_k$ sufficiently large in terms of k, implies (2.7) and completes the proof of the lemma. □

LEMMA 3. *Let T_n^k and X_n^k be defined by* (2.4) *and* (2.6) *of Lemma* 2. *Moreover, let x_n^k be defined by*

$$x_n^k = \exp\left\{\frac{1 + X_n^k}{e}\right\}. \tag{2.14}$$

Then we have

$$\lim_{m\to\infty}\left[x_n^k, x_{n+1}^k, \dots, x_m^k\right] = e^{1+T_n^k}. \tag{2.15}$$

Proof. We begin by observing that it suffices to show that there exists a sequence of integers $\{n'_k\}$ such that (2.15) holds for all $n \geq n'_k$. Indeed, assuming this we have, for any $l \geq n'_k$,

$$\begin{aligned}
\lim_{m\to\infty}\left[x_n^k, x_{n+1}^k, \dots, x_m^k\right] &= \left[x_n^k, \dots, x_l^k, \lim_{m\to\infty}\left[x_{l+1}^k, \dots, x_m^k\right]\right] \\
&= \left[x_n^k, \dots, x_l^k, e^{1+T_{l+1}^k}\right] \\
&= e^{1+T_n^k},
\end{aligned}$$

by (2.14) and (2.6). To exhibit the existence of such a sequence $\{n'_k\}$ we first observe that by (2.14), Lemma 1 and Lemma 2, any infinite exponential $\left[x_n^k, x_{n+1}^k, x_{n+2}^k, \dots\right]$, with $n \geq n_k$, where $\{n_k\}$ is a sequence defined in the statement of Lemma 2, is convergent. Let us denote the limit of such an infinite exponential by $e^{1+S_n^k}$. Then (2.15) will follow if we can show that

$$S_n^k = T_n^k, \tag{2.16}$$

for all $n \geq n'_k \geq n_k$ sufficiently large in terms of k.

To this end let us define, for integers $k \geq 0$ and $n \geq n_k$, t_n^k by

$$t_n^k = T_n^k - S_n^k. \tag{2.17}$$

We will deduce (2.16) from the following three inequalities:

$$S_n^k > S_n^l > 0 \qquad [k > l;\ n \geq \max(n_k, n_l)], \tag{2.18}$$

$$t_n^k \geq 0, \tag{2.19}$$

and

$$t_m^k \geq t_n^k \left(\frac{L_k(m-1)}{L_k(n)}\right)^{t_n^k/2} l_k(m-1) \quad [m > n \geq n'_k], \tag{2.20}$$

with $n'_k \geq n_k$ sufficiently large in terms of k, where in the case $k = 0$ $L_0(x) = x$. Indeed, assume (2.16) fails with $k = 0$ and some

$n \geq n_0'$. Then by (2.19) $t_n^0 > 0$, and hence by (2.20) and (2.4) we get

$$t_m^0 \geq t_n^0 \left(\frac{m-1}{n}\right)^{t_n^0/2} l_0(m-1) > l_0(m-1) = T_m^0,$$

for some $m > n$ sufficiently large in terms of t_n^0. But by (2.17) this implies that $S_m^0 < 0$, which contradicts (2.18). Thus (2.16), with $k = 0$, must hold for all $n \geq n_0'$. We now proceed by induction on k. Assume that (2.16) fails for some $k > 0$ and $n \geq n_k'$. Arguing as above we obtain the inequality

$$t_m^k > l_k(m-1),$$

for some $m > n$ sufficiently large in terms of t_n^k. This, together with (2.17) and (2.4) yield

$$S_m^k = T_m^k - t_m^k < T_m^{k-1} = S_m^{k-1}, \tag{2.21}$$

by the inductive hypothesis, provided $m \geq n_{k-1}'$, as we may assume. But since (2.21) contradicts (2.18) we conclude that (2.16) and hence (2.15) hold. Therefore it only remains to prove (2.18)–(2.20).

To this end, assuming as we may that the sequence $\{n_k\}$ is increasing, we have, for $k > l$ and $n \geq n_k \geq n_l$,

$$X_n^k > X_n^l > 0,$$

and hence

$$x_n^k > x_n^l > e^{1/e}, \tag{2.22}$$

by (2.7), (2.5) and (2.14). It was already shown by Euler [**E**] that the infinite exponential with constant exponents $e^{1/e}$ converges to e. This fact together with (2.22) yields (2.18). To prove (2.19) we observe that for $m > n \geq n_k$, we have

$$\left[x_n^k, x_{n+1}^k, \dots, x_m^k\right] < \left[x_n^k, \dots, x_m^k, e^{1+T_{m+1}^k}\right] = e^{1+T_n^k},$$

by (2.14) and (2.6). Hence $S_n^k \leq T_n^k$ and thus (2.19) holds by the definition (2.16) of t_n^k.

We begin proving (2.20) by observing that by the definition of S_n^k and (2.14) we have

$$e^{1+S_n^k} = \left[x_n^k, e^{1+S_{n+1}^k}\right] = e^{\left(1+X_n^k\right)e^{S_{n+1}^k}}.$$

Hence S_n^k satisfies the identity (2.6) with T_n^k replaced by S_n^k. Let us fix k and write S_n, T_n and t_n for S_n^k, T_n^k and t_n^k respectively. From our last observation it follows that

$$(1+S_n)e^{-S_{n+1}} = (1+T_n)e^{-T_{n+1}}.$$

Substituting $S_m = T_m - t_m$, $m = n,\ n+1$, into the last identity leads to

$$\frac{t_n}{1+T_n} = 1 - e^{-t_{n+1}}. \tag{2.23}$$

Now, by (2.19), (2.18), (2.4) and (2.1), we have

$$0 \le t_n \le T_n \ll_k \frac{1}{n}. \tag{2.24}$$

Hence

$$\begin{aligned} 1 - e^{-t_{n+1}} &= t_{n+1} \sum_{i=1}^{\infty} \frac{1}{i!}(-t_{n+1})^{i-1} \\ &< t_{n+1} \sum_{i=1}^{\infty} \left(-\frac{t_{n+1}}{2}\right)^{i-1} = \frac{t_{n+1}}{1+t_{n+1}/2}, \end{aligned}$$

provided $n \ge n_k'$ sufficiently large in terms of k. Using this bound for the right hand side of (2.23) we obtain

$$t_{n+1} > t_n \frac{1+t_{n+1}/2}{1+T_n}.$$

It now follows that for any integers $m > n \ge n_k'$ we have

$$t_m > t_n \prod_{i=n}^{m-1} \frac{1+t_{i+1}/2}{1+T_i}. \tag{2.25}$$

We use (2.25) in two steps. Firstly, by (2.25) and (2.24), we have

$$\begin{aligned} t_m > t_n \prod_{i=n}^{m-1} \frac{1}{1+T_i} &= t_n \exp\left\{ \sum_{i=n}^{m-1} \log \frac{1}{1+T_i} \right\} \\ &> t_n \exp\left\{ - \sum_{i=n}^{m-1} T_i \right\}, \end{aligned}$$

for $m > n >\geq n'_k$ sufficiently large in terms of k. Now, by (2.4) and (2.1),

$$\sum_{i=n}^{m-1} T_i = \sum_{i=n}^{m-1}\sum_{j=0}^{k} l_j(i-1) < \sum_{j=0}^{k}\int_{n-2}^{m-1} l_j(x)\,dx < \sum_{j=0}^{k} L_{j+1}(m-1).$$

Hence

$$\begin{aligned}(2.26)\qquad t_m &> t_n \exp\left\{-\sum_{j=0}^{k} L_{j+1}(m-1\right\}\\ &= t_n\frac{1}{(m-1)L_1(m-1)\dots L_k(m-1)}\\ &= t_n l_k(m-1),\end{aligned}$$

for any integers $m > n \geq n'_k$. We now reiterate the abpve argument this time using (2.26) instead of (2.19) on the right hand side of (2.25). To this end we observe that for $m > n \geq n'_k$ sufficiently large in terms of k we have $t_n l_k(i)/2 < T_i/2 \ll_k 1/i$ and

$$\sum_{j=n}^{m-1} l_k(i) > \int_n^{m-1} l_k(x)dx = L_{k+1}(m-1) - L_{k+1}(n).$$

Thus

$$\begin{aligned}t_m &> t_n \prod_{i=n}^{m-1}\frac{1+t_n l_k(i)/2}{1+T_i} = t_n\exp\left\{\sum_{i=n}^{m-1}\log\left(\frac{1+t_n l_k(i)/2}{1+T_i}\right)\right\}\\ &> t_n\exp\left\{\sum_{i=n}^{m-1}\left(\frac{1}{2}t_n l_k(i) - T_i\right)\right\}\\ &> t_n\exp\left\{\frac{1}{2}t_n\left(L_{k+1}(m-1) - L_{k+1}(n)\right) - \sum_{j=0}^{k} L_{j+1}(m-1)\right\}\\ &= t_n\left(\frac{L_k(m-1)}{L_k(n)}\right)^{t_n/2} l_k(m-1).\end{aligned}$$

This gives (2.20) and completes the proof of the lemma. □

3. Proofs of Theorems. *Proof of Theorem* 1. We may assume that for all n, $a_n \neq 1$, for otherwise both $[a_1, a_2, a_3, \dots]$ and $[\hat{a}_1, \hat{a}_2, \hat{a}_3, \dots]$ converge trivially. Now fix an integer n, and for $z \in C$, set

$$f(z) = \frac{d}{dz}[a_1, a_2, \dots, a_n, z], \tag{3.1}$$

and

$$g(z) = \frac{d}{dz}[\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n, z]. \tag{3.2}$$

We have, for any $m > n$,

(3.3)
$$\begin{aligned}[a_1, a_2, \dots, a_m] - [a_1, a_2, \dots, a_n] &= [a_1, \dots, a_n, [a_{n+1}, \dots a_m]] - [a_1, \dots, a_n, 1] \\ &= \int_1^{[a_{n+1}, \dots, a_m]} f(z)\, dz.\end{aligned}$$

Setting

$$u = [a_{n+1}, \dots, a_m], \tag{3.4}$$

and

$$w = [\hat{a}_{n+1}, \dots, \hat{a}_m], \tag{3.5}$$

we estimate the right hand side of (3.3) to obtain

$$\begin{aligned}&|[a_1, a_2, \dots, a_m] - [a_1, a_2, \dots, a_n]| \\ &= \left|\int_1^u f(z)\, dz\right| \\ &= \left|\int_0^1 f(1 + (u-1)t)\, d(1 + (u-1)t)\right| \\ &\leq |u - 1| \int_0^1 |f(1 + (u-1)t)|\, dt.\end{aligned} \tag{3.6}$$

Now, by (3.1), (1.5), (3.2) and (1.6), we have

$$\begin{aligned}f(z) &= b_1[a_1, a_2, \dots, a_n, z]\frac{d}{dz}[a_2, a_3, \dots, a_n, z] \\ &= \prod_{k=1}^{n} b_k[a_k, a_{k+1}, \dots, a_n, z],\end{aligned}$$

and

$$g(z) = \prod_{k=1}^{n} |b_k| \left[\hat{a}_k, \hat{a}_{k+1}, \dots , \hat{a}_n, z\right].$$

Hence, by (1.5), (1.6), (3.4) and (3.5), we obtain the inequality

$$\begin{aligned} |f(1-t+ut)| &= \prod_{k=1}^{n} |b_k[a_k, a_{k+1}, \dots , a_n, (1-t+ut)]| \\ &\leq \prod_{k=1}^{n} |b_k| \left[\hat{a}_k, \hat{a}_{k+1}, \dots , \hat{a}_n, (1-t+|u|t)\right] \\ &\leq g(1-t+wt), \end{aligned} \tag{3.7}$$

valid for $0 \leq t \leq 1$. Applying (3.7) to the right hand side of (3.6) we get

(3.8)

$$\begin{aligned} &|[a_1, a_2, \dots , a_m] - [a_1, a_2, \dots , a_n]| \leq |u-1| \int_0^1 g(1+(w-1)t)\, dt \\ &= \frac{|u-1|}{w-1} \int_0^1 g(1+(w-1)t)\, d(1+(w-1)t) = \frac{|u-1|}{w-1} \int_1^w g(z)\, dz \\ &= \frac{|u-1|}{w-1} \left([\hat{a}_1, \hat{a}_2, \dots , \hat{a}_m] - [\hat{a}_1, \hat{a}_2, \dots , \hat{a}_n]\right), \end{aligned}$$

by (3.2) and (3.5). We observe that $w > 1$ since $a_n \neq 1$ and hence $\hat{a}_n > 1$. Moreover,

(3.9)

$$\begin{aligned} |u-1| &= \left|e^{b_{n+1}[a_{n+2}, \dots , a_m]} - 1\right| = \left|\sum_{k=1}^{\infty} \frac{1}{k!} \left(b_{n+1}[a_{n+2}, \dots , a_m]\right)^k\right| \\ &\leq \sum_{k=1}^{\infty} \frac{1}{k!} \left(|b_{n+1}| \left[\hat{a}_{n+2}, \dots , \hat{a}_m\right]\right)^k = w - 1. \end{aligned}$$

The statement of the theorem now follows from (3.8) and (3.9) by the Cauchy criterion for convergence. $\square$

Proof of Theorem 2. By Theorem 1 it suffices to consider real exponents $a_n \geq 1$. In this case the sequence $[a_1, a_2, a_3, \dots]$ is monotonically increasing and it suffices to show that it is bounded. Define a sequence $\{c_n\} = \left\{c_n^{k_0+1}\right\}$ by

$$c_n = \exp\left\{\frac{1 + C_n^{k_0+1}}{e}\right\} \qquad [n \geq n_{k_0+1}],$$

where C^{k_0+1} and n_{k_0+1} are defined in the statement of Lemma 2. Setting $C_n = C^{k_0+1}$, we have, by (2.5), (2.1) and (1.7),

$$1 + \log\log c_n = \log(1 + C_n) = C_n + O\left(C_n^2\right) > \frac{1}{2}\sum_{j=0}^{k_0} l_j^2(n) \geq 1 + \log\log a_n,$$

for $n \geq n_0$ sufficiently large in terms of k_0 as we may assume. Therefore for $n \geq n_0$, $a_n \leq c_n$ and hence

$$[a_{n_0}, a_{n_0+1}, \dots, a_n] \leq [c_{n_0}, c_{n_0+1}, \dots, c_n].$$

Thus it suffices to show that the infinite exponential $[c_{n_0}, c_{n_0+1}, c_{n_0+2}, \dots]$ converges. By Lemma 1 this in term is equivalent to the existence of a sequence S_n, $n = n_0, n_0 + 1, n_0 + 2, \dots$, satisfying $S_n \geq -1$ and

$$1 + S_n \geq (1 + C_n)e^{S_{n+1}}. \tag{3.10}$$

But by Lemma 2

$$1 + C_n = 1 + C_n^{k_0+1} < 1 + X_n^{k_0+1} = \left(1 + T_n^{k_0+1}\right) e^{-T_{n+1}^{k_0+1}}.$$

Hence (3.10) is satisfied with $S_n = T_n^{k_0+1}$, $n \geq n_0$. This completes the proof of the theorem. □

Proof of Theorem 3. We argue by contradiction. Suppose to the contrary that the infinite exponential $[a_1, a_2, a_3, \dots]$ is convergent. Then, since $a_n > 1$, so is $[a_n, a_{n+1}, a_{n+2}, \dots]$ for any $n \geq 1$. Let us denote the limit of such an infinite exponential by e^{1+S_n}. Let us also define A_n by

$$a_n = \exp\left\{\frac{1 + A_n}{e}\right\}. \tag{3.11}$$

Then

$$e^{1+S_n} = \left[a_n, e^{1+S_{n+1}}\right] = e^{(1+A_n)e^{S_{n+1}}}. \tag{3.12}$$

In the remainder of this proof we will use n to denote an integer satisfying $n \geq n_0$. For such n, it is immediate from (1.8) that

$A_n > 0$, since $a_n > e^{1/e}$. Moreover, by (3.11), (1.8), (2.1), (2.5) and (2.7), we have

$$\begin{aligned}(3.13)\qquad A_n &\geq \log(1+A_n) = 1 + \log\log a_n \geq C_n^{k_0} + \frac{\epsilon}{2} l_{k_0}^2(n) \\ &> C_n^{k_0+1} > X_n^{k_0},\end{aligned}$$

for $n \geq n_0$ sufficiently large in terms of k_0 and ϵ, as we may assume. This gives

$$a_n > x_n^{k_0},$$

where $x_n^{k_0}$ is defined by (2.14). Therefore, by the definition of S_n and Lemma 3, we get

$$S_n > T_n^{k_0}. \tag{3.14}$$

We set

$$R_n = S_n - T_n^{k_0}$$

and

$$B_n = A_n - X_n^{k_0}.$$

Then by (3.14)

$$R_n > 0 \tag{3.15}$$

and by (3.13), (2.13), (2.5) and (2.1)

$$\begin{aligned} B_n &\geq C_n^{k_0} + \frac{\epsilon}{2} l_{k_0}^2(n) - X_n^{k_0} = \frac{\epsilon}{2} l_{k_0}^2(n) + O_{k_0}\left(\frac{1}{n^3}\right) \\ &= \epsilon\left(C_n^{k_0} - C_n^{k_0-1}\right) + O_{k_0}\left(\frac{1}{n^3}\right) \\ &= \epsilon\left(X_n^{k_0} - X_n^{k_0-1}\right) + O_{k_0}\left(\frac{1}{n^3}\right) \\ &> \frac{\epsilon}{2}\left(X_n^{k_0} - X_n^{k_0-1}\right), \end{aligned} \tag{3.16}$$

for $n \geq n_0$ sufficiently large in terms of k_0 and ϵ. Now, by (3.12), (2.6), (3.15) and (3.16), we have

$$\begin{aligned} &1 + T_n^{k_0} + R_n \\ &= \left(1 + X_n^{k_0} + B_n\right) e^{T_{n+1}^{k_0} + R_{n+1}} = \left(1 + T_n^{k_0}\right) e^{R_{n+1}} + B_n e^{T_{n+1}^{k_0} + R_{n+1}} \\ &> \left(1 + T_n^{k_0}\right)(1 + R_{n+1}) + \frac{\epsilon}{2}\left((1 + X_n^{k_0}) - (1 + X_n^{k_0-1})\right) e^{T_{n+1}^{k_0}} \\ &= \left(1 + T_n^{k_0}\right)(1 + R_{n+1}) + \frac{\epsilon}{2}\left(T_n^{k_0} - T_n^{k_0-1}\right). \end{aligned}$$

This together with (3.15) and (2.4) yield

$$R_n > R_{n+1} + \frac{\epsilon}{2} l_{k_0}(n-1).$$

Hence we obtain the bound

$$R_n > \frac{\epsilon}{2} \sum_{m=n-1}^{\infty} l_{k_0}(m).$$

But the last assertion is absurd in view of the definition (2.1) of $l_{k_0}(m)$. This contradicts our assumption and thus completes the proof of the theorem. $\square$

References

[B] D.F. Barrow, *Infinite exponentials*, American Math. Monthly, **43** (1936), 150–160.

[E] Leonhard Euler, *De formulis exponentialibus replicatis*, Acta Academiae Scientiarum Petropolitanae, **1** (1778), 38–60.

[G] Richard Guy, *Western Number Theory Problems*, 1991–12–19&22, 1992.

[K] Arthur Knoebel, *Exponentials reiterated*, American Math. Monthly, **88** (1981), 235–252.

[R] S. Ramanujan, Notebooks, Vol. 2, Tata Institute of Fundamental Research, Bombay, 1957.

[S] Donald L. Shell, *On the convergence of infinite exponentials*, Proceedings of the American Math. Society, **13** (1962), 678–681.

[T] W.J. Thron, *Convergence of infinite exponentials with complex elements*, Proceedings of the American Math. Society, **8** (1957), 1040–1043.

Received October 26, 1992 and in revised form March 4, 1994. I would like to thank Professor Adolf Hildebrand for both suggesting this problem and for his help which led to the present version of this paper.

University of Nevada
Las Vegas, NV 89154–4020
E-mail address: bachman@nevada.edu