CONCERNING HEREDITARILY INDECOMPOSABLE CONTINUA

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1. Introduction. A continuum is indecomposable if it is not the sum of two proper subcontinua. It is hereditarily indecomposable if each of its subcontinua is indecomposable. In [3] Knaster gave an example of a hereditarily indecomposable continuum which was not a point. In this paper we study some properties of the Knaster example and describe some other hereditarily indecomposable continua.

2. Chained hereditarily indecomposable continua are homeomorphic. The hereditarily indecomposable continuum given [3] by Knaster was a plane continuum which was described in terms of covering bands. For each positive number ϵ, it could be covered by an ϵ-chain. Moise used [5] a hereditarily indecomposable continuum to exhibit a continuum which was topologically equivalent to each of its nondegenerate subcontinua. He called it a pseudo-arc and noted that it was similar (if not in fact topologically equivalent) to Knaster's example. It could be chained. Bing used [2] such a continuum as an example of a homogeneous plane continuum. Anderson showed [1] that the plane is the sum of a continuous collection of such continua. Theorem 1 reveals that all of these continua are topologically equivalent.

We follow the definitions used in [2]. In particular, we recall the following. A *chain* $D = [d_1, d_2, \cdots, d_n]$ is a collection of open sets $d_1, d_2, \cdots, d_n$ such that d_i intersects d_j if and only if i is equal to $j-1, j$, or $j+1$. If the links are of diameter less than ϵ, the chain is called an ϵ-*chain.* We do not suppose that the links of a chain are necessarily connected.

If the chain $E = [e_1, e_2, \cdots, e_n]$ is a refinement of the chain $D = [d_1, d_2, \cdots, d_m]$, E is called *crooked* in D provided that if $k - h > 2$ and e_i and e_j are links of E in links d_h and d_k of D, respectively, then there are links e_r and e_s of E in links d_{k-1} and d_{h+1}, respectively, such that either $i > r > s > j$ or $i < r < s < j$.

EXAMPLE 1. *The pseudo-arc.* The following description of a chained hereditarily indecomposable continuum appeared in [2] and is much like one given

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earlier in [5]. In the plane let $D_1, D_2, \cdots$ be a sequence of chains between the distinct points p and q such that for each positive integer i, (a) D_{i+1} is crooked in D_i, (b) no link of D_i has a diameter of more than $1/i$, and (c) the closure of each link of D_{i+1} is a compact subset of a link of D_i. The common part of the sum of the links of D_1, the sum of the links of $D_2, \cdots$ is a pseudo-arc. That it is hereditarily indecomposable is shown in [2] and [5].

A continuum can be chained if for each positive number ϵ, the continuum can be covered by an ϵ-chain. A *composant* of a continuum W is a set H such that for some point p of W, H is the sum of all proper subcontinua of W containing p. We recall that a nondegenerate indecomposable compact continuum has uncountably many composants and no two of its composants intersect each other. The following result holds in a metric space.

THEOREM 1. *The compact nondegenerate hereditarily indecomposable continua M and M' are homeomorphic if each can be chained.*

In fact, if p and q are points of different composants of M while p' and q' are points of different composants of M', there is a homeomorphism carrying M into M', p into p', and q into q'.

Proof. Since M can be chained, there is a sequence $C_1, C_2, \cdots$ such that C_i is a $1/i$-chain covering M, each element of C_i intersects M, and C_{i+1} is a refinement of C_i.

First, we show that there is an integer n_2 so large that C_{n_2} is crooked in $C_1 = [c_{1,1}, c_{1,2}, \cdots, c_{1,t_1}]$. If this were not true there would be elements $c_{1,h}$ and $c_{1,k}$ of C_1 such that $k - h > 2$ and for infinitely many integers m, $C_m = [c_{m,1}, c_{m,2}, \cdots, c_{m,t_m}]$ would have two links $c_{m,i}$ and $c_{m,j}$ in $c_{1,h}$ and $c_{1,k}$ respectively such that if $c_{m,r}$ is in $c_{1,k-1}$ and between $c_{m,i}$ and $c_{m,j}$, then there is not a link of C_m in $c_{1,h+1}$ which is between $c_{m,r}$ and $c_{m,j}$. Denote by W_m the sum of $c_{m,i}$, $c_{m,r}$, and the elements of C_m between them, where we suppose that no element of C_m in $c_{1,k-1}$ is between $c_{m,i}$ and $c_{m,r}$. Let V_m be the sum of $c_{m,r}$, $c_{m,j}$, and the elements of C_m between them.

Let $a_1, a_2, \cdots$ be an increasing sequence of integers such that both W_{a_1}, $W_{a_2}, \cdots$ and $V_{a_1}, V_{a_2}, \cdots$ converge. But the limiting set W of $W_{a_1}, W_{a_2}, \cdots$ is a continuum which intersects $\bar{c}_{1,h}$ but not $\bar{c}_{1,k}$. Also, the limiting set V of $V_{a_1}, V_{a_2}, \cdots$ is a continuum which intersects $\bar{c}_{1,k}$ but not $\bar{c}_{1,h}$. Hence the assumption that there is no integer n_2 such that C_{n_2} is crooked in C_1 has led to the contradiction that the hereditarily indecomposable continuum M contains the decomposable continuum $W + V$.

Hence, there is a subsequence $C_{n_1}, C_{n_2}, C_{n_3}, \cdots$ of $C_1, C_2, C_3, \cdots$ such that $C_{n_{i+1}}$ is crooked in C_{n_i}.

Let p and q be points belonging to different composants of M. Then for each integer j, there is an integer k greater than j such that the subchain of C_{n_k} from p to q has a link that intersects the first link of C_{n_j} and has a link that intersects the last link of C_{n_j}. To see that this is so, let W_i be the sum of the links of the subchain of C_{n_i} from p to q. Since the limiting set of each subsequence W_1, W_2, $\cdots$ is a continuum in M containing $p + q$, each such limiting set is M. Hence, some $W_k (k > j)$ intersects both the first and last links of C_{n_k}, and the subchain of C_{n_k} corresponding to W_k has links intersecting the first and last links of C_{n_j}.

We find from Theorem 4 of [2] that there is a chain E_j such that the first link of E_j contains p, the last contains q, E_j is a consolidation of C_{n_j}, while each link of E_j lies in the sum of two adjacent links of C_{n_j} and is therefore of diameter less than $2/j$. Hence, there is no loss of generality in supposing that each of the chains $C_1, C_2, \cdots$ is from p to q. Therefore, there is a sequence $D_1, D_2, \cdots$ of chains from p to q such that for each positive integer i, (a) D_{i+1} is crooked in D_i, (b) the closure of each link of D_{i+1} is a subset of a link of D_i, (c) no link of D_i has a diameter of more than $1/i$, and (d) M is the common part of the sum of the links of D_1, the sum of the links of $D_2, \cdots$.

Similarly, we find that if p' and q' are points of different composants of M', there is a sequence $D_1', D_2', \cdots$ of chains from p' to q' such that (a) D_{i+1}' is crooked in D_i', (b) the closure of each element of D_{i+1}' is a subset of an element of D_i', (c) no element of D_i' has a diameter of more than $1/i$, and (d) D_i covers M'. Theorem 12 of [2] shows that there is a homeomorphism that carries M into M', p into p', and q into q'.

The preceding theorem shows that M is homogeneous and homeomorphic with each of its nondegenerate subcontinua. It also reveals that the continua studied by Knaster [3], Moise [5], Bing [2], and Anderson [1] are all topologically equivalent and are pseudo-arcs.

QUESTION. It would be interesting to know if each nondegenerate bounded hereditarily indecomposable plane continuum which does not separate the plane is homeomorphic to M. This question would be answered in the affirmative if it were shown that each bounded atriodic plane continuum which does not separate the plane can be chained (see Section 6, below).

3. Most continua are pseudo-arcs. Mazurkiewicz showed [4] that the continua contained in a circle plus its interior which were not hereditarily indecomposable were of the first category. We go even further to show that those which are not pseudo-arcs are of the first category.

The *space of compact continua* in the metric space S is the metric space $C(S)$ whose points are the compact continua of S, where the distance between two elements g_1, g_2 of $C(S)$ is the Hausdorff distance between them in S, namely L.U.B. (distance from x to g_i in S), $x \in g_1 + g_2$; $i = 1,2$. By saying that *most compact continua of S have a certain property*, we mean that there is a dense inner limiting (G_δ) subset W of $C(S)$ such that each element of W has the property (when regarded as a continuum in S). The collection of continua of S with this property is said to be of the second category.

The following theorem holds in a space S which is either a Hilbert space or a Euclidean n-space ($n > 1$).

THEOREM 2. *Most compact continua are pseudo-arcs.*

Proof. Let F_i be the collection of all compact continua f in S such that f can not be covered by a $1/i$-chain. If f_1, f_2, $\cdots$ is a sequence of elements of F_i converging to a compact continuum f_0, f_0 is an element of F_i because if a $1/i$-chain covers f_0, it covers some f_j. Hence, F_i is closed in $C(S)$.

Let G_i be the set of all compact continua K such that K contains a subcontinuum K' which is the sum of two continua K_1 and K_2 such that K_1 contains a point at a distance (in S) of $1/i$ or more from K_2, while K_2 contains a point at a distance of $1/i$ or more from K_1. Then G_i is a closed subset of $C(S)$. Furthermore, the collection of points of S is closed in $C(S)$.

The collection of all pseudo-arcs is dense in $C(S)$. For suppose that g is an element of $C(S)$ and ϵ is a positive number. There is a broken line ab whose distance (in $C(S)$) from g is less than $\epsilon/2$. Let D be an $\epsilon/2$-chain from a to b covering ab such that each element of D is the interior of a sphere. There is a pseudo-arc h containing $a + b$ which is covered by D. The distance from g to h is less than ϵ in $C(S)$.

Each element of $C(S)$ not belonging to ΣF_i can be chained and each element of $C(S)$ not belonging to ΣG_i is hereditarily indecomposable. Let W be the set of all elements g of $C(S)$ such that g is not a point of S, g is not an element of any F_i, and g is not an element of any G_i. By Theorem 1, each element of W is topologically equivalent to M. However, W is a dense inner limiting subset of $C(S)$.

4. Decomposition of a pseudo-arc. Like a simple closed curve, the pseudo-arc is homogeneous; and like an arc it is homeomorphic with each of its nondegenerate subcontinua. We now show that like both a simple closed curve and an arc, it is topologically equivalent to itself under a nontrivial monotone decomposition.

THEOREM 3. *If an upper semi-continuous collection of continua fills a chained compact continuum, the resulting decomposition space is chained.*

Proof. Suppose G is an upper semi-continuous collection of continua filling the chained continuum M and that the resulting decomposition space M' is given a metric. We show that for each positive number ϵ, an ϵ-chain in M' covers M'.

Let δ be a positive number so small that if two elements of G are no farther apart in M than δ, then the points in M' corresponding to them are no farther apart than $\epsilon/5$ in M'. Let $[c(1), c(2), \cdots, c(n)]$ be a δ-chain covering M. Let $n_1 = 1$, $n_2, n_3, \cdots, n_j = n$ be a monotone increasing sequence of integers such that an element of G intersects $c(n_i)$ and $c(n_{i+1})$, but none intersects $c(n_i)$ and $c(n_{i+1}+1)$. Denote by $D(i,j)$ the open subset of M' consisting of those points corresponding to elements of G which are covered by $c(i) + c(i+1) + \cdots + c(j)$. Then

$$D(n_1, n_5),\ D(n_4, n_8),\ D(n_7, n_{11}), \cdots,\ D(n_{3k+1}, n_j),$$

$$j - 5 \leq 3k + 1 < j - 3$$

is an ϵ-chain covering M'.

The following result follows from Theorems 1 and 3.

THEOREM 4. *If M is a pseudo-arc and G is an upper semi-continuous collection of proper subcontinua of M filling M, the resulting decomposition space is topologically equivalent to M.*

5. Other types of hereditarily indecomposable plane continua. The pseudo-arc can be imbedded in the plane. We now show that there are nondegenerate hereditarily indecomposable plane continua which are not topologically equivalent to the pseudo-arc. In fact, there are as many topologically different hereditarily indecomposable plane continua as there are plane continua.

A method which differs from the one below of showing that there are topologically different nondegenerate plane continua is to consider in 3-space the intersection of a plane with a hereditarily indecomposable continuum which separates 3-space. That there are such continua in 3-space will be shown in my paper, *Higher dimensional hereditarily indecomposable continua,* which proves that there are hereditarily indecomposable continua of all dimensions.

EXAMPLE 2. *A hereditarily indecomposable continuum that separates the plane.* A circular chain differs from a chain in that its first and last links intersects each other. In the plane let $D_1, D_2, \cdots$ be a sequence of circular chains such that (a) each element of D_i is the interior of a circle of diameter less than $1/i$, (b) the closure of each element of D_{i+1} lies in an element of D_i, (c) the sum of the elements of D_i is topologically equivalent to the interior of an annulus ring, (d) each complementary domain of the sum of the elements of D_{i+1} contains a complementary domain of the sum of the elements of D_i, and (e) if E_i is a proper subchain of D_i and E_{i+1} is a subchain of D_{i+1} contained in E_i, then E_{i+1} is crooked in E_i. Condition (c) is superfluous since it follows from condition (a). The required continuum M is the intersection of the sum of the elements of D_1, the sum of the elements of $D_2, \cdots$.

Suppose that a chain $D_i = [d_1, d_2, \cdots, d_n]$ satisfying the preceding conditions has been found. To see that there is a chain D_{i+1} satisfying the required conditions, it might be convenient first to consider a chain $D'_i = [d'_1, d'_2, \cdots, d'_{3n}]$ in D_i which follows the following pattern: $d'_1, d'_{n+1}, d'_{2n+1}$ are subsets of d_1; d'_2, d'_{n+2}, d'_{3n} are subsets of d_2; $d'_3, d'_{n+3}, d'_{3n-1}$ are subsets of $d_3; \cdots$; d'_n, d'_{2n}, d'_{2n+1} are subsets of d_n. Roughly speaking, D'_1 goes through D_i twice in one direction and once in the opposite direction. Then the circular chain D_{i+1} satisfying the required conditions is the sum of two chains one of which is crooked in $[d'_1, d'_2, \cdots, d'_{2n+1}]$ and the other of which is crooked in

$$[d'_{2n+1}, d'_{2n+2}, \cdots, d'_{3n}, d'_i].$$

That M separates the plane follows from conditions (c) and (d). We show that it is hereditarily indecomposable by showing that each of its proper subcontinua is indecomposable.

Suppose M' is a proper subcontinuum of M. Let m be an integer so large that some element of D_m does not intersect M'. If $E_k (k \geq m)$ is the collection of elements of D_k which intersect M', then $E_m, E_{m+1}, \cdots$ is a sequence of chains such that E_{k+1} is crooked in E_k. If M' were the sum of two proper subcontinua H and K, there would be a point p of H; a point q of K, and an integer w such that the distances from p to K and from q to H are each greater than $2/w$. Suppose e_i and e_j are elements of E_{w+1} containing p and q respectively. Since E_{w+1} is crooked in E_w, there are elements e_r and e_s in E_{w+1} such that e_r separates e_i from e_s in E_{w+1}, each point of e_s is nearer than $2/w$ to p, and each point of e_r is nearer to q than $2/w$. Then H would not be connected because it has a point in e_s, a point in

e_i, but none in e_r.

QUESTIONS. Theorem 1 shows that if M_1 and M_2 are two nondegenerate chained hereditarily indecomposable continua, they are topologically equivalent and each is homogeneous. Suppose W_1 and W_2 are two continua, each defined as described in Example 2. It would be interesting to know if W_1 would necessarily be homeomorphic to W_2. Also, is any such continuum homogeneous?

EXAMPLE 3. *Another hereditarily indecomposable continuum.* The following example of a hereditarily indecomposable plane continuum is somewhat of a combination of Examples 1 and 2.

If W is a nondegenerate indecomposable plane continuum, it contains a nondegenerate subcontinuum H such that no point of H is accessible from the complement of W. To see that this is true, consider two parallel lines L_1 and L_2, each of which separates W. Let G be an uncountable collection of mutually exclusive subcontinua of W each irreducible from L_1 to L_2, and let K be the sum of L_1, L_2, and the closure of the sum of the elements of G. If D is a complementary domain of K between L_1 and L_2, its closure does not intersect three elements of G. Hence, some element of G is not accessible from any complementary domain of K between L_1 and L_2. If H is a subcontinuum of this element of G which does not intersect $L_1 + L_2$, no point of H is accessible from the complement of W.

If W_0 is a nondegenerate hereditarily indecomposable plane continuum, there is a point p of W_0 such that if W' is a nondegenerate subcontinuum of W_0 containing p, then p is not accessible from the complement of W'. To find such a point, let $W_1, W_2, \cdots$ be a sequence of continua such that W_i is a subcontinuum of W_{i-1}, no point of W_{i+1} is accessible from the complement of W_i, and W_i is of diameter less than $1/i$. Then $W_0 \cdot W_1 \cdot W_2 \cdot \cdots$ is a point p; and if W' is a nondegenerate subcontinuum of W_0 containing p, it contains one of the W_i's. Hence, p is not accessible from the complement of W'.

Let M_1 be a pseudo-arc in the plane and M_2 be a hereditarily indecomposable plane continuum as described in Example 2. Let p be a point of M_1 such that p is not accessible from the complement of any nondegenerate subcontinuum of M_1. By a theorem of R. L. Moore, there is a continuous transformation T of the plane into itself such that $T^{-1}(p) =$ (M_2 plus its interior) and the inverse of each other point is a point. We show that $M_3 = (T^{-1}(M_1)$ minus the interior of M_2) is hereditarily indecomposable. That M_2 and M_3 are not homeomorphic follows from the fact that M_2 is irreducible with respect to separating the plane but M_3 is not.

If W is a subcontinuum of M_3 intersecting M_2 and containing a point of $M_3 - M_2$, it contains M_2. For suppose that it does not contain some point of M_2. Then there is an arc α from the exterior of M_2 to M_2 that does not intersect W. But $T(\alpha)$ would be an arc, revealing that p is accessible from the complement of $T(W)$. This is contrary to the definition of p.

Suppose that some subcontinuum M' of M_3 is the sum of two proper subcontinua H and K. Since M_2 is hereditarily indecomposable, M' is not a subset of M_2. Since M_1 is hereditarily indecomposable, we may suppose that $T(H) = T(M')$. But H would equal M' because T is one-one on the exterior of M_2 and H contains M_2 if it intersects it.

A variation of the method used in obtaining Example 3 may be used to get other topologically different hereditarily indecomposable plane continua. Instead of replacing a point of M_1 by a continuum homeomorphic to M_2, we can replace each of several points of M_1 by such a continuum.

THEOREM 5. *There are as many topologically different hereditarily indecomposable bounded plane continua as there are real numbers.*

Proof. Suppose $n_1, n_2, \cdots$ is a monotone increasing sequence of positive integers. The collection of such sequences has the power of the continuum. For each such sequence we describe a hereditarily indecomposable plane continuum such that no two of these continua are topologically equivalent.

The hereditarily indecomposable continuum associated with $n_1, n_2, \cdots$ will have one composant containing exactly n_1 continua each topologically equivalent to M_2 of Example 2, another composant containing exactly n_2 continua each topologically equivalent to $M_2, \cdots$; no other composant contains a continuum topologically equivalent to M_2.

Let M_1 be a pseudo-arc in the plane. Suppose

$$p_{1,1}, p_{1,2}, \cdots, p_{1,n_1}, p_{2,1}, \cdots$$

is a converging sequence of different points of M_1 such that $p_{i,j}$ is not accessible from the complement of any nondegenerate subcontinuum of M_1 containing it, and $p_{i,j}$ belongs to the composant containing $p_{r,s}$ if and only if $i = r$.

Suppose $M_{1,1}, M_{1,2}, \cdots, M_{1,n_1}, M_{2,1}, \cdots$ is a sequence of mutually exclusive continua in the plane all topologically equivalent to M_2 of Example 2 such that the sequence converges to a point and each of the continua lies in the exterior of each of the others. There is a continuous transformation T of the plane into itself such that $T^{-1}(p_{i,j})$ is ($M_{i,j}$ plus its interior), and $T^{-1}(q)$ is a point if q is not a $p_{i,j}$. The argument used in Example 3 shows that M_3 = ($T^{-1}(M_1)$ minus the sum of the interiors of the $M_{i,j}$'s) is a hereditarily indecomposable continuum. Furthermore, one composant of M_3 contains exactly n_1 mutually exclusive subcontinua, each topologically equivalent to M_2, another composant contains exactly n_2 such subcontinua, $\cdots$, while the other composants of M_3 contain no such continua.

6. Added in proof. R. D. Anderson answered the question at the end of Section 2 in the negative by announcing at the February, 1951, meeting of the American Mathematical Society in New York that there are nondegenerate bounded hereditarily indecomposable plane continua other than pseudo-arcs which do not separate the plane.

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