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THE COHOMOLOGY OF EXPANSIVE $\mathbb{Z}^d$-ACTIONS BY AUTOMORPHISMS OF COMPACT, ABELIAN GROUPS

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Based on the structure of the $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups and on techniques for proving the triviality of the first cohomology of higher rank abelian group actions we prove that, for $d > 1$, every real-valued Hölder cocycle of an expansive and mixing $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group in Hölder cohomologous to a homomorphism.

1. Introduction.

In this paper we explore one of the facets of a rather striking phenomenon, namely that the 'good' actions of the higher rank abelian groups, i.e. of $\mathbb{Z}^d$ and $\mathbb{R}^d$ for $d \geq 2$, are much more rare and 'rigid' than similar actions of $\mathbb{Z}$ or $\mathbb{R}$. While neither the total extent of this phenomenon nor the proper definition of a 'good' action are as yet clear, it definitely manifests itself in principal Anosov and partially hyperbolic actions of $\mathbb{R}^d$, $d \geq 2$ ([**KSp1**]), in $\mathbb{Z}^d$ actions by automorphisms of compact abelian groups ([**KSp2**] and the present paper), and for a large class of higher-dimensional shifts of finite type ([**S2**]).

An essential part of rigidity present in the higher rank abelian group actions is due to the triviality of the untwisted first cohomology for sufficiently regular classes of cocycles. Recall that, if Γ is a group, and $T : \gamma \mapsto T_\gamma$ is a continuous action of Γ on a compact, metrizable space X, i.e. a homomorphism from Γ into the group of homeomorphisms of X, then a map $c : \Gamma \times X \longmapsto \mathbb{Z}$ is a continuous (1-)*cocycle* for T if $c(\gamma, \cdot) : X \longmapsto \mathbb{R}$ is continuous for every $\gamma \in \Gamma$, and

$$(1.1) \qquad c(\gamma\gamma', x) = c(\gamma, T_{\gamma'}x) + c(\gamma', x)$$

for all $x \in X$ and $\gamma, \gamma' \in \Gamma$. The cocycle c is a *coboundary* if there exists a Borel map $b : X \longmapsto \mathbb{R}$ with

$$(1.2) \qquad c(\gamma, x) = b(T_\gamma x) - b(x)$$

for all $x \in X$, $\gamma \in \Gamma$. The function b in (1.2) is called the *cobounding function* of the coboundary c. Two cocycles $c_1, c_2 : \Gamma \times X \longmapsto R$ are *cohomologous* (*with transfer function* b) if their difference $c_1 - c_2$ is a coboundary with cobounding function b, and c_1, c_2 are *continuously cohomologous* if the transfer function b can be chosen to be continuous. Finally, a cocycle $c : \Gamma \times X \longmapsto \mathbb{R}$ is a *homomorphism* if $c(\gamma, \cdot)$ is constant for every $\gamma \in \Gamma$. If Γ is a discrete group then, given a certain notion of regularity for functions on X, such as smoothness, Hölder continuity, etc., we will call a cocycle $c : \Gamma \times X \longmapsto \mathbb{R}$ *regular* if, for each value $\gamma \in \Gamma$, the function $c(\gamma, \cdot)$ has the required kind of regularity. Having fixed a notion of regularity for transfer functions we will say that cocycles from a given class are *trivial* if they are cohomologous to homomorphisms, with transfer functions of required regularity.

To demonstrate the source of the difference between rank one and higher rank let us consider the untwisted ℓ_1-cohomology of the action of $\mathbb{Z}^d$ on itself by translation for $d = 1$ and $d = 2$. This is both a model and a building block for the cohomology trivialization results in the present paper as well as in **[KSp1]**, **[S2]**.

For $d = 1$, an ℓ_1 cocycle is represented by its value on the generator 1 of $\mathbb{Z}$, i.e. by an absolutely summable sequence $x = (x_n, n \in \mathbb{Z})$, and the coboundary condition (1.2) means that $x_n = y_{n+1} - y_n$ for another sequence $y = (y_n)$. Since x is absolutely summable we can define $y^+ = (y_n^+)$ where, for every $n \in \mathbb{Z}$,

$$y_n^+ = \sum_{i=-\infty}^{n} x_i,$$

and obtain that x is a coboundary. However, if we want in addition the cobounding function y to be in ℓ_1, or at least to vanish at infinity, an obstruction appears. Obviously, $y_n^+ \to \sum_{-\infty}^{\infty} x_n$ as $n \to +\infty$, and y^+ vanishes at infinity if and only if $\sum_{-\infty}^{\infty} x_n = 0$ or, equivalently, if $y^+ = y^-$, where $y_n^- = -\sum_{i=n+1}^{\infty} x_i$.

Even if this obstruction vanishes, the (uniquely defined) cobounding function vanishing at infinity may itself not be absolutely summable. However, this will be the case if x satisfies a reasonable decay condition at infinity: if x decays super-polynomially, exponentially, or super-exponentially, so does y.

For $d = 2$, a cocycle is given by its values on the two generators $(1, 0)$ and $(0, 1)$ of $\mathbb{Z}^2$, i.e. by two double sequences $x^{(1,0)} = \left(x_{m,n}^{(1,0)}\right)$ and $x^{(0,1)} = \left(x_{m,n}^{(0,1)}\right)$ in $\ell_1(\mathbb{Z}^2)$ satisfying the equation

$$x_{m,n+1}^{(1,0)} - x_{m,n}^{(1,0)} = x_{m+1,n}^{(0,1)} - x_{m,n}^{(0,1)} \tag{1.3}$$

for every $(m,n) \in \mathbb{Z}^2$. The cobounding relation (1.2) becomes

$$x^{(1,0)}_{m,n} = y_{m+1,n} - y_{m,n}, \quad x^{(0,1)}_{m,n} = y_{m,n+1} - y_{m,n}.$$

The cocycle equation (1.3) can be re-written as

$$x^{(1,0)}_{m,n+1} = x^{(1,0)}_{m,n} + x^{(0,1)}_{m+1,n} - x^{(0,1)}_{m,n},$$

and hence

$$\sum_{i=-m}^{m} x^{(1,0)}_{i,n+1} = \sum_{i=-m}^{m} x^{(1,0)}_{i,n} + x^{(0,1)}_{m+1,n} - x^{(0,1)}_{-m,n}.$$

Since $x^{(0,1)}$ is summable and hence vanishes at infinity one has, for every $n \in \mathbb{Z}$,

$$\sum_{m=-\infty}^{\infty} x^{(1,0)}_{m,n} = \sum_{m=-\infty}^{\infty} x^{(1,0)}_{m,n+1}.$$

Since $x^{(1,0)}$ is absolutely summable we deduce that $\sum_{m=-\infty}^{\infty} x^{(1,0)}_{m,n} = 0$ for every $n \in \mathbb{Z}$ (*the difference between rank one and higher rank lies precisely at this point*). Hence $y^+ = y^-$, where $y^+_{m,n} = \sum_{i=-\infty}^{m} x^{(1,0)}_{i,n}$ and $y^-_{m,n} = -\sum_{i=m+1}^{\infty} x^{(1,0)}_{i,n}$. Thus y^+ is a cobounding function for $x^{(1,0)}$ which vanishes at infinity due to the summability of $x^{(1,0)}$. The cocycle equation (1.3) shows that

$$x^{(0,1)}_{m,n} = y^+_{m,n+1} - y^+_{m,n},$$

so that y^+ cobounds the cocycle. By imposing super-polynomial, exponential, or super-exponential decay conditions on the cocycle we obtain similar conditions for the cobounding functions.

The proof of C^∞-cohomology trivialization for a mixing action of $\mathbb{Z}^d$, $d \geq 2$, by automorphisms of a finite-dimensional torus or, more generally, for a mixing action of $\mathbb{Z}^d{}_+$ by toral endomorphisms, is a more or less straightforward application of the above argument to the Fourier duals of the action and the cocycle ([**KSp1**], Section 4.1). An approximation argument using Livshitz' theorem [**Liv**] extends this result to Hölder cocycles if the action is Anosov (i.e. expansive). For non-expansive mixing actions the result is probably not true for Hölder cocycles, as Veech's example for rank one indicates [**V**].

When one passes from finite dimensional tori to a $\mathbb{Z}^d$-action α by automorphisms of a more general compact, abelian group X, the notion of a C^∞-structure becomes problematic, since the group is typically either not locally connected, or infinite dimensional, or both. There is, however, a natural Hölder structure associated with a given action, which coincides with the usual one for an expansive action on the torus (cf. Section 2, (2.3)). As we indicated above, even on the torus the proof of cohomology trivialization for Hölder cocycles is not totally straightforward. Thus we have to take a more indirect route, which also leads to trivialization results with continuous transfer functions for cocycles with the weakest possible regularity condition (summable variation—(2.2)). We abandon a direct reference to Fourier series in favour of a more geometric construction somewhat reminiscent of the proof of Livshitz' theorem [**Liv**]. In the Fourier series argument the obstruction to cocycle triviality is the sum of the Fourier coefficients along individual $\mathbb{Z}^d$-orbits of the dual action. In the general situation this summability may no longer hold. We overcome this difficulty by constructing, for a Hölder cocycle c, a tentative solution on the set of points asymptotic to a given one (the identity) along a certain double cone in $\mathbb{Z}^d$. An obstruction to extending this solution to a continuous transfer function is expressed in (2.11), which is a geometric counterpart of a sum of Fourier coefficients along an orbit of a regular element of the $\mathbb{Z}^d$-action. In the simplest possible terms this obstruction can be described as follows: if an element y of the group X is positive asymptotic to the identity element 0 with respect to some element $\mathbf{n} \in \mathbb{Z}^d$, then the difference of the values of any possible cobounding function at x and at $x+y$ has to be equal to the infinite sum $\sum_{k\geq 0}(c(\mathbf{n}, \alpha_{k,\mathbf{n}}(x)) - c(\mathbf{n}, \alpha_{k\mathbf{n}}(x+y)))$. A similar expression is obtained for negative asymptotic points. Since both expressions are defined if y is homoclinic to 0 with respect to $\mathbf{n}$, they must coincide for any such y in order for a cobounding function to exist. Proposition 2.6 not only shows that this condition is sufficient, but establishes a relativised version of it, which is used in an induction process.

As expansive $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups may have a very complicated structure, this argument cannot be applied to the action as a whole, but to a sequence of algebraic quotients of the original action. These quotients are obtained as duals of a suitable prime filtration of the dual group viewed as a Noetherian module. This structure is summarized in Theorems 3.1 and 3.2, and Lemma 3.3. The individual quotients which can appear in this decomposition are characterized in Corollaries 3.5 and 3.7. The key Proposition 2.6 allows us to carry out an inductive reduction of the cocycle to cocycles for the successive quotient actions, assuming that the geometric obstruction (2.11) vanishes for each of the successive quotients.

The vanishing of the obstructions is established in Section 4 for the four different types of quotient actions (Lemmas 4.2, 4.4, 4.6, and 4.8). It is interesting to note that in the last two of those cases we have to appeal to a slightly weaker version of the summability of Fourier coefficients.

We would also like to point out somewhat unequal contributions by the two authors of this paper. The original idea came in a conversation in

December 1991 during the second author's visit to Penn State, when we realized that techniques for proving trivialization of cohomology developed in [**KSp1**] could be combined with the structure theory of $\mathbb{Z}^d$ actions by automorphisms in [**S1**] and [**KiS2**] to produce trivialization results for such actions. We found a proof which worked in many, but not all cases. The rest is due to the second author who substantially modified the original approach and put the complete proofs in final form.

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2. Statement of the main theorem, and a Livshitz-type result.

Let T be a continuous action of $\mathbb{Z}^d$ on a compact, metric space (X, δ). We write $|\cdot|$ and $\langle\cdot\rangle$ for the Euclidean norm and inner product on $\mathbb{R}^d \supset \mathbb{Z}^d$, and put $\mathbf{B}(r) = \{\mathbf{n} \in \mathbb{Z}^d : |\mathbf{n}| \leq r\}$. If $f : X \longmapsto \mathbb{R}$ is a continuous function we set, for every $r \geq 0$,

$$(2.1)\ \omega_r^\delta(f, T, \varepsilon) = \sup_{\{(x,x') \in X \times X : \delta(T_\mathbf{n}(x), T_\mathbf{n}(x')) < \varepsilon \text{ for all } \mathbf{n} \in \mathbf{B}(r)\}} |f(x) - f(x')|,$$

and we say that f has *T-summable variation* if there exists an $\varepsilon > 0$ such that

$$\omega^\delta(f, T, \varepsilon) = \sum_{r=1}^{\infty} \omega_r^\delta(f, T, \varepsilon) < \infty. \tag{2.2}$$

The function f is *T-Hölder* if there exist constants $\varepsilon, \omega' > 0$ and ω with $0 < \omega < 1$ such that

$$\omega_r(f, T, \varepsilon) < \omega' \omega^r \tag{2.3}$$

for every $r > 0$. These notions are obviously independent of the specific metric δ on X, and every T-Hölder function has T-summable variation. If the $\mathbb{Z}^d$-action T is understood we simply say that f has *summable variation* or is *Hölder*. Note that, if $d > 1$ and f has T-summable variation (or is T-Hölder), then f will in general not have the corresponding property with respect to any of the $\mathbb{Z}$-actions $k \mapsto T_{k\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$. A cocycle $c : \mathbb{Z}^d \times X \longmapsto \mathbb{R}$ for T has *T-summable variation* (or is *T-Hölder*) if $c(\mathbf{n}, \cdot) : X \longmapsto \mathbb{R}$ has T-summable variation (or is T-Hölder) for every $\mathbf{n} \in \mathbb{Z}^d$.

Let X be a compact, additive, abelian group (always assumed to be metrizable), with identity element 0_X, and let $\mathrm{Aut}(X)$ be the group of continuous group-automorphisms of X. If $d \geq 1$, then a *$\mathbb{Z}^d$-action by automorphisms of X* is a homomorphism $\alpha : \mathbb{Z}^d \longmapsto \mathrm{Aut}(X)$. The action α is *ergodic* or *mixing* if it is so with respect to the normalized Haar measure λ_X of X, and α is *expansive* if there exists an open set $\mathcal{O} \subset X$ such that $\bigcap_{\mathbf{n}\in\mathbb{Z}^d} \alpha_{\mathbf{n}}(\mathcal{O}) = \{0_X\}$, where 0_X is the identity element of X. We shall prove the following theorem.

Theorem 2.1. *Let $d > 1$, and let α be an expansive and mixing $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X.*

(1) *Every cocycle $c : \mathbb{Z}^d \times X \longmapsto \mathbb{R}$ with α-summable variation is continuously cohomologous to a homomorphism;*

(2) *Every α-Hölder cocycle $c : \mathbb{Z}^d \times X \longmapsto \mathbb{R}$ is cohomologous to a homomorphism, with Hölder transfer function.*

The proof of Theorem 2.1 will occupy Sections 2–4, and will depend on the structure theory of $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups presented in Section 3, where we also discuss briefly the notions of functions with α-summable variation and of α-Hölder functions (Remark 3.10).

Let α be a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian (additive) group X, and let

$$\Delta_\alpha = \left\{ x \in X : \lim_{\mathbf{k}\to\infty} \alpha_{\mathbf{k}}(x) = 0_X \right\} \tag{2.4}$$

be the *homoclinic group* of α. For every nonzero element $\mathbf{n} \in \mathbb{Z}^d$ and every ξ with $0 < \xi < 1$ we define the cones

$$\begin{aligned} C^+(\mathbf{n}, \xi) &= \{\mathbf{m} \in \mathbb{Z}^d : \langle \mathbf{m}, \mathbf{n} \rangle \geq \xi \, |\mathbf{m}| \, |\mathbf{n}|\} , \\ C^-(\mathbf{n}, \xi) &= \{\mathbf{m} \in \mathbb{Z}^d : \langle \mathbf{m}, \mathbf{n} \rangle \leq -\xi \, |\mathbf{m}| \, |\mathbf{n}|\} , \end{aligned} \tag{2.5}$$

and consider the group

$$\begin{aligned} \Delta_\alpha(\mathbf{n}, \xi) = \Biggl\{ x \in X : & \lim_{\substack{\mathbf{k}\to\infty \\ \mathbf{k}\in C^+(\mathbf{n},\xi')}} \alpha_{\mathbf{k}}(x) = \\ & = \lim_{\substack{\mathbf{k}\to\infty \\ \mathbf{k}\in C^-(\mathbf{n},\xi')}} \alpha_{\mathbf{k}}(x) = 0_X \text{ for some } \xi' \in (0, \xi) \Biggr\} . \end{aligned} \tag{2.6}$$

Note that

$$\alpha_{\mathbf{m}}(\Delta_\alpha(\mathbf{n}, \xi)) = \Delta_\alpha(\mathbf{n}, \xi) \tag{2.7}$$

for every $\mathbf{m} \in \mathbb{Z}^d$.

Examples 2.2(1) Let F be a compact, abelian group, and let σ be the shift-action of $\mathbb{Z}^d$ on $F^{\mathbb{Z}^d}$ defined by

$$(\sigma_{\mathbf{n}}x)_{\mathbf{m}} = x_{\mathbf{m}+\mathbf{n}} \tag{2.8}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $x = (x_{\mathbf{m}}) = (x_{\mathbf{m}}, \mathbf{m} \in \mathbb{Z}^d) \in F^{\mathbb{Z}^d}$. Then

$$\Delta_\sigma(\mathbf{n}, \xi) \supset \Delta_\sigma \supset \{x = (x_{\mathbf{n}}) \in X : x_{\mathbf{n}} = 0_F \\ \text{for all but finitely many } \mathbf{n} \in \mathbb{Z}^d\},$$

and these groups are therefore dense in X; if F is finite, then

$$\Delta_\sigma = \{x = (x_{\mathbf{n}}) \in X : x_{\mathbf{n}} = 0_F \text{ for all but finitely many } \mathbf{n} \in \mathbb{Z}^d\}.$$

(2) Let F be a finite, abelian group,

$$X = \Big\{x = (x_{\mathbf{n}}) \in F^{\mathbb{Z}^d} : x_{(m,n)} + x_{(m+1,n)} + x_{(m,n+1)} = 0_F \\ \text{for every } (m,n) \in \mathbb{Z}^2\Big\},$$

and let α be the restriction of the shift-action σ in (2.8) to X. It is easy to see that $\Delta_\alpha = \{0_X\}$, and that $\Delta_\alpha((1,1), \xi)$ is dense in X for every $\xi \in \left(\frac{1}{\sqrt{2}}, 1\right)$.

(3) If α is an expansive and mixing $\mathbb{Z}^2$-action by automorphisms of a finite dimensional torus or solenoid X, then $\Delta_\alpha = \{0_X\}$. However, if $\mathbf{n} \in \mathbb{Z}^d$ is an element such that $\alpha_{\mathbf{n}}$ is expansive (such elements obviously exist), then $\Delta_\alpha(\mathbf{n}, \xi)$ is dense in X for some $\xi \in (0,1)$ (this can be proved by looking at the local product structure of $\alpha_{\mathbf{n}}$ at 0_X, which is described in some detail in Lemma 4.7).

Let α be an expansive $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X, and let $c : \mathbb{Z}^d \times X \longmapsto \mathbb{R}$ be a cocycle with α-summable variation. We fix a primitive element $\mathbf{n} \in \mathbb{Z}^d$ and $\xi \in (0,1)$, and define a cocycle $c^{(\mathbf{n})} : \Delta_\alpha(\mathbf{n}, \xi) \times X \longmapsto \mathbb{R}$ for the action of $\Delta_\alpha(\mathbf{n}, \xi)$ on X by translation by

$$c^{(\mathbf{n})}(y,x) = \sum_{k \in \mathbb{Z}} (c(\mathbf{n}, \alpha_{k\mathbf{n}}(x)) - c(\mathbf{n}, \alpha_{k\mathbf{n}}(x+y))) \tag{2.9}$$

for every $y \in \Delta_\alpha(\mathbf{n}, \xi)$ and $x \in X$. Since $y \in \Delta_\alpha(\mathbf{n}, \xi)$, (2.9) is well defined, and the cocycle equation *(1.1) implies that*

$$c^{(\mathbf{n})}(y,x) = c^{(\mathbf{n})}(\alpha_{\mathbf{m}}(y), \alpha_{\mathbf{m}}(x))$$

for every $\mathbf{m} \in \mathbb{Z}^d$ (cf. (2.7)). For $y \in \Delta_\alpha \subset \Delta_\alpha(\mathbf{n}, \xi)$ we obtain

$$c^{(\mathbf{n})}(y,x) = \lim_{\substack{\mathbf{m} \to \infty \\ \langle \mathbf{m}, \mathbf{n} \rangle = 0}} c^{(\mathbf{n})}(\alpha_{\mathbf{m}}(y), \alpha_{\mathbf{m}}(x)) = 0$$

for every $x \in X$, as a consequence of (2.1) and (2.4). In Example 2.2 (1), Δ_α is dense in $\Delta_\alpha(\mathbf{n}, \xi)$, and an elementary approximation argument yields that $c^{(\mathbf{n})}(y, \cdot) = 0$ for every $y \in \Delta_\alpha(\mathbf{n}, \xi)$. In general, however, Δ_α will not be dense in X (cf. Examples 2.2 (2)–(3)), and other techniques have to be employed in order to show that $c^{(\mathbf{n})}(y, \cdot)$ vanishes for all $y \in \Delta_\alpha(\mathbf{n}, \xi)$.

The next step of the argument is to show that the vanishing of $c^{(\mathbf{n})}$ implies that $c(\mathbf{n}, \cdot)$ is cohomologous to a constant. We begin with a definition. Let δ be an invariant metric on a compact, abelian group X (*invariant* means that $\delta(x, y) = \delta(x+z, y+z)$ for all $x, y, z \in X$), and let α a $\mathbb{Z}^d$-action by automorphisms of X

Definition 2.3. Let $\mathbf{0} \neq \mathbf{n} \in \mathbb{Z}^d$, and let $\xi \in (0,1)$. The $\mathbb{Z}^d$-action α has *weak* $(\mathbf{n}, \xi)$*-specification* if $\Delta_\alpha(\mathbf{n}, \xi)$ is dense in X, and if there exist, for every $\varepsilon > 0$, constants $s' \geq 1$, $t' \geq 0$, with the following property: for every $r \geq 0$, and for every $x \in \Delta_\alpha(\mathbf{n}, \xi)$ with $\delta(\alpha_{\mathbf{m}}(x), 0_X) < \varepsilon$ for every $m \in \mathbf{B}(s'r + t')$, one can find a $y \in \Delta_\alpha(\mathbf{n}, \xi)$ with

$$\begin{aligned} &\delta(\alpha_{\mathbf{k}}(y), \alpha_{\mathbf{k}}(x)) < \varepsilon \text{ for all } \mathbf{k} \in C^+(\mathbf{n}, \xi) + \mathbf{B}(r), \\ &\quad \delta(\alpha_{\mathbf{k}}(y)), 0_X < \varepsilon \text{ for all } \mathbf{k} \in C^-(\mathbf{n}, \xi) + \mathbf{B}(r). \end{aligned} \tag{2.10}$$

We say that α has *weak* $\mathbf{n}$*-specification* if it has weak $(\mathbf{n}, \xi)$-specification for some $\xi \in (0,1)$.

Examples 2.4(1) Let F be a compact, abelian group, and let σ be the shift-action (2.8) of $\mathbb{Z}^d$ on $X = F^{\mathbb{Z}^d}$. Then σ has weak $(\mathbf{n}, \xi)$-specification for every nonzero element $\mathbf{n} \in \mathbb{Z}^d$ and every $\xi \in (0,1)$.

(2) Let α be an expansive $\mathbb{Z}^d$-action by automorphisms of a finite-dimensional torus or solenoid X. For every $\mathbf{n} \in \mathbb{Z}^d$ for which $\alpha_{\mathbf{n}}$ is expansive, α has weak $\mathbf{n}$-specification. This is proved by looking at the local product structure of $\alpha_{\mathbf{n}}$ (cf. Lemma 4.7).

Other examples of $\mathbb{Z}^d$-actions with weak $\mathbf{n}$-specification will arise in Corollary 3.5 and 3.7. Before describing how weak $\mathbf{n}$-specification helps in proving that the function $c(\mathbf{n}, \cdot)$ is cohomologous to a constant if the cocycle $c^{(\mathbf{n})}$ in (2.9) vanishes we have to establish a preliminary result. If X is a compact, abelian group and $Y \subset X$ a closed subgroup we denote by λ_X and $\mathfrak{B}_X$ the normalized Haar measure and the Borel field of X, write $\mathfrak{B}_{X/Y} \subset \mathfrak{B}_X$ for the σ-algebra of Borel cosets of Y, and consider the conditional expectation $E_{\lambda_X}(f|\mathfrak{B}_{X/Y})$ of a function $f : X \longmapsto \mathbb{R}$ with respect to the σ-algebra $\mathfrak{B}_{X/Y}$.

Lemma 2.5. *Let α be a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X, and let $f : X \longmapsto \mathbb{R}$ be a function with α-summable variation. If $Y \subset X$ is a closed, α-invariant subgroup, then $E_{\lambda_X}(f|\mathfrak{B}_{X/Y})$ has α-*

summable variation. If f is α-Hölder, then $E_{\lambda_X}(F|\mathfrak{B}_{X/Y})$ is again α-Hölder.

Proof. We fix an invariant metric δ on X, set

$$X_\delta(r,\varepsilon)=\{x\in X:\delta(\alpha_{\mathbf{k}}(x),0_X)<\varepsilon \quad \text{for every } \mathbf{k}\in \mathbf{B}(r)\}$$

for every $r\geq 0$ and $\varepsilon>0$, and note that

$$\sup_{y\in X_\delta(r,\varepsilon)}\max_{x\in X}|f(x)-f(x+y)|=\omega_r^\delta(f,\alpha,\varepsilon).$$

If $f^y(x)=f(x+y)$, then

$$\left|E_{\lambda_X}\left(f|\,\mathfrak{B}_{X/Y}\right)(x)-E_{\lambda_X}\left(f^y|\mathfrak{B}_{X/Y}\right)(x)\right|\leq \omega_r^\delta(f,\alpha,\varepsilon)$$

for all $x\in X$ and $y\in X_\delta(r,\varepsilon)$, which proves that $E_{\lambda_X}(f|\mathfrak{B}_{X/Y})$ has summable variation (or is Hölder). □

The following proposition is similar to Livshitz' theorem ([**Liv**] and [**KSp1**], Theorem 2.14), which guarantees the vanishing of Hölder cocycles for an Anosov system, given that the cocycle vanishes on all periodic orbits. This proposition shows that the obstruction to reducing a cocycle to a cocycle on a quotient group is given by the expression (2.11).

Proposition 2.6. *Let α be a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X, $Y\subset X$ a closed, α-invariant subgroup, and assume that the restriction of α to Y has weak $(\mathbf{n},\xi)$-specification for some $\xi\in(0,1)$ and some nonzero $\mathbf{n}\in\mathbb{Z}^d$. Let $f:X\longmapsto\mathbb{R}$ be a function with α-summable variation. We set $\Delta(\mathbf{n},\xi)=\Delta_\alpha(\mathbf{n},\xi)\cap Y$ and define a cocycle $c_f^{(\mathbf{n})}:\Delta(\mathbf{n},\xi)\times X\longmapsto\mathbb{R}$ for the action of $\Delta(\mathbf{n},\xi)$ on X by translation by*

$$c_f^{(\mathbf{n})}(y,x)=\sum_{k\in\mathbb{Z}}(f\cdot\alpha_{k\mathbf{n}}(x)-f\cdot\alpha_{k\mathbf{n}}(x+y)) \tag{2.11}$$

for every $x\in X$ and $y\in\Delta(\mathbf{n},\xi)$. If $c_f^{(\mathbf{n})}(y,x)=0$ for all $y\in\Delta(\mathbf{n},\xi)$ and $x\in X$, then f is cohomologous—with bounded transfer function—to $E_{\lambda_X}(f\,|\mathfrak{B}_{X/Y})$.

Suppose furthermore that there exists a closed, α-invariant subgroup $Z\subset X$ such that $Y\cap Z=\{0_X\}$ and $X=Y+Z$. Then f is continuously cohomologous to $E_{\lambda_X}(f\,|\mathfrak{B}_{X/Y})$. Moreover, if f is α-Hölder, and if δ is a metric on X, then the transfer function $b:X\longmapsto\mathbb{R}$ can be chosen so that there exist positive constants ε, ω, ω' with $0<\omega<1$ and $|b(x+y)-b(x)|\leq$

$\omega'\omega^r$ *for all* $x \in X$, $r \geq 1$, *and* $y \in Y_\delta(r,\varepsilon) = \{y \in Y : \delta(\alpha_{\mathbf{k}}(y), 0_X) < \varepsilon$ for all $\mathbf{k} \in \mathbf{B}(r)\}$.

Proof. For every $x \in X$ and $y \in \Delta(\mathbf{n}, \xi)$ we put

$$c^+(y,x) = \sum_{k \geq 0} (f \cdot \alpha_{k\mathbf{n}}(x) - f \cdot \alpha_{k\mathbf{n}}(x+y)),$$
$$c^-(y,x) = \sum_{k < 0} (f \cdot \alpha_{k\mathbf{n}}(x) - f \cdot \alpha_{k\mathbf{n}}(x+y)),$$

and note that the maps $c^{\pm} : \Delta(\mathbf{n},\xi) \times X \longmapsto \mathbb{R}$ are well-defined cocycles, and that

$$c^+(y,x) + c^-(y,x) = c_f^{(\mathbf{n})}(y,x) = 0$$

for every $x \in X$ and $y \in \Delta(\mathbf{n}, \xi)$.

We fix an invariant metric δ on X and an $\varepsilon > 0$ such that $\sum_{r \geq 0} \omega_r^\delta(f, \alpha, \varepsilon) < \infty$, and use the weak $(\mathbf{n}, \xi)$-specification of α on Y to find constants $s' \geq 1$, $t' \geq 0$ with the following property: for every $r \geq 0$ and every $y \in \Delta(\mathbf{n}, \xi)$ with $\delta(\alpha_{\mathbf{m}}(y), 0_X) < \varepsilon$ for all $\mathbf{m} \in \mathbf{B}(s'r + t')$, there exists an element $y' \in \Delta(\mathbf{n},\xi)$ with $\delta(\alpha_{\mathbf{k}}(y'), \alpha_{\mathbf{k}}(y)) < \varepsilon$ for all $\mathbf{k} \in C^+(\mathbf{n},\xi) + \mathbf{B}(r)$, and $\delta(\alpha_{\mathbf{k}}(y'), 0_X) < \varepsilon$ for all $\mathbf{k} \in C^-(\mathbf{n},\xi) + \mathbf{B}(r)$. Then

$$\begin{aligned} &|c^+(y,x) - c^+(y',x)| \leq (1-\xi^2)^{-\frac{1}{2}} \sum_{r \geq M} \omega_r^\delta(f,\alpha,\varepsilon) = C'(M), \\ &|c^-(y',x)| = |c^+(y',x)| \leq C'(M), \text{ and } |c^+(y,x)| \leq 2C'(M) \end{aligned} \tag{2.12}$$

for every $y \in \Delta(\mathbf{n},\xi) \cap Y_\delta(s'M + t', \varepsilon)$ and $x \in X$. By varying M we see that

$$\lim_{\substack{y \to 0_X \\ y \in \Delta(\mathbf{n},\xi)}} \max_{x \in X} |c^+(y,x)| = 0. \tag{2.13}$$

Since $\Delta(\mathbf{n},\xi)$ is dense in Y, (2.13) allows us to extend c^+ uniquely to a continuous function $\mathbf{c}^+ : Y \times X \longmapsto \mathbb{R}$, and $\mathbf{c}^+$ is again a cocycle. We write $\theta : X \longmapsto X/Y$ for the quotient map and choose a Borel map $\theta' : X/Y \longmapsto X$ such that $\theta \cdot \theta'(x+Y) = x + Y$ for every $x \in X$ (cf. [**P**], Lemma I.5.1). The map $b : Y \longmapsto \mathbb{R}$, defined by $b(x) = \mathbf{c}^+(x - \theta' \cdot \theta(x), \theta' \cdot \theta(x))$ for every $x \in X$, is bounded and Borel and satisfies that $\mathbf{c}^+(y,x) = b(x+y) - b(x)$ for every $x \in X$ and $y \in Y$.

If there exists a closed, α-invariant subgroup $Z \subset X$ with $Y \cap Z = \{0\}$ and $Y + Z = X$, then we can write every $x \in X$ uniquely as $x = y(x) + z(Z)$ with $y(x) \in Y$ and $z(x) \in Z$, assume that

$$\delta(x, 0_X) = \max \{\delta(y(x), 0_X) + \delta(z(x), 0_X)\},$$

and set $\theta'(x+Y) = z(x)$. Then θ' is continuous, and $b : X \longmapsto \mathbb{R}$ is continuous.

If f is Hölder we assume without loss of generality that ε is sufficiently small and choose $0<\omega<1$, $\omega'>0$, such that (2.3) is satisfied. From (2.12) we see that there exists a positive constant $\bar{\omega}'$ such that $|c^+(y,x)| \leq \bar{\omega}'\omega^M$ for all $x \in X$ and $y \in \Delta(\mathbf{n},\xi) \cap Y_\delta(s'M+t',\varepsilon)$, and we conclude that $|\mathbf{c}^+(y,x)| \leq \bar{\omega}'\omega^M$ for all $x\in X$ and $y \in Y_\delta(s'M+t',\varepsilon)$. Hence $|b(x+y)-b(x)| \leq \bar{\omega}'\omega^M$ for all $x \in X$ and $y \in Y_\delta(s'M+t',\varepsilon)$.

Let $\Gamma = \mathbb{Z}\times Y$ with the product topology, and with group operation $(n,y)\cdot(n',y') = (n+n', \alpha_{n'\mathbf{n}}(y)+y')$, (n,y), $(n',y') \in \Gamma$, and let $\Gamma' \subset \Gamma$ be the subgroup consisting of all (n,y) with $n\in\mathbb{Z}$ and $y \in \Delta(\mathbf{n},\xi)$. We write T for the action of Γ on X given by $T_{(n,y)}(x) = \alpha_{n\mathbf{n}}(x+y)$ and define a continuous map $\psi : \Gamma \times X \longmapsto \mathbb{R}$ by setting, for every $(n,y)\in\Gamma$ and $x \in X$,

$$\psi((n,y),x) = \begin{cases} \sum_{j=0}^{n-1} f\cdot\alpha_{j\mathbf{n}}(x+y) + \mathbf{c}^+(y,x) & \text{if } n>0, \\ \mathbf{c}^+(y,x) & \text{if } n=0, \\ -\sum_{j=1}^{-n} f\cdot\alpha_{-j\mathbf{n}}(x+y) + \mathbf{c}^+(y,x) & \text{if } n<0. \end{cases}$$

A straightforward calculation shows that the restriction of ψ to $\Gamma'\times X$ is a cocycle for the restriction T' of T to Γ', and the definition of $\mathbf{c}^+$ as a continuous extension of c^+ implies that ψ is a cocycle for the Γ-action T. In particular,

$$\begin{aligned} & b(x+y)-b(x) \\ & = \mathbf{c}^+(y,x) = \psi((0,y),x) \\ & = \psi((-1,0),\alpha_{\mathbf{n}}(x+y)) + \psi((0,\alpha_{\mathbf{n}}(y)),\alpha_{\mathbf{n}}(x)) + \psi((1,0),x) \\ & = -\psi((1,0),(x+y)) + \psi((0,\alpha_{\mathbf{n}}(y)),\alpha_{\mathbf{n}}(x)) + \psi((1,0),x) \\ & = -f(x+y) + \mathbf{c}^+(\alpha_{\mathbf{n}}(y),\alpha_{\mathbf{n}}(x)) + f(x) \\ & = -f(x+y) + b\cdot\alpha_{\mathbf{n}}(x+y) - b\cdot\alpha_{\mathbf{n}}(x) \end{aligned}$$

for all $x\in X$, $y \in Y$, so that $f - b\cdot\alpha_{\mathbf{n}} + b$ is invariant under translation by Y. Hence $f-b\cdot\alpha_{\mathbf{n}}+b = E_{\lambda_X}(f-b\cdot\alpha_{\mathbf{n}}+b|\mathfrak{B}_{X/Y}) = E_{\lambda_X}(f|\mathfrak{B}_{X/Y}) - E_{\lambda_X}(b|\mathfrak{B}_{X/Y})\cdot\alpha_{\mathbf{n}} + E_{\lambda_X}(b|\mathfrak{B}_{X/Y})$, so that $f - E_{\lambda_X}(f|\mathfrak{B}_{X/Y}) = b'\cdot\alpha_{\mathbf{n}} - b'$ for some bounded Borel map $b' : X \longmapsto \mathbb{R}$. Finally, if the conditions of the last assertion are satisfied, then $b' = b - E_{\lambda_X}(b|\mathfrak{B}_{X/Y})$ is continuous or Hölder in the required sense (cf. Lemma 2.5). □

Corollary 2.7. *Let X', X'' be compact, abelian groups, α', α'' ergodic $\mathbb{Z}^d$-actions by automorphisms of X' and X'', respectively, and let $\mathbf{0} \neq \mathbf{n} \in \mathbb{Z}^d$ be* *an element such that α'' has weak $\mathbf{n}$-specification. Put $X = X' \times X''$, $\alpha = \alpha' \times \alpha''$, and assume that $f : X \longmapsto \mathbb{R}$ has α-summable variation. Suppose that there exists a Borel map $b : X \longmapsto \mathbb{R}$ such that $f = E_1(f) + b \cdot \alpha_{\mathbf{n}} - b$, where $E_1(f) = E_{\lambda_X}(f|\mathfrak{B}_{X/\{0_{X'}\times X''\}})$. Then there exists a continuous map $b' : X \longmapsto \mathbb{R}$ such that $f = E_1(f) + b' \cdot \alpha_{\mathbf{n}} - b'$.*

Proof. We set $\Delta = \{0_{X'}\} \times \Delta_{\alpha''}(\mathbf{n}, \xi)$ and define a cocycle $c_f^{(\mathbf{n})} : \Delta \times X \longmapsto \mathbb{R}$ by (2.11). Suppose that there exist elements $x \in X$ and $y \in \Delta$ such that $c_f^{(\mathbf{n})}(y, x) = a \neq 0$. Then there exists a neighbourhood $N(x) \subset X$ such that $\left|c_f^{(\mathbf{n})}(y, z)\right| > a/2$ for all $z \in N(x)$, and $\lambda_X(N(x))$ is obviously positive.

On the other hand, since $f = E_1(f) + b \cdot \alpha_{\mathbf{n}} - b$ for some Borel map $b : X \longmapsto \mathbb{R}$, and since there exists, for every $\varepsilon > 0$, a compact subset $C_\varepsilon \subset X$ such that $\lambda_X(C_\varepsilon) > 1 - \varepsilon$ and the restriction of b to C_ε is continuous, we obtain that $c_f^{(\mathbf{n})}(y, \cdot) = 0 \quad \lambda_X$-*a.e.*, which contradicts the conclusion of the first paragraph of this proof. Hence $c_f^{(\mathbf{n})}(y, x) = 0$ for all $y \in \Delta$ and $x \in X$, and the conclusion follows from Proposition 2.6. □

3. The structure of $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups.

Let X be a compact, abelian group with dual group $\hat{X}$. For all $x \in X$ and $a \in \hat{X}$ we denote by $a(x) = \langle x, a\rangle$ the value of the character a at x, and we write $\hat{\eta}$ for the automorphism of $\hat{X}$ dual to an automorphism $\eta \in \mathrm{Aut}(X)$, where $\hat{\eta}(a) = a \cdot \eta$ for all $a \in \hat{X}$.

Let $d \geq 1$, and let $\mathfrak{R}_d = \mathbb{Z}\left[u_1^{\pm 1}, \dots, u_d^{\pm 1}\right]$ be the ring of Laurent polynomials with integral coefficients in the commuting variables $u_1, \dots, u_d$. An element $f \in \mathfrak{R}_d$ will be written as $f = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}}$ with $c_f(\mathbf{n}) \in \mathbb{Z}$, $\sum_{\mathbf{n}\in\mathbb{Z}^d} |c_f(\mathbf{n})| < \infty$, and $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$ for every $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, and we denote by $\mathcal{S}(f) = \{\mathbf{n} \in \mathbb{Z}^d : c_f(\mathbf{n}) \neq 0\}$ the *support* of f. If α is a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X, then the dual group $\mathfrak{M} = \hat{X}$ of X becomes an $\mathfrak{R}_d$module under the $\mathfrak{R}_d$-action defined by

$$f \cdot a = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n}) \beta_{\mathbf{n}}(a) \tag{3.1}$$

for all $a \in \mathfrak{M}$ and $f \in \mathfrak{R}_d$, where $\beta_{\mathbf{n}} = \widehat{\alpha_{\mathbf{n}}}$ is the automorphism of $\mathfrak{M} = \hat{X}$ dual to $\alpha_{\mathbf{n}}$. In particular,

$$\widehat{\alpha_{\mathbf{n}}}(a) = \beta_{\mathbf{n}}(a) = u^{\mathbf{n}} \cdot a \tag{3.2}$$

for all $\mathbf{n} \in \mathbb{Z}^d$ and $a \in \mathfrak{M}$. Conversely, if $\mathfrak{M}$ is an $\mathfrak{R}_d$-module, and if

$$\beta_{\mathbf{n}}^{\mathfrak{M}}(a) = u^{\mathbf{n}} \cdot a \tag{3.3}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in \mathfrak{M}$, then we obtain a $\mathbb{Z}^d$-action

$$\alpha^{\mathfrak{M}} : \mathbf{n} \mapsto \alpha_{\mathbf{n}}^{\mathfrak{M}} = \widehat{\beta_{\mathbf{n}}^{\mathfrak{M}}} \tag{3.4}$$

on the compact, abelian group

$$X^{\mathfrak{M}} = \widehat{\mathfrak{M}} \tag{3.5}$$

dual to the $\mathbb{Z}^d$-action $\beta^{\mathfrak{M}} : \mathbf{n} \mapsto \beta_{\mathbf{n}}^{\mathfrak{M}}$ on $\mathfrak{M}$.

If $\mathfrak{M}$ is an $\mathfrak{R}_d$-module, then a prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ is *associated with* $\mathfrak{M}$ if $\mathfrak{p} = \{f \in \mathfrak{R}_d : f \cdot a = 0\}$ for some $a \in \mathfrak{M}$, and $\mathfrak{M}$ *is associated with a prime ideal* $\mathfrak{p} \subset \mathfrak{R}_d$ if $\mathfrak{p}$ is the only prime ideal in $\mathfrak{R}_d$ associated with $\mathfrak{M}$. For every ideal $\mathcal{J} \subset \mathfrak{R}_d$ we set

$$V_{\mathbb{C}}(\mathcal{J}) = \{\mathbf{c} = (c_1, \dots , c_d) \in \mathbb{C}^d : f(\mathbf{c}) = 0 \text{ for every } f \in \mathcal{J}\}. \tag{3.6}$$

A nonzero Laurent polynomial $f \in \mathfrak{R}_d$ is a *generalized cyclotomic polynomial* if there exist $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$, $\mathbf{n} \neq \mathbf{0}$, and a cyclotomic polynomial c in a single variable such that $f = u^{\mathbf{m}} c(u^{\mathbf{n}})$. The following assertions were proved in [**S1**] and [**KiS2**], Theorem 3.3.

Theorem 3.1. *Let α be a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X, and let $\mathfrak{M} = \hat{X}$ be the $\mathfrak{R}_d$-module arising from α via* (3.1)–(3.2).

(1) *The following conditions are equivalent.*
   - (a) *α is expansive;*
   - (b) *The $\mathfrak{R}_d$-module $\mathfrak{M}$ is Noetherian, and $V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{S}^d = \emptyset$ for every prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ associated with $\mathfrak{M}$, where $\mathbb{S} = \{c \in \mathbb{C} : |c| = 1\}$.*

(2) *The following conditions are equivalent.*
   - (a) *α is mixing;*
   - (b) *$\alpha_{\mathbf{n}}$ is ergodic for every nonzero element $\mathbf{n} \in \mathbb{Z}^d$;*
   - (c) *$\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is mixing for every prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ associated with $\mathfrak{M}$;*
   - (d) *None of the prime ideals associated with $\mathfrak{M}$ contains a generalized cyclotomic polynomial.*

(3) *If $\mathfrak{M}$ is Noetherian, the following conditions are equivalent.*
   - (a) *α is ergodic;*
   - (b) *$\alpha_{\mathbf{n}}$ is ergodic for some element $\mathbf{n} \in \mathbb{Z}^d$;*
   - (c) *$\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is ergodic for every prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ associated with $\mathfrak{M}$.*

We identify $\mathbb{Z}$ with the set of constant polynomials in $\mathfrak{R}_d$ and note that, for every prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$, $\mathfrak{p} \cap \mathbb{Z}$ is either equal to $p\mathbb{Z}$ for a unique rational prime $p = p(\mathfrak{p})$, or to $\{0\}$, in which case we set $p(\mathfrak{p}) = 0$. The next result is taken from [**KiS1**, Theorem 5.2 and Proposition 3.12] and [**S1**, Theorem 3.3].

Theorem 3.2. *Let α be a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X. The following are equivalent.*
(1) *X is zero-dimensional and α is expansive;*
(2) *The $\mathfrak{R}_d$-module $\mathfrak{M}$ defined in (3.1)–(3.2) is Noetherian, and $p(\mathfrak{p}) > 0$ for every prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ associated with $\mathfrak{M}$;*
(3) *There exists a finite, abelian group F and a continuous, injective group homomorphism $\phi : X \longmapsto F^{\mathbb{Z}^d}$ such that*

$$\phi \cdot \alpha_{\mathbf{n}} = \sigma_{\mathbf{n}} \cdot \phi \tag{3.7}$$

for every $\mathbf{n} \in \mathbb{Z}^d$, where σ is the shift-action (2.8) on $F^{\mathbb{Z}^d}$. If α is expansive, then α is ergodic if and only if $\mathfrak{R}_d/\mathfrak{p}$ is infinite for every prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ associated with $\mathfrak{M}$.

Let $\mathfrak{M}$ be a Noetherian $\mathfrak{R}_d$-module. Corollary VI.4.8 in [**La**] implies the existence of $\mathfrak{R}_d$-modules

$$\{0\} = \mathfrak{N}_0 \subset \mathfrak{N}_1 \subset \cdots \subset \mathfrak{N}_s = \mathfrak{M} \tag{3.8}$$

such that, for every $j = 1, \dots, s$, $\mathfrak{N}_j/\mathfrak{N}_{j-1} \cong \mathfrak{R}_d/\mathfrak{q}_j$ for some prime ideal $\mathfrak{q}_j \subset \mathfrak{R}_d$ containing one of the prime ideals associated with $\mathfrak{M}$ (Corollary 2.2 in [**S1**]). The sequence $\mathfrak{N}_0 \subset \cdots \subset \mathfrak{N}_s$ in (3.8) is a *prime filtration* of $\mathfrak{M}$. The following lemma helps to overcome the problem that the successive quotients in a prime filtration of $\mathfrak{M}$ may involve prime ideals which are not associated with $\mathfrak{M}$.

Lemma 3.3. *Let $\mathfrak{M}$ be a Noetherian $\mathfrak{R}_d$-module with associated primes $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. Then there exists a Noetherian $\mathfrak{R}_d$-module $\mathfrak{N} = \mathfrak{N}^{(1)} \oplus \cdots \oplus \mathfrak{N}^{(m)}$ and an injective $\mathbb{R}$-module homomorphism $\phi : \mathfrak{M} \longmapsto \mathfrak{N}$ such that each of the modules $\mathfrak{N}^{(j)}$ has a prime filtration $\{0\} = \mathfrak{N}_0^{(j)} \subset \cdots \subset \mathfrak{N}_{r_j}^{(j)} = \mathfrak{N}^{(j)} / \mathfrak{N}_{k-1}^{(j)} \cong \mathfrak{R}_d/\mathfrak{p}_j$ for $k = 1, \dots, r_j$.*

If $X = X^{\mathfrak{M}}$ and $Y = X^{\mathfrak{N}} = X^{\mathfrak{N}^{(1)}} \times \cdots \times X^{\mathfrak{N}^{(m)}}$, then the homomorphism $\psi : Y \longmapsto X$ dual to ϕ is surjective and satisfies that $\psi \cdot \alpha_{\mathbf{n}}^{\mathfrak{N}} = \psi \cdot \left(\alpha_{\mathbf{n}}^{\mathfrak{N}^{(1)}} \times \cdot \times \alpha_{\mathbf{n}}^{\mathfrak{N}^{(m)}}\right) = \alpha_{\mathbf{n}}^{\mathfrak{M}} \cdot \phi$ for every $\mathbf{n} \in \mathbb{Z}^d$.

Proof. Theorem VI.5.3 in [**La**] allows us to choose submodules $\mathfrak{W}_1, \dots, \mathfrak{W}_m$ of $\mathfrak{M}$ such that $\mathfrak{M}/\mathfrak{W}_i$ is associated with $\mathfrak{p}_i$ for $i = 1, \dots, m$, $\bigcap_{i=1}^m \mathfrak{W}_i = \{0\}$, and $\bigcap_{i \in S} \mathfrak{W}_i \neq \{0\}$ for every subset $S \subsetneq \{1, \dots, m\}$. In particular, the map $\phi : a \mapsto (a + \mathfrak{W}_1, \dots, a + \mathfrak{W}_m)$ from $\mathfrak{M}$ into $\mathfrak{k} = \bigoplus_{i=1}^m \mathfrak{M}/\mathfrak{W}_i$ is injective. We fix $j \in \{1, \dots, m\}$ for the moment and apply Lemma 3.4 in [**KiS2**] to find a prime filtration $\{0\} = \mathfrak{N}_0 \subset \cdots \subset \mathfrak{N}_s = \mathfrak{M}/\mathfrak{W}_j$ such that $\mathfrak{N}_k^{(j)}/\mathfrak{N}_{k-1}^{(j)} \cong \mathfrak{R}_d/\mathfrak{q}_k^{(j)}$ for every $k = 1, \dots, s_j$, where $\mathfrak{q}_k^{(j)} \subset \mathfrak{R}_d$ is a prime ideal containing $\mathfrak{p}_j$, and where there exists an $r_j \in \{1, \dots, s_j\}$ such that $\mathfrak{q}_k^{(j)} = \mathfrak{p}_j$ for $k = 1, \dots, r_j$, and $\mathfrak{q}_k^{(j)} \supsetneq \mathfrak{p}_j$ for $k = r_j + 1, \dots, s_j$. If $r_j < s_j$ we choose Laurent polynomials $g_k^{(j)} \in \mathfrak{q}_k^{(j)} \setminus \mathfrak{p}_j^{(j)}$ for $k = r_j + 1, \dots, s_j$, set $g^{(j)} = g_{r_j+1}^{(j)} \cdots g_{s_j}^{(j)}$, and note that the map $\psi^{(j)} : \mathfrak{M}/\mathfrak{W}_j \longmapsto \mathfrak{N}_{r_j}^{(j)}$ consisting of multiplication by $g^{(j)}$ is injective. Since $\mathfrak{N}_{r_j}^{(j)}$ has the prime filtration $\{0\} = \mathfrak{N}_0^{(j)} \subset \cdots \subset \mathfrak{N}_{r_j}^{(j)}$ whose successive quotients are all isomorphic to $\mathfrak{R}_d/\mathfrak{p}_j$, the module $\mathfrak{N} = \mathfrak{N}_{r_1}^{(1)} \oplus \cdots \oplus \mathfrak{N}_{r_m}^{(m)}$ has the required properties. The last assertion follows from duality. □

Let $\mathfrak{M}$ be a Noetherian $\mathfrak{R}_d$-module, and let $\{0\} = \mathfrak{N}_0 \subset \cdots \subset \mathfrak{N}_s = \mathfrak{M}$ be a prime filtration of $\mathfrak{M}$ with $\mathfrak{N}_k/\mathfrak{N}_{k-1} \cong \mathfrak{R}_d/\mathfrak{q}_k$ for $k = 1, \dots, s$. We write $X_k = \mathfrak{N}_k^{\perp} = \{x \in X^{\mathfrak{M}} = \widehat{\mathfrak{M}} : \langle x, a \rangle = 1 \text{ for every } a \in \mathfrak{N}_k\}$ for the annihilator subgroup of $\mathfrak{N}_k$ in $X^{\mathfrak{M}}$. and obtained closed, $\alpha^{\mathfrak{M}}$-invariant subgroups

$$\{0\} = X_s \subset \cdots \subset X_0 = X^{\mathfrak{M}} \tag{3.9}$$

such that

$$X_{k-1}/X_k = \widehat{\mathfrak{N}_k/\mathfrak{N}_{k-1}} \cong \widehat{\mathfrak{R}_d/\mathfrak{q}_k} \tag{3.10}$$

for every $k = 1, \dots, s$. In view of Theorem 3.1 we conclude that every expansive $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X has the property that there exist closed, α-invariant subgroups $\{0\} = X_s \subset \cdots \subset X_0 = X$ such that, for every $k = 1, \dots, s$, the $\mathbb{Z}^d$-action α^{X_{k-1}/X_k} induced by α on X_{k-1}/X_k is of the form $\alpha^{\mathfrak{R}_d/\mathfrak{q}_k}$ for some prime ideal $\mathfrak{q}_k \subset \mathfrak{R}_d$ (cf. (3.3)–(3.5)). In order to complete the picture we recall the explicit description of $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ for a prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ given in [**S1**] and [**KiS2**]: if $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and if σ is the shift-action (2.8) of $\mathbb{Z}^d$ on $\widehat{\mathfrak{R}_d} = \mathbb{T}^{\mathbb{Z}^d}$, then

$$X^{\mathfrak{R}_d}_{/\mathfrak{p}} = \left\{ x = (x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^d} : f(\sigma)(x) = 0_{\mathbb{T}^{\mathbb{Z}^d}} \text{ for every } f \in \mathfrak{p} \right\}, \tag{3.11}$$

where

$$f(\sigma)(x) = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) \sigma_{\mathbf{n}}(x) \tag{3.12}$$

for every $f = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}} \in \mathfrak{R}_d$, and $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is the restriction of σ to $X^{\mathfrak{R}_d/\mathfrak{p}}$. In the special case where $p = p(\mathfrak{p}) > 0$ we set $F_p = \{k/p (\text{mod } 1) : 0 \le k < p\} \subset \mathbb{T}$ and observe that $X^{\mathfrak{R}_d/\mathfrak{p}} \subset F^{\mathbb{Z}^d} \subset \mathbb{T}^{\mathbb{Z}^d}$. The obvious isomorphism of F_p with $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ allows us to regard $X^{\mathfrak{R}_d/\mathfrak{p}}$ as a closed, shift-invariant subgroup of $\mathbb{F}_p^{\mathbb{Z}^d}$.

Our next task is to investigate in more detail the structure of $\mathbb{Z}^d$-actions of the form $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$, where $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal. Recall that an element $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ is *primitive* if $\gcd\{n_1, \dots, n_d\} = 1$; more generally, a subgroup $\Gamma \subset \mathbb{Z}^d$ is *primitive* if the group $\mathbb{Z}^d/\Gamma$ is torsion-free. If $p = p(\mathfrak{p}) > 0$, then the following proposition shows that there exists a maximal primitive subgroup $\Gamma \subset \mathbb{Z}^d$ and a finite, abelian group G such that the restriction α^Γ of α to Γ is topologically and algebraically conjugate to the shift-action of Γ on G^Γ.

Proposition 3.4. *Let $\mathfrak{p} \subsetneq \mathfrak{R}_d$ be a prime ideal with $p = p(\mathfrak{p}) > 0$ such that the $\mathbb{Z}^d$-action $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$ on $X = X^{\mathfrak{R}_d/\mathfrak{p}} = \widehat{\mathfrak{R}_d/\mathfrak{p}} \subset F_p^{\mathbb{Z}^d}$ is ergodic (cf.* (3.3)–(3.5), Theorem 3.1, *and* (3.11)). *Then there exist an integer $r \in \{1, \dots, d\}$, a primitive subgroup $\Gamma \subset \mathbb{Z}^d$, and a finite set $Q \subset \mathbb{Z}^d$ with the following properties.*

(1) $\Gamma \cong \mathbb{Z}^r$;

(2) $\mathbf{0} \in Q$, *and* $Q \cap (Q + \mathbf{m}) = \emptyset$ *whenever* $\mathbf{0} \neq \mathbf{m} \in \Gamma$;

(3) *If $\bar{\Gamma} = \Gamma + Q = \{\mathbf{m} + \mathbf{n} : \mathbf{m} \in \Gamma, \mathbf{n} \in Q\}$, then the coordinate projection $\pi_{\bar{\Gamma}} : X \longmapsto \mathbb{F}_p^{\bar{\Gamma}}$, which restricts any point $x \in X \subset \mathbb{F}_p^{\mathbb{Z}^d}$ to its coordinates in $\bar{\Gamma}$, is a continuous group isomorphism; in particular, the Γ-action $\alpha^\Gamma : \mathbf{n} \mapsto \alpha_{\mathbf{n}}$, $\mathbf{n} \in \Gamma$, is a Bernoulli action with finite alphabet $\mathbb{F}_p^Q$.*

Proof. This is Noether's normalization lemma in disguise. We write ${\mathfrak{R}_d}^{(p)} = \mathbb{F}_p\left[u_1^{\pm 1}, \dots, u_d^{\pm 1}\right]$ for the ring of Laurent polynomials in $u_1, \dots, u_d$ with coefficients in the prime field $\mathbb{F}_p$, and define a surjective homomorphism $f \mapsto f_{/p}$ from $\mathfrak{R}_d$ to ${\mathfrak{R}_d}^{(p)}$ by reducing every coefficient of a Laurent polynomial $f \in \mathfrak{R}_d$ modulo p. Then $\mathfrak{q} = \{f_{/p} : f \in \mathfrak{p}\}$ is a prime ideal in ${\mathfrak{R}_d}^{(p)}$, and $\mathfrak{R}_d/\mathfrak{p} \cong {\mathfrak{R}_d}^{(p)}/\mathfrak{q}$.

We write $\mathbf{e}^{(i)}$ for the i-th unit vector in $\mathbb{Z}^d$ and claim that there exists a matrix $A \in GL(d, \mathbb{Z})$ and an integer r, $1 \leq r \leq d$, such that the elements $v_i = u^{A\mathbf{e}^{(i)}} + \mathfrak{q}$ are algebraically independent in the ring $\mathcal{R} = {\mathfrak{R}_d}^{(p)}/\mathfrak{q}$ for $i = 1, \dots, r$, and both $v_j = u^{A\mathbf{e}^{(j)}} + \mathfrak{q}$ and $v_j^{-1} = u^{-A\mathbf{e}^{(j)}} + \mathfrak{q}$ are algebraic integers over the subring $\mathbb{F}_p\left[v_1^{\pm}, \dots, v_{j-1}^{\pm 1}\right] \subset \mathcal{R}$ for $j = r+1, \dots, d$. Indeed, if $u_1' = u_1 + \mathfrak{q}, \dots, u_d' = u_d + \mathfrak{q}$ are algebraically independent elements of $\mathcal{R}$, then $\mathfrak{q} = \{0\}$, and the assertion holds with $r = d$, and with A equal to the $d \times d$ identity matrix. Assume therefore (after renumbering the variables, if necessary) that there exists an irreducible Laurent polynomial $f \in \mathfrak{q}$ of the form $f = g_0 + g_1 u_d + \cdots g_l u_d^l$, where $g_i \in \mathbb{F}_p\left[u_1^{\pm 1}, \dots, u_{d-1}^{\pm 1}\right]$ and $g_0 g_l \neq 0$. If the supports of g_0 and g_l are both singletons, then u_d and u_d^{-1} are both integral over the subring $\mathbb{F}_p\left[{u_1'}^{\pm 1}, \dots {u_{d-1}'}^{\pm 1}\right] \subset \mathcal{R}$. If the support of either g_0 or g_l is not a singleton one can find integers $k_1, \dots, k_d$ such that the substitution of the variables $w_i = u_i u_d^{k_i}$, $i = 1, \dots, d-1$, in f leads to a Laurent polynomial $g(w_1, \dots, w_{d-1}, u_d) = u_d^{k_d} f(u_1, \dots, u_d)$ of the form $g = g_0' + g_1' u_d + \cdots g_{l'}' u_d^{l'}$, where $g_i' \in \mathbb{F}_p\left[w_1^{\pm 1}, \dots, w_{d-1}^{\pm 1}\right]$, and where the supports of g_0' and $g_{l'}'$ are both singletons. We set

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 & k_1 \\ 0 & 1 & \cdots & 0 & k_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & k_{d-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

$w_i' = w_i + \mathfrak{q} = u^{B\mathbf{e}^{(i)}} + \mathfrak{q}$, $i = 1, \dots, d-1$, and note that w_d' and ${w_d'}^{-1}$ are integral over $\mathbb{F}_p\left[{w_1'}^{\pm 1}, \dots, {w_{d-1}'}^{\pm 1}\right] \subset \mathcal{R}$. If the elements $w_1', \dots, w_{d-1}'$ are algebraically independent in $\mathcal{R}$, then our claim is proved; if not, then we can apply the same argument to $w_1, \dots, w_{d-1}$ instead of $u_1, \dots, u_d$, and iteration of this procedure leads to a matrix $A \in GL(d, \mathbb{Z})$ and an integer $r \geq 0$ such that the elements $v_j' = u^{A\mathbf{e}^{(j)}} + \mathfrak{q} \in \mathcal{R}$ satisfy that $v_1', \dots, v_r'$ are algebraically independent, and v_j' and ${v_j'}^{-1}$ are integral over $\mathcal{R}^{(j-1)} = \mathbb{F}_p\left[{v_1'}^{\pm 1}, \dots, {v_{j-1}'}^{\pm 1}\right] \subset \mathcal{R}$ for $j > r$, where $\mathcal{R}^{(0)} = \mathbb{F}_p$ if $r = 0$ (in which case $\mathcal{R}$ must be finite). From Theorem 3.2 it is clear that the ergodicity of α implies that $r \geq 1$, and this completes the proof of our claim.

For the remainder of this proof we assume for simplicity that A is the $d \times d$ identity matrix, so that $v_i = u_i$ for $i = 1, \dots, d$ (this is—in effect—equivalent to replacing α by the $\mathbb{Z}^d$-action $\alpha' : \mathbf{n} \mapsto \alpha_{\mathbf{n}}' = \alpha_{A\mathbf{n}}$). The argument in the preceding paragraph gives us, for each $j = r+1, \dots, d$, an irreducible polynomial $f_j(x) = \sum_{k=0}^{l_j} g_k^{(j)} x^k$ with coefficients in the ring $\mathbb{F}_p\left[{u_1'}^{\pm 1}, \dots {u_{j-1}'}^{\pm 1}\right] \subset \mathfrak{R}_d^{(p)}$ such that $h_j(u_j) = h_j(u_1, \dots, u_{j-1}, u_j) \in \mathfrak{q}$ and the supports of $g_0^{(j)}$ and $g_{l_j}^{(j)}$ are singletons. Let $\Gamma \subset \mathbb{Z}^d$ be the group generated by $\{\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(r)}\}$, $Q = \{0\} \times \cdots \times \{0\} \times \{0, \dots, l_{r+1} - 1\} \times \{0, \dots, l_d - 1\} \subset \mathbb{Z}^d$, and let $\bar{\Gamma} = \Gamma + Q = \{\mathbf{m} + \mathbf{n} : \mathbf{m} \in \Gamma, \mathbf{n} \in Q\}$. We write $\pi_{\bar{\Gamma}} : X \longmapsto \mathbb{F}_p^{\bar{\Gamma}}$ for the coordinate projection which restricts every $x \in X$ to its coordinates in $\bar{\Gamma}$ and note that $\pi_{\bar{\Gamma}} : X \longmapsto \mathbb{F}_p^{\bar{\Gamma}}$ is a continuous group isomorphism. In other words, the restriction of α to the group $\Gamma \cong \mathbb{Z}^r$ is a Bernoulli shift with alphabet $\mathbb{F}_p^Q$. □

Corollary 3.5. *Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal such that $p = p(\mathfrak{p}) > 0$ and $r = r(\mathfrak{p}) \geq 1$, and $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is ergodic. Choose a primitive subgroup $\Gamma \cong \mathbb{Z}^r$ in $\mathbb{Z}^d$ according to Proposition 3.4 and fix a primitive element $\mathbf{n} \in \Gamma$. Then $\alpha^{\mathfrak{R}^d/\mathfrak{p}}$ has weak $\mathbf{n}$-specification. Furthermore, if $\mathfrak{N}$ is an $\mathfrak{R}_d$-module with a prime decomposition $\{0\} = \mathfrak{N}_0 \subset \cdots \subset \mathfrak{N}_s = \mathfrak{N}$ such that $\mathfrak{N}_j/\mathfrak{N}_{j-1} \cong \mathfrak{R}_d/\mathfrak{p}$ for every $j = 1, \dots, s$, then $\alpha^{\mathfrak{N}}$ has weak $\mathbf{n}$-specification.*

Proof. We assume for simplicity that $\Gamma \subset \mathbb{Z}^d$ is the subgroup generated by the unit vectors $\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(r)}$, and that $\mathbf{n} = \mathbf{e}^{(1)}$. Choose a finite set $Q \subset \mathbb{Z}^d$ with the properties stated in Proposition 3.4, set $\bar{\Gamma} = \Gamma + Q$, and note that the projection $\pi_{\bar{\Gamma}} : X^{\mathfrak{R}_d/\mathfrak{p}} \longmapsto \mathbb{F}_p^{\bar{\Gamma}}$ is a continuous group isomorphism.

From Proposition 3.4 it is clear that there exist $\xi \in (0,1)$ and $t' \geq 0$, which depend on the supports of the polynomials f_j, $j = r+1, \dots, d$, in the proof of that proposition, and on the set $Q \subset \mathbb{Z}^d$, such that the following condition is satisfied.

(*) For every $x = (x_{\mathbf{m}}) \in X^{\mathfrak{R}_d/\mathfrak{p}} \subset \mathbb{F}_p^{\mathbb{Z}^d}$, let $y^{\pm} \in X$ be the unique points with

$$y^+_{\mathbf{k}} = \begin{cases} x_{\mathbf{k}} & \text{if } \mathbf{k} = (k_1, \dots, k_d) \in \bar{\Gamma} \text{ with } k_1 \geq -t', \\ 0_{\mathbb{F}_p} & \text{if } \mathbf{k} = (k_1, \dots, k_d) \in \bar{\Gamma} \text{ with } k_1 < -t', \end{cases}$$

$$y^-_{\mathbf{k}} = \begin{cases} x_{\mathbf{k}} & \text{if } \mathbf{k} = (k_1, \dots, k_d) \in \bar{\Gamma} \text{ with } k_1 \leq -t', \\ 0_{\mathbb{F}_p} & \text{if } \mathbf{k} = (k_1, \dots, k_d) \in \bar{\Gamma} \text{ with } k_1 > -t'. \end{cases}$$

Then $\pi_{C^+(\mathbf{e}^{(1)},\xi)}(x) = \pi_{C^+(\mathbf{e}^{(1)},\xi)}(y^+)$ and $\pi_{C^-(\mathbf{e}^{(1)},\xi)}(x) = \pi_{C^-(\mathbf{e}^{(1)},\xi)}(y^-)$.

The weak $(\mathbf{e}^{(1)}, \xi)$-specification of $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is an easy consequence of (*).

If $\mathfrak{N}$ is an arbitrary $\mathfrak{R}_d$-module satisfying the assumptions of this corollary, we choose elements $b_1, \dots, b_s \in \mathfrak{N}$ such that $\mathfrak{N}_j = \mathfrak{R}_d \cdot b_1 + \dots + \mathfrak{R}_d \cdot b_j$ for $j = 1, \dots, s$, write $\theta : X \longmapsto (\mathbb{T}^s)^{\mathbb{Z}^d} = \widehat{\mathfrak{R}_d^s}$ for the injective homomorphism dual to the surjective map $\hat{\theta} : (h_1, \dots, h_s) \mapsto h_1 \cdot b_1 + \dots + h_s \cdot b_s$ from $\mathfrak{R}_d^s = \mathfrak{R}_d \oplus \dots \oplus \mathfrak{R}_d$ to $\mathfrak{N} = \mathfrak{R}_d \cdot b_1 + \dots + \mathfrak{R}_d \cdot b_s$, and identify X with $\theta(x) \subset (\mathbb{T}^s)^{\mathbb{Z}^d}$. Under this identification the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{N}}$ becomes the restriction to X of the shift-action σ of $\mathbb{Z}^d$ on $(\mathbb{T}^s)^{\mathbb{Z}^d}$.

We set $F_m = \{k/m : 0 \leq k < m\} \subset \mathbb{T}$, $m \geq 2$, and claim the following.

(1) The restriction to X of the projection map $\pi_{\bar{\Gamma}} : (\mathbb{T}^s)^{\mathbb{Z}^d} \longmapsto (\mathbb{T}^s)^{\bar{\Gamma}}$ is injective.

(2) There exist $\xi \in (0,1)$ and $t' \geq 0$ such that the following conditions are satisfied

For every $x = (x_{\mathbf{m}}) \in X \subset (\mathbb{T}^s)^{\mathbb{Z}^d}$, let $y^{\pm} \in X$ be the unique points with

$$y^+_{\mathbf{k}} = \begin{cases} x_{\mathbf{k}} & \text{if } \mathbf{k} = (k_1, \dots, k_d) \in \bar{\Gamma} \text{ with } k_1 \geq -t', \\ 0_{(\mathbb{T}^s)^Q} & \text{if } \mathbf{k} = (k_1, \dots, k_d) \in \bar{\Gamma} \text{ with } k_1 < -t', \end{cases}$$

$$y^-_{\mathbf{k}} = \begin{cases} x_{\mathbf{k}} & \text{if } \mathbf{k} = (k_1, \dots, k_d) \in \bar{\Gamma} \text{ with } k_1 \leq -t', \\ 0_{(\mathbb{T}^s)^Q} & \text{if } \mathbf{k} = (k_1, \dots, k_d) \in \bar{\Gamma} \text{ with } k_1 > -t'. \end{cases}$$

Then $\pi_{C^+(\mathbf{e}^{(1)},\xi)}(x) = \pi_{C^+(\mathbf{e}^{(1)},\xi)}(y^+)$ and $\pi_{C^-(\mathbf{e}^{(1)},\xi)}(x) = \pi_{C^-(\mathbf{e}^{(1)},\xi)}(y^-)$.

(3) There exists a continuous group isomorphism $\zeta : \pi_{\bar{\Gamma}}(X) \longmapsto \left(F_p^{s|Q|}\right)^{\mathbb{Z}^r}$ which intertwines the shift-actions of $\Gamma = \mathbb{Z}^r$ on $\pi_{\bar{\Gamma}}(X)$ and $\left(F_p^{s|Q|}\right)^{\mathbb{Z}^r}$.

In order to prove the first assertion we identify $(\mathbb{T}^s)^{\mathbb{Z}^d}$ with $\left(\mathbb{T}^{\mathbb{Z}^d}\right)^s$ in the obvious manner, write each $x \in X$ as $x = (x^{(1)}, \dots, x^{(s)})$ with $x^{(j)} \in \mathbb{T}\mathbb{Z}^d$, and denote by $\pi^{(j)} : x \mapsto x^{(j)}$ the j-th coordinate projection. Put $X_j = \mathfrak{N}_j^{\perp} \subset X$ and observe that $X_j = \{x = (x^{(1)}, \dots, x^{(s)}) \in X : x^{(1)} = \dots = x^{(j)} = 0\}$ and $\pi^{(j)}(X_{j-1}) \cong X_{j-1}/X_j$ for $j = 1, \dots, s$. Proposition 3.4 implies that any $x = (x^{(1)}, \dots, x^{(s)}) \in X$ with $\pi_{\bar{\Gamma}}(x) = 0$ must lie in X_1. Furthermore, since $\pi^{(2)}(X_1) \cong X_1/X_2 \cong \widehat{\mathfrak{N}_2/\mathfrak{N}_1} \cong \widehat{\mathfrak{R}_d/\mathfrak{p}}$, we know that $x^{(2)} \in X^{\mathfrak{R}_d/\mathfrak{p}} \subset \mathbb{T}^{\mathbb{Z}^d}$, and Proposition 3.4 implies that $x^{(2)} = 0$. By repeating this argument we eventually obtain that $x = 0$, i.e. that $\pi_{\bar{\Gamma}}$ is injective on X, as claimed in (1).

If we choose $\xi \in (0,1)$ as in (*), then (2) is an immediate consequence of the proof of (1).

For the third assertion we set $\mathfrak{R}_r = \mathbb{Z}[u_1^{\pm 1}, \dots, u_r^{\pm 1}] \subset \mathfrak{R}_d$, view $\mathfrak{N}$ as a Noetherian $\mathfrak{R}_r$-module $\mathfrak{N}'$, and observe that $\mathfrak{N}'$ is associated with the prime ideal $(p) = p\mathfrak{R}_r$. Fix an enumeration $Q = \{\mathbf{0} = \mathbf{m}^{(1)}, \dots \mathbf{m}^{(|Q|)}\}$ of Q, where $|Q|$ is the cardinality of Q, set $a_{(j-1)|Q|+k} = u^{\mathbf{m}^{(k)}} b_j$ for $j = 1, \dots, s$ and $k = 1, \dots, |Q|$, and put $\mathfrak{N}'_0 = \{0\}$ and $\mathfrak{N}'_l = \mathfrak{R}_r \cdot a_1 + \dots + \mathfrak{R}_r \cdot a_l$ for $l = 1, \dots, s|Q|$. The prime filtration $\{0\} = \mathfrak{N}'_0 \subset \dots \subset \mathfrak{N}'_{s|Q|} = \mathfrak{N}'$ satisfies that $\mathfrak{N}'_j/\mathfrak{N}_{j-1} \cong \mathfrak{R}_r/(p)$ for $j = 1, \dots, s|Q|$, and $\mathfrak{N}'_{j|Q|}$ is equal to $\mathfrak{N}_j$ (regarded as an R_r-module) for every $j = 0, \dots, s$. Write $Y_j = \mathfrak{N}_j'^{\perp}$ for the annihilator of $\mathfrak{N}'_j$ in $X = X^{\mathfrak{N}'} = \widehat{\mathfrak{N}'}$ for $j = 0, \dots, s|Q|$, and note that each Y_j is invariant under the $\mathbb{Z}^r$-action $\alpha' = \alpha^{\mathfrak{N}'}$, $X = Y_0 \supset \dots \supset Y_{s|Q|} = \{0\}$, and $Y_{j-1}/Y_j \cong X^{\mathfrak{R}_r/(p)}$ for $j = 1, \dots, s|Q|$. The dual of the surjective homomorphism $\hat{\omega} : \mathfrak{R}_r^{s|Q|} \longmapsto \mathfrak{N}'$ defined by $\hat{\omega}(f_1, \dots, f_{s|Q|}) = f_1 \cdot a_1 + \dots + f_{s|Q|} \cdot a_{s|Q|}$ is an injective homomorphism $\omega : X \longmapsto (\mathbb{T}^{s|Q|})^{\mathbb{Z}^r}$, and we identify X with $\omega(X) \subset (\mathbb{T}^{s|Q|})^{\mathbb{Z}^r}$ and α' with the restriction to X of the shift-action σ of $\mathbb{Z}^r$ on $(\mathbb{T}^{s|Q|})^{\mathbb{Z}^r}$. Each $x \in X$ can be written as $x = (x^{(1)}, \dots, x^{(s|Q|)})$ with $\eta^{(j)}(x) = x^{(j)} \in F_{p^j}^{\mathbb{Z}^r} \subset \mathbb{T}^{\mathbb{Z}^r}$, and $Y_j = \{x = (x^{(1)}, \dots, x^{(s|Q|)}) \in X : x^{(1)} = \dots = x^{(j)} = 0\}$ for every j. Note that $Y^{(j-1)}/Y^{(j)} \cong \eta^{(j)}(Y_{j-1}) = \widehat{\mathfrak{R}_r/(p)} \subset \mathbb{T}^{\mathbb{Z}^r}$ for $j = 1, \dots, s|Q|$, so that $x_{\mathbf{m}}^{(j)} \in F_p$ for every $j = 1, \dots, s|Q| - 1$, $x \in Y_{j-1}$, and $\mathbf{m} \in \mathbb{Z}^r$.

Put $F = F_{p^{s|Q|}}$, $F' = F/F_p$, write $\phi : F \longmapsto F'$ for the quotient map, choose a map $\psi : F' \longmapsto F$ such that $\phi \cdot \psi$ is the identity map on F', and define maps $\boldsymbol{\phi} : F^{\mathbb{Z}^r} \longmapsto F'^{\mathbb{Z}^r}$ and $\boldsymbol{\psi} : F'^{\mathbb{Z}^r} \longmapsto F^{\mathbb{Z}^r}$ by setting $(\boldsymbol{\phi}(u))_{\mathbf{m}} = \phi(u_{\mathbf{m}})$ and $(\boldsymbol{\psi}(v))_{\mathbf{m}} = \psi(v_{\mathbf{m}})$ for every $\mathbf{m} \in \mathbb{Z}^r$, $u = (u_{\mathbf{m}}) \in F^{\mathbb{Z}^r}$, and $u = (v_{\mathbf{m}}) \in F'^{\mathbb{Z}^r}$. The map

$$x = \left(x^{(1)}, \dots, x^{(s|Q|)}\right) \mapsto \left(\left(x^{(1)}, \dots, x^{(s|Q|-1)}\right), x^{(s|Q|)} - \boldsymbol{\psi} \cdot \boldsymbol{\phi}(x^{s|Q|})\right)$$

is a shift-commuting homeomorphism from X onto $X/X_{s|Q|-1} \times F_p^{\mathbb{Z}^r}$. Since

$X/X_{s|Q|-1} \subset \left(\mathbb{T}^{r|Q|-q}\right)^{\mathbb{Z}^r}$ is the dual group of $\mathfrak{N}_{s|Q|-1}$, we can repeat this argument and construct inductively a shift-commuting homeomorphism ζ' : $X \longmapsto \left(F_p^{\mathbb{Z}^r}\right)^{s|Q|}$. The proof of (3) is completed by setting $\zeta = \zeta' \cdot \pi_{\bar{\Gamma}}^{-1}$: $\pi_{\bar{\Gamma}}(X) \longmapsto \left(F_p^{\mathbb{Z}^r}\right)^{s|Q|} = \left(F_p^{s|Q|}\right)\mathbb{Z}^r$.

In order to prove the weak $\mathbf{e}^{(1)}$-specification of $\alpha^{\mathfrak{N}}$ we go back to the description of X as a closed, shift-invariant subgroup of $(\mathbb{T}^s)^{\mathbb{Z}^d}$ used for the proof of (1) and (2). From the proof of (3) we know that $p^{s|Q|}x = 0_X$ for every $x \in X$. In particular, $\pi_{\bar{\Gamma}}(X) \subset (F_{p^{s|Q|}})^{\bar{\Gamma}} = \left(F_{p^{s|Q|}}^{s|Q|}\right)^{\mathbb{Z}^r}$, and the shift-commuting homeomorphism $\zeta : \pi_{\bar{\Gamma}}(X) \longmapsto \left(F_p^{s|Q|}\right)^{\mathbb{Z}^r}$ is a *block-map*, which means that there exists a finite set $Q' \subset \mathbb{Z}^r$ such that, for every $v \in \pi_{\bar{\Gamma}}(X)$ and $\mathbf{k} \in \mathbb{Z}^r$, the coordinate $\zeta'(v)_{\mathbf{k}}$ of $\zeta'(v)$ is completely determined by the coordinates of v in the set $\mathbf{k} + Q'$. If $\xi \in (0,1)$ is the constant chosen in (2), and if $\zeta' = \zeta \cdot \pi_{\bar{\Gamma}}$, then (2) implies that there exists a constant $t'' \geq 0$ such that the following condition is satisfied.

For every $x = (x_{\mathbf{m}}) \in X \subset (\mathbb{T}^s)^{\mathbb{Z}^d}$, let $y^{\pm} \in X$ be the unique points with

$$(**)\qquad \begin{aligned} \zeta'(y^+)_{\mathbf{k}} &= \begin{cases} \zeta'(x)_{\mathbf{k}} & \text{if } \mathbf{k} = (k_1, \dots, k_d) \in \bar{\Gamma} \text{ with } k_1 \geq -t', \\ 0_{(\mathbb{T}^s)^Q} & \text{if } \mathbf{k} = (k_1, \dots, k_d) \in \bar{\Gamma} \text{ with } k_1 < -t', \end{cases} \\ \zeta'(y^-)_{\mathbf{k}} &= \begin{cases} \zeta'(x)_{\mathbf{k}} & \text{if } \mathbf{k} = (k_1, \dots, k_d) \in \bar{\Gamma} \text{ with } k_1 \leq -t', \\ 0_{(\mathbb{T}^s)^Q} & \text{if } \mathbf{k} = (k_1, \dots, k_d) \in \bar{\Gamma} \text{ with } k_1 > -t'. \end{cases} \end{aligned}$$

Then $\pi_{C^+(\mathbf{e}^{(1)},\xi)}(x) = \pi_{C^+(\mathbf{e}^{(1)},\xi)}(y^+)$ and $\pi_{C^-(\mathbf{e}^{(1)},\xi)}(x) = \pi_{C^-(\mathbf{e}^{(1)},\xi)}(y^-)$.

The weak $(\mathbf{e}^{(1)}, \xi)$-specification of $\alpha^{\mathfrak{N}}$ is again a straightforward consequence of (**). □

If the prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ satisfies that $p(\mathfrak{p}) = 0$, then the analysis of the action $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ becomes slightly more complicated. We write $\mathbb{K} = \hat{\mathbb{Q}}$ for the dual group of $\mathbb{Q}$ and denote by $\kappa : \mathbb{K} \longmapsto \mathbb{T}$ the surjective group homomorphism dual to the inclusion $\hat{\kappa} : \mathbb{Z} \longmapsto \mathbb{Q}$. If $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal with $p(\mathfrak{p}) = 0$ we regard $X^{\mathfrak{R}_d/\mathfrak{p}}$ as the subgroup (3.11) of $\mathbb{T}^{\mathbb{Z}^d}$, and define a closed, shift-invariant subgroup $\bar{X}^{\mathfrak{R}_d/\mathfrak{p}} \subset \mathbb{K}^{\mathbb{Z}^d}$ by

$$(3.13)\quad \bar{X}^{\mathfrak{R}_d/\mathfrak{p}} = \left\{x = (x_{\mathbf{n}}) \in \mathbb{K}^{\mathbb{Z}^d} : f(\sigma)(x) = 0_{\mathbb{K}^{\mathbb{Z}^d}} \text{ for every } f \in \mathfrak{p}\right\},$$

where σ is the shift-action (2.8) on $\mathbb{K}^{\mathbb{Z}^d}$ and $f(\sigma)$ is again defined by (3.12). The restriction of σ to $\bar{X}^{\mathfrak{R}_d/\mathfrak{p}}$ will be denoted by $\bar{\alpha}^{\mathfrak{R}_d/\mathfrak{p}}$. Define a continuous, surjective homomorphism $\boldsymbol{\kappa} : \mathbb{K}^{\mathbb{Z}^d} \longmapsto \mathbb{T}^{\mathbb{Z}^d}$ by $(\boldsymbol{\kappa}(x))_{\mathbf{n}} = \kappa(x_{\mathbf{n}})$ for every $x = (x_{\mathbf{m}}) \in \mathbb{K}^{\mathbb{Z}^d}$ and $\mathbf{n} \in \mathbb{Z}^d$, and write

$$\boldsymbol{\kappa}^{\mathfrak{R}_d/\mathfrak{p}} : \bar{X}^{\mathfrak{R}_d/\mathfrak{p}} \longmapsto X^{\mathfrak{R}_d/\mathfrak{p}} \tag{3.14}$$

for the restriction of $\boldsymbol{\kappa}$ to $\bar{X}^{\mathfrak{R}_d/\mathfrak{p}}$.

In order to obtain an algebraic description of $\bar{X}^{\mathfrak{R}_d/\mathfrak{p}}$, $\bar{\alpha}^{\mathfrak{R}_d/\mathfrak{p}}$, and $\boldsymbol{\kappa}^{\mathfrak{R}_d/\mathfrak{p}}$, and to define the corresponding objects $\bar{X}^{\mathfrak{N}}$, $\bar{\alpha}^{\mathfrak{N}}$, and $\boldsymbol{\kappa}^{\mathfrak{N}}$ for an arbitrary Noetherian $\mathfrak{R}_d$-module associated with $\mathfrak{p}$ we consider the ring

$$\overline{\mathfrak{R}}_d = \mathbb{Q}\left[u_1^{\pm 1}, \dots , u_d^{\pm 1}\right] = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{R}_d$$

of Laurent polynomials with rational coefficients in $u_1, \dots , u_d$, regard $\mathfrak{R}_d$ as the subring of $\overline{\mathfrak{R}}_d$ consisting of all polynomials with integral coefficients, and denote by $\bar{\mathfrak{p}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{p} \subset \bar{\mathfrak{R}}_d$ the prime ideal in $\bar{\mathfrak{R}}_d$ corresponding to $\mathfrak{p}$. Since $p(\mathfrak{p}) = 0$, every $\mathfrak{R}_d$-module $\mathfrak{N}$ associated with $\mathfrak{p}$ is embedded injectively in the $\overline{\mathfrak{R}}_d$-module $\overline{\mathfrak{N}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{N}$ by

$$\hat{\imath}^{\mathfrak{N}} : a \mapsto 1 \otimes_{\mathbb{Z}} a \tag{3.15}$$

for every $a \in \mathfrak{N}$, and $\overline{\mathfrak{N}}$ associated with $\bar{\mathfrak{p}}$. Since $\mathfrak{R}_d \subset \overline{\mathfrak{R}}_d$, $\overline{\mathfrak{N}}$ is an $\mathfrak{R}_d$-module, and we can define the $\mathbb{Z}^d$-action $\alpha^{\overline{\mathfrak{N}}}$ on $X^{\overline{\mathfrak{N}}}$ as in (3.1)–(3.2). Note that the set of prime ideals associated with the $\mathfrak{R}_d$-module $\overline{\mathfrak{N}}$ is the same as that of $\mathfrak{N}$; in particular, $\alpha^{\overline{\mathfrak{N}}}$ is ergodic if and only if $\alpha^{\mathfrak{N}}$ is ergodic and, for every $\propto\in \mathbb{Z}^d$, $\alpha_{\mathbf{n}}^{\overline{\mathfrak{N}}}$ is ergodic if and only if $\alpha_{\mathbf{n}}^{\mathfrak{N}}$ is ergodic. The homomorphism

$$\imath^{\mathfrak{N}} : X^{\overline{\mathfrak{N}}} \longmapsto X^{\mathfrak{N}} \tag{3.16}$$

dual to

$$\hat{\imath} : \mathfrak{N} \longmapsto \overline{\mathfrak{N}} \tag{3.17}$$

is surjective, and for $\mathfrak{N} = \mathfrak{R}_d/\mathfrak{p}$ we obtain that

$$\begin{aligned} X^{\overline{\mathfrak{R}_d/\mathfrak{p}}} &= \bar{X}^{\mathfrak{R}_d/\mathfrak{p}}, \\ \alpha^{\overline{\mathfrak{R}_d/\mathfrak{p}}} &= \bar{\alpha}^{\mathfrak{R}_d/\mathfrak{p}}, \\ \imath^{\mathfrak{R}_d/\mathfrak{p}} &= \boldsymbol{\kappa}^{\mathfrak{R}_d/\mathfrak{p}}. \end{aligned} \tag{3.18}$$

Proposition 3.6. *Let $\mathfrak{p} \subsetneq \mathfrak{R}_d$ be a prime ideal with $p = p(\mathfrak{p}) = 0$ such that the $\mathbb{Z}^d$-action $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$ on $X = X^{\mathfrak{R}_d/\mathfrak{p}} = \widehat{\mathfrak{R}_d/\mathfrak{p}} \subset \mathbb{F}_p^{\mathbb{Z}^d}$ is ergodic. Then there exists an integer $r \in \{0, \dots , d\}$ with the following properties.*

(1) *If $r = 0$ then X has finite topological dimension, i.e. X is a finite-dimensional torus or solenoid.*

(2) *If $r \geq 1$ there exist a primitive subgroup $\Gamma \subset \mathbb{Z}^d$ and a finite set $Q \subset \mathbb{Z}^d$ such that*
   (i) $\Gamma \cong \mathbb{Z}^d$,
   (ii) $\mathbf{0} \in Q$, *and* $Q \cap (Q + \mathbf{m}) = \emptyset$ *whenever* $\mathbf{0} \neq \mathbf{m} \in \Gamma$,
   (iii) *If* $\bar{\Gamma} = \Gamma + Q = \{\mathbf{m} + \mathbf{n} : \mathbf{m} \in \Gamma, \mathbf{n} \in Q\}$, *then the coordinate projection* $\pi_{\bar{\Gamma}} : \bar{X}^{\mathfrak{R}_d/\mathfrak{p}} \longmapsto \mathbb{K}^{\bar{\Gamma}}$, *which restricts any point* $x \in X^{\mathfrak{R}_d/\mathfrak{p}} \subset \mathbb{K}^{\mathbb{Z}^d}$ *to its coordinates in* $\bar{\Gamma}$, *is a continuous group isomorphism; in particular, the* Γ*-action* $\mathbf{n} \mapsto \bar{\alpha}_{\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{p}}$, $\mathbf{n} \in \Gamma$, *is a Bernoulli action with alphabet* $\mathbb{K}^Q$.

Proof. The proof is completely analogous to that of Proposition 3.4. We find a matrix $A \in GL(d, \mathbb{Z})$ and an integer $r \in \{0, \dots, d\}$ with the following properties: if $v_j = u^{A\mathbf{e}^{(j)}}$ and $v_j' = v_j + \mathfrak{p}$ for $j = 1, \dots, d$, then $v_1', \dots, v_r'$ are algebraically independent elements of $\mathcal{R} = \mathfrak{R}_d/\mathfrak{p}$, and there exists, for each $j = r+1, \dots, d$, an irreducible polynomial $f_j(x) = \sum_{k=0}^{l_j} g_k^{(j)} x^k$ with coefficients in the ring $\mathbb{Z}\left[v_1^{\pm 1}, \dots, v_{j-1}^{\pm 1}\right] \subset \mathfrak{R}_d$ such that $f_j(v_1, \dots, v_{j-1}, v_j) \in \mathfrak{q}$ and the supports of $g_0^{(j)}$ and $g_{l_j}^{(j)}$ are singletons.

If $r = 0$, then $V_{\mathbb{C}}(\mathfrak{p})$ is the orbit of a single point $c \in \mathbb{C}^d$ under the Galois group, and X is a finite-dimensional torus or solenoid ([**S1**], Section 5). If $r > 0$ we assume again that A is the $d \times d$ identity matrix, so that $v_j = u_j$ for $j = 1, \dots, d$ and $\Gamma \cong \mathbb{Z}^r$ is generated by $\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(r)}$, set $Q = \{0\} \times \cdots \times \{0\} \times \{0, \dots, l_{r+1} - 1\} \times \cdots \times \{0, \dots, l_d - 1\} \subset \mathbb{Z}^d$, and complete the proof in the same way as that of Proposition 3.4, using (3.13) instead of (3.11). □

Corollary 3.7. *Let* $\mathfrak{p} \subset \mathfrak{R}_d$ *be a prime ideal such that* $p(\mathfrak{p}) = 0$, $r(\mathfrak{p}) \geq 1$, *and* $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ *is ergodic, choose a primitive subgroup* $\Gamma \cong \mathbb{Z}^r$ *in* $\mathbb{Z}^d$ *according to* Proposition 3.6, *and fix a primitive element* $\mathbf{n} \in \Gamma$. *Then* $\hat{\alpha}^{\mathfrak{R}_d/\mathfrak{p}}$ *has weak* $\mathbf{n}$*-specification. More generally, if* $\mathfrak{N}$ *is an* $\mathfrak{R}_d$*-module with a prime filtration* $\{0\} = \mathfrak{N}_0 \subset \cdots \subset \mathfrak{N}_2 = \mathfrak{N}$ *such that* $\mathfrak{N}_j/\mathfrak{N}_{j-1} \cong \mathfrak{R}_d/\mathfrak{p}$ *for every* $j = 1, \dots, s$, *then* $\alpha^{\overline{\mathfrak{N}}}$ *has weak* $\mathbf{n}$*-specification. Finally, if* $Fix(\alpha_{k\mathbf{n}}^{\mathfrak{N}}) = \left\{x \in X^{\overline{\mathfrak{N}}} : \alpha_{k\mathbf{n}}^{\overline{\mathfrak{N}}}(x) = x\right\}$ *for every* $k \geq 1$, *then* $\bigcap_{k \geq 1} Fix(\alpha_{k\mathbf{n}}^{\mathfrak{N}})$ *is dense in* $X^{\overline{\mathfrak{n}}}$.

Proof. Put $\overline{\mathfrak{N}}_j = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{N}_j$ and note that $\overline{\mathfrak{N}}_j/\overline{\mathfrak{N}}_{j-1} \cong \overline{\mathfrak{R}_d/\mathfrak{p}} \cong \overline{\mathfrak{R}}_d/\bar{\mathfrak{p}}$ for every $j = 1, \dots, s$. We write $\bar{X}_j = \overline{\mathfrak{N}}_j^{\perp}$ for the annihilator of $\overline{\mathfrak{N}}_j$ in $\bar{X} = X^{\overline{\mathfrak{N}}}$ for $j = 1, \dots, s$. Each $\bar{X}_j$ is invariant under the $\mathbb{Z}^d$-action $\bar{\alpha} = \alpha^{\overline{\mathfrak{N}}}$, $\bar{X} = \bar{X}_0 \supset \cdots \supset \bar{X}_s = \{0\}$, and $\bar{X}_{j-1}/\bar{X}_j \cong X^{\overline{\mathfrak{R}_d/\mathfrak{p}}}$ for $j = 1, \dots, s$. Choose elements $a_1, \dots a_s$ in $\mathfrak{N}$ such that $\mathfrak{N}_j = \mathfrak{N}_{j-1} + \mathfrak{R}_r \cdot a_j$ and hence $\overline{\mathfrak{N}}_j = \overline{\mathfrak{N}}_{j-1} + \overline{\mathfrak{R}}_r \cdot \imath(a_j)$ for $j = 1, \dots, s$. The dual of the surjective homomorphism $\hat{\theta} : \overline{\mathfrak{R}}_d^s \longmapsto \overline{\mathfrak{N}}$ defined by $\theta(f_1, \dots, f_s) = f_1 \cdot a_1 + \cdot + f_s \cdot a_s$ is an injective homomorphism $\theta : \bar{X} \longmapsto (\mathbb{K}^s)^{\mathbb{Z}^d}$, and we identify $\bar{X}$ with $\theta(\bar{X}) \subset (\mathbb{K}^s)^{\mathbb{Z}^d}$ and $\bar{\alpha}$ with the restriction to $\bar{X}$ of the shift-action σ of $\mathbb{Z}^d$ on $(\mathbb{K}^s)^{\mathbb{Z}^r}$. Every element $x \in \bar{X}$ will be written as $x = (x^{(1)}, \dots, x^{(s)})$ with $x^{(j)} = \pi^{(j)}(x) \in \mathbb{K}^{\mathbb{Z}^d}$ for $j = 1, \dots, s$.

We choose a set $Q \subset \mathbb{Z}^d$ according to Proposition 3.6, set $\bar{\Gamma} = \Gamma + Q$, and claim that the coordinate projection $\pi_{\bar{\Gamma}} : \bar{X} \longmapsto (\mathbb{K}^s)^{\bar{\Gamma}}$, which sends any $x \in \bar{X} \subset (\mathbb{K}^s)^{\mathbb{Z}^d}$ to its coordinates in $\bar{\Gamma}$, is a continuous group isomorphism. Since $\pi^{(1)}(\bar{X}) = X^{\overline{\mathfrak{R}_d/\mathfrak{p}}}$, Proposition 3.6 shows that the coordinate projection $\pi_{\bar{\Gamma}} : \pi^{(1)}(\bar{X}) \longmapsto \mathbb{K}^{\bar{\Gamma}}$ is a continuous group isomorphism. Furthermore, since $\pi^{(2)}(\bar{X}_1) = X^{\overline{\mathfrak{R}_d/\mathfrak{p}}}$ and hence $\pi_{\bar{\Gamma}} \cdot \pi^{(2)}(\bar{X}_1) \longmapsto \mathbb{K}^{\bar{\Gamma}}$, and since each element in $\pi^{(2)}(\bar{X}_1)$ is completely determined by its coordinates in $\bar{\Gamma}$, we can prescribe, for any $x \in \bar{X}$, the coordinates of $x^{(1)}$ and $x^{(2)}$ in $\bar{\Gamma}$ arbitrarily and thereby specify x uniquely up to an element in $\bar{X}_2$. By applying this argument s times we obtain that $\pi_{\bar{\Gamma}} : \bar{X} \longmapsto (\mathbb{K}^s)^{\bar{\Gamma}}$ is indeed a continuous group isomorphism. In particular, if $\mathbf{n} \in \Gamma$ is a primitive element, then $\bar{\alpha}_{\mathbf{n}}$ is the shift on $Z^{\mathbb{Z}}$, where Z is a compact group isomorphic to $\mathbb{K}^{\mathbb{Z}^{r-1}}$, and the set of $\bar{\alpha}_{\mathbf{n}}$-periodic points is dense in $\bar{X}$.

The weak $\mathbf{n}$-specification of $\bar{\alpha}^{\mathfrak{R}_d/\mathfrak{p}}$ and $\bar{\alpha}$ is proved exactly as in Corollary 3.5. □

Remarks 3.8. (1) The integer $r = r(\mathfrak{p})$ appearing in the Propositions 3.4 and 3.6 will be called the *free rank* of the $\mathbb{Z}^d$-action $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$ on X. If X is zero-dimensional, then the proof of Proposition 3.4 shows that the free rank of α is equal to the smallest integer $r \geq 1$ such that $\mathbb{Z}^d$ has a primitive subgroup $\Gamma \cong \mathbb{Z}^r$ for which the restriction of α to Γ is expansive, and to the smallest integer $r \geq 1$ such that there exists a primitive subgroup $\Gamma \cong \mathbb{Z}^r$ of $\mathbb{Z}^d$ such that α^Γ has finite entropy.

If X is connected the restriction of α to the group $\Gamma \subset \mathbb{Z}^d$ obtained in Proposition 3.6 has infinite entropy; however, $\alpha^{\Gamma'}$ has finite entropy if $\Gamma' \subset \mathbb{Z}^d$ is a subgroup which is isomorphic to $\mathbb{Z}^{r'}$ for any $r' > r(\mathfrak{p})$.

(2) Even if the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{N}}$ in Proposition 3.6 is expansive, the action $\alpha^{\overline{\mathfrak{N}}}$ is nonexpansive. By proving a more intricate version of Proposition 3.6 one can analyze the structure of the group $X^{\mathfrak{N}}$ directly, without passing to $X^{\overline{\mathfrak{N}}}$. However, the weak $\mathbf{n}$-specification of $\alpha^{\mathfrak{N}}$ is a little more difficult to see than that of $\alpha^{\overline{\mathfrak{N}}}$, which explains the apparent detour of Proposition 3.6 and Corollary 3.7. For $\mathfrak{N} = \mathfrak{R}_d/\mathfrak{p}$ one can show that the projection $\pi_{\bar{\Gamma}} : X^{\mathfrak{R}_d/\mathfrak{p}} \longmapsto \mathbb{T}^{\bar{\Gamma}}$ is still surjective, but no longer injective, and that the kernel of $\pi_{\bar{\Gamma}}$ is of the form $Y^{\bar{\Gamma}}$ for some compact, zero-dimensional group Y (cf. Example 3.9 (2)).

(3) There is considerable freedom in the choice of the subgroup Γ in Proposition 3.4 and 3.6, which will be exploited later in the proof of Theorem 2.1 (2).

Examples 3.9 (1) Let $\mathfrak{p} = (2, 1+u_1+u_2) = 2\mathfrak{R}_2 + (1+u_1+u_2)\mathfrak{R}_2 \subset \mathfrak{R}_2$. Then $p(\mathfrak{p}) = 2$, $r(\mathfrak{p}) = 1$, and we may set $\Gamma = \{(k,k) : k \in \mathbb{Z}\} \cong \mathbb{Z}$ and $Q = \{(0,0),(1,0)\} \subset \mathbb{Z}^2$ in Proposition 3.4. In this example $X = X^{\mathfrak{R}_2/\mathfrak{p}} = \{x = (x_{\mathbf{m}}) \in \mathbb{F}_2^{\mathbb{Z}^d} : x_{(m_1,m_2)} + x_{(m_1+1,m_2)} + x_{(m_1,m_2+1)} = 0_{\mathbb{F}_2}$ for all $(m_1,m_2) \in \mathbb{Z}^2\}$, and the projection $\pi_{\bar{\Gamma}} : X \longmapsto \mathbb{F}_2^{\bar{\Gamma}}$ sends the shift $\alpha_{(1,1)}^{\mathfrak{R}_2/\mathfrak{p}} = \sigma_{(1,1)}$ on X to the shift on $\mathbb{F}_2^{\bar{\Gamma}} \cong (\mathbb{Z}/4\mathbb{Z})^{\mathbb{Z}}$. Note that, although $\alpha_{(1,1)}$ acts expansively on X, other elements of $\mathbb{Z}^2$ may not be expansive; for example, $\alpha_{(1,0)}$ is nonexpansive.

(2) Let $\mathfrak{p} = (3+u_1+2u_2) \subset \mathfrak{R}_2$. Then $p(\mathfrak{p}) = 0$, $r(\mathfrak{p}) = 1$, and Γ and Q may be chosen as in Example (1). Note that $X^{\mathfrak{R}_2/\mathfrak{p}} = X = \{x = (x_{\mathbf{m}}) \in \mathbb{T}^{\mathbb{Z}^d} : x_{(m_1,m_2)} + x_{(m_1+1,m_2)} + x_{(m_1,m_2+1)} = 0_{\mathbb{T}}$ for all $(m_1,m_2) \in \mathbb{Z}^2\}$; the coordinate projection $\pi_{\bar{\Gamma}} : X \longmapsto \mathbb{T}^{\bar{\Gamma}}$ in Proposition 3.5 is not injective; for every $x \in X$, the coordinates $x_{(m_1,m_2)}$ with $m_1 \geq m_2$ are completely determined by $\pi_{\bar{\Gamma}}(x)$, but each of the coordinates $x_{(k,k+1)}$, $k \in \mathbb{Z}$, has two possible values. Similarly, if we know the coordinates $x_{(m_1,m_2)}$, $m_1 \geq m_2 - r$ of a point $x = (x_{\mathbf{m}}) \in X$ for any $r \geq 0$, then there are exactly two (independent) choices for each of the coordinates $x_{k,k+r+1)}$, $k \in \mathbb{Z}$. This shows that the kernel of the surjective homomorphism $\pi_{\bar{\Gamma}} : X \longmapsto \mathbb{T}^{\bar{\Gamma}}(\mathbb{T}^2)^{\mathbb{Z}}$ is isomorphic to $\mathbb{Z}_2^{\Gamma}$, where $Y = \mathbb{Z}_2$ denotes the group of dyadic integers.

If $\mathfrak{p}$ is replaced by the prime ideal $\mathfrak{q}' = (1+3u_1+2u_2) \subset \mathfrak{R}_2$, then Γ and Q remain unchanged, but the kernel of $\pi_{\bar{\Gamma}}$ becomes isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_3)^{\Gamma}$, where $\mathbb{Z}_3$ is the group of tri-adic integers.

We end this section with a brief discussion of the notions of Hölder continuity and summable variation.

Remark 3.10. If α is an expansive $\mathbb{Z}^d$-action by automorphisms of a compact, abelian, zero-dimensional group X, then Theorem 3.2 allows us to embed X as a closed, shift-invariant subgroup of $F^{\mathbb{Z}^d}$ for some finite, abelian group F and to assume that α is the shift-action of $\mathbb{Z}^d$ on $X \subset F^{\mathbb{Z}^d}$. Any function $f : X \longmapsto \mathbb{R}$ which depends on only finitely many coordinates is α-Hölder, and other examples of Hölder functions and functions with summable variation can be constructed quite easily. If $X = X^{\mathfrak{R}_d/\mathfrak{p}}$ and $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$ for some prime ideal $p \subset \mathfrak{R}_d$ with $\mathfrak{p} \cap \mathbb{Z} = \{0\}$ (so that X is connected), then we regard X as the closed, shift-invariant subgroup (3.11) of $\mathbb{T}^{\mathbb{Z}^d}$. For every $t \in \mathbb{T}$ we write $|t| = \min_{k\in\mathbb{Z}} |t-k|$ for the usual arc-length distance of t from 0. Then one can prove that there exist constants $\varepsilon > 0$ and $c \in (0,1)$ with the following property: for every finite set $F \subset \mathbb{Z}^d$ we can find a $K > 0$ such that, for every $r \geq 1$,

$$\begin{aligned} (3.19) \qquad & \{x \in X : |x_{\mathbf{m}}| < \varepsilon \text{ for every } \mathbf{m} \in \mathbf{B}(r)\} \\ & \subset \{x \in X : |x_{\mathbf{m}}| < Kc^r \text{ for every } \mathbf{m} \in F\}. \end{aligned}$$

If the free rank $r(\mathfrak{p})$ is equal to 0, then $X = X^{\mathfrak{R}_d/\mathfrak{p}}$ is a finite-dimensional torus or solenoid, (3.19) can be obtained from the proof of Lemma 4.7. If $r(\mathfrak{p}) > 0$, then (3.19) follows from the estimate used in the calculation of entropy in [**LiSW**]. If $F \subset \mathbb{Z}^d$ is a finite set and $f : \mathbb{T}^F \longmapsto \mathbb{R}$ a function which is Hölder in the usual sense, then (3.19) shows that the map $f' = f \cdot \pi_F : X \longmapsto \mathbb{R}$ is α-Hölder. The last statement can be generalized as follows: if α is an expansive $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X, then there exists an integer $n \geq 1$ and a continuous, injective embedding $\phi : X \longmapsto (\mathbb{T}^n)^{\mathbb{Z}^d}$ of X as a closed, shift-invariant subgroup of $(\mathbb{T}^n)^{\mathbb{Z}^d}$ which sends α to the shift-action σ of $\mathbb{Z}^d$ on X. If $F \subset \mathbb{Z}^d$ is a finite set and $f : (\mathbb{T}^n)^F \longmapsto \mathbb{R}$ a Hölder function, then one can again show that the map $f' = f \cdot \pi_F : X \longmapsto \mathbb{R}$ is α-Hölder. There exist, of course, α-Hölder functions on $\phi(X) \subset (\mathbb{T}^n)^{\mathbb{Z}^d}$ which do not depend on only finitely many coordinates.

4. The proof of Theorem 2.1.

For the proof of Theorem 2.1 we need several lemmas.

Lemma 4.1. *Let $d > 1$, and let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal such that $p(\mathfrak{p}) = 0$ and the $\mathbb{Z}^d$-action $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is expansive, mixing, and has free rank $r = r(\mathfrak{p}) \geq 1$. We define the $\mathbb{Z}^d$-action $\bar{\alpha} = \bar{\alpha}^{\mathfrak{R}_d/\mathfrak{p}}$ on $\bar{X} = \bar{X}^{\mathfrak{R}_d/\mathfrak{p}}$ as in* (3.13) *and set, for every nonzero element $\mathbf{n} \in \mathbb{Z}^d$, Fix$(\bar{\alpha}_{\mathbf{n}}) = \{x \in \bar{X} : \bar{\alpha}_{\mathbf{n}}(x) = x\}$. Let $\Gamma \cong \mathbb{Z}^r$ be a subgroup of $\mathbb{Z}^d$ with the properties described in* Proposition 3.6, *and let $\mathbf{n} \in \Gamma$ be a primitive element. Then there exists an integer $K \geq 1$ with the following property: for every $k \geq K$ there exists an element $\mathbf{m} \in \mathbb{Z}^d$ such that the restriction of $\bar{\alpha}_{\mathbf{m}}$ to Fix$(\bar{\alpha}_{k\mathbf{n}}) \subset \bar{X}$ is ergodic.*

Proof. We assume for simplicity that Γ is generated by $\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(r)}$, and that $\mathbf{n} = \mathbf{e}^{(1)}$. If k is greater than the maximum K of the degrees in the variable u_1 of the polynomials f_j, $j = r+1, \dots, d$, occurring in the proof of Proposition 3.6, then the explicit description of the group $\bar{X} = \bar{X}^{\mathfrak{R}_d/\mathfrak{p}}$ in (3.13) allows us to conclude that $\mathrm{Fix}(\bar{\alpha}_{k\mathbf{n}})$ is (isomorphic to) $\mathbb{K}^{\bar{\Gamma}_k}$ and hence connected, where $\gamma_k = \{0, \dots, k-1\} \times \mathbb{Z}^{r-1} \times \{0\} \times \dots \times \{0\}$ and $\bar{\Gamma}_k = \Gamma_k + Q$. If $r \geq 2$, then the restriction of $\bar{\alpha}^{\mathfrak{R}_d/\mathfrak{p}}_{\mathbf{e}^{(2)}}$ to $\bar{Y}_k = \mathrm{Fix}\,(\bar{\alpha}_{k\mathbf{e}^{(1)}})$ is obviously ergodic, which implies our assertion.

Consider therefore the case $r = 1$, and assume that the $\mathbb{Z}^d$-action induced by $\bar{\alpha}$ on $\bar{Y}_k$ is nonergodic. By Lemma 2.2 in [**KiS1**] there exists a nonzero element $a \in \widehat{\bar{Y}_k} = \overline{\mathfrak{R}}_d \big/ \left(\bar{\mathfrak{p}} + (u_1^k - 1) \cdot \overline{\mathfrak{R}}_d\right)$ and an integer $N \geq 1$ such that $\left(u_j^N - 1\right) \cdot a = 0$ for $j = 1, \dots, d$. We choose an $f \in \overline{\mathfrak{R}}_d$ such that $a = f + \bar{\mathfrak{p}} + (u_1^k - 1) \cdot \overline{\mathfrak{R}}_d$ and obtain that $\left(u_j^N - 1\right) f \in \bar{\mathfrak{p}} + (u_1^k - 1) \cdot \overline{\mathfrak{R}}_d$ for $j = 1, \dots, d$. Since $\bar{Y}_k$ is connected, $mf \notin \bar{\mathfrak{p}} + (u_1^k - 1) \cdot \overline{\mathfrak{R}}_d$ for every $m \geq 1$, and we choose $m \geq 1$ so that $mf \in \mathfrak{R}_d$ and $\left(u_j^N - 1\right) f \in \mathfrak{p} + (u_1^k - 1) \cdot \mathfrak{R}_d$ for $j = 1, \dots, d$.

We have found a nonzero element $b = mf + \mathfrak{p} + (u_1^k - 1) \cdot \mathfrak{R}_d \in \mathfrak{R}_d/\mathfrak{p} + (u_1^k - 1) \cdot \mathfrak{R}_d$ such that $\left(u_j^N - 1\right) \cdot b = 0$ for $j = 1, \dots, d$. Set $Y_k = Fix\,(\alpha_{k\mathbf{e}^{(1)}}) \subset X = X^{\mathfrak{R}_d/\mathfrak{p}}$ and note that, according to (3.13), $Y_k = \boldsymbol{\kappa}^{\mathfrak{R}_d/\mathfrak{p}}\,(\bar{Y}_k)$, where $\boldsymbol{\kappa}^{\mathfrak{R}_d/\mathfrak{p}}$ is defined in (3.14). In particular, Y_k is a connected, uncountable, α-invariant subgroup of X, and the restriction of α to Y_k must obviously be expansive, and hence ergodic by Theorem 3.3 and 3.7 in [**S1**]. On the other hand, $\widehat{Y_k} = \mathfrak{R}_d/\mathfrak{p} + (u_1^k - 1) \cdot \mathfrak{R}_d$, and the existence of the element $b = mf$ found above shows that α is nonergodic on Y_k. This contradiction implies that $\bar{\alpha}$ is ergodic on $\bar{Y}_k$.

Now assume that the restriction of $\bar{\alpha}_{\mathbf{m}}$ to $\bar{Y}_k$ is nonergodic for every $\mathbf{m} \in \mathbb{Z}^d$. The argument in the preceding paragraph shows that we can find, for every $\mathbf{m} \in \mathbb{Z}^d$, a Laurent polynomial $g \in \mathfrak{R}_d$ such that $(u^{\mathbf{m}} - 1)\,g \in \mathfrak{p} + (u_1^k - 1) \cdot \mathfrak{R}_d$, which implies that $\alpha_{\mathbf{m}}$ is nonergodic on Y_k. Since $\widehat{Y_k} = \mathfrak{R}_d/\mathfrak{p} + (u_1^k - 1) \cdot \mathfrak{R}_d$ is a Noetherian $\mathfrak{R}_d$-module, Theorem 3.1 implies that α is nonergodic on Y_k, which is absurd. We conclude that there must exist an element $\mathbf{m} \in \mathbb{Z}^d$ such that $\bar{\alpha}_{\mathbf{m}}$ is ergodic on $\bar{Y}_k$, as claimed. □

Lemma 4.2. *Let $d > 1$, $\mathfrak{p} \subset \mathfrak{R}_d$ a prime ideal such that $p(\mathfrak{p}) = 0$ and $r(\mathfrak{p}) \geq 1$, and let $\Gamma \cong \mathbb{Z}^r$ be a subgroup of $\mathbb{Z}^d$ with the properties described in* Proposition 3.6. *Suppose furthermore that $\mathfrak{K}$, $\mathfrak{L}$ are Noetherian $\mathfrak{R}_d$-modules, and that $\mathfrak{L}$ has a prime filtration $\{0\} = \mathfrak{L}_0 \subset \cdots \subset \mathfrak{L}_s = \mathfrak{L}$ such that $\mathfrak{L}_j/\mathfrak{L}_{j-1} \cong \mathfrak{R}_d/\mathfrak{p}$ for $j = 1, \dots, s$. Put $X_1 = X^{\mathfrak{K}}$, $X_2 = X^{\mathfrak{L}}$, $X = X_1 \times X_2$ and $\alpha = \alpha^{\mathfrak{K}} \times \alpha^{\mathfrak{L}}$. Then every cocycle $c : \mathbb{Z}^d \times X \longmapsto \mathbb{R}$ with α-summable variation is continuously cohomologous to the cocycle $E_1(c) : \mathbb{Z}^d \times X \longmapsto \mathbb{R}$ defined by $E_1(c)(\mathbf{k}, (x, x')) = \int c(\mathbf{k}, (x, y))\,d\lambda_{X_2}(y)$ for every $\mathbf{k} \in \mathbb{Z}^d$, $x \in X_1$, and $x' \in X_2$.*

Proof. We choose a primitive subgroup $\Gamma \subset \mathbb{Z}^d$ for $\mathfrak{p}$ with the properties described in Proposition 3.6, and fix a primitive element $\mathbf{n} \in \Gamma$. But $\bar{X} = X_1 \times X^{\bar{\mathfrak{L}}}$, $\bar{\alpha} = \alpha^{\mathfrak{K}} \times \alpha^{\bar{\mathfrak{L}}}$, $\bar{Y} = \{0_{X_1}\} \times \bar{Y}_{s-1}$, where $\bar{Y}_j = \bar{\mathfrak{L}}_j^{\perp} \subset X^{\bar{\mathfrak{L}}}$ for every $j = 0, \dots, s$, and define a continuous, surjective homomorphism $\rho : \bar{X} \longmapsto X$ by $\rho(x, x') = (x, \imath^{\mathfrak{L}}(x'))$ for every $(x, x') \in \bar{X}$, where $\imath^{\mathfrak{L}} : X^{\bar{\mathfrak{L}}} \longmapsto X_2$ is defined by (3.16). For every $j = 0, \dots, s$ we set $Y_j = \mathfrak{L}_j^{\perp} \subset X_2$ and note that $\imath^{\mathfrak{L}}\,(\bar{Y}_j) = Y_j$.

Apply Corollary 3.7, choose $\xi \in (0, 1)$ such that $\bar{\alpha}^{\mathfrak{R}_d/\mathfrak{p}}$ has weak $(\mathbf{n}, \xi)$-specification, put $\Delta = \Delta_{\bar{\alpha}}(\mathbf{n}, \xi) \cap \bar{Y}$, and consider the cocycle $c_{\bar{h}}^{(\mathbf{n})} : \Delta \times \bar{X} \longmapsto \mathbb{R}$ defined by (2.11) with $\bar{\alpha}$ replacing α, and with $\bar{h} = c(\mathbf{n}, \rho(\cdot))$.

We claim that $c_{\bar{h}}^{(\mathbf{n})}(y, x) = 0$ for all $y \in \Delta$ and $x \in \bar{X}$. From Theorem 7.3 in [**KiS1**] we know that $\bigcup_{k \geq 1} Fix\,(\alpha_{k\mathbf{n}}^{\mathfrak{K}})$ is dense in X_1, and the density of

$\bigcup_{k\geq 1} Fix\left(\alpha^{\bar{\mathfrak{L}}}_{k\mathbf{n}}\right)$ in $X^{\bar{\mathfrak{L}}}$ was proved in Corollary 3.7. Since the restriction of $\alpha^{\bar{\mathfrak{L}}}$ to $\bar{Y}_{s-1}$ is (algebraically ad topologically isomorphic to) $\alpha^{\overline{\mathfrak{R}_d/\mathfrak{p}}} = \bar{\alpha}^{\mathfrak{R}_d/\mathfrak{p}}$, Corollary 3.7 also implies that $\bigcup_{k\geq 1}\left(Fix\left(\alpha^{\bar{\mathfrak{L}}}_{k\mathbf{n}}\right)\cap \bar{Y}_{s-1}\right)$ is dense in $\bar{Y}_{s-1}$. If there exist $y\in\Delta$ and $x\in\bar{X}$ such that $c^{(\mathbf{n})}_{\bar{h}}(y,x)\neq 0$, then we can also find an integer $k\geq K$ and elements $y\in Fix\,(\bar{\alpha}_{k\mathbf{n}})\cap\bar{Y}$, $x\in Fix\,(\bar{\alpha}_{k\mathbf{n}})$, such that $H_k(x)-H_k(x+y)\neq 0$, where $H_k=\sum_{j=0}^{k-1}\bar{h}\cdot\bar{\alpha}_{j\mathbf{n}}$. The cocycle equation (1.1) implies that $H_k(x)-H_k(x+y)=H_k\cdot\bar{\alpha}_{\mathbf{m}}(x)-H_k\cdot\bar{\alpha}_{\mathbf{m}}(x+y)$ for every $\mathbf{m}\in\mathbb{Z}^d$, so that the continuous function $F_k : (Fix\,(\bar{\alpha}_{k\mathbf{n}})\cap\bar{Y})\times Fix\,(\bar{\alpha}_{k\mathbf{n}})\longmapsto\mathbb{R}$, defined by $F_k(y,x)=H_k(x)-H_k(x+y)$, is invariant under the $\mathbb{Z}^d$-action $\bar{\alpha}'$ induced by $\bar{\alpha}$ on $(Fix\,(\bar{\alpha}_{k\mathbf{n}})\cap\bar{Y})\times Fix\,(\bar{\alpha}_{k\mathbf{n}})$. If k is large enough, then Lemma 4.1 guarantees the existence of an element $\mathbf{m}\in\mathbb{Z}^d$ such that $\bar{\alpha}_{\mathbf{m}}$ is ergodic—and hence mixing—on $Fix\,(\bar{\alpha}_{k\mathbf{n}})\cap\bar{Y}$. This implies that the $\bar{\alpha}'$-invariant function F_k must be constant in the first variable, so that $H_k(x)-H_k(x+y)=F_k(y,x)=F_k(0,x)=0$ for every $(y,x)\in(Fix\,(\bar{\alpha}_{k\mathbf{n}})\cap\bar{Y})\times Fix\,(\bar{\alpha}_{k\mathbf{n}})$. As explained above, this contradiction shows that the cocycle $c^{(\mathbf{n})}_{\bar{h}}$ vanishes on $\Delta\times\bar{X}$.

Since $\bar{\alpha}$ has weak $(\mathbf{n},\xi)$-specification on $\bar{Y}\cong X^{\overline{\mathfrak{R}_d/\mathfrak{p}}}=\bar{X}^{\mathfrak{R}_d/\mathfrak{p}}$ by Corollary 3.7, Proposition 2.6 implies that there exists a bounded Borel map $\bar{b}:\bar{X}\longmapsto\mathbb{R}$ such that $\bar{h}=E_{\lambda_{\bar{X}}}\left(\bar{h}|\mathfrak{B}_{\bar{X}/\bar{Y}}\right)+\bar{b}\cdot\bar{\alpha}_{\mathbf{n}}-\bar{b}$. We regard $b=E_{\lambda_{\bar{X}}}\left(\bar{b}|\rho^{-1}(\mathfrak{B}_X)\right)$ as a bounded Borel map from X to $\mathbb{R}$ and obtain that $E_{\lambda_X}\left(h|\mathfrak{B}_{X/Y}\right)=h+b\cdot\alpha_{\mathbf{n}}-b$, where $Y=\{0_{X_1}\}\times Y_{s-1}\subset X$ and $h=c(\mathbf{n},\cdot)$.

Since $X/Y=\widehat{\mathfrak{K}\oplus\mathfrak{L}_{s-1}}$ we can regard $E_{\lambda_X}\left(h|\mathfrak{B}_{X/Y}\right)$ as a function on X/Y with summable variation and repeat the above argument with $\mathfrak{L}_{s-1}$ replacing $\mathfrak{L}=\mathfrak{L}_s$. After s steps we obtain that h is cohomologous—with bounded transfer function—to the function $h_1=E_{\lambda_X}\left(h|\mathfrak{B}_{X/\{0_{X_1}\}\times X_2}\right)$. Put $\bar{h}_1=h_1\cdot\rho=E_{\lambda_{\bar{X}}}\left(\bar{h}|\mathfrak{B}_{\bar{X}/\{0_{X_1}\}\times X^{\bar{\mathfrak{L}}}}\right):\bar{X}\longmapsto\mathbb{R}$ and note that $\bar{h}_1=\bar{h}+\bar{b}'\cdot\bar{\alpha}_{\mathbf{n}}-\bar{b}'$ for some (bounded) Borel map $\bar{b}':\bar{X}\longmapsto\mathbb{R}$. Since $\bar{\alpha}$ has weak $(\mathbf{n},\xi)$-specification on $\{0_{X_1}\}\times X^{\bar{\mathfrak{L}}}$ by Corollary 3.7, Corollary 2.7 implies that there exists a continuous function $\bar{b}'':\bar{X}\longmapsto\mathbb{R}$ such that $\bar{h}_1=\bar{h}+\bar{b}''\cdot\bar{\alpha}_{\mathbf{n}}-\bar{b}''$. We regard $b''=E_{\lambda_{\bar{X}}}\left(\bar{b}''|\rho^{-1}(\mathfrak{B}_X)\right)$ as a continuous map from X to $\mathbb{R}$ and obtain that $h_1=h+b''\cdot\alpha_{\mathbf{n}}-b''$.

The proof is concluded by noting that the ergodicity of $\alpha_{\mathbf{n}}$ and the cocycle equation (1.1) together imply that c is cohomologous to $E_1(c)$, with continuous transfer function b''. □

Lemma 4.3. *Let $d>1$, and let $\mathfrak{p}\subset\mathfrak{R}_d$ be a prime ideal such that the $\mathbb{Z}^d$-action $\alpha=\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is mixing, $p=p(\mathfrak{p})>0$, and $r=r(\mathfrak{p})>1$. Choose a subgroup $\Gamma\cong\mathbb{Z}^r$ of $\mathbb{Z}^d$ with the properties described in* Proposition 3.4, *and let $\mathbf{n}\in\Gamma$ be a primitive element. Then there exists an integer $K\geq 1$*

such that the $\mathbb{Z}^d$-action induced by α on the closed, α-invariant subgroup $Fix\,(\alpha_{k\mathbf{n}}) \subset X$ is ergodic for every $k \geq K$.

Proof. This is proved in the same way as Lemma 4.1, by using Proposition 3.4 instead of Proposition 3.6. □

Lemma 4.4. *Let $d > 1$, $\mathfrak{p} \subset \mathfrak{R}_d$ a prime ideal such that $p(\mathfrak{p}) > 0$ and $r(\mathfrak{p}) > 1$, and let $\Gamma \cong \mathbb{Z}^r$ be a subgroup of $\mathbb{Z}^d$ with the properties described in* Proposition 3.4. *Suppose furthermore that $\mathfrak{K}$, $\mathfrak{L}$ are Noetherian $\mathfrak{R}_d$-modules, and that $\mathfrak{L}$ has a prime filtration $\{0\} = \mathfrak{L}_0 \subset \cdots \subset \mathfrak{L}_s = \mathfrak{L}$ such that $\mathfrak{L}_j/\mathfrak{L}_{j-1} \cong \mathfrak{R}_d/\mathfrak{p}$ for $j = 1, \dots, s$. Put $X_1 = X^{\mathfrak{K}}$, $X_2 = X^{\mathfrak{L}}$, $X = X_1 \times X_2$ and $\alpha = \alpha^{\mathfrak{K}} \times \alpha^{\mathfrak{L}}$. Then every cocycle $c : \mathbb{Z}^d \times X \longmapsto \mathbb{R}$ with α-summable variation is continuously cohomologous to the cocycle $E_1(c) : \mathbb{Z}^d \times X \longmapsto \mathbb{R}$ defined as in* Lemma 4.3.

Proof. The argument is slightly simpler than, but otherwise completely analogous to, the proof of Lemma 4.2, and uses Proposition 3.4, Corollary 3.5 and Lemma 4.3, instead of Proposition 3.6, Corollary 3.7 and Lemma 4.1. □

Lemma 4.5. *Let $d > 1$, $\mathfrak{p} \subset \mathfrak{R}_d$ a prime ideal such that $p = p(\mathfrak{p}) > 0$, $r(\mathfrak{p}) = 1$, and $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is mixing, and choose a subgroup $\Gamma \cong \mathbb{Z}$ in $\mathbb{Z}^d$ with the properties described in* Proposition 3.4 *and a primitive element $\mathbf{n} \in \Gamma$. If $h : X = X^{\mathfrak{R}_d/\mathfrak{p}} \longmapsto \mathbb{R}$ is a function with α-summable variation and Fourier transform $\hat{h} : \mathfrak{R}_d/\mathfrak{p} \longmapsto \mathbb{C}$, then*

$$\lim_{\substack{\mathbf{m}\to\infty \\ \langle \mathbf{m},\mathbf{n}, \rangle = 0}} \sum_{k\in\mathbb{Z}} \left| \hat{h}\left(u^{k\mathbf{n}+\mathbf{m}} \cdot a\right) \right| = 0 \tag{4.1}$$

for every nonzero element $a \in \mathfrak{R}_d/\mathfrak{p}$.

Proof. Let $Q \in \mathbb{Z}^d$ be the set defined in Proposition 3.4 and let $F = \mathbb{F}_p^Q$. The coordinate projection $\pi_{\bar{\Gamma}} : X \longmapsto \mathbb{F}_p^{\bar{\Gamma}} \cong F^{\mathbb{Z}} = Y$ defines a continuous group isomorphism $\eta : X \longmapsto Y$, and we denote by σ the shift on $Y = F^{\mathbb{Z}}$ and note that $\eta \cdot \alpha_{\mathbf{n}} = \sigma \cdot \eta$. There exist constants $L, L' \geq 0$ (which depend on Q and on the polynomials f_j, $j = 2, \dots, d$, in the proof of Proposition 3.4), such that, for every $r \geq 1$, $\pi_{\{-r,\dots,r\}}(\eta(x)) = \pi_{\{-r,\dots,r\}}(\eta(x'))$ whenever $\pi_{\mathbf{B}(r+L)}(x) = \pi_{\mathbf{B}(r+L)}(x')$, and $\pi_{\mathbf{B}(r)}(x) = \pi_{\mathbf{B}(r)}(x')$ whenever $\pi_{\{-L'r,\dots,L'r\}}(\eta(x)) = \pi_{\{-L'r,\dots,L'r\}}(\eta(x'))$ (here π_E again denotes the coordinate projection onto a set of coordinates E). For every $r \geq 1$ we set

$$\omega_r(h) = \sup_{\{(x,x')\in X\times X:\pi_{\mathbf{B}(r)}(x)=\pi_{\mathbf{B}(r)}(x')\}} |h(x)-h(x')|,$$
$$\omega'_r(h') = \sup_{\{(y,y')\in Y\times Y:\pi_{\{-r,\dots,r\}}(y)=\pi_{\{-r,\dots,r\}}(y')\}} |h'(y)-h'(y')|,$$

where $h' : Y \longmapsto \mathbb{R}$ is defined by $h = h' \cdot \eta$. Since h has summable variation, $\sum_{r\geq 1}\omega_r(h) < \infty$; furthermore $\omega_r(h) \geq \omega'_{L'r}(h')$ for every $r \geq 1$, so that $\sum_{r\geq 1}\omega'_r(h') < \infty$.

For every $r \geq 1$ we can find a function $h'_r : Y \longmapsto \mathbb{R}$, which only depends on the coordinates $\{-r,\dots,r\} \in \mathbb{Z}$, such that $|h'(y) - h'_r(y)| \leq \omega'_r(h')$ for every $y \in Y$. If $\hat{\eta} : \hat{Y} \longmapsto \mathfrak{R}_d/\mathfrak{p}$ is the isomorphism dual to $\eta : X \longmapsto Y$, then $\left|\hat{h}\left(\hat{\eta}(\chi)\right) - \widehat{h'_r}(\chi)\right| \leq \omega'_r(h')$ for every $\chi \in \hat{Y}$ and $r \geq 1$. We set

$$S(h,r) = \left\{a \in \mathfrak{R}_d/\mathfrak{p} : \left|\hat{h}(a)\right| > \omega'_r(h')\right\}$$

and observe that

$$\text{(4.2)} \qquad \begin{aligned} \hat{\eta}^{-1}(S(h,r)) = S(h',r) &= \left\{\chi \in \hat{Y} : \left|\hat{h}\left(\hat{\eta}(\chi)\right)\right| > \omega'_r(r)\right\} \\ &\subset S(h'_r) = \left\{\chi \in \hat{Y} : \widehat{h'_r}(\chi) \neq 0\right\}. \end{aligned}$$

For every nonzero element $a \in \mathfrak{R}_d/\mathfrak{p}$,

$$\begin{aligned} S(r,a) &= \left\{k \in \mathbb{Z} : \widehat{h'_r}\left(\hat{\eta}^{-1}(a)\cdot\sigma^k\right) \neq 0\right\} \\ &\supset T(r,a) = \left\{k \in \mathbb{Z} : \left|\hat{h}\left(u^{k\mathbf{n}}\cdot a\right)\right| > \omega'_r(h')\right\} \end{aligned}$$

and

$$|T(r,a)| \leq |S(r,a)| \leq 2r+1$$

whenever $0 \neq a \in \mathfrak{L}$ and $r \geq 1$, where $|S|$ denotes the cardinality of a set S. In particular, if $\omega'_0(h') = \max_{y\in Y}|h'(y)|$ and $T(0,a) = \emptyset$, then

$$\text{(4.3)} \qquad \sum_{k\in\mathbb{Z}}\left|\hat{h}\left(u^{k\mathbf{n}}\cdot a\right)\right| \leq \sum_{r\geq 1}\left(\omega'_{r-1}(h') - \omega'_r(h')\right)|T(r,a)| < \infty$$

for every nonzero element $a \in \mathfrak{R}_d/\mathfrak{p}$.

We fix a nonzero element $a \in \mathfrak{R}_d/\mathfrak{p}$. Since h'_r depends only on the coordinates $\{-r,\dots,r\}$, (4.2) shows that $|S(h,r)| = |S(h',r)| \leq |S(h'_r)| \leq |F|^{2r+1}$. Furthermore, since α is mixing, $u^{\mathbf{m}}\cdot a \neq u^{\mathbf{m}'}\cdot a$ whenever $\mathbf{m} \neq \mathbf{m}' \in \mathbb{Z}^d$, so that there exist, for every $r \geq 1$, at most $|F|^{2r+1}$ elements $\mathbf{m} \in \mathbb{Z}^d$ with $T(r, u^{\mathbf{m}}\cdot a) \neq \emptyset$. In particular we can find, for every $M \geq 1$, an integer

$M' \geq 1$ such that $T(r, u^{\mathbf{m}} \cdot a) = \emptyset$ whenever $r \leq M$, $\langle \mathbf{m}, \mathbf{n} \rangle = 0$, and $\|\mathbf{m}\| > M'$, in which case (4.3) implies that

$$\sum_{k \in \mathbb{Z}} \left| \hat{h}\left(u^{k\mathbf{n}+\mathbf{m}} \cdot a\right) \right| \leq \sum_{r \geq M} \left(\omega'_{r-1}(h') - \omega'_r(h')\right)(2r+1)$$
$$\leq 2 \sum_{r \geq M-1} \omega'_r(h').$$

By letting $M \to \infty$ we obtain (4.1). $\square$

Lemma 4.6. *Let $d > 1$, and let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal with $p = p(\mathfrak{p}) > 0$ and $r = r(\mathfrak{p}) = 1$. Suppose furthermore that $\mathfrak{K}$, $\mathfrak{L}$ are Noetherian $\mathfrak{R}_d$-modules with the following properties.*

(1) *The $\mathbb{Z}^d$-actions $\alpha^{\mathfrak{K}}$ and $\alpha^{\mathfrak{L}}$ are expansive and mixing;*
(2) *Every prime ideal $\mathfrak{q} \subset \mathfrak{R}_d$ associated with $\mathfrak{K}$ satisfies that either $p(\mathfrak{q}) = r(\mathfrak{q}) = 0$, or $p(\mathfrak{q}) > 0$ and $r(\mathfrak{q}) = 1$;*
(3) *$\mathfrak{L}$ has a prime filtration $\{0\} = \mathfrak{L}_0 \subset \cdots \subset \mathfrak{L}_s = \mathfrak{L}$ with $\mathfrak{L}_j/\mathfrak{L}_{j-1} \cong \mathfrak{R}_d/\mathfrak{p}$ for $j = 1, \ldots, s$.*

Let $X_1 = X^{\mathfrak{K}}$, $X_2 = X^{\mathfrak{L}}$, $X = X_1 \times X_2 = X^{\mathfrak{K} \oplus \mathfrak{L}}$, $\alpha = \alpha^{\mathfrak{K}} \times \alpha^{\mathfrak{L}}$, and let $c : \mathbb{Z}^d \times X \longmapsto \mathbb{R}$ be a cocycle with α-summable variation. Then c is continuously cohomologous to the cocycle $E_1(c) : \mathbb{Z}^d \times X \longmapsto \mathbb{R}$ defined as in Lemma 4.2.

Proof. We choose a primitive subgroup $\Gamma \cong \mathbb{Z}$ in $\mathbb{Z}^d$ for $\mathfrak{p}$ with properties stated in Proposition 3.4, fix a primitive element $\mathbf{n} \in \Gamma$ and a $\xi \in (0,1)$ such that $\alpha^{\mathfrak{L}}$ has weak $(\mathbf{n}, \xi)$-specification (Corollary 3.5), set $h = c(\mathbf{n}, \cdot)$, and put $Y_j = \mathfrak{L}_j^{\perp} \subset X_2$ for $j = 0, \ldots, s$. Then $X_2 = Y_0 \supset \cdots \supset Y_s = \{0\}$. We set $Y = \{0_{X_1}\} \times Y_{s-1} \subset X$, $\Delta = \{0_{X_1}\} \times (\Delta_{\alpha^{\mathfrak{L}}}(\mathbf{n}, \xi) \cap Y_{s-1}) \subset X$, and consider the cocycle $c_h = c_h^{(\mathbf{n})} : \Delta \times X \longmapsto \mathbb{R}$ defined in (2.11). We claim that $c_h^{(\mathbf{n})}(y, x) = 0$ for every $y \in \Delta$ and $x \in X$; since the set $\bigcup_{k \geq 1} Fix\,(\alpha_{k\mathbf{n}})$ is dense in X by Corollary 7.4 or Theorem 7.5 in [**KiS1**] this is easily seen to be equivalent to the assertion that $c_h^{(\mathbf{n})}(y, x) = 0$ for all $y \in \Delta$ and $x \in \bigcup_{k \geq 1} Fix\,(\alpha_{k\mathbf{n}})$.

Suppose that there exist $k \geq 1$ and $z \in Fix\,(\alpha_{k\mathbf{n}})$ such that

$$c_h^{(\mathbf{n})}(y, z) \neq 0 \tag{4.4}$$

for some $y \in \delta$. The conditions (1)–(2) imply that $\alpha_{\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{q}}$ is ergodic and has finite entropy for every prime ideal $\mathfrak{q} \subset \mathfrak{R}_d$ associated with $\mathfrak{K} \oplus \mathfrak{L}$, so that $Fix\,(\alpha_{k\mathbf{n}})$ is finite (cf. [**S1**]).

For every $j \in \mathbb{Z}$ we define a function $g_j : Y \longmapsto \mathbb{R}$ with summable variation by $g_j(y) = h\,(\alpha_{j\mathbf{n}}(z) + y)$ for every $y \in Y$, and consider the cocycle c :

$\Delta \times Y \longmapsto \mathbb{R}$ defined by

$$c(y,x) = \sum_{j\in\mathbb{Z}} g_j \cdot \alpha_{j\mathbf{n}}(x) - g_j \cdot \alpha_{j\mathbf{n}}(x+y)$$

for $y \in \Delta$, $x \in Y$. Since $Fix\,(\alpha_{k\mathbf{n}})$ is finite, $\Omega(\mathbf{n})' = \{\mathbf{m} \in \mathbb{Z}^d\colon \langle \mathbf{m}, \mathbf{n}\rangle = 0$ and $\alpha_{\mathbf{m}}(z) = z\}$ has finite index in $\Omega(\mathbf{n}) = \{\mathbf{m} \in \mathbb{Z}^d : \langle \mathbf{m}, \mathbf{n}\rangle = 0\}$. As $Y \cong X^{\mathfrak{R}_d/\mathfrak{p}}$, Lemma 4.5 shows that

$$\lim_{\substack{\mathbf{m}\to\infty \\ \mathbf{m}\in\Omega'(\mathbf{n})}} \sum_{k\in\mathbb{Z}} \left|\hat{g}_k\left(u^{k\mathbf{n}+\mathbf{m}} \cdot a\right)\right| = 0 \tag{4.5}$$

for every nonzero element $a \in \hat{Y}$. The cocycle equation (1.1) implies that

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}(g_j \cdot \alpha_{j\mathbf{n}}(x) - g_j \cdot \alpha_{j\mathbf{n}}(x+y)) \\ &\qquad = \sum_{j\in\mathbb{Z}} (g_j \cdot \alpha_{j\mathbf{n}+\mathbf{m}}(x) - g_j \cdot \alpha_{j\mathbf{n}+\mathbf{m}}(x+y))\end{aligned} \tag{4.6}$$

for every $\mathbf{m} \in \Omega'(\mathbf{n})$. By combining (4.5) and (4.6) we obtain that the cocycle c vanishes. This contradicts (4.4) and proves that the cocycle $c_h^{(\mathbf{n})}$: $\Delta \times X \longmapsto \mathbb{R}$ vanishes.

The proof is completed in exactly the same manner as that of Lemma 4.2. Proposition 2.6 implies that h is cohomologous to $E_{\lambda_X}\left(h|\mathfrak{B}_{X/Y}\right)$, with bounded transfer function, and by viewing $E_{\lambda_X}\left(h|\mathfrak{B}_{X/Y}\right)$ as a function on X/Y with summable variation we can apply the above argument again and obtain after s steps that h is cohomologous $E_{\lambda_X}\left(h|\mathfrak{B}_{X/(\{0_{X_1}\}\times X_2)}\right)$. Since $\alpha^{\mathfrak{L}}$ has weak $\mathbf{n}$-specification by Corollary 3.5, we conclude as in the proof of Lemma 4.2 that h is continuously cohomologous to $E_{\lambda_X}\left(h|\mathfrak{B}_{X/(\{0_{X_1}\}\times X_2)}\right)$, and that c is therefore continuously cohomologous to $E_1(c)$. □

Lemma 4.7. *Suppose that $d > 1$, and that $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal such that $p(\mathfrak{p}) = r(\mathfrak{p}) = 0$ and $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is expansive and mixing. Then there exist primitive elements $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$ with the following property: if $h : X = X^{\mathfrak{R}_d/\mathfrak{p}} \longmapsto \mathbb{R}$ is a function with α-summable variation and Fourier transform $\hat{h} : \mathfrak{R}_d/\mathfrak{p} \longmapsto \mathbb{C}$, then*

$$\lim_{l\to\infty} \sum_{k\in\mathbb{Z}} \left|\hat{h}\left(u^{k\mathbf{n}+l\mathbf{m}} \cdot a\right)\right| = 0 \tag{4.7}$$

for every nonzero element $a \in \mathfrak{R}_d/\mathfrak{p}$.

Proof. There exists a point $c = (c_1, \dots, c_d) \in \overline{\mathbb{Q}}^d \subset \mathbb{C}^d$ such that $\mathfrak{p} = \{f \in \mathfrak{R}_d : f(c) = 0\}$ and $V_{\mathbb{C}}(\mathfrak{p})$ is the orbit of c under the Galois group of $\left[\overline{\mathbb{Q}} : \mathbb{Q}\right]$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. We write $\mathbb{Q}(c)$ for the algebraic number field generated by $\{c_1, \dots, c_d\}$, $\mathfrak{o}(c)$ for the ring of integers in $\mathbb{Q}(c)$, and $\mathfrak{P}$, $\mathfrak{P}_f$, $\mathfrak{P}_\infty$ for the sets of places, finite places, and infinite places of $\mathbb{Q}(c)$. for each $\nu \in \mathfrak{P}$ we write $\mathbb{Q}(c)_\nu$ for the corresponding completion of $\mathbb{Q}(c)_\nu$, choose a Haar measure λ_ν on the locally compact field $\mathbb{Q}(c)_\nu$ (regarded as an additive group) and a compact set $C_\nu \subset \mathbb{Q}(c)_\nu$ with nonempty interior, and define the valuation $|\cdot|_\nu : \mathbb{Q}(c)_\nu \longmapsto \mathbb{R}^+$ by $|a|_\nu = \lambda_\nu(aC_\nu)/\lambda_\nu(C_\nu)$ for every $a \in \mathbb{Q}(c)_\nu$, where $aC_\nu = \{ay : y \in C_\nu\}$. With this choice of $|\cdot|_\nu$, $\nu \in \mathfrak{P}$, we have that $\prod_{\nu \in \mathfrak{P}} |a|_\nu = 1$ for every $a \in \mathbb{Q}(c)$. If $\nu \in \mathfrak{P}_f$ we write $\mathfrak{o}_\nu = \{y \in \mathbb{Q}(c)_\nu : |y|_\nu \leq 1\}$ for the maximal compact subring of $\mathbb{Q}(c)_\nu$ and choose a prime element $\pi_\nu \in \mathfrak{o}_\nu$, i.e. an element such that $\pi_\nu \mathfrak{o}\nu$ is the maximal ideal in $\mathfrak{o}_\nu$. For $\nu \in \mathfrak{P}_\infty$ we set $\mathfrak{o}_\nu = \mathfrak{o}(c)$.

Let $\mathfrak{P}_f(c) = \{\nu \in \mathfrak{P}_f : |c_i|_\nu \neq 1 \text{ for some } i \in \{1, \dots, d\}\}$, $S(c) = \mathfrak{P}_f(c) \cup \mathfrak{P}_\infty$, and denote by $\jmath : \mathbb{Q}(c) \longmapsto \prod_{\nu \in S(c)} \mathbb{Q}(c)_\nu$ the diagonal embedding $a \mapsto (a, \dots, a)$, $a \in \mathbb{Q}(c)$. If

$$R_c = \{a \in \mathbb{Q}(c) : |a|_\nu \leq 1 \text{ for every } \nu \in \mathfrak{P} \setminus S(c)\},$$

and if $\eta_c : f \mapsto f(c)$, $f \in \mathfrak{R}_d$, is the evaluation map, then $\eta_c(\mathfrak{R}_d) \cong \mathfrak{R}_d/\mathfrak{p}$, and $\eta_c(\mathfrak{R}_d)$ is a subgroup of finite index in R_c (Lemma 5.1 in [**S1**]). We continue as in Section 5 in [**S1**]. The subgroup $\jmath(R_c) \subset Z = \prod_{\nu \in S(c)} \mathbb{Q}(c)_\nu$ is discrete and co-compact, and $Y = \widehat{R_c} = Z/\jmath(R_c)$. A typical element of $y \in Y$ will be written as $y = (y_\nu) = (y_\nu, \nu \in S(c)) + \jmath(R_c)$, where $y_\nu \in \mathbb{Q}(c)_\nu$ for every $\nu \in S(c)$. For every $\nu \in S(c)$ and $\varepsilon' > 0$ we set $Q_\nu(\varepsilon') = \{y \in \mathbb{Q}(c)_\nu : |y|_\nu \leq \varepsilon'\}$, and put $Q(\varepsilon') = \prod_{\nu \in S(c)} Q_\nu(\varepsilon')$. Since $\jmath(R_c)$ is a discrete subgroup of Z there exists, for every sufficiently small $\varepsilon' > 0$, a neighbourhood $N(\varepsilon')$ of 0_Y in Y which is homeomorphic to $Q(\varepsilon') \subset Z$, and we identify these neighbourhoods and regard $Q(\varepsilon')$ as a neighbourhood of 0_Y in Y.

In order to understand how an element of R_c defines a character on Y we follow [**W**]: there exists, for every $\nu \in \mathfrak{P}$, a character $\chi_\nu \in \mathfrak{o}_\nu^\perp \subset \widehat{\mathbb{Q}(c)_\nu}$ with the following properties.

(1) If $\nu \in \mathfrak{P}_f$, then $\chi_\nu(\pi_\nu^{-1}\mathfrak{o}_\nu) = \{e^{2\pi i k|\pi_\nu|_\nu} : k \in \mathbb{Z}\} \subset \mathbb{S}$;

(2) $\prod_{\nu \in S} \chi_\nu(b) = 1$ for every $b \in \mathbb{Q}(c)$.

For every $z = (z_\nu, \nu \in S) \subset Z$ we set $\psi(z) = \prod_{\nu \in S} \chi_\nu(z_\nu)$ and obtain that $\psi(\jmath(a)) = 1$ for every $a \in R_c$, so that $\psi \in \jmath(R_c)^\perp \subset \hat{Z}$. Hence ψ induces a character $\chi \in \hat{Y}$. Every character in $\hat{Y}$ is of the form $\chi^{(a)} : y \mapsto \chi(a \cdot y) = \prod_{\nu \in S} \chi_\nu(a \cdot y_\nu)$ for some $a \in R_c$, where $y = (y_\nu) \in Y$ and $a \cdot y = (ay_\nu)$.

We define a $\mathbb{Z}^d$-action α' by automorphisms of Y by $\alpha_{\mathbf{n}}(y) = c^{\mathbf{n}} \cdot y$ for every $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ and $y \in Y$, where $c^{\mathbf{n}} = c_1^{n_1} \cdots c_d^{n_d}$, write $\beta' : \mathbf{n} \mapsto \beta'_{\mathbf{n}} = \widehat{\alpha'_{\mathbf{n}}}$ for the $\mathbb{Z}^d$-action on $\hat{Y} = R_c$ dual to α', and observe that

$\beta'_{\mathbf{n}}(b) = c^{\mathbf{n}}b$ for every $\mathbf{n} \in \mathbb{Z}^d$ and $b \in R_c$. The homomorphism $\tau : Y \longmapsto X$ dual to the inclusion map $\mathfrak{R}_d/\mathfrak{p} \cong \eta_c(\mathfrak{R}_d) \hookrightarrow R_c$ is surjective and finite-to-one, and $\alpha_{\mathbf{n}} \cdot \tau = \tau \cdot \alpha'_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$. We shall prove (4.7) for the function $f' = f \cdot \tau : Y \longmapsto \mathbb{R}$, which has α'-summable variation, and for every nonzero $a \in R_c$; this will imply that (4.7) holds for the function f.

If $V_{\mathbb{C}}(\mathfrak{p}) = \{c = c(1), \dots , c(t)\}$ with $c(j) = (c(j)_1, \dots , c(j)_d)$ for every $j = 1, \dots , t$, then Theorem 3.1 (1) implies that $c(j) \notin \mathbb{S}^d$ for $j = 1, \dots , t$. Since α and α' are mixing, $\alpha'_{\mathbf{k}}$ is ergodic whenever $\mathbf{0} \neq \mathbf{k} = (k_1, \dots , k_d) \in \mathbb{Z}^d$, so that the eigenvalues $c(j)^{\mathbf{k}} = c(j)_1^{k_1} \cdots c(j)_d^{k_d}$, $j = 1, \dots , t$, cannot all have modulus 1. An element $\mathbf{k} \in \mathbb{Z}^d$ will be called *hyperbolic* if *none* of these eigenvalues has modulus 1, which is equivalent to saying that $\left|c^{\mathbf{k}}\right|_\nu \neq 1$ for all $\nu \in \mathfrak{P}_\infty$. Note that there exist finitely many one-dimensional linear subspaces $L_j \subset \mathbb{R}^d$ such that every $\mathbf{n} \in \mathbb{Z}^d \setminus \bigcup L_j$ is hyperbolic.

We claim that there exist primitive, hyperbolic elements

$$\mathbf{m} = (m_1, \dots , m_d), \qquad \mathbf{n} = (n_1, \dots , n_d)$$

in $\mathbb{Z}^d$ with the following properties.

(1) For every $\nu \in \mathfrak{P}_\infty$, either $\left|c^{\mathbf{m}}\right|_\nu > 1$ and $\left|c^{\mathbf{n}}\right|_\nu > 1$, or $\left|c^{\mathbf{m}}\right|_\nu < 1$ and $\left|c^{\mathbf{n}}\right|_\nu < 1$;

(2) There exists at least one $\nu \in \mathfrak{P}_\infty$ such that $|c^{\mathbf{n}}|_\nu > 1$;

(3) If $|c^{\mathbf{n}}|\nu > 1$ for all $\nu \in \mathfrak{P}_\infty$ then there exists a $\nu' \in \mathfrak{P}_f(c)$ with $|c^{\mathbf{m}}|_{\nu'} \prod_{\nu \in \mathfrak{P}_\infty} |c^{\mathbf{m}}|_\nu < 1$ and $|c^{\mathbf{n}}|'_\nu \prod_{\nu \in \mathfrak{P}_\infty} |c^{\mathbf{n}}|_\nu < 1$.

In order to prove that such choices are possible we set $a_k = \prod_{j=1}^{t} c(j)_k \in \mathbb{Q}$ for $k = 1, \dots , d$, and let $\varepsilon > 0$. Then there exist relatively prime integers k_1, k_2 such that $\left|a_1^{k_1} a_2^{k_2} - 1\right| < \varepsilon$, and we set $\mathbf{k} = \mathbf{k}(\varepsilon) = (k_1, k_2, 0, \dots , 0) \in \mathbb{Z}^d$. If $a_1^{k_1} a_2^{k_2} = 1$ then the product of the eigenvalues of the ergodic automorphism $\alpha'_{\mathbf{k}}$ is equal to 1, and we conclude that there exist j, $j' \in \{1, \dots , t\}$ such that $\left|c(j)^{\mathbf{k}}\right| < 1 < \left|c(j')^{\mathbf{k}}\right|$, which is equivalent to saying that there exists valuations ν, $\nu' \in \mathfrak{P}_\infty$ with $\left|c^{\mathbf{k}}\right|_\nu < 1 < \left|c^{\mathbf{k}}\right|_{\nu'}$. Put $\mathbf{k}' = m\mathbf{k} + \mathbf{e}^{(1)}$ for some large $m > 1$, and divide the entries of $\mathbf{k}'$ by their highest common factor, so that $\mathbf{k}'$ becomes primitive. Then $\left|c^{\mathbf{k}'}\right|_\nu > 1$ for every $\nu \in \mathfrak{P}_\infty$ with $\left|c^{\mathbf{k}}\right|_\nu > 1$, and $\left|c^{\mathbf{k}'}\right|_\nu < 1$ for every $\nu \in \mathfrak{P}_\infty$ with $\left|c^{\mathbf{k}}\right|_\nu < 1$. If either $\mathbf{k}$ or $\mathbf{k}'$ are nonhyperbolic we choose a primitive hyperbolic element $\mathbf{n}' \in \mathbb{Z}^d$ and set $\mathbf{n} = \mathbf{n}' + m\mathbf{k}$, $\mathbf{m} = \mathbf{n}' + m\mathbf{k}'$ for a suitably large $m \geq 1$ (if these elements turn out not to be primitive, divide them by the highest common factor of their entries). Then $\mathbf{m}$, $\mathbf{n}$ are hyperbolic and satisfy (1)–(2).

Now assume that, for every primitive $\mathbf{k} \in \mathbb{Z}^d$, either $\left|c^{\mathbf{k}}\right|_\nu \geq 1$ for all $\nu \in \mathfrak{P}_\infty$, or $\left|c^{\mathbf{k}}\right|_\nu \leq 1$ for all $\nu \in \mathfrak{P}_\infty$, in which case the preceding paragraph shows that $a_1^{k_1} a_2^{k_2} \neq 1$ whenever $(0,0) \neq (k_1, k_2) \in \mathbb{Z}^2$, and that 1 is a limit point of $\left\{a_1^{k_1}a_2^{k_2} : (0,0) \neq (k_1,k_2) \in \mathbb{Z}^2\right\}$. For every $\varepsilon > 0$ we can choose $\mathbf{k}(\varepsilon) = (k_1(\varepsilon), k_2(\varepsilon), 0, \dots, 0) \in \mathbb{Z}^d$ as above, but with considerably more freedom, such that $\mathbf{k}(\varepsilon)$ is hyperbolic, and $\left|c^{\mathbf{k}(\varepsilon)}\right|_\nu \geq 1$ for every $\nu \in \mathfrak{P}_\infty$. Then $1 < \prod_{\nu\in\mathfrak{P}_\infty} \left|c^{\mathbf{k}(\varepsilon)}\right|_\nu = a_1^{k_1(\varepsilon)}a_2^{k_2(\varepsilon)} < 1+\varepsilon$. However, $\prod_{\nu\in S(c)} \left|c^{\mathbf{k}}\right|_\nu = 1$, which implies that, if ε is small, some of the valuations $\left|c^{\mathbf{k}}\right|_\nu$, $\nu \in \mathfrak{P}_f(c)$, must be very large, and others very small, as their product is close, but not equal, to 1. In particular there exists, for every $M > 1$, a primitive, hyperbolic element $\mathbf{k} \in \mathbb{Z}^d$ such that $\left|c^{\mathbf{k}}\right|_{\nu'} \prod_{\nu\in\mathfrak{P}_\infty} \left|c^{\mathbf{k}}\right|_\nu < M^{-1}$ for some $\nu' \in \mathfrak{P}_f(c)$. By increasing M we obtain distinct primitive, hyperbolic elements $\mathbf{n}$, $\mathbf{m}$ in $\mathbb{Z}^d$ and a $\nu' \in \mathfrak{P}_f(c)$ such that $|c^{\mathbf{m}}|_\nu > 1$ and $|c^{\mathbf{n}}|_\nu > 1$ for all $\nu \in \mathfrak{P}_\infty$, and $|c^{\mathbf{m}}|_{\nu'} \prod_{\nu\in\mathfrak{P}_\infty} |c^{\mathbf{m}}|_\nu < 1$, $|c^{\mathbf{n}}|_{\nu'} \prod_{\nu\in\mathfrak{P}_\infty} |c^{\mathbf{n}}|_\nu$ for some $\nu' \in \mathfrak{P}_f(c)$.

Having found primitive, hyperbolic elements $\mathbf{m}$, $\mathbf{n}$ in $\mathbb{Z}^d$ satisfying (1)–(3) we estimate the Fourier coefficients $\left|\hat{f}'\left(c^{k\mathbf{n}}\right)\right|$, $k \in \mathbb{Z}$. Choose an invariant metric δ on Y and $\varepsilon > 0$, and find a (small) $\varepsilon' > 0$ such that

$$Y_\delta(\varepsilon) = \{y \in Y : \delta(y, 0_Y) < \varepsilon\} \supset Q(\varepsilon') \tag{4.8}$$

and hence

$$\bigcap_{\mathbf{m}\in\mathbf{B}(r)} \alpha'_{\mathbf{m}}(Y_\delta(\varepsilon)) \supset Q\left(\varepsilon'/\|c\|_\nu^{(r)}\right), \tag{4.9}$$

where

$$\|c\|_\nu^{(r)} = \max_{i=1,\dots d} \max\left\{\left|c_i^r\right|_\nu, \left|c_i^{-r}\right|_\nu\right\}$$

for every $r \geq 1$. We claim that, for every $r \geq 1$ and $a \in R_c$,

$$\left|\hat{f}'(a)\right| \leq \omega_r^\delta(f', \alpha', \varepsilon) \tag{4.10}$$

whenever

$$a \in P\left(\varepsilon'/\|c\|_\nu^{(r)}\right) = \left\{b \in R_c : \chi^{(b)}\left(Q\left(\varepsilon'/\|c\|_\nu^{(r)}\right)\right) = \mathbb{S}\right\}. \tag{4.11}$$

Indeed, write $Y^{(a)} = \ker\left(\chi^{(a)}\right) \subset Y$ for the kernel of the homomorphism $\chi^{(a)} : Y \longmapsto \mathbb{S}$ and choose a Borel set

$$B^{(a)} \subset Q\left(\varepsilon'/\|c\|_\nu^{(r)}\right) \subset \bigcap_{\mathbf{m}\in\mathbf{B}(r)} \alpha'_{\mathbf{m}}(Y_\delta(\varepsilon))$$

which intersects each coset of $Y^{(a)} \subset Y$ in exactly one point ([**P**], Lemma 1.5.1). Next choose a probability measure μ on Y such that $\mu\left(B^{(a)}\right) = 1$

and $\int_{Y^{(a)}} \mu(B+y)\,d\lambda_{Y^{(a)}}(y) = \lambda(B)$ for all $B \in \mathfrak{P}_Y$, and set $h'(y) = \int f'(y+z)\,d\mu(z)$ for every $y \in Y^{(a)}$. Then $|h'(y) - f'(y)| \le \omega_r^\delta(f', \alpha, \varepsilon)$ for all $y \in Y$, $0 = \int_Y \chi^{(a)}\,d\lambda_Y = \int_{Y^{(a)}} \int_Y \chi^{(a)}(y+z)\,d\mu(z)\,d\lambda_Y(y) = \int_Y \chi^{(a)}(z)\,d\mu(z)$, and $\hat{f}'(a) = \int \left(\int h'(z)\overline{\chi^{(a)}(z)}\,d\lambda_y(z)\,d\lambda_Y(y) = 0\right)$, which proves (4.10).

The next step is to investigate, for every $\zeta > 0$, the set

$$P(\zeta) = \left\{a \in R_c : \chi^{(a)}(Q(\zeta)) = \mathbb{S}\right\}.$$

There exists, for every $\nu \in \mathfrak{P}_\infty$ a $t_\nu(\zeta) > 0$ such that

$$\chi_\nu(Q_\nu(\zeta)) = \{e^{2\pi i t} : |t| \le t_\nu(\zeta)\} \subset \mathbb{S};$$

for $\nu \in \mathfrak{P}_f(c)$ there exists a unique integer $m(\nu,\zeta) \ge 1$ such that $Q_\nu(\zeta) = \pi_\nu^{m(\nu,\zeta)}\mathfrak{o}_\nu$ and hence $\chi_\nu(Q_\nu(\zeta)) = \left\{e^{2\pi i k \left|\pi_\nu^{m(\nu,\zeta)}\right|_\nu} : k \in \mathbb{Z}\right\}$, and we set $t_\nu(\zeta) = \left|\pi_\nu^{-m(\nu,\zeta)}\right|_\nu$. A nonzero element $a \in R_c$ lies in $P(\zeta)$ if

$$\sum_{\nu \in \mathfrak{P}_\infty} t_\nu(\zeta)|a|_\nu \ge \frac{1}{2}, \tag{4.12}$$

or if there exists a $\nu \in \mathfrak{P}_f(c)$ with

$$\left(\sum_{\nu' \in \mathfrak{P}_\infty} t_{\nu'}(\zeta)|a|_{\nu'}\right) \cdot t_\nu(\zeta) \ge \frac{1}{2}. \tag{4.13}$$

We also note that, for every $\nu \in S$ and $r \ge 1$,

$$t_\nu\left(\zeta/\|c\|_\nu^{(r)}\right) = t_\nu(\zeta)/\|c\|_\nu^{(r)}. \tag{4.14}$$

For every nonzero element $a \in R_c$ we put $P(a,r,\varepsilon') = \{k \in \mathbb{Z}\colon c^{k\mathbf{n}}a \in P(r,\varepsilon')\}$. Then we can estimate the cardinality $|P(a,r,\varepsilon')|$ as follows. According to property (2) of $\mathbf{n}$ there exists a valuation $\nu \in \mathfrak{P}_\infty$ with $|c^{\mathbf{n}}|_\nu > 1$, and we denote by $M = M(a,\nu)$ the smallest nonnegative real number such that $2\,|c^{r\mathbf{n}}|_\nu^M\, t_\nu(\varepsilon')|a|_\nu \ge \|c\|_{\nu'}^{(r)}$ for every $r \ge 1$. If there exists a $\nu' \in \mathfrak{P}_\infty$ such that $|c^{\mathbf{n}}|_\nu < 1$, then we can find a smallest nonnegative real number $M' = M'(a,\nu') \ge 1$ such that $2\,|c^{-r\mathbf{n}}|_{\nu'}^{M'}\, t_{\nu'}(\varepsilon')|a|_{\varepsilon'} \ge \|c\|_{\nu'}^{(r)}$ for every $r \ge 1$. If no such ν' exists, then the property (3) of $\mathbf{n}$ and (4.14) imply that there is a valuation $\nu' \in \mathfrak{P}_f(c)$ and a smallest nonnegative real number $M' = M'(a,\nu')$ such that $2\left(\sum_{\xi \in \mathfrak{P}_\infty} t_\xi(\varepsilon')|a|_\xi\right) \cdot |c^{-r\mathbf{n}}|_{\nu'}^{M'} \ge \|c\|_{\nu'}^{(r)}$ for every $r \ge 1$. In either case (4.12)–(4.13) imply that $|P(a,r,\varepsilon')| \le r(M(a,\nu) + M'(a,\nu'))$ for all $r \ge 1$, and hence that $\left|\left\{k \in \mathbb{Z} : \left|\hat{h}'\left(c^{k\mathbf{n}}a\right)\right| > \omega_r^\delta(h', \alpha, \varepsilon)\right\}\right| \le r(M(a,\nu) +$

$M'(a,\nu'))$ for all $r \geq 1$. Since h' has summable variation this shows that $\sum_{k\in\mathbb{Z}} \left|\hat{h}'\left(c^{\mathbf{n}k}a\right)\right| < \infty$ for every nonzero element $a \in R_c$.

If we replace a by $c^{-l\mathbf{m}}a$ for $l \geq 1$ then $M\left(c^{-l\mathbf{m}}a,\nu\right) \leq M(a,\nu)$ and $M'\left(c^{-l\mathbf{m}}a,\nu'\right) \leq M'(a,\nu')$, and there exists an $l' \geq 1$ with $M\left(c^{-l\mathbf{m}}a,\nu\right) = M'\left(c^{-l\mathbf{m}}a,\nu'\right) = 0$ for every $l \geq l'$. Hence $\lim_{l\to\infty} \sum_{k\in\mathbb{Z}} \left|\hat{h}'\left(c^{k\mathbf{n}-l\mathbf{m}}a\right)\right| = 0$, which proves (4.7) (after replacing $\mathbf{m}$ by $-\mathbf{m}$). □

Lemma 4.8. *Let $d > 1$, and let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal with $p(\mathfrak{p}) = r(\mathfrak{p}) = 0$. Suppose furthermore that $\mathfrak{K}$, $\mathfrak{L}$ are Noetherian $\mathfrak{R}_d$-modules with the following properties.*

(1) *The $\mathbb{Z}^d$-actions $\alpha^{\mathfrak{K}}$ and $\alpha^{\mathfrak{L}}$ are expansive and mixing;*

(2) *Every prime ideal $\mathfrak{q} \subset \mathfrak{R}_d$ associated with $\mathfrak{K}$ satisfies that either $p(\mathfrak{q}) = r(\mathfrak{q}) = 0$, or $p(\mathfrak{q}) > 0$ and $r(\mathfrak{q}) = 1$;*

(3) *$\mathfrak{L}$ has a prime filtration $\{0\} = \mathfrak{L}_0 \subset \cdots \subset \mathfrak{L}_s = \mathfrak{L}$ with $\mathfrak{L}_j/\mathfrak{L}_{j-1} \cong \mathfrak{R}_d/\mathfrak{p}$ for $j = 1,\dots,s$.*

Let $X_1 = X^{\mathfrak{K}}$, $X_2 = X^{\mathfrak{L}}$, $X = X_1 \times X_2 = X^{\mathfrak{K}\oplus\mathfrak{L}}$, $\alpha = \alpha^{\mathfrak{K}} \times \alpha^{\mathfrak{L}}$, and let $c : \mathbb{Z}^d \times X \longmapsto \mathbb{R}$ be a cocycle with α-summable variation. Then c is continuously cohomologous to the cocycle $E_1(c) : \mathbb{Z}^d \times X \longmapsto \mathbb{R}$ defined as in Lemma 4.2.

Proof. The proof of this lemma is completely analogous to that of Lemma 4.6, except that we use Lemma 4.7 instead of 4.5. □

Proof of Theorem 2.1 (1). Let $\mathfrak{M} = \hat{X}$ be the Noetherian $\mathfrak{R}_d$-module arising from (3.1)–(3.2) (cf. Theorem 3.2), and let $\{\mathfrak{p}_1,\dots,\mathfrak{p}_m\}$ be the set of prime ideals associated with $\mathfrak{M}$. Lemma 3.3 implies the existence of a Noetherian $\mathfrak{R}_d$-module $\mathfrak{N}$ with associated primes $\{\mathfrak{p}_1,\dots,\mathfrak{p}_m\}$ and of an injective module-homomorphism $\phi : \mathfrak{M} \longmapsto \mathfrak{N}$ such that $\mathfrak{N} = \mathfrak{N}^{(1)} \oplus \cdots \oplus \mathfrak{N}^{(m)}$, where each $\mathfrak{N}^{(j)}$ has a prime filtration $\{0\} = \mathfrak{N}_0^{(j)} \subset \cdots \subset \mathfrak{N}_{r_j}^{(j)}$ with $\mathfrak{N}_k^{(j)}/\mathfrak{N}_{k-1}^{(j)} \cong \mathfrak{R}_d/\mathfrak{p}_j$ for $k = 1,\dots,r_j$. We assume without loss in generality that exist integers s, s', s'', $0 \leq s \leq s' \leq s'' \leq m$, such that $p(\mathfrak{p}_j) = r(\mathfrak{q}_j) = 0$ for $j = 1,\dots,s$, $p(\mathfrak{p}) > 0$ and $r(\mathfrak{q}_j) = 1$ for $j = s+1,\dots,s'$, $p(\mathfrak{p}) = 0$ and $r(\mathfrak{q}_j) \geq 1$ for $j = s'+1,\dots,s''$, and $p(\mathfrak{p}) > 0$ and $r(\mathfrak{p}_j) \geq 2$ for $j; s''+1,\dots,m$. From Theorem 3.1 we know that $\alpha^{\mathfrak{N}^{(j)}}$ is mixing for $j = 1,\dots,m$. We write $\psi : X' = X^{\mathfrak{N}} \longmapsto X = X^{\mathfrak{M}}$ for the surjective homomorphism dual to ϕ and define a cocycle $c' : \mathbb{Z}^d \times X' \longmapsto \mathbb{R}$ with summable variation by $c'(\mathbf{k},\cdot) = c(\mathbf{k},\psi(\cdot))$. Repeated application of Lemma 4.4 shows that c' is continuously cohomologous to the cocycle $E_{(1,\dots,s'')}(c')$, where $E_{(1,\dots,j)}(c') : \mathbb{Z}^d \times X^{\mathfrak{N}} \longmapsto \mathbb{R}$ is defined as in the last paragraph of Section 2 for every $j = 1,\dots,m$. By applying Lemma 4.2 repeatedly we see that $E_{(1,\dots,s'')}(c')$ (and hence c') is continuously cohomologous to $E_{(1,\dots,s')}(c')$, and

Lemma 4.6 implies that c' is continuously cohomologous to $E_{(1,\dots,s)}(c')$. Finally we use Lemma 4.8 to conclude that c' is continuously cohomologous to the homomorphism $\mathbf{k} \mapsto \int_{\lambda_{X'}} c'(\mathbf{k},\cdot)d\lambda_{X'}$. We choose a continuous function $b' : X' \longmapsto \mathbb{R}$ such that $c'(\mathbf{k},x) = \int_{\lambda_{X'}} c'(\mathbf{k},\cdot)d\lambda_{X'} + b'\cdot\alpha^{\mathfrak{N}}_{\mathbf{k}} - b'$ for every $\mathbf{k} \in \mathbb{Z}^d$, set $b = E_{\lambda_{X'}}(b|\psi^{-1}(\mathfrak{P}_X))$, and obtain that $c(\mathbf{k},x) = \int_{\lambda_X} c(\mathbf{k},\cdot)d\lambda_X + b\cdot\alpha_{\mathbf{k}} - b$ for every $\mathbf{k} \in \mathbb{Z}^d$, which completes the proof. $\square$

Proof of Theorem 2.1 (2). Let $\mathfrak{p}_1,\dots,\mathfrak{p}_m$ be the prime ideals associated with the $\mathfrak{R}_d$-module $\mathfrak{M} = \hat{X}$, and assume as in the proof of part (1) of this theorem that there exist integers s, s', s'', $0 \le s \le s' \le s'' \le m$, with the properties described there. We fix $j \in \{s+1,\dots,m\}$ for the moment and consider the subgroup $\Gamma = \Gamma_j \cong \mathbb{Z}^{r(\mathfrak{p}_j)}$ associated with the prime ideal $\mathfrak{p}_j$ by Proposition 3.4 or 3.6. Fix the polynomials $f_{r(\mathfrak{p}_j)+1},\dots,f_d$ in the proof of the relevant proposition, denote by $\mathcal{C}(f_i) \subset \mathbb{R}^d$ the convex hull of the support $\mathcal{S}(f_i) \subset \mathbb{Z}^d$, and write $H_i^{(k)}$, $k = 1,\dots,l_j$, for the finitely many distinct hyperplanes which are parallel to the faces of $\mathcal{C}(f_i)$. Then any primitive subgroup $\Gamma_j \cong \mathbb{Z}^r$ in $\mathbb{Z}^d$ will satisfy the conditions (1)–(3) in Proposition 3.4 or 3.6, if it is not contained in any of these hyperplanes. By varying $j \in \{s+1,\dots,m\}$ we obtain a finite collection of hyperplanes to be avoided. Furthermore, if we fix $j \in \{1,\dots,s\}$, and if $V_{\mathbb{C}}(\mathfrak{p}_j)$ is the variety of $\mathfrak{p}_j$ (cf. Lemma 4.7), then there exist finitely many elements $\mathbf{v}_j^{(k)} \in \mathbb{R}^d$, $k = 1,\dots,l_j$, such that $|c_1|^{v_1}\cdots|c_d|^{v_d} \ne 1$ for all $c = (c_1,\dots,c_d) \in V_{\mathbb{C}}(\mathfrak{p}_j)$ whenever $\mathbf{v} = (v_1,\dots,v_d) \in \mathbb{R}^d$ is not orthogonal to any of the $\mathbf{v}_j^{(k)}$, $k = 1,\dots,l_j$. In particular, $\alpha_{\mathbf{m}}^{\mathfrak{R}_d/\mathfrak{p}_j}$ is expansive whenever $\mathbf{m}$ is not orthogonal to any $\mathbf{v}_j^{(k)}$, $k = 1,\dots,l_j$, and by varying $j \in \{1,\dots,s\}$ we obtain a finite collection of hyperplanes of $\mathbb{R}^d$ which have to be avoided. By taking into account all these restrictions we can choose a primitive element $\mathbf{n} \in \mathbb{Z}^d$ and, for every $j = s+1,\dots,m$, a primitive subgroup $\Gamma_j \cong \mathbb{Z}^{r(\mathfrak{p}_j)}$ in $\mathbb{Z}^d$ which satisfies the requirements of Proposition 3.4 or 3.6, such that $\alpha^{\mathfrak{R}_d/\mathfrak{p}_j}$ is expansive for $j = 1,\dots,s$, and $\mathbf{n} \in \Gamma_j$ for $j = 2+1,\dots,m$.

We choose $\phi : \mathfrak{M} \longmapsto \mathfrak{N} = \mathfrak{N}^{(1)} \oplus \cdots \oplus \mathfrak{N}^{(m)}$ as in the proof of Theorem 2.1 (1) and write $\psi : X^{\mathfrak{N}} \longmapsto X$ for the dual surjection. For every $j \in \{s'+1,\dots,s''\}$ we define $\overline{\mathfrak{N}^{(j)}}$, $\alpha^{\overline{\mathfrak{N}^{(j)}}}$, $X^{\overline{\mathfrak{N}^{(j)}}}$, and $\imath^{\overline{\mathfrak{N}^{(j)}}} : X^{\overline{\mathfrak{N}^{(j)}}} \longmapsto X^{\mathfrak{N}^{(j)}}$ as in the discussion preceding Proposition 3.6, set

$$
\begin{aligned}
\bar{X} &= X^{\mathfrak{N}^{(j)}} \times \cdots \times X^{\mathfrak{N}^{(s')}} \times X^{\overline{\mathfrak{N}^{(s'+1)}}} \times \cdots \times X^{\overline{\mathfrak{N}^{(s'')}}} \\
&\qquad \times X^{\mathfrak{N}^{(s''+1)}} \times \cdots \times X^{\mathfrak{N}^{(m)}}, \\
\bar{\alpha} &= \alpha^{\mathfrak{N}^{(j)}} \times \cdots \times \alpha^{\mathfrak{N}^{(s')}} \times \alpha^{\overline{\mathfrak{N}^{(s'+1)}}} \times \cdots \times \alpha^{\overline{\mathfrak{N}^{(s'')}}} \\
&\qquad \times \alpha^{\mathfrak{N}^{(s''+1)}} \times \cdots \times \alpha^{\mathfrak{N}^{(m)}},
\end{aligned}
$$

and consider the surjective homomorphism $\psi \cdot \imath : \bar{X} \longmapsto X$ induced by applying either the identity map or $\imath^{\mathfrak{N}^{(j)}}$ to each factor of $\bar{X}$, and by composing the resulting map from $\bar{X}$ to $X^{\mathfrak{N}}$ with $\psi : X^{\mathfrak{N}} \longmapsto X$. Our choice of $\mathbf{n}$ guarantees that $\bar{\alpha}_{\mathbf{n}}$ is ergodic and has weak $(\mathbf{n}, \xi)$-specification on $\bar{X}$ for some $\xi \in (0,1)$ (Corollaries 3.5 and 3.7). We put $h = c(\mathbf{n}, \cdot)$ and know from Theorem 2.1 (1) that there exists a continuous function $b : X \longmapsto \mathbb{R}$ such that $h - \int_X h \, d\lambda_X = b \cdot \alpha_{\mathbf{n}} - b$. Put $\bar{h} = h \cdot \imath : \bar{X} \longmapsto \mathbb{R}$, $\bar{b} = b \cdot \imath : \bar{X} \longmapsto \mathbb{R}$, and note that $\bar{h} - \int_{\bar{X}} \bar{h} \, d\lambda_{\bar{X}} = \bar{b} \cdot \bar{\alpha}_{\mathbf{n}} - \bar{b}$. Hence the cocycle $c_{\bar{h}}^{(\mathbf{n})} : \Delta_{\bar{\alpha}}(\mathbf{n}, \xi) \times \bar{X} \longmapsto \mathbb{R}$ in (2.11) vanishes, and Proposition 2.6 (with $X = Y = \bar{X}$) implies that $\bar{b}$ is $\bar{\alpha}$-Hölder (note that $\bar{b}$ is determined uniquely up to a constant). By Lemma 2.5, b is Hölder, and the ergodicity of $\alpha_{\mathbf{n}}$ implies that c is cohomologous to a homomorphism, with transfer function b. □

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