PACIFIC JOURNAL OF MATHEMATICS
Vol. 170 No. 2 1995

EQUISINGULARITY THEORY FOR PLANE CURVES WITH EMBEDDED POINTS

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It is studied the foundations of a theory of equisingularity for plane algebroid or complex analytic curves, reduced everywhere except at one point. A definition of equivalence, or equisingularity, for two such curves, involving resolution by means of quadratic transformations is given as well as several definitions of the concept of equisingular one-parameter family of such singularities are proposed. The theory is related to both the theory of equisingularity for plane, reduced curves and that for ideals with finite support. The different notions of equisingularity for families are compared and a number of examples are presented.

Introduction.

The theory of equisingularity for plane, reduced algebroid curves is classical (cf., e.g., the treatise by Enriques and Chisini [**EC**]), it was renewed and expanded by Zariski (cf.[**Z**] and the bibliography cited there) and continued by other authors (cf., e.g., [**C**]). Basically, one attempts to determine when two singularities are equally complicated, and given a family of singularities to find criteria insuring that all its members are "equally singular". In this planar case, it's remarkable that many seemingly different definitions turn out to be equivalent (cf. e.g., [**T1**, 5.3.1]). For skew curves the situation is considerably more complicated. Anyway, it is reasonable to consider, more generally, curves which are not necesarily reduced (see, e.g., [**A**, p.33]). This seems particularly interesting in the case of skew-curves. In this situation, a reduced (even smooth) curve can degenerate into one with embedded points. In the paper [**BG**] it is systematically considered families of such curves, and the behavior of various numerical invariants of the curves in the family.

In the present article, we intend to study in some detail the case of plane algebroid (or locally analytic) curves, reduced everywhere but at one point (or a finite number of points, in the analytic case). We introduce a notion of "equivalence," completely analogous to that for reduced curves, which requires that the desingularization process of the curves, via quadratic transformations, be the same. In this non-reduced case, that process essentially amounts to desingularize the corresponding reduced curve and an ideal, having support corresponding to the embedded points. This is studied in Section 2. Before, in Section 1, we recall the necessary basic facts about reduced curves and $\mathfrak{M}$-primary ideals in a regular, two-dimensional local ring (A, $\mathfrak{M}$).

As usual, it is important to have a good notion of "equisingular family of plane curves with embedded points". There are two obvious such definitions (say, for a one-parameter family, parametrized by an open set T in $\mathbb{C}$). One is to require that the different members X_t be equivalent to each other in the sense introduced in §2. Other is based on "resolving in a nice way" the ideal of the total space of the family. A little more precisely, if the one-parameter family is given by $\pi : X \to T$, where X is a surface in an open $Z \subset \mathbb{C}^3$, defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, try to "improve" $\mathcal{I}$ by taking a monoidal transform of Z with center a curve W, such that at each $w \in W$ the stalk $\mathcal{I}_w$ is not principal; take the proper transform $\mathcal{I}_1$ of $\mathcal{I}$, and repeat the process if necessary (i.e. if $\mathcal{I}$ does not define a divisor of the blown-up variety Z_1). This was studied first by F. Pham (in [**P**]), see (3.5) and (3.6) for the precise definitions. We call these families T-equisingular and I-equisingular respectively. It turns out that these two approaches are equivalent, the proof (in §5) is based on that of Risler in [**R**], where he considers families of ideals with finite support. However, these definitions, in a sense, are not completely satisfactory from a geometric standpoint. In fact, with Pham's procedure, one gets a sequence of "ambient spaces" $Z = Z_0, Z_1, \dots, Z_r$, each one a blowing-up of the previous one along a smooth curve and on each space an ideal $\mathcal{I}_i \subset \mathcal{O}_{Z_i}$, the "proper transform" of $\mathcal{I}_{i-1}$. Of course $\mathcal{I}_i$ defines a subspace $X_i \subset Z_i$, and there is a natural projection $X_i \to T$, for each i, these would be the successive "proper transforms" of the "original family" $X \to T$. But, even if the original family is I -(or T-) equisingular, the morphisms $X_i \to T$ are not necessarily flat. If, as usual, we demand the "flatness condition" to talk about families, this means that we cannot really simultaneously "desingularize" the flat family $X \to T$ in a reasonable way obtaining flat families throughout the process. So, a stronger equisingularity condition would require, moreover, the flatness of the morphisms $X_i \to T$ mentioned above. We call this C-equisingularity. In general this is strictly stronger than I-equisingularity, but there is an important case where these notions are equivalent. Namely, when all the ideals $\mathcal{I}\,\mathcal{O}_{X_t}$ (i.e. the ideals induced by $\mathcal{I}$ on the different fibers) are complete, or integrally closed. This is proved in §5. This condition turns out to be equivalent (under the assumption of I-equisingularity) to requiring that the sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ be integrally closed. To our understanding, the latter fact is not completely trivial. One has to prove first the corresponding statement for families of ideals with finite support, which seems to be rather technical. This is done in §4. Other properties of equisingular families are discussed in §5. Perhaps other criteria developed by Zariski in the reduced case will remain valid, with suitable modifications, in the present situation. We hope to address these questions in the future, as well as to study the case of skew-curves; although here the theory is probably much more complicated.

In the present paper we work over an algebraically closed field of zero characteristic and, in the last three sections, over $\mathbb{C}$, with the language of local Analytic Geometry, in order to simplify the presentation. In section 0 we fix the terminology and conventions.

I want to thank B. Johnston and J. Verma who introduced me to the theory of complete ideals in two dimensional rings (also initiated by Zariski) by means of a series of very nice seminar talks, J. Castellanos for several fruitful discussions on the theory of equisingularity, and E. Casas Alvero, who discovered an error (concerning the weighted tree of a curve, cf. (1.1)) in a previous version of this paper.

0. Notation and Conventions.

Throughout this paper, the following notation is used:

1. $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ denote the natural, integral, real and complex numbers respectively; we consider 0 to be an element of $\mathbb{N}$.
2. All rings are commutative with an identity, which is respected by homomorphisms. If A is a ring, and I and M are ideals of A, then $\nu_M(I)$ is the M-order of the ideal I, i.e, $\max\{r : I \subset M^r\}$ (this could be ∞). If $f \in A, \nu_M(f)$ is the M-order of the principal ideal (f), The symbol $r(I)$ denotes the radical of the ideal I.
3. If A is a ring and I an ideal of A, the integral closure (or completion) of I is the set $\bar{I} = \{b \in A : b^n + \sum_{i=1}^{n} a_i b_i = 0, a_i \in I^i$ for all i and n a suitable natural number $\}$, which is an ideal of A. The ideal I is integrally close or complete if $I = \bar{I}$.
4. If X is a scheme or an analytic space, $\mathcal{O}_X$ denotes the structural sheaf. If $\mathcal{I} \subset \mathcal{O}_X$ is a sheaf of ideals, we also say that $\mathcal{I}$ is an $\mathcal{O}_X$-Ideal (with capitals) or just an Ideal, if X is clear. The support of an Ideal $\mathcal{I}$ is $\{x \in X : \mathcal{I}_x \neq \mathcal{O}_{X,x}\}$.
5. We follow the standard terminology of Algebraic and Analytic Geometry as presented, e.g., in [**H**] and [**HI**] (appendix by B. Moonen).

1. Review of Known Results on Reduced Curves.

1.1. Let us recall some basic facts on the theory of equisingularity for plane, reduced curves (cf. [**Z**]). Let C be an algebroid plane reduced curve, defined by $f = 0$ (i.e. $C = \operatorname{Spec} k[[x,y]]/(f), f \in k[[x,y]]$ being a series without multiple factors). A weighted tree will be one in which to each vertex it is associated a vector w in $\mathbb{N}^n$ (where n may vary from vertex to vertex). Let C be as above, and consider an ordering γ of its braches, or irreducible components, $\gamma = (\gamma_1, \cdots, \gamma_s)$. We associate to the pair (C, γ) a weighted tree $T_1(C, \gamma)$ as follows. We take the quadratic transformation $X_1 \to \mathcal{A}^2 = \operatorname{Spec} k[[x,y]]$, i.e, the blowing up of $\mathcal{A}^2$ with center the closed point; we consider the proper transform C' of C (defined by $\mathcal{J} := \mathcal{E}^{-n}(f)\mathcal{O}_{X_1}$, where $\mathcal{E}$ is the sheaf of ideals defining the exceptional divisor E and n the multiplicity of C), let $\{P_1, \ldots, P_r\} = E \cap C'$, and $C_i = \operatorname{Spec}\left(\widehat{\mathcal{O}}_{X_1,P_i}/\widehat{\mathcal{J}}_{P_i}\right), i = 1, \ldots, r$. Then, the algebroid curve $C_1 \amalg \ldots \amalg C_r$ (having r connected component) is, by definition the (algebroid) proper transform of C. Next we repeat the process using the fact that $\widehat{\mathcal{O}}_{X_1,P_i} \approx k[[x,y]]$ and C_i is defined by a single, reduced equation. The tree $T_1(C)$ is obtained by considering a vertex for each connected algebroid curve that appears in this process, with the obvious incidence relations, and to each vertex (say, that corresponding to a curve with t irreducible components $\delta_1, \cdots, \delta_t$, arranged in the order induced by γ) we associate the vector $(m_1, \cdots, m_t)$, where m_i is the multiplicity of δ_i. (Actually, this is naturally a directed tree with a root, or "lowest vertex".) Then, given two plane reduced algebroid curves C, D, one says that they are equivalent, or (Zariski) equisingular, if (for suitable orderings γ, δ of their branches), their corresponding trees $T_1(C, \gamma)$ and $T_1(D, \gamma)$ are isomorphic (as weighted trees). It is well known ([Z]) that this definition is equivalent to many other "reasonable" possible definitions. However, in case C has more than one branch, it would be erroneous to describe Zaraiski-equisingularity by means of the tree introduced above, but where the weight associated to a vertex is the multiplicity of the corresponding curve. For instance, the curves $C : (y^3 - x^4)(y^4 - x^3) = 0$ and $D : (y^2 - x^3)(y^5 - x^4) = 0$ have isomorphic weighted graphs (using the definition just given), but they are not Zariski-equivalent. The tree $T_1(C, \gamma)$ will have, eventually, vertices with weight one only, i.e., eventually , we eliminate the singularities of C by means of quadratic transformations. We let $\sigma(C) = \min\{m \in \mathbb{N} :$ a sequence of m suitable quadratic transformations desingularizes $C\}$.

1.2. Let us recall some well-known facts about $\mathfrak{M}$-primary ideals in $k[[x,y]]$, $\mathfrak{M} = (x,y)$ (cf. [**ZS**, appendix 5]; [**L2**]). One defines the transforms of such an ideal I as follows: let $X_1 \to \mathcal{A}^2, \mathcal{E}, E$ be as in (1.1), then the support of $\mathcal{I}_1 = \mathcal{E}^{-m} I\mathcal{O}_{X_1} (m = \mathfrak{M}-$ order of $I)$ has finitely many points $Q_1, \ldots, Q_s$ (necessarily in E). Then $I_i = \mathcal{I}_1\widehat{\mathcal{O}}_{X_1,Q_i}, i = 1, \ldots, s$, are the (first order) strict, or proper transforms of I (of course, here we assume $s \geqslant 1$, if $\{Q_1, \ldots, Q_s\} = \emptyset$, the proper transforms are undefined); whereas $I\widehat{\mathcal{O}}_{X_1,Q_i}, i = 1, \ldots, s$ are the (first order) total transforms of I. Repeating the process one gets the higher order transforms. It is well known that any first order proper transform I_i of I satisfies: $e(I_i) < e(I)$ (multiplicities in the appropriate ambient ring, see, e.g., [**R**, p. 5, Cor.1]); also $\nu(I_i) \leq \nu(I)$, where ν is the order with respect to the maximal ideal. We may associate to I a weighted directed tree $\tau(I)$ (where the weights are non-negative integers), each vertex corresponding to each proper transform of I, with obvious incidence relations, and where to each vertex we associate the order of the corresponding ideal. Then, $\tau(I)$ is a finite tree (because of the formula $e(I_i) < e(I)$ quoted above). Let $\sigma_1(I)$ be the number of quadratic transforms necessary in order to obtain the whole tree $\tau(I)$, if I is the unit ideal we set $\sigma_1(I) = 0$.

One defines, following Risler ([**R**, Section 2]): two $\mathfrak{M}$-primary ideals I, J are equivalent, or equisingular, if $\tau(I) \approx \tau(J)$, as weighted trees. It is well known that $\tau(I) = \tau(\bar{I})$, where $\bar{I}$ is the integral closure of I in $R = k[[x,y]]$. One might hope that if I, J are $\mathfrak{M}$-primary ideals and $\tau(I) = \tau(J)$, then $\bar{I}$ and $\bar{J}$ are isomorphic, in the sense that $k[[x,y]]/\bar{I} \approx k[[x,y]]\bar{J}$ (isomorphism of k-algebras). However this is not true, as the example of $I = (y^2, xy, x^4)$ and $J = (y^2, x^2y, x^3)$ (both complete ideals) show. They are not isomorphic (they have distinct Hilbert-Samuel functions) but $\tau(I) = \tau(J)$, see [Br], p. 78; in the same section there are other examples.

2. Equivalent Plane Curves.

2.1. By a plane algebroid curve we mean a closed subscheme C of $\mathcal{A}^2 = \operatorname{Spec} k[[x,y]]$ of dimension one. Thus, letting $R = k[[x,y]], \mathfrak{M} = (x,y), C = \operatorname{Spec}(R/I)$ and I is either a principal ideal or an ideal having a (minimal) primary decomposition $I = \mathcal{P}_1 \cap \ldots \cap \mathcal{P}_r \cap \mathrm{M}$, where each $\mathcal{P}_i$ has height 1, $r \geqslant 1$, and $r(\mathrm{M}) = \mathfrak{M}$. If $C - \{\mathfrak{M}\}$ is reduced we'll say that C is generically reduced, in this case $\mathcal{P}_1 \cap \ldots \cap \mathcal{P}_r$ is a radical ideal (necessarily principal, generated by series $f \in R$ without multiple factors). Note that using the fact that R is a unique factorization domain, we may write for the ideal I of a plane curve:

$$I = (f) \cdot \mathcal{N} \subset k[[x,y]] \tag{2.1.1}$$

where $f \in R$ and either $r(\mathcal{N}) = \mathfrak{M}$ or $\mathcal{N} = R$ (in the latter case there are no embedded points). It is easy to check that $\mathcal{N} = (I : I_0)$, where $I_0 = \mathcal{P}_1 \cap \ldots \cap \mathcal{P}_r$ (using the previously introduced notation). It is also possible to obtain the factorization (2.1.1) from the Hilbert-Burch Theorem on ideals of homological dimension equal to one, cf. [**Bu**] or [**BG**, Statz 6.1]; this approach shows that $\mathcal{N}$ is a determinantal ideal ([**BG**, Section 6]).

2.2. In the sequel we shall consider algebroid plane curves C which are generically reduced. Let C be defined by an ideal $I = (f) \cdot \mathcal{N}$ (as in (2.1.1)). Then C_{red} (the reduced curve associated to C) will be defined by the principal ideal (f). We may introduce infinitely near points as follows. Consider $\pi_1 : X_1 \to \operatorname{Spec} R$, the blowing-up of center $\{\mathfrak{M}\}$, let E be the exceptional divisor, defined by the ideal $\mathcal{E} \subset \mathcal{O}_{X_1}$. To define the proper transform of C, let n (resp. m) be order of f (resp. of $\mathcal{N}$). Consider

$$I_1 = (\mathcal{E}^{-n-m})(f) \cdot \mathcal{N} \subset \mathcal{O}_{X_1}. \tag{2.2.2}$$

Then, $E \cap \operatorname{supp}(I_1)$ is a finite set $\{P_1, \dots, P_r\}$. Then, by definition, $P_1, \dots, P_r$ are the points (infinitely close to the origin of C) in the first neighborhood of C; the first order proper transform of C is the disjoint union of $C_1^{(1)}, \dots, C_r^{(1)}$, where $C_i^{(1)}$ is the subscheme of Spec $\left(\widehat{\mathcal{O}}_{X_1,P_i}\right) \approx \mathcal{A}_2$ defined by $I_1\widehat{\mathcal{O}}_{X_1,P_i}$. This ideal is called the proper transform of the ideal (2.2.1) at P_i. Now assuming that C has indeed an embedded point at the origin, there are three possibilities for each "connected component" C_i (we'll check this in a moment): (a) C_i is again a curve with an embedded point at the origin, (b) C_i is a reduced curve, (c) C_i is a subscheme of Spec $\left(\widehat{\mathcal{O}}_{X_1,P_i}\right) \approx \mathcal{A}^2$ whose support is the "origin", i.e. the closed point of $\mathcal{A}^2$. In each case, we may continue the process: in case (a), C_i is in the same conditions as C, and we repeat what we did, in (b) and (c), we are in the situations described in §1 , we take proper transform as in that section.

Let us check the claims just made. We may choose coordinates (x, y) such that $I = (f) \cdot \mathcal{N}$, where, if $F = in(f)$ and G is the greatest common divisor of $\{in(g) : g \in \mathcal{N}$ and $\nu_{(x,y)}(g) = \nu_{(x,y)}(\mathcal{N})\}$ ($in(h)$ denotes the initial form of $h \in R$ in $gr_{\mathfrak{M}}(R) \approx k[U, V]$) then $U = in(x)$ does not divide FG. Write $F = \prod_{i=1}^{u} (\alpha_i U - V)^{n_i}$, $G = \prod_{j=1}^{v} (\beta_j U - V)^{m_j}$. Now, E (the exceptional divisor) is isomorphic to $\mathbb{P}^1$; if $P \in E$ has homogeneous coordinates $(1 : l)$, and I_1 is as in (2.2.2), then: $\widehat{\mathcal{O}}_{X_1,P} \approx k[[x, x(y-l)]]$ and

$$I^{(l)} := I_1\widehat{\mathcal{O}}_{X_1,P} = (f')\mathcal{N}' \tag{2.2.3}$$

where $f' = x^{-n}f(x, x(y-l)), n = \nu(f)$; and $\mathcal{N}'$ is generated by $\{g' : g' = x^{-m}g(x, x(y-l))$ and $g \in \mathcal{N}\}$. It immediately follows that $I^{(l)}$ is the unit ideal if $l \notin \{\alpha_1, \dots, \alpha_u, \beta_1 \dots, \beta_v, \}$;if $l = \alpha_i = \beta_j$ (for suitable i, j) then, at P, we have situation (a); if $l = \alpha_i$ but $\alpha_i \notin \{\beta_i, \dots, \beta_v\}$, we are in situation (b); if $l = \beta_j \notin \{\alpha_1, \dots, \alpha_u\}$, we are in situation (c).

Thus, given our algebroid plane curve C, generically reduced, by taking proper transform we get $C_1^{(1)}, \dots, C_{r_1}^{(1)}$ in the first neighborhood, repeating we get $C_1^{(2)}, \dots, C_{t_2}^{(2)}$ in the second, etc. Each $C_j^{(i)}$ is a generically reduced curve (possibly reduced) or a (possibly non-reduced) point.

2.3. We'll see that any algebroid plane curve C, generically reduced, can be desingularized by means of quadratic transformations. More precisely, this means the following. Consider the components of the proper transform of C (in the first neighborhood), $C_1^{(1)}, \dots, C_{r_1}^{(1)}$. For each $C_i^{(1)}$ consider if possible the proper transform , with components $C_{i1}^{(2)} \dots C_{i,r_i}^{(2)}$ It might be impossible to do this, precisely in case when we are in case (c), (2.2), and the proper transform gives the unit ideal. Repeat the process with each $C_{i_1,\dots,i_s}^{(j)}$ thus obtained, and so on. The claim is that, eventually, each component $C_{i_i,\dots,i_s}^{(j)}$ will be either a smooth algebroid curve or a zero-dimensional scheme, whose transform leads to the unit ideal.

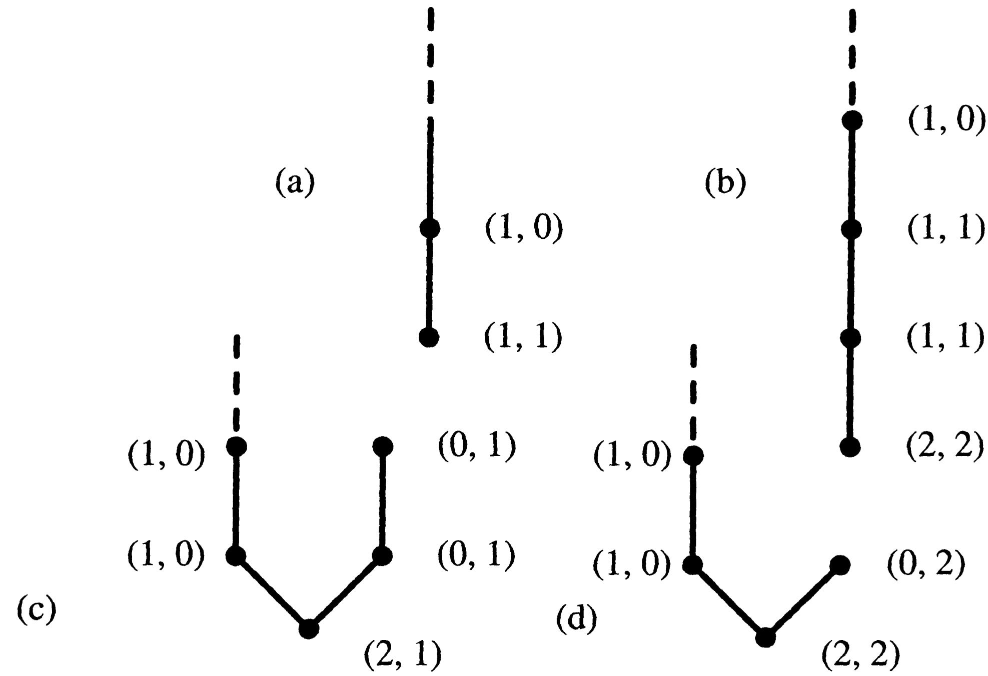
This is an immediate consequence of the desingularization theorems for curves and for ideals mentioned in Section 1. In fact, if C, defined by $I = (f)\mathcal{N}$ (as in (2.1.1)) is a generically reduced curve, define $\sigma_0(C) := \sigma(f) + \sigma_1(\mathcal{N})$. Then $\sigma_0(C) = 0$ means that $\nu(f) = 1$ ($f = 0$ is smooth) and $\nu(\mathcal{N}) = 0$. A simple calculation shows that in the first neighborhood there is just one component, which is a smooth curve. If $\sigma_0(C) > 0$, then concerning a component $C_i^{(1)}$ in the first neighborhood, either we are in case (a), i.e., this is a curve with an embedded point, clearly satisfying $\sigma_0(C_i^{(1)}) < \sigma_0(C)$, and we can use the induction hypothesis, or we are in cases (b) or (c), i.e., we are in the situation of Section 1. But then we already know that we may desingularize by means of quadratic transformations.

2.4. We may associate to a generially reduced curve C, together with an ordering γ of the branches of C_{red}, a bi-weighted directed tree $T_2(C, \gamma)$ as follows. Take the different proper transforms of C (in the first neighborhood, second, etc), $\{C_{i_1,\dots,i_r}^{(j)}\}$. Consider a vertex for each such connected component, join the vertex P and Q if the component corresponding to Q is obtained from that corresponding to P by a single proper transformation. Attach to each vertex P an ordered $(\mathbf{i},j)$ ($\mathbf{i}$ a vector, j and integer) as follows. If P corresponds to $C_{i_1,\dots,i_r}^{(j)}$, and (in the appropriate local ring, which is isomorphic to $k[[u,v]]$) this curve is defined by the ideal $(f')\mathcal{N}'$, with $\mathcal{N}'$ primary to the maximal ideal, let $\mathbf{i}= (m_1, \cdots, m_t)$ where m_s is the multiplicity of the s-th branch of the curve defined by f', in the order induced by γ, and $j = \nu(\mathcal{N}')$. According to the results of (2.3), this tree will contain at least an infinite branch, each such having, eventually bi-weights (1,0) only; and possibly some finite branches, its "final" point having weight $(0, r)$, for some $r \geqslant 1$. We also say that this is the tree associated to the ideal I of (2.2.1).

Examples. We exhibit the trees corresponding to the following ideals. The calculations are left to the reader.

(a) $I = (y)(x, y)$, (b) $I = (y^2 - x^3)(y^2, x^3)$, (c) $I = (y^2 - x^3)(y^3, x)$,

(d) $I = (y^2 - x^3)(y^4, x^2)$



2.5. Definition. Two (algebroid, plane, generically reduced) curves C, D are said to be equivalent or equisingular, (notation: $C \equiv D$) if there is an isormorphism of bi-weighted trees $T_2(C, \gamma) \approx T_2(D, \delta)$, for suitable orderings γ, δ of the branches of C, D respectively.

2.6. Remarks. (a) It follows from the given definition that if $C \equiv D$, then C is reduced if and only if D is reduced. For reduced curves, definition (2.5) agrees with Zariski-equisingularity (see (1.1))

(b) Clearly, if generically reduced curves C and D are isomorphic (as schemes), then they are equisingular. The converse is not true. E.g., as remarked in (1.2), if C is defined by the ideal $I = (f)\mathcal{N}$ (cf.(2.2.1)) and $\bar{C}$ by $\bar{I} = (f)\bar{\mathcal{N}}$ ($\bar{\mathcal{N}}$= integral closure of $\mathcal{N}$), then $T_2(C) = T_2(\bar{C})$, hence $C \equiv \bar{C}$; but if $\mathcal{N}$ is not integrally closed C is not isomorphic to $\bar{C}$. In fact, it is easily proved that if C (defined by $(f)\mathcal{N}$) is isomorphic to D (defined by $(g)\mathrm{M}$), then $R/\mathcal{N} \approx R/\mathrm{M}$ (as k-algebras). Even if $\mathcal{N}, \mathrm{M}$ are integrally closed $\mathfrak{M}$-primary ideals of R, C and D are defined by $(f)\mathcal{N}$ and $(g)\mathrm{M}$ respectively and $C \equiv D$, then it does not necessarily follow that C and D are isomorphic. Again we may use the examples (1.2): let C (defined by $(y - x)(y^2, xy, x^4)$) and D (defined by $(y - x)(y^2, x^2y, x^3)$), they are equivalent, $C \equiv D$, but not isomorphic, although here $\mathcal{N}, \mathrm{M}$ are both integrally closed.

2.7. In the article [**BG**], the authors introduce the invariants ε and δ for a (not necessarily plane) generically reduced algebroid curve. Given $C = \operatorname{Spec}(A), A = R/I$, a generically reduced curve, then $N=$ nilradical of A is an A-module of finite length ε (also $\varepsilon = \dim_k N$, as a vector space), this is the ε-invariant. The δ-invariant is $\delta = \delta^u - \varepsilon$, where $\delta^u = \dim(\bar{A}_o/A_o)$, where $A_o = A/N$ (the associated reduced ring) and the bar indicates integral closure in the total ring of fractions. This seems to be the "correct" δ-invariant in the non-reduced case. The ε-invariant is not an invariant of the equisingularity class of C, i.e., $C \equiv D$ does not always imply $\varepsilon(C) = \varepsilon(D)$. In fact, we have:

2.8. Proposition. *Let $C = \operatorname{Spec} A, A = R/I, \quad I = (f)\mathcal{N}$ (as in (2.1.2)) be a generically reduced plane curve. Then, $N=$ nilradical of A is a principal ideal, $N = (a)A, \alpha \in A$, and if r=smallest i such that $\alpha^{i+1} = 0$ and $s = \dim_k(A/\mathcal{N})$ (as a k-vector space), then $s \leqslant \varepsilon(C) \leqslant rs$*

Proof. Clearly, we have $A_o := A/N = R/(f)$, hence an exact sequence:

$$0 \to (\alpha) \to A \to A_o \to 0$$

where α is the class of f in A. Thus, N is principal. Since $A = R/f\mathcal{N}$, one easily checks that if $g_1, \dots, g_s$ are elements of R inducing a basis of $A/\mathcal{N}$, then $\{f^i g_j : i = 1, \dots, r, j = 1, \dots, s\}$ induces a generating set for the vector space $(\alpha) \subset A$. Thus, $\varepsilon(C) \leq rs$. On the other hand, we may choose $g_1 = 1$; then the elements $f, fg_2, \dots, fg_s$, induce linearly independent vectors in A (were $\sum_{i=1}^s l_i f g_i \in (f) \cdot \mathcal{N}, l_i \in k$ for all i, then $\sum_{i=1}^s l_i g_i \in \mathcal{N}$, hence $l_i = 0$ for each i, since $g_1, \dots, g_s$ are linearly independent $\mod \mathcal{N}$); thus $s \leq \varepsilon(C)$. □

2.9. If $I = (f)\mathcal{N}$ (as in (2.1.1)), where $f \in \mathcal{N}$, then the number r of (2.8) is equal to 1, in this case we get(for the curve C defined by I): $\varepsilon(C) = s = sr = \dim A/\mathcal{N}$. If $f \in \mathcal{N}$ and $\mathcal{N}$ is not integrally closed (e.g. $\mathcal{N} = (y^2, x^3)$, here x^2y is integral over $\mathcal{N}$ but not in $\mathcal{N}$), then $\varepsilon(C) > \varepsilon(\bar{C})$, $\bar{C}$ defined by $(f)\bar{\mathcal{N}}$; however $C \equiv \bar{C}$, as was seen before. It is possible to find curves C such that $s < \varepsilon(C) < rs$. For instance if C is defined by $(y^2 - x^2)(x^3, y^3, xy)$, then $r = 2, s = 5$, but $\varepsilon(C) = 6$.

2.10. What was presented in this section can be developed, with minor changes, in the context of complex local analytic geometry. The translation of the results above to the case of germs of curves (C, O) of complex curves is straightforward. Sometimes it is convenient to consider more global objects. Namely, a (locally) plane curve will be a one-dimensional complex subspace C of a smooth surface S. The curve C might have several singularities, and even isolated points. By the bi-weighted tree of such a curve we mean the following. Let $\{P_i : i \in 1\}$ be the points where $\mathcal{O}_{C,P_i}$ is not regular and one-dimensional, fix orderings γ_i of the branches of (C_{red}, P_i) for each i. Then, the bi-weighted tree $T_2(C, \gamma)$ is the disjoint union of the trees $T_2((C, P_i), \gamma_i), i \in 1$. If the curve is reduced, the "second weight" will be always zero, we may drop it, and consider the resulting (single) weighted tree. It $\mathcal{I}$ is the sheaf of ideals of $\mathcal{O}_S$ defining $C \subset S$, we also write $T_2(\mathcal{I}, \gamma) := T_2(C, \gamma)$.

3. Basic Properties of Families of Curves and Ideals.

3.1. In this section we shall use the language of local analytic geometry, to simplify the presentation and to be in agreement with the main references that we shall quote. The germ determined by an analytic space X at $P \in X$ is denoted by (X, P), or simply X if "the center" P is clear from the context.

3.2. Definition. A one-parameter family of generically reduced locally plane curves is a commutative diagram:

(3.2.1)
$$\begin{array}{ccc} X & \longrightarrow & Z \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & T & \end{array}$$

where Z is a smooth three-fold, X is a closed surface in Z, T is an open neighborhood of $O \in \mathbb{C}, \pi$ is smooth and surjective, p is flat, $X_t := p^{-1}(t)$ is a one dimensional subspace of X for all $t \in T$, moreover each point $P \in X_t$ such that $\mathcal{O}_{X_t,P}$ is not one- dimensional and reduced is isolated in X_t. Sometimes we say that the family (3.2.1) is a deformation of the "special fiber" $X_o \subset Z_o$.

3.3. Definition. A one-dimensional family of plane ideals, parametrized by an open set $T \subset \mathbb{C}^1$, is an ordered pair $(\mathcal{I}, \pi)$ where π is a smooth morphism $\pi : Z \to T$ (where Z is a smooth three-fold) and $\mathcal{I}$ is on $\mathcal{O}_Z$-Ideal (cf. [R], Section 2). The family is flat if the morphism $p : Y \to T$(induced by π, where Y is the subspace of Z defined by $\mathcal{I}$) is flat. A family of ideals induces a sheaf of ideals $\mathcal{I}(t) := \mathcal{I}\mathcal{O}_{Z_t} \subset \mathcal{O}_{Z_t}$, for all $t \in T$. A one parameter family of curves (cf. (3.2)) induces a family of ideals by taking $\mathcal{I} \subset \mathcal{O}_Z$ to be the $\mathcal{O}_Z$-Ideal defining $X \subset Z$; this is a flat family of ideals.

3.4. Theorem. *Consider a family of curves* (3.2.1) *and points* $P \in Z$ *and* $O \in T$ *such that* $\pi(P) = O$ *(note that* $\mathcal{O}_{Z,P} \approx \mathbb{C}\{x, y, t\}$ *and* $\mathcal{O}_{Z_o,P} \approx \mathbb{C}\{x, y\}$*); let* $I_o = (f)(g_1, \dots, g_m)$ *define the inclusion* $(X_o, P) \subset (Z_o, P)$*, where* $r(g_1, \dots, g_m) = (x, y)$*. Then the inclusion* $(X, P) \subset (Z, P)$ *is defined by an ideal* $I \subset \mathbb{C}\{x, y, t\}$*, of the form* $I = (F)(G_1, \dots, G_m)$*, where* $F = f + \sum_{i=1}^{r} t_i f_i, G_j = g_j + \sum_{l=1}^{r} t_l g_{jl}$ *(for suitable* $f_1, \dots, g_{ml}$ *in* $\mathbb{C}\{x, y, t\}$*)*

in such a way that $(F)\mathbb{C}\{x,y,t\}$ *(resp.* $(G_1,\dots,G_m)\mathbb{C}\{x,y,t\}$*) defines, by taking suitable representatives of the corresponding germs, a family of reduced curves (resp. a flat family of ideals with finite support).*

The proof is given in [**BG**, p. 112].

In [**P**], F. Pham proposed the following definition of equisingular family if ideals. If $(\mathcal{I},\pi)$ is a family of ideals as in (3.3), let $|\mathcal{I}|$ denote the support of $\mathcal{I}(=\{z\in Z:\mathcal{I}_z\neq \mathcal{O}_{Z,z}\})$ and $[\mathcal{I}]=\{z\in Z:\mathcal{I}_z \text{ is not smooth }\}$ (the singular locus of $\mathcal{I}$), both regarded as reduced subspaces of Z. Here, $\mathcal{I}_z$ smooth means that in a suitable system of coordinates near z, say $u_1,\dots,u_n$, we have $\mathcal{I}_z=vu_1^r$, for some $r\geqslant 1, v$ being a unit near z.

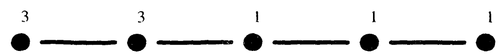
3.5. (Pham's Definition.) A family $(\mathcal{I},\pi)$ of plane ideals (as in (3.3)) is *equisingular* if the following conditions hold: (T_o) : $\pi|[\mathcal{I}]$ is smooth, (T_1) : blow up Z along $[\mathcal{I}]$ to get $Z_1' \xrightarrow{\eta_1} Z$, then $\pi\eta_1|\left[\mathcal{I}\mathcal{O}_{Z_1'}\right]$ is smooth, (T_2) : blow-up Z_1', along $[\mathcal{I}\mathcal{O}_{Z_1}]$ to get $Z_2' \xrightarrow{\eta_2} Z_1'$, then $\pi\eta_1\eta_2|\left[\mathcal{I}\mathcal{O}_{Z_2'}\right]$ is smooth, and so on until, eventually , (T_m): if $\eta_r: Z_r'\to Z_{r-1}'$ is the blowing-up of Z_{r-1}' with center $\left[\mathcal{I}\mathcal{O}_{Z_{r-1}'}\right]$, then $\pi\eta_1\dots\eta_r|[\mathcal{I}\mathcal{O}_{Z_r}]$ is smooth and $\mathcal{I}\mathcal{O}_{Z_r}$ is a normal crossings divisor.

3.6. In the case where π induces a finite morphism $[\mathcal{I}]\to T$, this definition may be rephrased in terms of proper transforms. Recall that if $W\subset Z$ is a smooth connected curve, defined by the Ideal $\mathcal{J}\subset\mathcal{O}_Z$, if $\mathcal{I}$ is an Ideal of $\mathcal{O}_Z$ and for $w\in W$ we denote by $\nu(w)$ the $\mathcal{J}_w$-order of the ideal $\mathcal{I}_w\subset\mathcal{O}_{Z,w}$, then there is an analytic open dense set $U\subset W$ such that for $w\in U,\nu(w)$ is constant; this number ν is called the generic order of $\mathcal{I}$ along W. Then, if we take the blowing-up $Z_1\to Z$ of Z with center W, the $\mathcal{O}_{Z_1}$ - Ideal $\mathcal{I}_1:=\mathcal{E}^{-\nu}\mathcal{I}\mathcal{O}_{Z_1}$, is called the proper transform of $\mathcal{I}$ (where $\mathcal{E}$ is the $\mathcal{O}_{Z_1}$-Ideal defining the exceptional divisor).

Then, definition (3.5) translates as follows, as is easily checked: here we impose conditions $P_o,\dots,P_r$: (P_o) : $\pi|[\mathcal{I}]$ is etale, (P_1): we blow up Z along a connected component of $[\mathcal{I}]$, to get $Z_1\xrightarrow{\rho_1} Z$, we demand that $\pi\rho_1|[\mathcal{I}_2]$ be etale, where $\mathcal{I}_1$ is the proper transform of $\mathcal{I}_1,(P_2)$: we blow-up Z_1 along a connected component of $[\mathcal{I}_1]$ to get $Z_2\xrightarrow{\rho_2} Z_1$, we demand that $\pi\rho_1\rho_2|[\mathcal{I}_2]$ be etale (where $\mathcal{I}_2$ is the proper transform of $\mathcal{I}_1$), etc; finally in (P_r) it must be true that the proper transform $\mathcal{I}_r$ of $\mathcal{I}_{r-1}$ satisfies: $[\mathcal{I}_r]$ is empty. In [R], Section 2, the following result is proved:

3.7. Theorem (Risler). *Let* $(\mathcal{I},\pi)$ *be a one-parameter family of plane ideals with finite supports, i.e., such that the induced morphism* $\pi:|\mathcal{I}|\to T$ *is finite. Then, the family is equisingular (in the sense of* (3.5)*) if and only if* $\tau(\mathcal{I}(0))\approx\tau(\mathcal{I}(t))$, *for* $t\in T$ *(*τ *indicating "tree", cf.* (1.2) *and* (2.10)*).*

3.8. Remarks. (a) If a family of ideals (with finite supports) $(\mathcal{I}, \pi)$ is equisingular in the sense of (3.5) (or (3.6)), it does not follow that it is flat (see (3.3)). In fact, in this case (i.e., when p is finite) flatness is equivalent to: length $(\mathcal{O}_{Z_t}/\mathcal{I}(t)) := l(\mathcal{I}(t))$ (the colength of $\mathcal{I}(t)$) is constant. But, e.g., the family given by $\mathcal{I} = (x^2, txy, y^2) \subset \mathbb{C}\{x, y, t\}$ is equisingular, but $l(\mathcal{I}(0)) = 4, l(\mathcal{I}(t)) = 3$ for $t \neq 0$, hence it is not flat . (b) Even if an equisingular family $(\mathcal{I}, \pi)$ is equisingular and flat, it does not follow that the proper transform is flat. More precisely, if $(\mathcal{I}, \pi)$ is an equisingular family of ideals with finite supports (as in (3.7)), such that $Y \xrightarrow{p} T$ is flat (where $Y \subset Z$ is the subspace defined by $\mathcal{I}$, and p is induced by π), W is a connected component of $|\mathcal{I}|, Z_1 \to Z$ the blowing-up of Z with center W and $Y_1 \subset Z_1$ is the subspace defined by the proper transform $\mathcal{I}_1$ of $\mathcal{I}$, then the induced morphism $Y_1 \to T$ is not necessarily flat. In fact, consider for instance the one parameter family given by $\mathcal{I} = (x^3, txy^5 + (1-t)x^2y^4, xy^6, y^7) \subset \mathbb{C}\{x, y, t\}$, here $|\mathcal{I}|$ is the t-axis. It is easy to check that the tree $\tau(\mathcal{I}(t))$ is always isomorphic to



and $l(\mathcal{I}(t)) = 17$, for all t, hence it is equisingular and flat. However, $\mathcal{I}_1$ corresponds to $(x^3, (tx + t(1-t)x^2)y^3, y^4)$, and $l(\mathcal{I}_1(0)) = 11, l(\mathcal{I}_1(t)) = 10$ for $t \neq 0$; hence $Y_1 \to T$ is not flat. In view of this, it makes sense to introduce:

3.9. Definition. A one-parameter equisingular family of ideals with finite supports $(\mathcal{I}, \pi)$ (cf. (3.7)) is said to be properly flat if (using the notation of (3.6)) the induced morphism $Y_i \to T$, where $Y_i \subset Z_i$ is the subspace defined by the i-th proper transform $\mathcal{I}_i$ of $\mathcal{I}$, is flat, for $i = 1, \dots, r$.

We have:

3.10. Theorem. *Let $(\mathcal{I}, \pi)$ be an equisingular one-parameter family of ideals, as in (3.7); assume $\mathcal{I}(t)\mathcal{O}_{Z_t,P}$ is integrally closed, for all t and all P in $|\mathcal{I}|$ Then, $(\mathcal{I}, \pi)$ is properly flat.*

Proof. It is an easy consequence of the following facts: (a) The fiber over t of $Y_i \to T$ is identified to the subscheme $(Z_i)_t$ (cf. (3.6)) defined by the i-th proper transform of $\mathcal{I}(t)$, this is checked in [**R**]. (b) The "Hoskins-Deligne formula: if $I \subset \mathbb{C}\{x, y\}$ is an (x, y) primary ideal, integrally closed, $I_0 = I, I_1, \dots, I_s$ are all the non-trival proper transforms of I (which are again complete ideals), and ν_i = order of I_i (in the appropriate ambient ring), then $l(J) = \sum_{i=0}^{s} \binom{\nu_i + 1}{2}$ (cf. [**L**, §3]), (c) for a finite morphism, flatness is equivalent to the constancy of the length of the fibers. □

4. Families of Complete Ideals.

In this section we study in greater detail families of ideals which are complete or integrally closed. We'll see that in this case the "pathologies" of Remark 3.8 are not possible. Our basic setup is as follows.

4.1. We consider a germ $(\mathcal{I}, \pi)$ of a 1-parameter family of plane ideals, i.e., $\pi : (Z, 0) \to (T, 0)$ is a germ of morphism, where Z (resp.T) is an open neighborhood of the origin 0 of $\mathbb{C}^3$ (resp. $\mathbb{C}^1$), and $\mathcal{I} \subset \mathcal{O}_Z$ is a sheaf of ideals, which corresponds, in the usual way, to an ideal $I \subset \mathbb{C}\{x, y, t\}$; we assume that the support of $\mathcal{I}$ is the intersection of Z with the t-axis. We assume that $(\mathcal{I}, \pi)$ is equisingular (cf. (3.5) and (3.7)). Our main goal is to prove :

4.2. Theorem. *Let $(\mathcal{I}, \pi)$ be as in* (4.1), *assume $\mathcal{I} = \bar{\mathcal{I}}$ (i.e., $\mathcal{I}$ is integrally closed, cf.* [**T1**, p. 327]*). Then, $\mathcal{I}(t) = \overline{\mathcal{I}(t)}$ for all $t \in T, t$ small enough (where, as usual, $\mathcal{I}(t) = \mathcal{I}\mathcal{O}_{Z_t}, Z_t = \pi^{-1}(t), t \in T$).*

We need several preliminary facts, some essentially well known (Lemmas 4.3 to 4.8).

4.3. Lemma. *Let $(\mathcal{I}, \pi)$ be as in* (4.1). *Then, $e(\mathcal{I}(0)) = e(\mathcal{I}(t))$, for all t in T (where $e(\mathcal{I}(t)) = e(\mathcal{I}(t), \mathcal{O}_{Z_t, P_t}), P_t = (0, 0, t)$).*
Proof. This is true because the weighted tree $\tau(\mathcal{I}(t))$ is constant, and $e(\mathcal{I}(t))$ can be expressed in terms of the weights $\nu_i : e = \sum_i \nu_i^2$ (cf. [**R**, p.6]). □

4.3.'Lemma. *Let $(\mathcal{I}, \pi)$ and I be as in* (4.1). *Then, $l'(I) = h(I)$ (where l' is the analytic spread, cf.* [**1**, Section (1)]*; $h(I)$ is the height of I).*
Proof. This is an immediate consequence of Proposition 5.1 of [**T1**]. □

4.4. Lemma. *Assume $(\mathcal{I}, \pi)$ and I are as in* (4.1). *Then, $\bar{I}$ has no embedded primes.*
Proof. This is Theorem 3 of [**L1**, (Section 1)]. □

4.5. Proposition. *Let $(\mathcal{I}, \pi)$ be as in* (4.2) *(i.e, we assume $\mathcal{I} = \bar{\mathcal{I}}$). Then, this is a flat family.*
Proof. By definition, we have to show that if $Y \subset Z$ is the subspace defined by T and $p : Y \to T$ is the induced morphism, then p is flat. But by (4.4), Y will be a Cohen-Macaulay curve, and the morphism p is finite and surjective, hence flat ([**F**, p. 154]). □

4.6. Lemma. *Let $I \subset \mathbb{C}\{x,y,t\}$ be as in* (4.1), $M = (x,y)\mathbb{C}\{x,y,t\}$, *assume* $I = \bar{I}$. *Then,* $IM = \overline{IM}$ (*in* $\mathbb{C}\{x,y,t\}$).

Proof. First, let us note that we may find discrete valuation rings $V_1, \dots, V_s$ of F (the field of fractions of $A = \mathbb{C}\{x,y,t\}$) such that $I = \bar{I} = \bigcap_{i=1}^{s}(IV_i \cap A)$. In fact, it suffices to take any morphism $f : X \to \operatorname{Spec} A$, birational and proper, such that $I\mathcal{O}_X$ is invertible and X is normal; if $D_1, \dots, D_s$ are the irreducible components of the divisor defined by $I\mathcal{O}_X$ take $V_i = \mathcal{O}_{X,P_i}$, where P_i is the generic point of D_i. Here we take f to be the normalized blowing-up of $\operatorname{Spec} A$ with center IM. With this choice, moreover $\overline{MI}V_i = MIV_i$. Since $M\mathcal{O}_{V_i}$ is principal, $i = 1, \dots, s$, from the finiteness of s if we take α, β in $\mathbb{C}$, "generally chosen", then $z = \alpha x + \beta y$ will satisfy:

$$MV_i = (z)V_i, \; i = 1, \dots, s. \tag{4.6.1}$$

Now consider $A_0 = A/(z)A$. If the coefficients α, β are suitable chosen ("general enough") the fact that I has no embedded primes (i.e, that A/I is $\mathbb{C}\{t\}$–flat) will imply that $I_0 = IA_0$ again has no embedded primes (this is elementary, but it will be proved in Lemma 4.7). Note that A_0 is again regular, two dimensional. Since MA_0 is principal, MIA_0 has no embedded primes, hence it has height one and so it is principal; hence a complete ideal in the normal ring A_0 ([**T**, 1.2]).

Let $J = \overline{MI}$. We claim that $J \subset MI + (z)$. In fact, in $A_0, (MI)A_0 \subseteq JA_0$, but clearly JA_0 is integral over $(MI)A_0$, which is complete, so $MIA_0 = JA_0$; this implies the claim. Next we claim: $J = MI + z(J : z)$. In fact, if $a \in J$, then by above's inclusion $a = b + rz$, with $b \in MI$ and $r \in A$, then $rz = a - b \in J$, i.e., $r \in (J : z)$, so $a \in MI + z(J : z)$. The other inclusion is clear. So, to show that $J := \overline{MI} = MI$, it suffices to show: $(J : z) \subset I$, which we do next. Let $r \in (J : z)$; then $rz \in J$. Then, with the notation of the beginning of the proof, $rzV_i \subset JV_i, i = 1, \dots, s$, i.e. $rzV_i \subset MIV_i$, hence $rzV_i \subset IzV_i$, for all i, then $r \in IV_i \cap A$ for all i, i.e., $r \in I$, as wanted. □

Note. I am indebted to Jugal Verma, who presented a proof of this Lemma in the case of an (x,y)-primary ideal in $k[[x,y]]$, this proof as well as the proof of (4.8) are directly inspired by his.

As promised, we prove the result on embedded primes used in (4.6).

4.7. Lemma. *Let $A = \mathbb{C}\{x,y,t\}, I$ an ideal of A such that $r(I) = (x,y)A$ and having no embedded primes. Then for α, β generally chosen in $\mathbb{C}$, the ideal $I(A/(\alpha x + \beta y))$ has no embedded primes.*

Proof. Let $A_{\alpha\beta} = A/(\alpha x + \beta y), I_{\alpha,\beta} = IA_{\alpha\beta}, R = \mathbb{C}\{t\}$. Since $A_{\alpha,\beta}/I_{\alpha,\beta}$ is a finite R-algebra, $I_{\alpha\beta}$ will have no embedded primes if and only if $A_{\alpha\beta}/I_{\alpha\beta}$

is R-flat. Let $B = A/I$, note that $A_{\alpha\beta}/I_{\alpha\beta} \approx B/(f_{\alpha\beta})$, where $f_{\alpha\beta}$ is the class of $\alpha x + \beta y$ in B; so we must show that $B/(f_{\alpha\beta})$ is R-flat (α, β suitably chosen in $\mathbb{C}$).

Let N be the nilradical of B. Clearly since $r(I) = (x, y)$, we have $N = (x, y)B$. We have an exact sequence

$$O \to N \to B \to R \to O \tag{4.7.1}$$

and since $(x, y)^m \subset I$ for m large enough, B (and hence N) are finite R-modules. Since I has no embedded primes, B is Cohen-Macaulay and hence R-flat. It follows that N is also R-flat, hence free. If $N = 0$ then $I = (x, y)$ and the conclusion of the Lemma is clear.

Assume $N \neq 0$. We claim that for α, β generally chosen in $\mathbb{C}$, $f_{\alpha,\beta} \in N$ is part of a free basis of N as an R-module. This will imply that $B/(f_{\alpha\beta})$ is R-flat. In fact, writing $f = f_{\alpha\beta}$, we have an exact sequence of R-modules:

$$0 \to N/(f) \to B/(f) \to B/N \to 0,$$

where $B/N \approx R$. If f is part of a basis of N, then $N/(f)$ is R-free, which implies: $B/(f)$ is R-free, as stated. To check the claim (and conclude the proof), consider $N \otimes_R \mathbb{C}$. If the classes $\bar{x}, \bar{y}$ in $N \otimes_R \mathbb{C}$ of x, y respectively are both zero, then $N \otimes_R \mathbb{C} = 0$. But since N is R-free, this implies $N = 0$, contrary to our assumption. So, $\bar{x}$ or $\bar{y}$ is not zero, and for α, β in $\mathbb{C}$ "generally chosen", $\alpha\bar{x} + \beta\bar{y} \neq 0$ in $N \otimes_R \mathbb{C}$. But then, by Nakayama's lemma, $\alpha x + \beta y \in N$ will be part of a free basis of N as an R-module, as we wanted to show. □

4.8. Lemma. *Let $(\mathcal{I}, \pi)$ be as in (4.2) (i.e. $\mathcal{I} = \bar{\mathcal{I}}$). Let $q : Z_1 \to Z$ be the blowing-up of Z along the support of $\mathcal{I}$ and $\mathcal{I}_1$ the proper transform of $\mathcal{I}$. Then, $\mathcal{I}_1$ is integrally closed.*

Proof. Let I be as in (4.1), P a point of the exceptional divisor of $q, \mathcal{I}_1$ the proper transform of $\mathcal{I}$. If we choose coordinates in a suitable way, we shall get

$$\mathcal{O}_{Z_{1,P}} \approx \mathbb{C}\{x, y_1, t\}, (\mathcal{I}_1)_P \approx I_1 = x^{-\nu} I\mathbb{C}\{x, y_1, t\},$$

where $y = xy_1$ and ν is the (x, y)-order of I. Clearly it sufices to show that I_1 is a complete ideal (P being arbitrary, with $q(P)$ sufficiently close to $O \in T$), or to show that $J = I\mathbb{C}\{x, y_1, t\}$ is complete. Now, let $h \in \bar{J}$, i.e., there is a relation

$$h^n + a_1 h^{n-1} + \cdots + a_n = 0,\ a_i \in J^i. \tag{4.8.1}$$

Let $\mathrm{M} = (x, y)$. We may write $h = g/x^r, a_i = b_i/x^r, b_i \in I^i\mathrm{M}^r$, for some fixed integer r. Substituting in (4.8.1) and multiplying by x^{rn}, we get:

$$g^n + \sum_{i=0}^{n-1} b_{i+1} x^{ir} g^{n-i-1} = 0.$$

Since $b_i(x^r)^{i-1} \subseteq (I^i\mathrm{M}^r)(x^r)^{i-1} \subseteq (I\mathrm{M}^r)^i$, we see that $g \in (I\mathrm{M}^r)^- = I\mathrm{M}^r$, by (4.7) (applied r times). Thus, $h = \frac{g}{x^r} \in I\frac{\mathrm{M}^r}{x^r}$, which is an element of $I\mathbb{C}\{x, y_1, t\} = J$. This shows that J is complete. □

4.9. Before presenting a proof of Theorem 4.2 we need to recall a few more facts about ideals in a two dimensional noetherian regular local ring. Let R be such a ring, $\mathfrak{M}$ its maximal ideal, I an $\mathfrak{M}$-primary ideal. Recall the definition of a contracted ideal. Consider $X \to \operatorname{Spec} R$, the blowing-up with center $\mathfrak{M}$, and the associated isomorphism $R \to H^0(X, \mathcal{O}_X)$. The ideal I is contracted if $H^0(X, I\mathcal{O}_X) \cap R = I$. It is known that if I is complete then I is contracted, and if I is contracted then the fact that $\mathcal{I}_1$ (the proper transform of I on X) is integrally closed implies that I is integrally closed. We shall need, specially, the following translation into analytic language: if $\mathcal{I} \subset \mathcal{O}_{X_0}$ is a sheaf of ideals (where X_0 is a neighborhood of 0 in $\mathbb{C}^2$), having $\{0\}$ as its support, and $\mathcal{I}_P$ (its stalk at $P \in X_0$) is a contracted ideal of $\mathcal{O}_{X_0,P} \approx \mathbb{C}\{x, y\}$, the fact that the proper transform $\mathcal{J}$ of $\mathcal{I}$ to X (the blowing-up of X_0 with center P) is integrally closed implies that $\mathcal{I}$ is integrally closed (of course the only non-trivial thing is the completeness of $\mathcal{I}_0$, since $\mathcal{I}_Q = \mathcal{O}_{X_0,Q}$ for $Q \neq 0$). Finally let us remark that if I is an $\mathfrak{M}$-primary ideal in a two dimensional regular local ring R, $r(I) = \mathfrak{M}, \mu(I)$ is the minimum number of generators of $I, \nu(I)$ its order, then $\mu(I) \leq \nu(I) + 1$, with equality if and only if I is contracted ([**L2**, Corollary (3.2)]).

4.10. *Proof of* 4.2. Let $\mathcal{I}$ be our integrally closed sheaf of ideals defining our equisingular family. We want to see that $\mathcal{I}(t)$ is integrally closed, for all t near 0. We may assume, changing coordinates if necessary, that the support of $\mathcal{I}$ is the intersection of Z with the t-axis. Since then the support of $\mathcal{I}(t)$ is the point $(0, 0, t) \in Z$, it suffices to see that the ideal $\mathcal{I}(t)_{(0,0,t)}$ is complete (in the ring $\mathcal{O}_{Z_t,(0,0,t)}$) This is certainly true if $t \neq 0$. In fact, this is equivalent to checking that $I \otimes_R F$ is integrally closed in $\mathbb{C}\{x, y, t\} \otimes_R F$, where $R = \mathbb{C}\{t\}$ is the field of fractions of $R, I = \mathcal{I}_0$. But it is well known that integral closure commutes with localization (in particular, with F). (Another proof can be obtained using the characterization, $\bar{\mathcal{I}} = f_*(\mathcal{I}\mathcal{O}_V) \cap \mathcal{O}_Z$, where $f : V \to Z$ is the normalized blowing-up of Z with center $\mathcal{I}$, see [**T1**, 1.31].) So, the hard part is to check that $\mathcal{I}(0)_P := I_P, P = (0, 0, 0)$, is complete.

We shall proceed by induction on $e(\mathcal{I}(0)_P, \mathcal{O}_{Z_0,P})$. Consider the blowing-up $q : Z' \to Z$ with center the support of $\mathcal{I}$ (i.e., the t-axis), and the proper transform $\mathcal{I}_1$ of $\mathcal{I}$. Let $P_1, \dots P_r$ be the points of Z_1 lying over $P \in Z$, since the family is equisingular, $\mathcal{I}_1\mathcal{O}_{Z'_0,P_i}$ (with $Z'_0 = (q\pi)^{-1}(0)$) can be identified with the proper transform of $\mathcal{I}(0)$ (via the blowing-up of Z_0 at P). Since the multiplicity drops by taking proper transform (cf. [R], p.5), and since $\mathcal{I}_1$ is again integrally closed (by (4.8)), by induction the conclusion is true for $\mathcal{I}_1$, i.e., $\mathcal{I}_1(0)$ is integrally closed. By (4.9), were the ideal $I_0 := \mathcal{I}(0)_P$ (in $\mathcal{O}_{Z_{0,P}} \approx \mathbb{C}\{x,y\}$) contracted, then $\mathcal{I}(0)$ would be integrally closed, finishing the proof. We claim that the ideal I_0 is contracted. In fact, (letting $I_t := \mathcal{I}(t)_{(0,0,t)}$), if $t \neq 0, \nu(I_0) + 1 = \nu(I_t) + 1 = \mu(I_t) \leq \mu(I_0) \leq \nu(I_0) + 1$, the 1st equality because $(\mathcal{I}, \pi)$ is an equisingular family, the second because I_t is complete hence contracted, if $t \neq 0$; the third by Nakayama's lemma. Hence, $\nu(I_0) + 1 = \mu(I_0)$ and I_0 is contracted, as desired. □

4.11. Proposition. *Let* $(\mathcal{I}, \pi)$ *be as in* (4.1). *Then, for all* t *near* $0, \bar{\mathcal{I}}(t) = \overline{\mathcal{I}(t)}$.

Proof. Consider the (germ of) family of ideals $(\bar{\mathcal{I}}, \pi)$. We claim that this is equisingular, i.e. $\tau(\bar{\mathcal{I}}(0)) = \tau(\bar{\mathcal{I}}(t))$, t near 0, where τ denotes "tree". As remarked in the proof of (4.10), for $t \neq 0, \bar{\mathcal{I}}(t) = \overline{\mathcal{I}(t)}$ ("integral closure commutes with localization"). For $t = 0$, we easily get

$$\bar{\mathcal{I}}(0) \subseteq \overline{\mathcal{I}(0)} \tag{4.11.1}$$

so since ideals with the same integral closure have the same tree, $\tau(\bar{\mathcal{I}}(0)) = \tau(\overline{\mathcal{I}(0)}) = \tau(\mathcal{I}(0)) = \tau(\mathcal{I}(t)) = \tau(\overline{\mathcal{I}(t)}) = \tau(\bar{\mathcal{I}}(t))$. So, $(\bar{\mathcal{I}}, \pi)$ is equisingular and we may apply Theorem 4.10. We get: $\bar{\mathcal{I}}(0)$ is complete; from this (4.11.1) must be an equality, and this completes the proof. □

4.12. Proposition. *If* $(\mathcal{I}, \pi)$ *is an equisingular family of ideals with* $\mathcal{I} = \bar{\mathcal{I}}$, *then it is properly flat.*

Proof. From (4.2) it follows that $\mathcal{I}(t)$ is integrally closed, for all t near 0. Now use (3.10). □

Theorem 4.2 has the following converse:

4.13. Theorem. *Let* $(\mathcal{I}, \pi)$ *be an equisingular family of plane ideals as in* (4.1). *Assume* $\mathcal{I}(t) = \overline{\mathcal{I}}(t)$ *for all* $t \in T$. *Then,* $\mathcal{I} = \bar{\mathcal{I}}$.

Proof. We may work with suitably small neighborhoods (still denoted by Z and T) of $P \in \operatorname{Supp}\mathcal{I}$ and $0 = \pi(P) \in T$ respectively ,we may assume that the coordinates x, y, t on Z are such that $\operatorname{Supp}\mathcal{I} := T_1$ is the intersection of Z with the t-axis and that π induces an isomorphism $T_1 \approx T$. Consider the exact sequence of $\mathcal{O}_Z$-modules $0 \to \mathcal{I} \to \bar{\mathcal{I}} \to \mathrm{C} \to 0$ (C the cokernel of the inclusion $\mathcal{I} \subset \bar{\mathcal{I}}$). We shall check that for all $z = (0,0,t) \in T_1, \mathrm{C}_z = 0$,this will imply: $\mathcal{C} = 0$. Consider the induced exact sequence of $R_t := \mathcal{O}_{T,t}$-modules.

$$0 \to \mathcal{I}_z \to \bar{\mathcal{I}}_z \to \mathrm{C}_z \to 0,$$

tensoring with $\mathbb{C}_t : R_t/r(R_t)$ over R_t, we get an exact sequence of $\mathcal{O}_{Z_t,z}$ modules:

$$\mathcal{I}(t)_z \overset{\alpha}{\to} \overline{\mathcal{I}(t)_z} \to \mathrm{C}_z \otimes \mathbb{C}_t \to 0$$

where we used (4.2) and the flatness of $\mathcal{O}_{Z,z}/\bar{\mathcal{I}}_z$(cf. (4.12)) to identify $\overline{\mathcal{I}_z} \otimes \mathbb{C}_t$ with $\overline{\mathcal{I}_z \mathcal{O}_{Z_t}}$. By assumption, $\mathcal{I}(t)_z$ is integrally closed. Since $\overline{\mathcal{I}(t)_z}$ is integral over $\mathcal{I}(t)_z, \alpha$ must be the identity. So, $\mathrm{C} \otimes \mathbb{C}_t = 0$. Now, note that $\mathrm{Supp}\, \mathrm{C} \subset T_1$, which implies that C_z is a finite R_t-module. So we may apply Nakayama's lemma to conclude that $\mathrm{C}_z = 0$, as wanted. □

5. Equisingular Families of Curves.

5.1. In this section we use again the language of Complex Analytic Geometry. Throughout we work with one parameter families of generally reduced curves, in the sense of (3.2), whose notation we shall use. We let $\mathcal{I}(X)$ denote the sheaf of ideals of $\mathcal{O}_Z$ defining the inclusion $X \subset Z$, and we shall make the following assumption:

5.1.1. The morphism $[\mathcal{I}(X)] \to T$ induced by π is finite, where $[\mathcal{I}]$ denotes "singular locus of $\mathcal{I}$"(cf. (3.4)).

We want to introduce and compare three equisingularity conditions to be imposed on such a family.

5.2. Definition. A family (3.2.1) is said to be I-equisingular if the family of ideals $(\mathcal{I}(X), \pi)$ is equisingular, in the sense of (3.5) (or (3.6), see also (3.7)).

5.3. Definition. A family (3.2.1) is said to be T-equisingular if for any of pair points t, t' in the same connected component of T, it is possible to introduce orderings $\gamma_t, \gamma_{t'}$ of the branches of X_t and $X_{t'}$ respectively, in such a way that the trees $T_2(X_t, \gamma_t)$ and $T_2(X_{t'}, \gamma_{t'})$ are isomorphic (cf. (2.4), (2.10)).

5.4. A family (3.2.1) is said to be C-equisingular if it is I-equisingular and moreover (using the notation of (3.6)) when we consider any morphism $\pi^{(i)} : Z_i \to T$, where $\pi^{(i)} = \pi\rho_1 \ldots \rho_i, i = 1, \ldots r$, and $X_i \subset Z_i$ is the subspace defined by the proper transform $\mathcal{I}_i$ of $\mathcal{I}$, then the induced morphism $X_i \to T$ is flat (i.e, for each i, the proper transform $\mathcal{I}_i$ of $\mathcal{I}$ defines a one- parameter family of (not necessarily pure dimensional) curves, in the sense of (3.2), and eventually we get a simultaneous desingularization).

We have the following:

5.5. Theorem. *Consider a family* (3.2.1), *subject to the finiteness condition* (5.1.1). *Then:*

(a) *It is I-equisingular if and only if it is T-equisingular*

(b) *If it is C-equisingular, it is I-equisingular (and hence T-equisingular, by* (a)*).*

Proof. (b) is clear. Let us check (a).

(i) To prove either implication, we may restrict our attention to the family induced by π over suitable (arbitrarily small) neighborhoods of $P \in Z, 0 \in T$, where these points are arbitrarily chosen, subject to the condition $\pi(P) = 0$. We do not introduce new notation, i.e, X still denotes the neighborhood of P, etc. Moreover, if the inclusion of germs $(X, P) \subset (Z, P)$ corresponds to the ideal $I \subset \mathbb{C}\{x, y, t\} \approx \mathcal{O}_{Z,P}$ then by Theorem 3.4, $I = (F)(G), (G) = (G_1, \dots, G_r)$, with the properties listed in that theorem. Thus we may assume that $\mathcal{I} = \mathcal{F}\mathrm{G}$ on $\mathcal{O}_Z$, where $\mathcal{F}$ defines a family of reduced curves and G a flat family of zero dimensional ideals (unless $\mathrm{G} = \mathcal{O}_Z$, which happens if and only if we deal with a family of reduced curves).

(ii) Assume now that our family is I-equisingular. We work in the conditions described in (i). First we state some easily verified facts:

1. If $\mathcal{F}', \mathrm{G}'$ are sheaves of ideals of $\mathcal{O}_Z$, then $[\mathcal{F}'\mathrm{G}'] = [\mathcal{F}'] \cup [\mathrm{G}']$.
2. Let $\mathcal{F}', \mathrm{G}'$ be as in (1), then $[\mathcal{F}'\mathrm{G}'] \to T$ is etale if and only if both $[\mathcal{F}'] \to T$ and $[\mathrm{G}'] \to T$ are etale, and for any connected components C, D of $[\mathcal{F}']$ and $[\mathrm{G}']$ respectively, either $C \cap D = \emptyset$ or $C = D$ (all the morphisms mentioned above are induced by $\pi : Z \to T$, by restriction).

Note that on $\mathcal{O}_{Z_1}$ (notation of (3.6)) the factorization $\mathcal{I} = \mathcal{F}\mathrm{G}$ of **(i)** induces: $\mathcal{I}_1 = \mathcal{F}_1\mathrm{G}_1$ (the index "1" indicating proper transform), and similarly for proper transforms over $Z_2, \dots, Z_r$. By using (2) repeatedly, the equisingularity of $\mathcal{I}$ implies equisingularity of $\mathcal{F}$ and G (cf. (3.6)). But this implies that, for all $t \in T$, we may find an ordering γ_t of the branches of X_t so that $T_1(\mathcal{F}(t)) = T_1(\mathcal{F}(0)), \tau(\mathrm{G}(t)) = \tau(\mathrm{G}(0))$ (here $T_1(\mathcal{F}(t)) := T_1((X_t)_{\mathrm{red}}, \gamma_t)$, cf (2.10)). In fact, the statement for $\mathcal{F}$ is well-known (a form of Zariski's definition of equisingularity for families of reduced plane curves), the one for G is Risler's Theorem 3.7. Using (2) again, it immediately follows that the bi-weighted trees $T_2(X_t, \gamma_t)$ corresponding to $\mathcal{I}$ are all isomorphic to $T_2(X_0, \gamma_0)$, as was to be shown.

(iii) Now assume that the family is T-equisingular. We work in the conditions described in (i). It is immediately checked that the hypothesis "$\mathcal{I}$ is T-equisingular", i.e., $T_2(\mathcal{I}(t)) = T_2(\mathcal{I}(0)), t \in T$, implies that $T_1(\mathcal{F}(t)) =$

$T_1(\mathcal{F}(0))$ and $\tau(\mathrm{G}(t)) = \tau(\mathrm{G}(0))$, all $t \in T$ (here and in the sequel, $T_2(\mathcal{I}(t)) := T_2(X_t, \gamma_t), T_1(\mathcal{F}(t)) := T_1((X_{\text{red}})_t, \gamma_t)$, for a suitable ordering of the branches of X_t). This implies that the families of ideals $(\mathcal{F}, \pi)$ and (G, π) are equisingular, in the sense of (3.6): the first one (which defines a family of reduced curves), by Zariski's equisingularity theory; the second by Risler's Theorem. From now on, we assume $\mathrm{G} \neq \mathcal{O}_Z$, otherwise we're done at this point. To check that $\mathcal{I}$ also satisfies (3.6) (i.e, that out family of curves is I-equisingular), consider first the morphism $[\mathcal{I}] = [\mathcal{F}\mathrm{G}] \to T$ induced by π (stage (0) of (3.6)). We claim this is etale. But, as remarked in (1), (2) above, $[\mathcal{I}] = [\mathcal{F}] \cup [\mathrm{G}]$, and $[\mathcal{F}] \to T$ and $[\mathrm{G}] \to T$ are etale, because $\mathcal{F}$ and G satisfy (3.6). Now $P \in [\mathcal{F}] \cap [\mathrm{G}]$; if $[\mathcal{F}] \neq [\mathrm{G}]$ then the fiber of $[\mathcal{I}] \to T$ would have, for $t \in T$ near 0, more than one point; clearly this forces: $T_2(\mathcal{I}(0)) \neq T_2(\mathcal{I}(t))$. Thus, $[\mathcal{F}] = [\mathrm{G}]$, and by (2), $[\mathcal{I}] \to T$ is etale. We may continue in a similar way on higher order neighborhoods. E.g., consider $\rho_1 : Z_1 \to Z$, and a point $Q \in [\mathcal{I}_1]$ lying over P (there are finitely many such). Near Q, we have one of three possibilities: $(\alpha)\, \mathcal{I}_1$ defines a family of reduced plane curves, $(\beta)\, \mathcal{I}_1$ defines a family of 0-dimensional ideals, and $(\gamma)\, \mathcal{I}_1 = \mathcal{F}_1\mathrm{G}_1$ where $\mathcal{F}_1$ defines a family of reduced curves and G_1 one of 0-dimensional ideals. In either case, $\mathcal{I}_1(t)$ is the proper transform of $\mathcal{I}(t)$, for all t (here, in $\mathcal{I}_1(t)$ we use the ordering of the branches induced by γ_1). This is because the T-equisingularity assumption forces: $\nu(F(x,y,t)) = \nu(F(x,y,0)), \nu(G(x,y,t)) = \nu(G(x,y,0)), t$ near 0. From this it follows that in case (α) $T_1(\mathcal{I}(t))$ is constant, in case $(\beta)\, \tau\, (\mathcal{I}_1(t))$ is constant, and in case (γ) $T_2(\mathcal{I}_1(t))$ is constant. In the first two cases $[\mathcal{I}_1] \to T$ is etale (Zariski's and Risler's theories respectively),in case (γ) we are in the same conditions as before, and we may repeat the argument just given to get the same conclusion. Repeating the process (or, if one prefers, by using induction on $\sigma_0((X_t)), t$ general, cf. (2.3)), we check the I-equisingularity of our family, completing the proof of Theorem 5.5. □

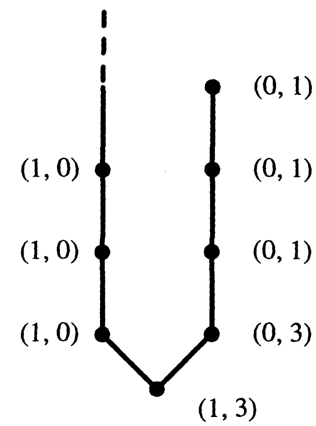
5.6. Remarks (a). Given a smooth morphism $\pi : Z \to T$ as in Definition 3.2, if $\mathcal{I} \subset \mathcal{O}_Z$ is a sheaf of ideals which admits a factorization $\mathcal{I} = \mathcal{F}\mathrm{G}$, where $\mathcal{F}$ defines a family of reduced curves and $\mathrm{G}(t) \subset \mathcal{O}_{Z_t}$ defines a zero dimensional subspace of Z_t for all t, then $\mathcal{I}$ is a flat family of ideals (i.e, $\mathcal{I}$ defines a flat family of curves, in the sense of (3.2)) if and only if G is a flat family of ideals. This is checked in [**BG**, 6.2 and 6.3].

(b) In the course of the proof of Theorem 5.5 we have essentially verified that if a family of curves (defined by an ideal $\mathcal{I} \subset \mathcal{O}_Z$) is I-equisingular, then (using the notation of Definition 3.6), for each morphism $Z_{i+1} \to Z_i, \mathcal{I}_{i+1}(t)$ can be identified to the proper transform of $\mathcal{I}_i(t)$ via the induced morphism of fibers $(Z_{i+1})_t \to (Z_i)_t$, which is the blowing up of $(Z_i)_t$ with center $[\mathcal{I}_i(t)]$.

Thus the morphisms $Z_r \to Z_{r-1} \to \ldots \to Z$ induce a desingularization of the ideals $\mathcal{I}(t) \subset \mathcal{O}_{Z_t}$, fiberwise.

An I-equisingular family is not necessarily C-equisingular. In fact we have:

5.7. Example. Consider the family (3.2.1) defined by the ideal (of $\mathbb{C}\{x, y, t\}$) $I = (y-x)(x^3, txy^5 + (1-t)x^2y^4, xy^6, y^7)$ (i.e. T, Z are small neighborhoods of the origins of $\mathbb{C}$ and $\mathbb{C}^3$ respectively, $X \subset Z$ is defined by the $\mathcal{O}_Z$ Ideal $\mathcal{I}$ corresponding to I). We claim that this family is I-equisingular but not C-equisingular. In fact, first of all $p : X \to T$ is flat, because the ideal $(G) = (x^3, txy^5 + (1-t)x^2y^4, xy^6y^7)$ defines a flat family of ideals (cf.(3.8) (b), then use Remark 5.6(a)). Now, $[\mathcal{I}]$ is the t-axis, we consider the blowing-up Z_1 of Z with center $[\mathcal{I}]$. If $\mathcal{I}_1$ is the proper transform of $\mathcal{I}, X_1$ its corresponding subspace of Z_1, then X_1 has two connected components V and W, where V is the proper transform of the trivial family of reduced curves defined by $y - x = 0$, and W is defined by the proper transform of the ideal (G), i.e. by $(x_1^3, (tx_1 + (1-t)x_1^2)y^3, y^4)$. By (3.8) (b), the projection $W \to T$ fails to be flat at a point lying over $0 \in T$. Thus, X_1 is not flat over T. However, the original family of curves is T-equisingular, hence I-equisingular. In fact, for all t near 0, the tree $T_2(\mathcal{I}(t))$ is isomorphic to



The phenomenon of this example cannot occur if we are dealing with families involving complete ideals. Precisely, we have:

5.8. Proposition. *Consider a family of curves* (3.2.1), *where* $X \subset Z$ *is defined by an Ideal* $\mathcal{I} \subset \mathcal{O}_Z$; *assume* $\mathcal{I}(t) := \mathcal{I}\mathcal{O}_{Z_t}$ *is integrally closed for all* $t \in T$, *and that the family is* I*-equisingular. Then, it is* C*-equisingular.*

Proof. It suffices to check the statement for sufficiently small neighborhoods of an arbitrary point $P \in Z$ and $0 = \pi(P) \in T$ respectively. Here, we may write $\mathcal{I} = \mathcal{F}\mathrm{G}$ respectively as in the proof Theorem 5.5, part (i) (i.e, $\mathcal{F}$ locally principal, $\mathrm{G}(t)$ with zero dimensional support) for all t. Using the fact that in a normal domain D an ideal J is integrally closed if and only if uJ is integrally closed, $u \in D, u \neq 0$, it follows that $\mathrm{G}(t)$ is integrally closed, for all $t \in T$. Now the theorem is an immediate consequence of Theorem 3.10 and Remark 5.6(a). □

The following proposition explains when the hypotheses of Theorem 5.7 are met.

5.9. Proposition. *Consider an I-equisingular family of curves* (3.2.1), *where $X \subset Z$ is defined by an Ideal $\mathcal{I} \subset \mathcal{O}_Z$. Then $\mathcal{I}(t)$ is integrally closed in $\mathcal{O}_{Z_t}$, for all $t \in T$, if and only if $\mathcal{I}$ is integrally closed.*
Proof. By writing (locally) $\mathcal{I} = \mathcal{F}\mathrm{G}$, resp. where $\mathcal{F}$ defines a family of reduced curves and G one of zero-dimensional ideals (cf. (3.4)), the proposition is an immediate consequence of the fact that $\mathcal{I}$ is complete if and only if G is complete (cf. the proof of (5.7)) and Theorems 4.2 and 4.13. □

5.10. Remark. Since the integral closure $\bar{\mathcal{I}}$ of an Ideal $\mathcal{I} \subset \mathcal{O}_Z$ is always defined ([**T1**, 1.3.1]) it is possible to associate to each I-equisingular family of curves (3.2.1) an I-equisingular family of curves satisfying the conditions of Proposition 5.8 (hence, it will also be C-equisingular). Namely, take the family defined by $\bar{\mathcal{I}}$. In fact, by writing (locally) $\mathcal{I} = \mathcal{F}\mathrm{G}$ (as in the proof of (5.8)), we have $\bar{\mathcal{I}} = \mathcal{F}\bar{\mathrm{G}}$. Since $\tau(\mathrm{G}(t)) = \tau(\overline{\mathrm{G}(t)})$, for all $t \in T$, it follows that G is an equisingular family of 0-dimensional ideals, and since $[\mathrm{G}] = [\bar{\mathrm{G}}]$ (and the same holds for their proper transforms), it rapidly follows that $\mathcal{F}\mathrm{G} = \mathcal{I}$ is I-equisingular. Now use Proposition 5.9.

Next we'll see how to associate to an equisingular family of plane curves (3.2.1) another whose members are reduced curves. In fact, this is achieved simply by substituting X by X_{red}, its associated reduced space. Precisely, we have:

5.11. Theorem. *Given an I-equisingular family of plane curves* (3.2.1), *let $P \in X$ be a point such that (X_t, P) is one-dimensional. Assume $P \in S$, the singular locus of X_{red}. Then, X_{red} is equisingular at P along S (in Zariski's sense), and there is a neighborhood U of P in X_{red} such that for all $t \in T$, sufficiently near $0 = \pi(P)$, $(X_{\mathrm{red}})_t \cap U$ is a reduced curve C_t, $C_t \cap S$ consists of a single point P_t which is the only singularity of C_t and the germs (C_t, P_t) and (C_o, P) are equivalent (cf.* (1.1)), *for all t near* 0.
Proof. Restrict the given family to suitably small neighborhoods of P and $0 = \pi(P) \in T$, which are still denoted by Z, X and T respectively. According

to Theorem 3.4, we have a factorization $\mathcal{I} = \mathcal{F}\mathrm{G}$ of the Ideal defining $X \subset Z$, corresponding to the factorization $I = (F)(G)$ of that theorem (where $I = \mathcal{I}_P \subset \mathcal{O}_{Z,P} = \mathbb{C}\{x, y, t\}$). By the equisingularity assumption, $[\mathcal{I}] = [\mathcal{F}] \cup [\mathrm{G}]$ is etale over T; if our neighborhoods are small enough we may assume π induces an isomorphism $[\mathcal{I}] \to T$. It is clear that $X_{\mathrm{red}} \subset Z$ is defined by the ideal $\mathcal{F}$ and that the singular locus of X_{red} is just $[\mathcal{F}] = [\mathcal{I}]$ (e.g., using the constancy of $T_2(X_t)$ we see that the multiplicity of X_{red} along $[\mathcal{F}]$ is constant). The fibers C_t of the induced morphism $X_{\mathrm{red}} \to T$ are all reduced (this is clear for the general fiber, and since X_{red} is defined on Z by a single equation, it is Cohen-Macaulay; this easily implies that the special fiber is also reduced). The constancy of the bi-weighted trees $T_2(X_t)$ immediately implies the constancy of the tree $T_1(C_t)$ (cf. (1.1)); it is well known that this is equivalent to Zariski's equisingularity of X_{red} along $S = [\mathcal{F}]$ (e.g., "$T_1(C_t)$ constant" implies the constancy of the number of branches and the "δ invariant", then use [**T2**, 5.3.1]). This proves the theorem. □

5.12. Remark. *In* [**BG**] *it is shown* (Korollar 2.3.5) *that given a family* (3.2.1) (of curves not necessarily locally plane), with X equidimensional, then the induced morphism $\widetilde{X} \to T$, (where $\widetilde{X} \to X_{\mathrm{red}}$ is the normalization) has the property that all the fibers $\widetilde{X}_t$ are smooth if and only if ε_t is constant or equivalently, δ_t is constant (cf. (2.7)). As we saw, in the case where our *family* (3.2.1) *is* I*-equisingular, the family* $X_{\mathrm{red}} \to T$ *is Zariski equisingular* family of reduced curves; it is well-known (cf. [**T2**, 5.3.1]) that this implies that the normalization of X_{red} simultaneously desingularizes the fibers. Thus, with notation of (5.10), $\varepsilon(X_t, P_t) = \varepsilon(X_0, P_0)$, t near $0 \in T$ and the same holds for the δ invariant. Since equivalent curves may have a different ε-invariant, it follows that we cannot "connect" two equivalent curves by means of an I-equisingular family parametrized by a connected space. The observation might be of interest if one wishes to develop a theory of moduli for non-reduced curve singularities.

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Received January 15, 1993.

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