THE HEAVY SPHERE SUPPORTED BY A CONCENTRATED FORCE

WERNER LEUTERT

1. Introduction. In the linear three-dimensional theory of elasticity only a few particular solutions are known which describe the action of a concentrated force on an isotropic homogeneous solid. The fundamental particular solution which expresses the displacement due to a force at a point within an indefinitely extended solid was given first by Lord Kelvin [5]. It was found again at a later date by Boussinesq [1] along with other particular solutions which can be derived from it and which lead to the solution of the problem of a concentrated force acting on an infinite solid bounded by a plane. Michell [4] obtained the displacements and stresses in an infinite cone acted on by a concentrated force at the vertex by using Boussinesq's results. The solids considered by these authors all extend to infinity.

In this paper a particular solution describing the action of a concentrated force on a finite solid will be considered.

2. The problem. Let there be given an isotropic homogeneous sphere of radius a, which is supported by a radial concentrated force at the south pole. Our problem is the determination of the displacement vector at any point of the sphere in the case of equilibrium, that is, in the case in which the magnitude of the force is equal to the total weight of the sphere.

3. General theory. In the linear theory of elasticity for an isotropic homogeneous medium, the components u, v, w of the displacement vector $\mathfrak{u}$ with respect to a cartesian coordinate system x, y, z satisfy the differential equations of Lamé [2],

$$\Delta \mathfrak{u} + \alpha \operatorname{grad} \operatorname{div} \mathfrak{u} + \beta \mathfrak{X} = 0 , \tag{1}$$

where

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} .$$

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The vector X with components X, Y, Z respectively denotes the body force per unit volume, and

$$\alpha = \frac{\lambda + G}{G}, \qquad \beta = \frac{1}{G}$$

are two constants depending on the material considered. The first component of (1) is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \alpha\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right] + \beta X = 0,$$

which explains the vector notation used. We restrict our attention to the physically important case

$$1 < \alpha, \qquad 0 < \beta .$$

The components F_x, F_y, F_z of the distributed force per unit surface area F which is necessary to maintain the displacement u throughout the solid are given by

$$\beta F_x = \left(\frac{\partial \mathsf{u}}{\partial x}, \mathsf{n}\right) + \frac{\partial u}{\partial n} + (\alpha - 1)\, n_x \operatorname{div} \mathsf{u}$$

$$\beta F_y = \left(\frac{\partial \mathsf{u}}{\partial y}, \mathsf{n}\right) + \frac{\partial v}{\partial n} + (\alpha - 1)\, n_y \operatorname{div} \mathsf{u} \tag{2}$$

$$\beta F_z = \left(\frac{\partial \mathsf{u}}{\partial z}, \mathsf{n}\right) + \frac{\partial w}{\partial n} + (\alpha - 1)\, n_z \operatorname{div} \mathsf{u} \,;$$

n_x, n_y, n_z are the components of the exterior unit normal n. The first line in (2) may be written in the form

$$\beta F_x = n_x \frac{\partial u}{\partial x} + n_y \frac{\partial v}{\partial x} + n_z \frac{\partial w}{\partial x} + n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y} + n_z \frac{\partial u}{\partial z} + (\alpha - 1)\, n_x\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right].$$

4. **Particular solutions when no body forces are present.** Boussinesq showed [1] that particular solutions of (1) for $X = 0$ may be obtained from a scalar function $\phi(x, y, z)$ by putting

$$u = \frac{\partial^2\phi}{\partial x\partial z} \qquad v = \frac{\partial^2\phi}{\partial y\partial z} \qquad w = \frac{\partial^2\phi}{\partial z^2} - \frac{\alpha + 1}{\alpha}\Delta\,\phi, \tag{3}$$

provided ϕ is a biharmonic function,

$$\Delta\,\Delta\,\phi = 0\,. \tag{4}$$

Let

$$r^2 = x^2 + y^2 + z^2\,;$$

then

$$\phi = r \tag{5}$$

represents the action of a concentrated force in the z direction at the origin within an infinite solid [5].

The function

$$\phi = z\,\log(r + z) - r \tag{6}$$

leads to Boussinesq's solution [1] for an infinite solid bounded by the (x, y)-plane and acted upon by a concentrated force at the origin in the z direction. Michell's solution [4] can be obtained by a linear combination of (5) and (6).

The function

$$\phi = (r^2 - 3z^2)\,\log(r + z) + 3zr \tag{7}$$

was used by the author [3] to describe the displacements in a spherical shell under concentrated radial forces.

Since for $X = 0$ the system (1) is linear homogeneous with constant coefficients, particular solutions can be obtained from (5)–(7) by partial differentiation.

If (5) and (6) are differentiated with respect to z, two new particular solutions

$$\phi = \frac{z}{r}\,, \tag{8}$$

$$\phi = \log(r + z) \tag{9}$$

result. From (9) the particular solution

$$\phi = \frac{1}{r} \tag{10}$$

can be derived. A linear combination of (10) and the derivative of (8) with respect to z yields

$$\phi = \frac{z^2}{r^3} \,. \tag{11}$$

5. A particular solution for constant body force in the z-direction. If u, v, w are computed from

$$\phi = \frac{P\alpha\beta}{64\pi(\alpha+1)(3\alpha-1)a^3}\,[-(2\alpha-1)\,r^4 - 6r^2z^2 + (4\alpha+7)\,z^4 - 16(\alpha+1)\,az^3 + 8(\alpha+1)\,ar^2z\,]\,, \tag{12}$$

according to (3), it can be verified that (1) is satisfied for

$$X = Y = 0\,, \qquad Z = -\frac{3P}{4\pi a^3} \tag{13}$$

Equation (4) is no longer valid for the ϕ of (12).

6. Solution of the problem. The south pole of the sphere is taken as the origin of the coordinate system, with the z axis directed vertically upward. The sphere is then represented by the equation

$$r^2 \le 2az\,. \tag{14}$$

The components of the exterior unit normal n are

$$an_x = x\,, \qquad an_y = y\,, \qquad an_z = z - a\,.$$

It can be verified that the function

$$\phi = \frac{P\alpha\beta}{64\pi(\alpha+1)(3\alpha-1)\,a^3}\,[-(2\alpha-1)\,r^4 - 6r^2z^2 \tag{15}$$

$$+ (4\alpha + 7)\ z^4 - 16(\alpha + 1)\ az^3 + 8(\alpha + 1)\ ar^2 z\ ]$$

$$+ \frac{P\beta}{96\ \pi(\alpha + 1)\ ar^3}[9(\alpha + 1)\ zr^4 - 12(2\alpha + 1)\ ar^4$$

$$- 48\alpha a^2 zr^2 + 16(\alpha - 1)\ a^3 r^2 + 16\alpha a^3 z^2]$$

$$+ \frac{P\beta}{32\pi a}\left[r^2 - 3z^2 + 4az - \frac{16a^2}{\alpha + 1}\right]\log(r + z)$$

satisfies (1) provided the body forces are distributed according to (13). The particular solution (14) consists of a linear combination of (5)–(11) added to (12). On the surface of the sphere $r^2 = 2az$ it is found that the distributed force F per unit surface area (2) vanishes on the whole surface except at the origin, where the particular solution (15) has a singularity.

Because the resulting body force must be in equilibrium with the resulting exterior force, it follows from (13) that the latter is radial upward and of magnitude P.

Since ϕ in (15) possesses a singularity at the origin, the corresponding displacements and stresses will be infinite at that point. To avoid this difficulty we can imagine the material near the origin cut out and the concentrated force P replaced by the statically equivalent forces distributed over the surface of the small cavity.

The displacements belonging to (15) can be computed by using (3).

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UNIVERSITY OF MARYLAND