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CHARACTERS OF SUPERCUSPIDAL REPRESENTATIONS OF $SL(N)$

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Let Θ_π be the character of an irreducible supercuspidal representation π of the special linear group $SL_n(F)$, where F is a p-adic field of characteristic zero and residual characteristic greater than n. In this paper, we investigate the existence of a regular elliptic adjoint orbit $\mathcal{O}_\pi$ such that, up to a nonzero constant, Θ_π (composed with the exponential map) coincides on a neighbourhood of zero with the Fourier transform of the invariant measure on $\mathcal{O}_\pi$. When such an orbit $\mathcal{O}_\pi$ exists, the coefficients in the local expansion of Θ_π as a linear combination of Fourier transforms of nilpotent adjoint orbits are given as multiples of values of the correponding Shalika germs at $\mathcal{O}_\pi$. Let q be the order of the residue class field of F. If n and $q-1$ are relatively prime, we show that there is an elliptic orbit $\mathcal{O}_\pi$ as above attached to every irreducible supercuspidal π. When n and $q-1$ have a common divisor, necessary and sufficient conditions for existence of an orbit $\mathcal{O}_\pi$ are given in terms of the number of representations in the Langlands L-packet of π.

1. Introduction.

Let $d(\pi)$ be the formal degree of π. Our aim is to determine the conditions under which there exists a regular elliptic element X_π in the Lie algebra of $SL_n(F)$ such that

$$\Theta_\pi(\exp X) = d(\pi)\, \widehat{\mu}_{\mathcal{O}(X_\pi)}(X) \tag{1.1}$$

for all regular elements X in some neighbourhood of zero in the Lie algebra. Here $\widehat{\mu}_{\mathcal{O}(X_\pi)}$ denotes the Fourier transform of the orbital integral associated to the $\operatorname{Ad} SL_n(F)$-orbit $\mathcal{O}_\pi = \mathcal{O}(X_\pi)$ of X_π. An $\operatorname{Ad} SL_n(F)$-orbit $\mathcal{O}$ is said to be nilpotent if it consists of nilpotent elements. Harish-Chandra ([**HC2**]) proved that there exist constants $c_{\mathcal{O}}(\pi)$ such that

$$\Theta_\pi(\exp X) = \sum_{\mathcal{O} \text{ nilpotent}} c_{\mathcal{O}}(\pi)\, \widehat{\mu}_{\mathcal{O}}(X),$$

for regular elements X in some neighbourhood of zero. If (1.1) holds, the coefficients in Harish-Chandra's expansion have the form

$$c_{\mathcal{O}}(\pi) = d(\pi)\,\Gamma_{\mathcal{O}}(X_\pi), \qquad \mathcal{O} \text{ nilpotent} \tag{1.2}$$

where $\Gamma_{\mathcal{O}}$ is the Shalika germ associated to the orbit $\mathcal{O}$.

In an earlier paper ([**Mu1**]), under the assumption $p > n$, (1.1) and (1.2) were proved for all irreducible supercuspidal representations of $GL_n(F)$. As shown by Howe ([**H**]) and Moy ([**Mo**]), the equivalence classes of irreducible supercuspidal representations of $GL_n(F)$ correspond bijectively with the conjugacy classes of admissible characters of multiplicative groups of degree n extensions of F. If θ is such a character, π_θ denotes an element of the corresponding equivalence class of representations. An irreducible supercuspidal representation of $SL_n(F)$ is a component of the restriction π'_θ of some π_θ to $SL_n(F)$. Moy and Sally ([**MS**]) studied the decompositions of the representations π'_θ.

Moy and Sally realized certain (not necessarily irreducible) components of π'_θ as representations induced from finite-dimensional representations of open compact subgroups. The inducing data for one of these components $\overline{\pi}$ is the restriction of the inducing data for π_θ to $SL_n(F)$. If X_{π_θ} is the element of the Lie algebra of $GL_n(F)$ appearing in (1.1) for $\pi = \pi_\theta$, set

$$S_\theta = X_{\pi_\theta} - \frac{\operatorname{tr}(X_{\pi_\theta})}{n} I_n,$$

where I_n is the $n \times n$ identity matrix. §2 is devoted to proving (Proposition 2.6)

$$f_\theta(1)^{-1} \int_K f_\theta(k^{-1} \exp X k)\, dk = \int_K \psi_0(\operatorname{tr}\ S_\theta \operatorname{Ad} k^{-1}(X))\, dk,$$

where f_θ is a particular matrix coeffient of $\overline{\pi}$, ψ_0 is a nontrivial character of F, K is a certain open compact subgroup, and X is any nilpotent element in the Lie algebra of $SL_n(F)$. Many results in §2 are proved by modifying similar results in §3 of [**Mu1**].

In Theorem 3.2, using Proposition 2.6 and results of Harish-Chandra, we show that (1.1) holds for $\pi = \overline{\pi}^g$, g in $GL_n(F)$, with $X_\pi = \operatorname{Ad} g(S_\theta)$. It then follows that (1.2) also holds (Corollary 3.5). Necessary and sufficient conditions for the representations $\overline{\pi}^g$ to be irreducible are determined in ([**MS**]). When these conditions are satisfied, the irreducible components of π'_θ are all of the form $\overline{\pi}^g$ ([**MS**]), and thus (1.1) and (1.2) hold. These irreducible components make up an L-packet of supercuspidal representations, and the associated X_π's make up a set of representatives for the orbits within the stable orbit of S_θ. These results are summarized in Corollary 3.6. If n and $q-1$ are relatively prime (recall that q is the order of the residue class field of F), the representations $\overline{\pi}^g$ are irreducible for all admissible characters θ ([**MS**]), and therefore (1.1) and (1.2) hold for all irreducible supercuspidal representations of $SL_n(F)$.

The case where $\overline{\pi}$ is reducible is considered in §4. The irreducible components of π'_θ still form an L-packet of supercuspidal representations, and we can associate the stable orbit of S_θ to this L-packet. However, as proved in Theorem 4.5, if π is an element of the L-packet, (1.1) does not hold for any X_π. As shown in §3, appropriate direct sums of elements in the L-packet (that is, the representations $\overline{\pi}^g$) satisfy (1.1) and (1.2) with X_π in the stable orbit of S_θ.

Suppose n is prime. Although (1.1) may not hold, modulo determination of the values of the Shalika germs on the regular elliptic set, the coefficients $c_{\mathcal{O}}(\pi)$ appearing in the local character expansion of an irreducible supercuspidal representation are known for all nilpotent orbits $\mathcal{O}$. For details, see remarks at the end of §4. In this case, Assem([**As**]) has obtained explicit formulas for the functions $\widehat{\mu}_{\mathcal{O}}$.

Results of type (1.1) and (1.2) have also been proved for supercuspidal representations of the unramified 3×3 unitary group ([**Mu2**]) and other classical groups ([**Mu3**]).

2. Preliminary results.

Let $n \geq 2$ be an integer which is prime to the residual characteristic p of F. Let $G = GL_n(F)$ and $G' = SL_n(F)$. To each admissible character θ of a degree n extension of F, Howe ([**H**]) associated a finite-dimensional representation κ_θ of an open, compact mod centre subgroup K_θ of G. The induced representation $\pi_\theta = \mathrm{Ind}_{K_\theta}^G \kappa_\theta$ is irreducible and supercuspidal. In this way, Howe defined an injection from the set of conjugacy classes of admissible characters of degree n extensions of F into the set of equivalence classes of irreducible supercuspidal representations of G. Moy ([**Mo**]) showed that this map is a bijection. That is, every irreducible supercuspidal representation of G is equivalent to some π_θ.

From this point onward, we assume that p is greater than n. The main result of this section, Proposition 2.6, is the analogue of Proposition 3.10 of [**Mu1**] for a certain (not necessarily irreducible) component of the restriction of π_θ to G'.

Let E be a finite extension of F such that the degree of E over F is prime to p. We shall write O_E for the ring of integers in E, $\mathfrak{p}_E$ for the maximal prime ideal in O_E, and ϖ_E for a prime element in O_E. Let $N_{E/F}$ and $\mathrm{tr}_{E/F}$

be the norm and trace maps from E to F.

Fix an additive character ψ_F of F having conductor $\mathfrak{p}_F$, that is, $\psi_F \mid \mathfrak{p}_F \equiv 1$ and $\psi_F \mid O_F \not\equiv 1$. In later sections, Fourier transforms will be taken relative to the additive character ψ_0 of F defined by $\psi_0(x) = \psi_F(\varpi x)$. Set $\psi_E = \psi_F \circ \mathrm{tr}_{E/F}$.

If $\theta : E^\times \to C^\times$ is a continuous quasi-character of $E^\times$, the conductoral exponent $f_E(\theta)$ of θ is the smallest non-negative integer i such that $1 + \mathfrak{p}_E^i$ is contained in the kernel of θ.

Let θ be an admissible character ([**H**] or [**Mo**]) of the multiplicative group of a degree n extension E of F. In §3 of [**Mu1**], an element of E was associated to each such θ. In this paper, we call that element X_θ. For completeness, we restate the definition here.

Lemma 2.1 ([**H**]). *There exists a unique tower of fields*

$$F = E_0 \subset E_1 \subset \cdots \subset E_r = E$$

and quasi-characters χ, $\phi_1, \ldots, \phi_r$ *of* $F^\times$, $E_1^\times, \ldots, E_r^\times$ *respectively, with* ϕ_s *generic over* E_{s-1} *and such that*

$$\theta = (\chi \circ N_{E/F})(\phi_1 \circ N_{E/E_1}) \cdots \phi_r.$$

The conductoral exponents are unique and satisfy

$$f_E(\phi_1 \circ N_{E/E_1}) > \cdots > f_E(\phi_r).$$

For the definition of generic, see [**Mo**], [**MS**] or [**Mu1**]. Set

$$\ell(s) = \left[\frac{f_{E_s}(\phi_s) + n - 1}{n}\right], \qquad 1 \le s \le r-1.$$

Because $p > n$, the function $x \mapsto \phi_s\left(\sum_{0 \le m \le n-1} x^m/m!\right)$ is a character of $\mathfrak{p}_{E_s}^{\ell(s)}$, $s = 1, \ldots, r-1$. Thus there exists $c_s \in E_s$ such that

$$\phi_s\left(\sum_{m=0}^{n-1} x^m/m!\right) = \psi_{E_s}(c_s x), \qquad x \in \mathfrak{p}_{E_s}^{\ell(s)}.$$

If $f_E(\phi_r) > 1$, c_r is defined as are $c_1, \ldots, c_{r-1}$. If $f_E(\phi_r) = 1$, c_r is taken to be a root of unity in O_E such that $c_r + \mathfrak{p}_E$ generates $O_E/\mathfrak{p}_E$ over $O_{E_{r-1}}/\mathfrak{p}_{E_{r-1}}$. c_s is not defined the same way as the element c_s of [**MS**], though it does satisfy the definition in [**MS**]. X_θ is given by

$$X_\theta = \varpi_F^{-1}(c_1 + \cdots + c_r).$$

Lemma 2.2 ([**Mu1**], Lemma 3.4). $E = F[X_\theta]$.

Thus X_θ is a regular elliptic element of $\mathfrak{g}$.

Let $M_s = End_{E_s}E^+$. For $i \geq 0$, set

$$\mathcal{A}_s^i = \left\{ X \in M_s \mid X\mathfrak{p}_E^j \subset \mathfrak{p}_E^{j+i}\ \forall j \right\}.$$

This definition is extended to all integers via $\mathcal{A}_s^{e_s+i} = \varpi_{E_s}\mathcal{A}_s^i$, where e_s is the ramification degree of E_s over F. $\mathfrak{p}_E^0$ is understood to mean O_E.

Let $j_s = f_E(\phi_s \circ N_{E/E_s})$. If $j_s > 1$, set $i_s = j_s/2]$ and $m_s = [(j_s+1)/2]$. If $j_r = 1$, set $i_r = m_r = 1$. Define

$$\widetilde{K}_\theta = \begin{cases} (1+\mathcal{A}_{r-1}^{m_r})(1+\mathcal{A}_{r-2}^{m_{r-1}})\cdots(1+\mathcal{A}_0^{m_1}), & \text{if } j_r > 1; \\ (\mathcal{A}_{r-1}^0)^\times(1+\mathcal{A}_{r-2}^{m_{r-1}})\cdots(1+\mathcal{A}_0^{m_1}) & \text{if } j_r = 1. \end{cases}$$

$\overline{K}_\theta$ is defined similarly, except with i_s replacing m_s. In [**Mu1**], the notation K'_θ was used instead of $\overline{K}_\theta$. However, in this paper, A' denotes $A \cap G'$, where A is a subset of G. The inducing subgroup for π_θ is $K_\theta = E^\times\overline{K}_\theta$. $K_E = (\mathcal{A}_0^0)^\times$ is an open compact subgroup of G. If C is an open subset of K'_E and $\mathfrak{g}'$ is the Lie algebra of G', set

$$\mathcal{I}(X,Y;C) = \int_C \psi_0(\mathrm{tr}(X\,\mathrm{Ad}\,k^{-1}(Y)))\,dk, \qquad X,Y \in \mathfrak{g}.$$

Here, tr denotes trace. As in [**Mu1**], given X in $\mathfrak{g}$ define

$$H_X = \left\{ k \in K_E \mid 1 + \mathrm{Ad}\,k^{-1}(X) \in \widetilde{K}_\theta \right\}$$
$$H_X^0 = \left\{ k \in K_E \mid 1 + \mathrm{Ad}\,k^{-1}(X) \in \overline{K}_\theta \right\}.$$

It is easily seen from our description of $\mathcal{A}_0^1$ in §3 of [**Mu1**] that $\det(1 + \mathcal{A}_0^1) \subset 1 + \mathfrak{p}_F$. Because p does not divide n ($p > n$), given $x \in 1 + \mathfrak{p}_F$, there exists a unique $y \in 1+\mathfrak{p}_F$ such that $y^n = x$ ([**Ha**], p. 217). Given $h \in 1+\mathcal{A}_0^1$, let $d(h)$ be the scalar matrix y times the identity matrix, where $y \in 1 + \mathfrak{p}_F$ is such that $y^n = \det h^{-1}$. Thus $\det d(h) \det h = 1$. Viewing $\mathcal{A}_s^m$, $m \geq 1$, as a subset of $\mathcal{A}_0^1$, define

$$B_s^m = \{ d(h)h \mid h \in 1 + \mathcal{A}_s^m \}.$$

Let $\mathcal{N}$ be the nilpotent subset of $\mathfrak{g}$. Since a nilpotent matrix has trace zero, $\mathcal{N}$ is also the nilpotent subset of $\mathfrak{g}'$.

Lemma 2.3. *Assume $X \in \mathcal{N}$.*

(1) *If $j_r > 1$, then $\mathcal{I}(X_\theta, X; K'_E) = \mathcal{I}(X_\theta, X; H'_X)$.*

(2) *If* $j_r = 1$ *and* $X \in \mathcal{A}_0^1$, *then* $\mathcal{I}(X_\theta, X; K'_E) = \mathcal{I}(X_\theta, X; H'_X)$.
(3) *If* $j_r = 1$ *and* $X \notin \mathcal{A}_0^1$, *then* $\mathcal{I}(X_\theta, X; K'_E) = \mathcal{I}(X_\theta, X; H^{0\prime}_X)$.

Proof. The proofs of Lemmas 3.7–9 of [**Mu1**] can be modified slightly to obtain a proof of this lemma.

First, assume that $r = 1$. In this case $\widetilde{K_\theta} = 1 + \mathcal{A}_0^{m_1}$. $X \in \mathcal{A}_0^i - \mathcal{A}_0^{i+1}$ for some integer i. If $i \geq m_1$, then $H'_X = K'_E$. If $j_1 = 1$ and $i = 0$, then $H^{0\prime}_X = K'_E$. Therefore, we assume that $i < m_1$ if $j_1 > 1$, and $i < 0$ if $j_1 = 1$. Since $H'_X = \emptyset$ if $j_1 > 1$, and $H^{0\prime}_X = \emptyset$ if $j_1 = 1$, we must show that $\mathcal{I}(X_\theta, X; K'_E) = 0$. Let $\ell = [(j_1 - i + 1)/2]$. At this point, in [**Mu1**], an extra integration over $1 + \mathcal{A}_0^\ell$ was introduced. Since $1 + \mathcal{A}_0^\ell$ is not a subset of K'_E, we introduce an integration over the subgroup B_0^ℓ of K'_E. $\mathcal{I}(X_\theta, X; K'_E)$ is a nonzero multiple of

$$\int_{K'_E} \int_{B_0^\ell} \psi_0 \left(\operatorname{tr} \left(X_\theta \operatorname{Ad}(kb)^{-1}(X) \right) \right) db\, dk.$$

It suffices to show that the inner integral vanishes for all $k \in K'_E$. Given $b \in B_0^\ell$, write $b = d(h)h$, $h \in 1 + \mathcal{A}_0^\ell$. Since $d(h)$ is a scalar matrix, $\operatorname{Ad}(kb)^{-1}(X) = \operatorname{Ad}(kh)^{-1}(X)$ for all $k \in K'_E$. Therefore, the inner integral equals

$$\int_{1+\mathcal{A}_0^\ell} \psi_0 \left(\operatorname{tr} \left(X_\theta \operatorname{Ad}(kh)^{-1}(X) \right) \right) dh,$$

which, as shown in the proofs of Lemmas 3.7–9 of [**Mu1**], equals zero.

Assume $r \geq 2$. When $i \geq 1$, this case is argued as in the proof of Lemma 3.7 of [**Mu1**], except that the integrals over K_E and $1 + \mathcal{A}_s^m$, for appropriately chosen m, are replaced by integrals over K'_E and B_s^m. Since $b \in B_s^m$ has the form $d(h)h$ for some $h \in 1 + \mathcal{A}_s^m$ and $d(h)$ is scalar, the integral over B_s^m equals the integral over $1 + \mathcal{A}_s^m$, and thus has the vanishing properties required to prove the lemma. The proof for $i \leq 0$ is obtained the same way as Lemmas 3.8 and 3.9 of [**Mu1**]. □

The next lemma will be used in the case $j_r = 1$.

Lemma 2.4. *Let* $\bar{\psi}$ *be a nontrivial character of a finite field* $\mathbf{F}$. *Let* $\bar{G} = GL_m(\mathbf{F})$ *and* $\bar{G}' = SL_m(\mathbf{F})$, $m \geq 2$. *Suppose that* $|\cdot|$ *denotes cardinality, and* tr *is the trace map on the Lie algebra of* $\bar{G}$. *Let* S, *resp.* X, *be a regular elliptic, resp. arbitrary, element of the Lie algebra of* $\bar{G}$. *Then*

$$|\bar{G}|^{-1} \sum_{x \in \bar{G}} \bar{\psi}(\operatorname{tr}(S \operatorname{Ad} x^{-1}(X))) = |\bar{G}'|^{-1} \sum_{x \in \bar{G}'} \bar{\psi} \left(\operatorname{tr} \left(S \operatorname{Ad} x^{-1}(X) \right) \right).$$

Proof. It suffices to show that

$$\sum_{x \in \bar{G}'} \bar{\psi} \left(\operatorname{tr} \left(S \ \operatorname{Ad} (xy)^{-1}(X) \right) \right)$$

is independent of the choice of $y \in \bar{G}$. $\mathbf{E} = \mathbf{F}[S]$ is a degree m extension of $\mathbf{F}$. Since the norm map $N_{\mathbf{E}/F}$ from $\mathbf{E}^\times$ to $\mathbf{F}^\times$ is onto, there exists $\alpha \in \mathbf{E}^\times$ such that $N_{\mathbf{E}/F}(\alpha) = \det y$. Identifying α with an element of $\bar{G}$ which commutes with S,

$$\begin{aligned}\bar{\psi}\left(\operatorname{tr}\left(S\ \operatorname{Ad}(xy)^{-1}(X)\right)\right) &= \bar{\psi}\left(\operatorname{tr}\left(\operatorname{Ad}\alpha(S)\operatorname{Ad}\left(\alpha y^{-1}x^{-1}\right)(X)\right)\right) \\ &= \bar{\psi}\left(\operatorname{tr}\left(S\ \operatorname{Ad}(\alpha y^{-1}x^{-1})(X)\right)\right).\end{aligned}$$

Because $\det(\alpha y^{-1}) = N_{\mathbf{E}/F}(\alpha)\det y^{-1} = 1$, αy^{-1} can be absorbed into the sum over $x \in \bar{G}'$. $\square$

Suppose $\pi_\theta = Ind_{K_\theta}^G \kappa_\theta$. Let ρ_θ be the character of κ_θ. Define $f_\theta : G \to C$ by

$$f_\theta(x) = \begin{cases}\rho_\theta(x) & \text{if } x \in K_\theta, \\ 0 & \text{otherwise.}\end{cases}$$

The representation

$$\bar{\pi} = Ind_{K'_\theta}^{G'}(\kappa_\theta | K'_\theta)$$

is a supercuspidal representation of G' and is a component of the restricton of π_θ to G' ([**MS**]). The restriction of f_θ to G' is a matrix coefficient of $\bar{\pi}$. Define

$$S_\theta = X_\theta - \frac{(\operatorname{tr}_{E/F} X_\theta)}{n} I_n, \tag{2.5}$$

where I_n is the $n \times n$ identity matrix.

Proposition 2.6. *Let $X \in \mathcal{N}$. Then*

$$f_\theta(1)^{-1}\int_{K'_E} f_\theta(k^{-1}\exp Xk)\,dk = \mathcal{I}(S_\theta, X; K'_E).$$

Proof. Because $\operatorname{tr} X = 0$, and X_θ and S_θ differ by a scalar matrix,

$$\mathcal{I}(X_\theta, X; K'_E) = \mathcal{I}(S_\theta, X; K'_E).$$

Thus in the statement of the proposition S_θ can be replaced by X_θ.

The proof of this proposition is a slight modification of the proof of Proposition 3.10 of [**Mu1**].

The representation κ_θ is a tensor product $(\chi \circ \det) \otimes \kappa_1 \otimes \cdots \otimes \kappa_r$. ρ_s, $1 \le s \le r$, denotes the character of κ_s.

As observed in [**Mu1**], if $X \in \mathcal{N}$, then

$$\exp X \in K_\theta \iff \exp X \in \overline{K}_\theta$$

Thus

$$f_\theta(1)^{-1}\int_{K'_E} f_\theta(k^{-1}\exp Xk)\,dk = \rho_\theta(1)^{-1}\int_{H_X^{0\prime}} \rho_\theta(k^{-1}\exp Xk)\,dk.$$

Case 1: As shown in [**Mu1**], if $X \in \mathcal{N}$, then

$$\frac{\rho_\theta(\exp X)}{\rho_\theta(1)} = \begin{cases} \psi_0(\mathrm{tr}(X_\theta X)), & \text{if } \exp X \in \widetilde{K}_\theta, \\ 0 & \text{if } \exp X \in \overline{K}_\theta - \widetilde{K}_\theta. \end{cases}$$

Therefore

$$\begin{aligned} f_\theta(1)^{-1}\int_{K'_E} f_\theta(k^{-1}\exp Xk)\,dk &= \int_{H'_X} \psi_0(\mathrm{tr}(X_\theta\ \mathrm{Ad}\,k^{-1}(X)))\,dk \\ &= \mathcal{I}(X_\theta, X; H'_X) = \mathcal{I}(X_\theta, X; K'_E). \end{aligned}$$

The last equality is Lemma 2.3(1).

Case 2: $j_r = 1$. The representations κ_s, $1 \le s \le r-1$ and κ_r are considered separately.

A certain cuspidal representation of the finite general linear group

$$\left(calA^0_{r-1}\right)^* / 1 + \mathcal{A}^1_{r-1}$$

is used to produce the representation K_r. Lemma 2.4 shows that the Green functions attached to elliptic Cartan subgroups are the same the finite general linear and special linear groups. As shown in Proposition 3.10 of [**Mu1**], if $x \in \mathcal{N}$ is such that $\exp X \in \overline{K}_\theta$, then

$$\frac{\rho_r(\exp X)}{\rho_r(1)} = \int_{\left(calA^0_{r-1}\right)^*} \Psi_F\left(\mathrm{tr}(c_r\,\mathrm{Ad}\,h^{-1}(X))\right)dh.$$

By Lemma 2.4, we may replace $\left(calA^0_{r-1}\right)^*$ with $\left(calA^0_{r-1}\right)^* \cap G'$ in the above integral.

For $1 \le i \le r-1$, define

$$K_s = (1+\mathcal{A}^{i_r}_{r-1})\cdots(1+\mathcal{A}^{i_{s+1}}_s) \quad \text{and} \quad L_s = (1+\mathcal{A}^{i_s}_{s-1})\cdots(1+\mathcal{A}^{i_1}_0).$$

Set $L_0 = \{1\}$. As was shown in [**Mu1**], if $X \in \mathcal{N}$ is such that $\exp X \in \overline{K}_\theta$,

$$\frac{\rho_s(\exp X)}{\rho_s(1)} = \begin{cases} \int_{1+\mathcal{A}^{i_s}_{s-1}} \psi_F(\mathrm{tr}(c_s\,\mathrm{Ad}\,h^{-1}(X)))\,dh \\ \qquad \text{if } \exp X \in K_s(1+\mathcal{A}^{m_s}_{s-1})L_{s-1} \cup (\overline{K}_\theta - K_sL_s), \\ 0 \qquad \text{otherwise.} \end{cases}$$

Arguing as in the proof of Lemma 2.3,

$$\int_{1+\mathcal{A}_{s-1}^{i_s}} \psi_F\left(\operatorname{tr}\left(c_s \operatorname{Ad} h^{-1}(X)\right)\right) dh = \int_{B_{s-1}^{i_s}} \psi_F\left(\operatorname{tr}\left(c_s \operatorname{Ad} b^{-1}(X)\right)\right) db.$$

Let $X \in \mathcal{N}$. If $X \in \mathcal{A}_0^1$ and $\exp X \in \overline{K}_\theta - \widetilde{K}_\theta$, then $\exp X \in K_s L_s - K_s(1 + \mathcal{A}_{s-1}^{m_s})L_{s-1}$ for some s, so $\rho_s(\exp X) = 0$. Thus $\rho_\theta(\exp X) = 0$. All remaining $X \in \mathcal{N}$ such that $\exp X \in \widetilde{K_\theta}$ satisfy one of the following:
(i) $X \in \mathcal{A}_0^1$ and $\exp X \in \widetilde{K}_\theta$
(ii) $X \in \mathcal{A}_0^0 - \mathcal{A}_0^1$ and $\exp X \in \overline{K}_\theta$.
For these X,

$$\text{(2.7)} \quad \frac{\rho_\theta(\exp X)}{\rho_\theta(1)} = \left(\int_{((\mathcal{A}_{r-1}^0)^\times \cap G')} \psi_F\left(\operatorname{tr}\left(c_r \operatorname{Ad} h^{-1}(X)\right)\right) dh\right) \cdot \prod_{s=1}^{r-1} \int_{B_{s-1}^{i_s}} \psi_F\left(\operatorname{tr}\left(c_s \operatorname{Ad} b^{-1}(X)\right)\right) db = \int_{L_\theta} \psi_0\left(\operatorname{tr}\left(X_\theta \operatorname{Ad} h^{-1}(X)\right)\right) dh.$$

To obtain the second equality argue as in [**Mu1**] (following equation (3.14)). Here

$$L_\theta = ((\mathcal{A}_{r-1}^0)^\times \cap G') \prod_{s=1}^{r-1} B_{s-1}^{i_s}$$

is a subgroup of K'_E. It follows from (2.7) that for $X \in \mathcal{N} \cap \mathcal{A}_0^1$,

$$f_\theta(1)^{-1} \int_{K'_E} f_\theta(k^{-1} \exp X k)\, dk = \rho_\theta(1)^{-1} \int_{H'_X} \rho_\theta(k^{-1} \exp X k)\, dk = \int_{H'_X} \int_{L_\theta} \psi_0\left(\operatorname{tr}\left(X_\theta \operatorname{Ad}(kh)^{-1}(X)\right)\right) dh\, dk = \mathcal{I}(X_\theta, X; H'_X).$$

The last equality holds because H'_X is invariant under translation by L_θ. A similar equality holds for $X \in \mathcal{N} \cap (\mathcal{A}_0^0 - \mathcal{A}_0^1)$, except with H_X replaced by $H_X^{0\prime}$. If $X \in \mathcal{N}$ and $X \notin \mathcal{A}_0^0$, then $f_\theta(k^{-1} \exp X k) = 0$ for all $k \in K'_E$, and $H_X^{0\prime} = \emptyset$. Apply Lemma 2.3(2) and (3) to complete the proof. □

3. The character of $\overline{\pi}$ as a Fourier transform.

Let θ be an admissible character of the multiplicative group of a degree n extension E of F. Define

$$G_E = E^\times G' = \{\, x \in G \mid \det x \in N_{E/F}(E^\times) \,\}.$$

As in §2, $\overline{\pi}$ denotes the supercuspidal representation of G' defined by

$$\overline{\pi} = \operatorname{Ind}_{K'_\theta}^{G'}(\kappa_\theta \mid K'_\theta).$$

Then ([**MS**])

$$\pi_\theta \mid G' = \bigoplus_{g \in G/G_E} \overline{\pi}^g, \tag{3.1}$$

where $\overline{\pi}^g(x) = \overline{\pi}(g^{-1}xg)$, $x \in G'$, $g \in G$. Two of the main results, Theorem 3.2 and Corollary 3.5, are proved for the representations $\overline{\pi}^g$, $g \in G/G_E$. As a consequence (Corollary 3.6), (1.1) and (1.2) hold for the irreducible components of $\pi_\theta \mid G'$ whenever there are exactly $|F^\times/N_{E/F}(E^\times)|$ such components.

Given f in $C_c^\infty(\mathfrak{g}')$, the space of locally constant, compactly supported, complex-valued functions on $\mathfrak{g}'$, let $\widehat{f}$ be the function in $C_c^\infty(\mathfrak{g}')$ defined by

$$\widehat{f}(X) = \int_{\mathfrak{g}'} \psi_0(\operatorname{tr}(XY)) f(Y)\, dY.$$

The Haar measure dY on $\mathfrak{g}'$ is assumed to be self-dual with respect to $\widehat{\ }$. Given X in $\mathfrak{g}'$, $\mathcal{O}(X)$ denotes the $\operatorname{Ad} G'$-orbit of X. If $\mu_{\mathcal{O}(X)}$ is the distribution given by integration over the orbit $\mathcal{O}(X)$, the Fourier transform $\widehat{\mu}_{\mathcal{O}(X)}$ is given by $\widehat{\mu}_{\mathcal{O}(X)}(f) = \mu_{\mathcal{O}(X)}(\widehat{f})$, f in $C_c^\infty(\mathfrak{g}')$. Let $\mathfrak{g}'_{reg}$ be the regular subset of $\mathfrak{g}'$. Recall ([**HC2**]) that $\widehat{\mu}_{\mathcal{O}(X)}$ can be realized as a locally integrable function (also called $\widehat{\mu}_{\mathcal{O}(X)}$) on $\mathfrak{g}'$ which is locally constant on $\mathfrak{g}'_{reg}$. If a representative of an orbit $\mathcal{O}$ is not specified, the notation $\mu_{\mathcal{O}}$ and $\widehat{\mu}_{\mathcal{O}}$ will be used for the corresponding orbital integral and its Fourier transform.

Fix a Haar measure dx on G'. If X is a regular elliptic element in $\mathfrak{g}'$, the measure on $\mathcal{O}(X)$ is normalized to equal dx. Formal degrees of supercuspidal representations are computed relative to dx. Haar measure on any compact group is normalized so that the total volume of the group equals one.

Let $\mathfrak{g}'^*$ be an open $\operatorname{Ad} G$-invariant subset of $\mathfrak{g}'$ containing zero such that $\exp : \mathfrak{g}'^* \to G'$ is defined and $\exp(\operatorname{Ad} x(X)) = x \exp X x^{-1}$ for x in G and X in $\mathfrak{g}'^*$. Fix an integer $\ell \geq 1$ such that $\mathfrak{g}(\mathfrak{p}^\ell) \subset \mathcal{A}_0^{j_1}$. Choose an integer i large enough that, if $V_\pi = \mathfrak{g}(\mathfrak{p}^i)'$,

(i) $V_\pi \subset \mathfrak{g}'^*$,

(ii) $i \geq \max\{\ell,\ n(\ell + e(F/Q_p))/(p-n+1)\}$.

Theorem 3.2. *Let S_θ be as in* (2.5). *Then, if $g \in G$ and $X \in \mathrm{Ad}\, g(V_\pi) \cap \mathfrak{g}'_{reg}$,*

$$\Theta_{\overline{\pi}^g}(\exp X) = d(\overline{\pi}^g)\, \widehat{\mu}_{\mathcal{O}(\mathrm{Ad}\, g(S_\theta))}(X).$$

Proof. By definition of $\overline{\pi}^g$,

$$\Theta_{\overline{\pi}^g}(x) = \Theta_{\overline{\pi}}(g^{-1}xg) \qquad x \in G',$$

and $d(\overline{\pi}^g) = d(\overline{\pi})$. Also

$$\widehat{\mu}_{\mathcal{O}(\mathrm{Ad}\, g(Y))}(X) = \widehat{\mu}_{\mathcal{O}(Y)}(\mathrm{Ad}\, g^{-1}(X)), \qquad X \in \mathfrak{g}'_{reg},\ Y \in \mathfrak{g}'.$$

Therefore, it is sufficient to prove the theorem for $g = 1$.

Let K_0 be any open compact subgroup of G'. As shown in Lemma 4.1(1) of [**Mu1**], Harish-Chandra's integral formula for $\widehat{\mu}_{\mathcal{O}(S_\theta)}(X)$ ([**HC2**], Lemma 19) can be rewritten as:

(3.3)

$$\begin{aligned}\widehat{\mu}_{\mathcal{O}(S_\theta)}(X) &= \int_{G'} \int_{K_0} \left[\int_{K'_E} \psi_0 \left(\mathrm{tr}\left(S_\theta\, \mathrm{Ad}(kxh)^{-1}(X)\right)\right) dh \right] dk\, dx \\ &= \int_{G'} \int_{K_0} \mathcal{I}\left(S_\theta, \mathrm{Ad}(kx)^{-1}(X); K'_E\right) dk\, dx, \qquad X \in \mathfrak{g}'_{reg}.\end{aligned}$$

Since $f_\theta\, |K'_\theta$ is a matrix coefficient of $\overline{\pi}$, Harish-Chandra's integral formula for $\Theta_{\overline{\pi}}$, ([**HC1**, p. 60]), can be rewritten as ([**Mu1**], Lemma 4.1(2)):

$$\text{(3.4)} \quad \Theta_{\overline{\pi}}(\exp X) = \frac{d(\overline{\pi})}{f_\theta(1)} \int_{G'} \int_{K_0} \left[\int_{K'_E} f_\theta\left((kxh)^{-1}(\exp X)kxh\right) dh \right] dk\, dx, \qquad X \in \mathfrak{g}'^{*} \cap \mathfrak{g}'_{reg}.$$

Fix $x \in G'$ and $k \in K'_E$. Then there exist $Y \in \mathcal{N}$ and $Z \in V_\pi$ such that $\mathrm{Ad}(kx)^{-1}(X) = Y + Z$. This follows from (see Lemma 4.2 of [**Mu1**])

$$\mathrm{Ad}\, x^{-1}(\mathfrak{g}(\mathfrak{p}^t)') \subset \mathcal{N} + \mathfrak{g}(\mathfrak{p}^t)', \qquad x \in G',\ t \geq 1.$$

As shown in the proof of Theorem 4.3 of [**Mu1**],

$$f_\theta(h^{-1}\exp(Y+Z)h) = f_\theta(h^{-1}(\exp Y)h), \qquad h \in K'_E.$$

It follows from $\mathrm{tr}\, \mathcal{A}_0^1 \subset \mathfrak{p}_F$ and $\varpi_F \mathcal{A}_0^t = \mathcal{A}_0^{t+e}$, $e = e(E/F)$, that $\mathrm{tr}\, \mathcal{A}_0^m \subset \mathfrak{p}_F^{[(m-1)/e]+1}$. As a consequence of $\varpi_F X_\theta \in \mathcal{A}_0^{-j_1+1}$ and $Z \in \mathcal{A}_0^{j_1}$, we have

$X_\theta Z \in \varpi_F^{-1}\mathcal{A}_0^1$, $\operatorname{tr}_{E/F} X_\theta = \operatorname{tr} X_\theta \in \mathfrak{p}_F^{[-j_1/e]+1}$ and $\operatorname{tr} Z \in \mathfrak{p}_F^{[(j_1-1)/e]+1}$. Therefore,

$$\psi_0(\operatorname{tr}(S_\theta Z)) = \psi_0(\operatorname{tr}(X_\theta Z))\psi_0(\operatorname{tr}_{E/F} X_\theta \operatorname{tr} Z)^{-1} = 1.$$

Thus

$$\psi_0(\operatorname{tr}(S_\theta(Y+Z))) = \psi_0(\operatorname{tr}(S_\theta Y)).$$

We can now apply Proposition 2.6 to the inner integrals in (3.3) and (3.4), completing the proof. □

Let $(\mathcal{N})'$ be the set of nilpotent $\operatorname{Ad} G'$-orbits in $\mathfrak{g}'$. Suppose π is an admissible representation of G' of finite length. If $\mathcal{O} \in (\mathcal{N})'$, $c_{\mathcal{O}}(\pi)$ denotes the coefficient of $\widehat{\mu}_{\mathcal{O}}$ in Harish-Chandra's local character expansion of π at the identity ([**HC2**]):

$$\Theta_\pi(\exp X) = \sum_{\mathcal{O} \in (\mathcal{N})'} c_{\mathcal{O}}(\pi)\widehat{\mu}_{\mathcal{O}}(X),$$

for $X \in \mathfrak{g}'_{reg}$ sufficiently close to zero. For $\mathcal{O} \in (\mathcal{N})'$, let $\Gamma_{\mathcal{O}} : \mathfrak{g}'_{reg} \to R$ be the Shalika germ corresponding to $\mathcal{O}$ ([**HC2**]).

Corollary 3.5. *Let $g \in G$. Then*

$$c_{\mathcal{O}}(\overline{\pi}^g) = d(\overline{\pi}^g)\,\Gamma_{\mathcal{O}}(\operatorname{Ad} g(S_\theta)), \qquad \mathcal{O} \in (\mathcal{N})'.$$

Proof. As follows from Lemma 21 of [**HC2**], there exists an open neighbourhood V of zero in $\mathfrak{g}'$ such that:

$$\widehat{\mu}_{\mathcal{O}(\operatorname{Ad} g(S_\theta))}(X) = \sum_{\mathcal{O} \in (\mathcal{N})'} \Gamma_{\mathcal{O}}(\operatorname{Ad} g(S_\theta))\widehat{\mu}_{\mathcal{O}}(X), \qquad X \in V \cap \mathfrak{g}'_{reg}.$$

The corollary is now a consequence of Theorem 3.2 and the linear independence of the functions $\widehat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in (\mathcal{N})'$ ([**HC2**]). □

An irreducible supercuspidal representation of G' is a component of $\pi_\theta \mid G'$, for some admissible character θ of $E^\times$, where E is a degree n extension of F ([**MS**]). Each π_θ decomposes with multiplicity one upon restriction to G' ([**T**]). An L-packet of supercuspidal representations of G' consists of the irreducible components of the restriction of an irreducible supercuspidal representation of G to G' ([**GK**]).

Suppose θ is such that $j_r = 1$. Since ϕ_r is a character of $E^\times$ which is trivial on $1 + \mathfrak{p}_E$, ϕ_r may be viewed as a character $\bar{\phi}_r$ of $\mathbf{E}^\times$, where $\mathbf{E}$ is the residue class field of E. Let N_1 be the kernel of the norm map from $\mathbf{E}^\times$ to $\mathbf{E}_{r-1}^\times$. As in [**MS**], we define $\bar{\phi}_r \mid N_1$ to be *regular* if the number of distinct conjugates of $\bar{\phi}_r \mid N_1$ under the action of the Galois group of $\mathbf{E}$ over $\mathbf{E}_{r-1}$ is equal to $[\mathbf{E} : \mathbf{E}_{r-1}]$.

Corollary 3.6. *Let π be an irreducible supercuspidal representation of G'. Choose θ such that π is a component of $\pi_\theta \mid G'$. Suppose one of the following conditions holds:*

(i) $j_r > 1$,

(ii) $j_r = 1$ *and* $\bar{\phi}_r \mid N_1$ *is regular.*

Then there exists a regular elliptic $X_\pi \in \mathfrak{g}'$ *such that*

(1) $\Theta_\pi \circ \exp = d(\pi)\widehat{\mu}_{\mathcal{O}(X_\pi)}$ *on some open neighbourhood of zero intersected with* $\mathfrak{g}'^* \cap \mathfrak{g}'_{reg}$,

(2) $c_{\mathcal{O}}(\pi) = d(\pi)\,\Gamma_{\mathcal{O}}(X_\pi)$, $\mathcal{O} \in (\mathcal{N})'$,

(3) *The L-packet of* π *is* $\{\, \pi^g \mid g \in G/G_E \,\}$. (1) *and* (2) *hold for* π^g *with* $X_{\pi^g} = \operatorname{Ad} g(X_\pi)$.

Proof. As proved in [**MS**], conditions (i) and (ii) are necessary and sufficient for each of the representations $\overline{\pi}^g$, $g \in G/G_E$, to be irreducible. In that case (see (3.1)), the representations $\overline{\pi}^g$ are the members of the L-packet of π, and (1), (2), and (3) are restatements of Theorem 3.2 and Corollary 3.5. □

Remark 3.7. Moy and Sally showed that if n and $q-1$ are relatively prime, then, whenever $j_r = 1$, $\bar{\phi}_r \mid N_1$ is regular ([**MS**],Cor. 3.15). Therefore, (1) and (2) hold for all irreducible supercuspidal representations of G' when n and $q-1$ are relatively prime.

Two elements X_1 and X_2 of $\mathfrak{g}'$ are *stably conjugate* if there exists g in G such that $X_2 = \operatorname{Ad} g(X_1)$. The *stable orbit* $\mathcal{O}_{st}(X)$ of X in $\mathfrak{g}'$ consists of the set of stable conjugates of X. Given θ, since $E = F[S_\theta]$ and $S_\theta \in \mathfrak{g}'_{reg}$,

$$\mathcal{O}_{st}(S_\theta) = \cup_{g \in G/G_E} \mathcal{O}(\operatorname{Ad} g(S_\theta)).$$

To the L-packet of supercuspidal representations of G' consisting of the components of $\pi_\theta \mid G'$, we associate the stable orbit $\mathcal{O}_{st}(S_\theta)$. Of course, the choice of θ is not unique. However, as discussed in §4 of [**MS**], any two choices for theta must satisfy certain conjugacy conditions. Corollary 3.6 deals with those L-packets which contain $|F^\times/N_{E/F}(E^\times)| = |G/G_E|$ representations. In this case, the representations in the L-packet correspond to the $\operatorname{Ad} G'$-orbits in the associated stable orbit via Corollary 3.6(3). If an L-packet contains more than $|F^\times/N_{E/F}(E^\times)|$ representations, we do not have such a correspondence. The elements if the L-packet are the irreducible components of the representations $\overline{\pi}^g$, g in G/G_E. This case is discussed in more detail in the next section.

4. The case $\overline{\pi}$ reducible.

Let π be an irreducible supercuspidal representation of G'. Choose an admissible character θ such that π is a component of $\pi_\theta \,|\, G'$. Let E be the associated degree n extension of F. Define

$$G(\pi) = \{\, g \in G \,|\, \pi^g \sim \pi \,\}.$$

Here, $\sim$ denotes equivalence of representations. Set $\pi'_\theta = \pi_\theta \,|\, G'$. By [**T**]

$$\pi'_\theta = \bigoplus_{g \in G/G(\pi)} \pi^g.$$

In this section, we assume that the L-packet of π contains more than $|F^\times / N_{E/F}(E^\times)|$ representations. That is,

$$|G/G(\pi)| > |F^\times / N_{E/F}(E^\times)|. \tag{4.1}$$

This is equivalent to the representation $\overline{\pi}$ being reducible ([**MS**]). The purpose of this section is to prove that $\Theta_\pi \circ \exp$ is not a multiple of the Fourier transform of a semisimple orbit on any neighbourhood of zero (Theorem 4.5). In order for (4.1) to hold, it is necessary that n and $q-1$ have a nontrivial common divisor (see Remark 3.7).

Let $X \in \mathfrak{g}'$. We assume that the measures on the orbits in the stable orbit $\mathcal{O}_{st}(X)$ of X are normalized so that

$$\mu_{\mathcal{O}(X)}(f^g) = \mu_{\operatorname{Ad} g^{-1} \cdot \mathcal{O}(X)}(f), \qquad f \in C_c^\infty(\mathfrak{g}'),\ g \in G.$$

Here $f^g(X) = f(\operatorname{Ad} g^{-1}(X))$, $X \in \mathfrak{g}'$.

Lemma 4.2. *$c_{\mathcal{O}}(\pi^g) = c_{\operatorname{Ad} g^{-1} \cdot \mathcal{O}}(\pi)$, $\mathcal{O} \in (\mathcal{N})'$, $g \in G$.*

Proof. The above compatibility conditions on the measures on $\mathcal{O}$ and $\operatorname{Ad} g \cdot \mathcal{O}$, $\mathcal{O} \in (\mathcal{N})'$, imply that

$$\widehat{\mu}_{\operatorname{Ad} g \cdot \mathcal{O}}(X) = \widehat{\mu}_{\mathcal{O}}(\operatorname{Ad} g^{-1}(X)), \qquad X \in \mathfrak{g}'_{reg}.$$

The lemma follows from a comparision of the local character expansions of π and π^g and the linear independence of the functions $\widehat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in (\mathcal{N})'$, on neighbourhoods of zero intersected with $\mathfrak{g}'_{reg}$. □

Given $\mathcal{O} \in (\mathcal{N})'$, let $\mathcal{O}_{st}$ be the stable orbit containing $\mathcal{O}$. $\mathcal{O}_{st}$ is an $\operatorname{Ad} G$-orbit in $\mathfrak{g}'$. Define a measure $\mu_{\mathcal{O}_{st}}$ on $\mathcal{O}_{st}$ by:

$$\mu_{\mathcal{O}_{st}} = \sum_{\widetilde{\mathcal{O}} \subset \mathcal{O}_{st}} \mu_{\widetilde{\mathcal{O}}}.$$

Lemma 4.2 holds for any smooth admissible representation of G' of finite length. Therefore, since $(\pi'_\theta)^g \sim \pi'_\theta$ for all g in G, the coefficients $c_{\mathcal{O}}(\pi'_\theta)$ coincide for all orbits $\mathcal{O}$ contained in a stable orbit $\mathcal{O}_{st}$. Let $c_{\mathcal{O}_{st}}(\pi'_\theta)$ denote their common value. Then

$$\theta_{\pi_\theta}(\exp X) = \sum_{\mathcal{O}_{st}\subset\mathcal{N}} c_{\mathcal{O}_{st}}(\pi'_\theta)\,\widehat{\mu}_{\mathcal{O}_{st}}(X),$$

for X in $\mathfrak{g}'_{reg}$ sufficiently close to zero.

Lemma 4.3. *Choose $g \in G$ such that π is a component of $\overline{\pi}^g$. Let $\mathcal{O} \in (\mathcal{N})'$.*
(1) *If $\operatorname{Ad} g \cdot \mathcal{O} = \mathcal{O}$ for all $g \in G_E$, then*

$$c_{\mathcal{O}}(\pi) = d(\pi)\,\Gamma_{\mathcal{O}}(\operatorname{Ad} g(S_\theta)).$$

(2) *If $\operatorname{Ad} g \cdot \mathcal{O} = \mathcal{O}$ for all $g \in G$, that is, $\mathcal{O} = \mathcal{O}_{st}$, then*

$$c_{\mathcal{O}}(\pi) = d(\pi)d(\pi'_\theta)^{-1}c_{\mathcal{O}_{st}}(\pi'_\theta).$$

Proof. (1) Since π'_θ decomposes with multiplicity one, $\overline{\pi}^g$ also decomposes with multiplicity one. Thus ([**T**])

$$\overline{\pi}^g = \bigoplus_{x\in G_E/G(\pi)} \pi^x.$$

Applying Corollary 3.5 and Lemma 4.2,

$$\begin{aligned} c_{\mathcal{O}}(\overline{\pi}^g) = d(\overline{\pi}^g)\,\Gamma_{\mathcal{O}}(\operatorname{Ad} g(S_\theta)) &= \sum_{x\in G_E/G(\pi)} c_{\mathcal{O}}(\pi^x) \\ &= \sum_{x\in G_E/G(\pi)} c_{\operatorname{Ad} x^{-1}\cdot\mathcal{O}}(\pi) = |G_E/G(\pi)|\,c_{\mathcal{O}}(\pi) \\ &= d(\pi)^{-1}d(\overline{\pi}^g)\,c_{\mathcal{O}}(\pi), \end{aligned}$$

to obtain (1).

(2) Assume $\mathcal{O} = \mathcal{O}_{st}$. By linear independence of the Fourier transforms of nilpotent orbits, and Lemma 4.2,

$$\begin{aligned} c_{\mathcal{O}_{st}}(\pi'_\theta) &= \sum_{g\in G/G(\pi)} c_{\mathcal{O}}(\pi^g) = \sum_{g\in G/G(\pi)} c_{\operatorname{Ad} g^{-1}\cdot\mathcal{O}}(\pi) \\ &= |G/G(\pi)|\,c_{\mathcal{O}}(\pi) = d(\pi'_\theta)d(\pi)^{-1}c_{\mathcal{O}}(\pi). \end{aligned}$$

□

Remark. As (4.1) was not used in the proof, Lemma 4.3 holds for all irreducible supercuspidal representations of G'. In general there exist $\mathcal{O} \in (\mathcal{N})'$ which are stable under $\operatorname{Ad} G_E$, but not under $\operatorname{Ad} G$.

Let $(\mathcal{N}_{reg})'$ denote the set of regular (maximal dimension) nilpotent $\operatorname{Ad} G'$-orbits in $\mathfrak{g}'$. Define $w(\pi)$ to be the number of orbits $\mathcal{O}$ in $(\mathcal{N}_{reg})'$ such that $c_{\mathcal{O}}(\pi)$ is nonzero.

Lemma 4.4. *The L-packet of* π *contains* $w(\pi)^{-1}|F^\times/(F^\times)^n|$ *representations.*

Proof. Up to a positive constant depending on the normalization of the measure on $\mathcal{O} \in (\mathcal{N}_{reg})'$, $c_{\mathcal{O}}(\pi)$ equals the multiplicity with which some Whittaker model occurs in π ([**Ro**]). As shown in Remark 2.9 of [**T**], for each $\mathcal{O} \in (\mathcal{N}_{reg})'$, there exists exactly one $g \in G/G(\pi)$ such that $c_{\mathcal{O}}(\pi^g) \neq 0$. The determinant map factors to an isomorphism between $G/F^\times G'$ and $F^\times/(F^\times)^n$ and $(\mathcal{N}_{reg})'$ is the disjoint union of the orbits $\operatorname{Ad} g \cdot \mathcal{O}$, $g \in G/F^\times G'$ ([**Re**]). Thus

$$\sum_{g \in G/G(\pi)} w(\pi^g) = |F^\times/(F^\times)^n|.$$

By Lemma 4.2, $w(\pi^g) = w(\pi)$. Therefore

$$w(\pi)|G/G(\pi)| = |F^\times/(F^\times)^n|.$$

□

Theorem 4.5. *Assume that* (4.1) *holds.* $d(\pi)^{-1}\Theta_\pi \circ \exp | V \cap \mathfrak{g}'_{reg}$ *is not of the form* $\widehat{\mu}_{\mathcal{O}(X)} | V \cap \mathfrak{g}'_{reg}$, *for any* $X \in \mathfrak{g}'_{reg}$ *and open neighbourhood* V *of zero in* $\mathfrak{g}'$.

Proof. Suppose that $\Theta_\pi \circ \exp$ and $\lambda \widehat{\mu}_{\mathcal{O}(X)}$ coincide on $V \cap \mathfrak{g}'_{reg}$ for some constant λ and neighbourhood V, where $X \in \mathfrak{g}'_{reg}$. Then

$$c_{\mathcal{O}}(\pi) = \lambda \Gamma_{\mathcal{O}}(X), \qquad \mathcal{O} \in (\mathcal{N})'.$$

(To see this, argue as in the proof of Corollary 3.5.) Since $c_{\{0\}}(\pi) = d(\pi)\Gamma_{\{0\}}(X) \neq 0$, ([**HC2**]), $\lambda = d(\pi)$. Also, X is elliptic, because $\Gamma_{\{0\}}$ vanishes off the regular elliptic set ([**HC2**]). Let L be the degree n extension of F such that $L^\times$ is isomorphic to the stabilizer of X in G.

By Theorem 6.3(i) of [**Re**], if $\mathcal{O} \in (\mathcal{N}_{reg})'$ and $g \in G$,

$$\Gamma_{\operatorname{Ad} g \cdot \mathcal{O}}(X) = \begin{cases} \Gamma_{\mathcal{O}}(X), \text{ if } \det g \in N_{L/F}(L^\times), \\ 0 \text{ otherwise.} \end{cases}$$

Suppose $\det g \in N_{L/F}(L^\times)$. By Lemma 4.2,

$$c_{\operatorname{Ad} g \cdot \mathcal{O}}(\pi) = d(\pi^g)\, \Gamma_{\operatorname{Ad} g \cdot \mathcal{O}}(X) = d(\pi)\, \Gamma_{\mathcal{O}}(X) = c_{\mathcal{O}}(\pi), \qquad \mathcal{O} \in (\mathcal{N}_{reg})'.$$

Since there exists an $\mathcal{O} \in (\mathcal{N}_{reg})'$ such that $c_{\mathcal{O}}(\pi) \neq 0$ ([**T**]), $w(\pi) = |N_{L/F}(L^\times)/(F^\times)^n|$. Thus, given the relation between $w(\pi)$, $|G/G(\pi)|$ and $|F^\times/(F^\times)^n|$ described in the proof of Lemma 4.4,

$$|G/G(\pi)| = |F^\times/N_{L/F}(L^\times)|.$$

For g such that $\det g \in N_{L/F}(L^\times)$, the relation

$$\begin{aligned} c_{\mathcal{O}}(\pi^g) = c_{\operatorname{Ad} g^{-1}\cdot\mathcal{O}}(\pi) = d(\pi^g)\,\Gamma_{\operatorname{Ad} g^{-1}\cdot\mathcal{O}}(X) &= d(\pi)\,\Gamma_{\mathcal{O}}(X) \\ &= c_{\mathcal{O}}(\pi), \qquad \mathcal{O} \in (\mathcal{N}_{reg})', \end{aligned}$$

together with the fact that there is exactly one $g \in G/G(\pi)$ such that $c_{\mathcal{O}}(\pi^g)$ is nonzero ([**T**]), implies that $g \in G(\pi)$. We can now conclude that

$$G(\pi) = \{\, g \in G \mid \det g \in N_{L/F}(L^\times) \,\}.$$

Choose $\mathcal{O} \in (\mathcal{N}_{reg})'$ such that $c_{\mathcal{O}}(\pi) \neq 0$. Fix $x \in G$ such that π is a component of $\overline{\pi}^x$. Then

$$\overline{\pi}^x = \bigoplus_{g \in G_E/G(\pi)} \pi^g.$$

$$c_{\mathcal{O}}(\overline{\pi}^x) = \sum_{g \in G_E/G(\pi)} c_{\mathcal{O}}(\pi^g) = \sum_{g \in G_E/G(\pi)} d(\pi^g)\,\Gamma_{\operatorname{Ad} g^{-1}\cdot\mathcal{O}}(X) = d(\pi)\Gamma_{\mathcal{O}}(X),$$

the final equality resulting from $\Gamma_{\operatorname{Ad} g^{-1}\cdot\mathcal{O}}(X) = 0$ whenever $g \in G_E - G(\pi)$ (because $\det g \notin N_{L/F}(L^\times)$). By Corollary 3.5,

$$c_{\mathcal{O}}(\overline{\pi}^x) = d(\overline{\pi}^x)\Gamma_{\mathcal{O}}(\operatorname{Ad} x(S_\theta)).$$

Since $d(\overline{\pi}^x) = |G_E/G(\pi)|\, d(\pi)$,

$$\Gamma_{\mathcal{O}}(X) = |N_{E/F}(E^\times)/N_{L/F}(L^\times)|\,\Gamma_{\mathcal{O}}(\operatorname{Ad} x(S_\theta)).$$

Repka ([**Re**]) computed $\Gamma_{\mathcal{O}}$ on the regular set in G'. Lifting the Shalika germs from the group to the Lie algebra, and substituting the values of $\Gamma_{\mathcal{O}}(X)$ and $\Gamma_{\mathcal{O}}(\operatorname{Ad} x(S_\theta))$, we obtain

$$\begin{aligned} (4.6) \quad & |N_{L/F}(O_L^\times)/(O_F^\times)^n|\; (q^{n/e_L} - 1)q^{n/(2e_L)}\; |\eta(X)|^{-1/2} \\ &= |N_{E/F}(E^\times)/N_{L/F}(L^\times)|\; |N_{E/F}(O_E^\times)/(O_F^\times)^n| \\ &\qquad \cdot (q^{n/e_E} - 1)q^{n/(2e_E)}\; |\eta(\operatorname{Ad} x(S_\theta))|^{-1/2}. \end{aligned}$$

Here $\eta : \mathfrak{g}_{reg} \to C$ is the discriminant function ([**HC2**]), and e_L and e_E are the ramification degrees of L and E over F, respectively.

$N_{L/F}(L^\times)$ is a subset of $N_{E/F}(E^\times)$ $(G(\pi) \subset G_E)$. Since $N_{L/F}(L^\times)$ contains an element of valuation n/e_L and the valuation of any element of $N_{E/F}(E^\times)$ is a multiple of n/e_E, e_L is a divisor of e_E. As a consequence,

$$N_{E/F}(O_E^\times) = (O_F^\times)^{e_E} \subset (O_F^\times)^{e_L} = N_{L/F}(O_L^\times) \subset N_{E/F}(O_E^\times),$$

so $(O_F^\times)^{e_L} = (O_F^\times)^{e_E}$. Thus

$$|N_{E/F}(E^\times)/N_{L/F}(L^\times)| = e_E/e_L.$$

Therefore (4.6) becomes

$$e_L\,(q^{n/e_L} - 1)q^{n/(2e_L)}\,|\eta(X)|^{1/2} = e_E\,(q^{n/e_E} - 1)q^{n/(2e_E)}\,|\eta(\operatorname{Ad} x(S_\theta))|^{1/2}.$$

$q^{n/(2e_L)}|\eta(X)|^{1/2}$ and $q^{n/2e_E}|\eta(\operatorname{Ad} x(S_\theta))|^{1/2}$ are powers of q. Because q is a power of p, $p > n$, and e_L and e_E divide n, e_L and e_E are relatively prime to q. Therefore (4.6) implies

$$e_L\,(q^{n/e_L} - 1) = e_E\,(q^{n/e_E} - 1).$$

That is,

$$e_E/e_L = (q^{n/e_E} - 1)^{-1}(q^{n/e_L} - 1) = 1 + q^{n/e_E} + \cdots + q^{n(e_E - e_L)/(e_E e_L)} > n,$$

which is impossible. □

Remarks. Suppose $n = \ell$ is prime (not necessarily dividing $q-1$). Let π be any irreducible supercuspidal representation of G'.

(1) If $\mathcal{O} \in (\mathcal{N})' - (\mathcal{N}_{reg})'$, then $\operatorname{Ad} g \cdot \mathcal{O} = \mathcal{O}$ for every $g \in G$ ([**Re**]). Thus, by Lemma 4.3,

$$c_{\mathcal{O}}(\pi) = d(\pi)\,\Gamma_{\mathcal{O}}(S_\theta) = d(\pi)d(\pi'_\theta)^{-1}c_{\mathcal{O}_{st}}(\pi'_\theta).$$

Lemma 4.3(2) was first observed by Assem([**As**]) in the case where π'_θ has ℓ irreducible components.

(2) The elements of an L-packet containing ℓ representations correspond to G/G_E (Corollary 3.6), and, if π belongs to the L-packet,

$$c_{\mathcal{O}}(\pi) = d(\pi)\,\Gamma_{\mathcal{O}}(\operatorname{Ad} g(S_\theta))\ ,\ \mathcal{O} \in (\mathcal{N}_{reg})',$$

where g is a representative of the corresponding coset. If ℓ divides $q-1$, there exist L-packets containing ℓ^2 representations ([**MS**]). As noted in ([**As**]), the elements of such an L-packet correspond to the orbits in $(\mathcal{N}_{reg})'$, each π being identified with the unique $\mathcal{O} \in (\mathcal{N}_{reg})'$ such that $c_{\mathcal{O}}(\pi)$ is nonzero (up to a constant depending on normalization of $\mu_{\mathcal{O}}$ this nonzero coefficient equals one ([**Ro**])).

(3) Modulo determination of the values of Shalika germs, (1) and (2) combine to give the values of the coefficients $c_{\mathcal{O}}(\pi)$, $\mathcal{O} \in (\mathcal{N})'$, for supercuspidal representations of $SL_\ell(F)$.

(4) The functions $\widehat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in (\mathcal{N})'$, were computed by Assem ([**As**]). Thus, whenever the coefficients $c_{\mathcal{O}}(\pi)$ are known, substitution of Assem's formulas into the local character expansion of π yields a formula for the character Θ_π on a neighbourhood of the identity element.

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