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SYMMETRIC MINIMAL SURFACES IN $\mathbb{R}^3$

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Contents

1. Introduction.

A minimal surface in $\mathbb{R}^3$ is called symmetric if its isometry group G is not trivial. Here we define G to be the group of orientation preserving intrinsic isometries on the minimal surface. The catenoid is an example of a symmetric minimal surfaces with isometry group $G = SO(2) \ltimes \mathbb{Z}_2$. The purpose of this article is to answer the following question: What group can be the symmetry group of a minimal surface in $\mathbb{R}^3$?

In [**CMW**], Choi, Meeks and White proved that, if a minimal surface has a catenoid end, then any intrinsic local isometry of the minimal surface may be extended to a extrinsic isometry; thus an element in the symmetry group can be extended to a rigid motion of Euclidean space. As a corollary (Corollary 2.2), one has: If M is a minimal surface in $\mathbb{R}^3$ with finite total curvature and embedded ends, and at least one of its ends is catenoidal, then the symmetry group of M is a closed subgroup of $SO(3)$. This corollary shows that for minimal surfaces with catenoid ends, intrinsic and extrinsic symmetry are identical. Again, since we only consider orientation preserving isometries, reflection symmetries will be ignored.

Therefore the question is: Given G a closed subgroup of $SO(3)$, is there a complete immersed minimal surface in $\mathbb{R}^3$ whose symmetry group is G? Jorge and Meeks [**JM**] constructed a family of minimal surfaces whose symmetry group is the dihedral group D_n $(n > 2)$. Barbanel [**B**] and Lopez [**L**] found examples of minimal surfaces with trivial symmetry group and with symmetry group $C_2 \cong \mathbb{Z}_2$.

We will prove the following main theorem (Theorem 4.9): If $G \subset SO(3)$ is a closed subgroup. and $G \not\cong SO(2)$, $SO(3)$, then there is a complete genus 0 minimal surface with finite total curvature and embedded ends, whose symmetry group is G. We use a method which is similar to the representation of minimal surfaces in terms of "spinors" in [**KS**] and [**S**].

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2. Preliminaries.

Let M be a complete minimal surface with finite total curvature and embedded ends. Schoen pointed out that (cf. [**Sc**]) each end of M can be expressed as a graph, after properly choosing the coordinate in $\mathbb{R}^3$, with

$$x_3(x_1,\ x_2) = a\ \log\ r + \frac{b_1x_1}{r^2} + \frac{b_2x_2}{r^2} + O(r^{-2}),$$

where $a \geq 0$, and $r^2 = x_1^2 + x_2^2$.

We can compactify the surface by adding a point to each end. The resulting closed Riemann surface is denoted by M^*. Let g be the genus of M^*. Then the total curvature of M is

$$c(M) = \int_M KdS = -4\pi[(k-1) - g],$$

where k is the number of the ends [**JM**].

One has the following result on the rigidity of the minimal surfaces:

Lemma 2.1 (H. I. Choi, W. H. Meeks and B. White[**CMW**]). *If M is a minimal surface in $\mathbb{R}^3$ and M contains a compact minimal annulus A whose boundary curves lie on opposite sides of a plane P, then any isometry of M can be extended to a Euclidean motion in the ambient space $\mathbb{R}^3$. Thus the symmetry groups of such minimal surfaces are closed subgroups of $E(3)$, the Euclidean group of isometries of $\mathbb{R}^3$.*

From [**Sc**] one can see that, when the minimal surface is of finite total curvature and with each end embedded, then all its ends must be either planar or catenoidal. Let $\mathrm{Aut}(M)$ denote the group of all orientation preserving isometries of M. As a corollary of the above lemma, one has

Corollary 2.2. *If M is a complete minimal surface of finite total curvature with each end embedded and with at least one catenoid end, then* $\mathrm{Aut}(M)$ *is a subgroup of $SO(3)$.*

We next develop an effective way to work with $\mathrm{Aut}(M)$ in case M^* is the Riemann sphere $S^2 = \mathbb{C} \cup \infty = \mathbb{C}P^1$. We will use $z = \frac{z_1}{z_2} \in \mathbb{C} \cup \infty$ for the meromorphic coordinate, where $[z_1\ z_2]$ are homogeneous coordinates for $\mathbb{C}P^1$, and we often use $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{C})$ to represent $\pm \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2(\mathbb{C})$. It is well known that all Möbius transformations form a group $\mathcal{M}$ isomorphic to $PSL_2(\mathbb{C})$, in the following sense: $A = \pm \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2(\mathbb{C})$ corresponds to the Möbius transformation $\mu(z) = \dfrac{az+b}{cz+d}$.

Let $f(z)$ be a meromorphic function on S^2. There are two homogeneous polynomials of the same degree, $p(z_1, z_2)$ and $q(z_1, z_2)$, relatively prime to each other with $q(z_1, z_2) \not\equiv 0$, such that

$$f(z) = \frac{p(z_1, z_2)}{q(z_1, z_2)}. \tag{1}$$

Denote by $\mathcal{S}_{2\times 2}(\mathbb{C})$ the set of all 2×2 symmetric complex matrices. Define a linear isomorphism $\Phi : \mathbb{C}^3 \to \mathcal{S}_{2\times 2}(\mathbb{C})$ by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} -x_1 - ix_2 & x_3 \\ x_3 & x_1 - ix_2 \end{bmatrix}. \tag{2}$$

Furthermore, let

$$B(x) = -\det(\Phi(x)) = x_1^2 + x_2^2 + x_3^2. \tag{3}$$

An element of $SO(3;\mathbb{C})$ is a linear transformation in $\mathbb{C}^3$ which preserves $B(x)$. For any $A \in SL_2(\mathbb{C})$, we define an action of A on $\mathcal{S}_{2\times 2}(\mathbb{C})$ as

$$X \mapsto AXA^T, \quad \text{for} \quad X \in \mathcal{S}_{2\times 2}(\mathbb{C}).$$

This is a linear action on $\mathcal{S}_{2\times 2}(\mathbb{C})$, and

$$\det(AXA^T) = \det(X). \tag{4}$$

Thus we have a homomorphism $h : SL_2(\mathbb{C}) \to SO(3;\mathbb{C})$ so that $\Phi(h(A)x) = A\Phi(x)A^T$, for any $x \in \mathbb{C}^3$ and $A \in SL_2(\mathbb{C})$. It is not hard to prove that $ker(h) = \left\{ \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \right\}$. By studying the stablizers of elements in $\mathcal{S}_{2\times 2}(\mathbb{C})$, one may see that h is surjective. Thus h induces an isomorphism $\widetilde{h} : PSL_2(\mathbb{C}) \to SO(3;\mathbb{C})$.

Let $\mathbb{R}^3 = \{x \in \mathbb{C}^3 \mid \overline{x} = x\}$ be the real subspace of $\mathbb{C}^3$. Then

$$SO(3) = \{g \in SO(3;\mathbb{C}) \mid g\mathbb{R}^3 = \mathbb{R}^3\}. \tag{5}$$

Note that

$$\Phi(\mathbb{R}^3) = \left\{ \begin{bmatrix} -w_1 & w_3 \\ w_3 & w_2 \end{bmatrix} \mid \overline{w_1} = w_2, \overline{w_3} = w_3; w_1,\ w_2,\ w_3 \in \mathbb{C} \right\}.$$

Thus one can see that $A \in h^{-1}(SO(3))$ if and only if $A = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}$, where $|\alpha|^2 + |\beta|^2 = 1$, that is $h^{-1}(SO(3)) = SU(2)$. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in S^2$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SU(2)$. Let $z = \dfrac{x_1 + ix_2}{1 - x_3}$ be the complex coordinate of x under the stereographic projection. Then if $z' = \dfrac{x_1' + ix_2'}{1 - x_3'}$, where $x' = h(A)x$, then one may find that $z' = \dfrac{az + b}{cz + d}$. Thus one sees that the action of $SO(3)$ on S^2 is equivariant with the action of $SU(2) \subset SL_2(\mathbb{C})$ under h.

Furthermore, since the symmetric space

$$SO(3;\mathbb{C})/SO(3) = SL_2(\mathbb{C})/SU(2) = \mathbb{H}^3$$

is the 3-dimensional hyperbolic space, any compact subgroup of $SO(3;\mathbb{C})$ induces an action on $\mathbb{H}^3$. By Cartan's theorem (cf. [**Or**]), this group action must admit a fixed point in $\mathbb{H}^3$. Thus, we have:

Any compact subgroup of $SO(3;\mathbb{C})$ must be conjugate to a subgroup of $SO(3)$.

Let $\gamma \in \mathcal{M}$ be a Möbius transformation. A meromorphic function $f(z)$ is said to be γ-invariant if $f(\gamma \circ z) = \gamma \circ f(z)$. If $f(z)$ is γ-invariant for all $\gamma \in G \subset \mathcal{M}$, then $f(z)$ is said to be G-invariant. On the other hand, given a meromorphic function $f(z)$, we denote its symmetry group by

$$\mathrm{Aut}(f) = \{\gamma \in \mathcal{M} \mid f \circ \gamma = \gamma \circ f\}.$$

Note that when $\mathrm{Aut}(f) = PSL_2(\mathbb{C})$ we have $f(z) = z$.

Since $\mathcal{M} \cong SO(3;\mathbb{C}) \cong PSL_2(\mathbb{C})$, any Möbius transformation γ corresponds to a pair of linear transformations on $\mathbb{C}^2$, defined as γ^+ and γ^-, with $\gamma^+ = -\gamma^-$. A homogeneous polynomial $p(z_1,\ z_2)$ is said to be γ-invariant if there are constants $c_{\gamma^+} = \pm c_{\gamma^-}$ such that $p(\gamma^+(z_1,\ z_2)) = c_{\gamma^+} p(z_1,\ z_2)$, and $p(\gamma^-(z_1,\ z_2)) = c_{\gamma^-} p(z_1,\ z_2)$. A 1-form θ in $\mathbb{C}^2$ is said to be γ-invariant if there are constants $c'_{\gamma^+} = \pm c'_{\gamma^-}$ such that $\gamma^{+*}\theta = c'_{\gamma^+}\theta$, and $\gamma^{-*}\theta = c'_{\gamma^-}\theta$. The following lemma by Doyle and McMullen can be used to find all meromorphic functions on S^2 which are invariant under a given group $G \subset \mathcal{M}$.

Lemma 2.3 (P. Doyle & C. McMullen[**DM**]). *A homogeneous 1-form θ in $\mathbb{C}^2$ is γ-invariant if and only if there exist two γ-invariant homogeneous polynomials $p(z_1, z_2)$ and $q(z_1, z_2)$ which satisfy* $\deg p = \deg q - 2 = \deg \theta - 2$ *and $c_p(\gamma) = c_q(\gamma)$ for any γ, and*

$$\theta = p(z_1, z_2)(z_1 dz_2 - z_2 dz_1) + dq(z_1, z_2).$$

Corollary 2.4. *A meromorphic function on S^2 is γ-invariant if and only if it has the form*

$$f(z) = \frac{p(z_1, z_2)z_1 + q_{z_2}(z_1, z_2)}{p(z_1, z_2)z_2 - q_{z_1}(z_1, z_2)},$$

where p and q are two γ-invariant homogeneous polynomials satisfying $\deg p = \deg q - 2$, *and $c_p = c_q$.*

Proof. One only has to see that $f(z) = \dfrac{p(z_1, z_2)}{q(z_1, q_2)}$ is γ-invariant if and only if the vector field $X_f(z_1, z_2) = p(z_1, z_2)\dfrac{\partial}{\partial z_1} + q(z_1, z_2)\dfrac{\partial}{\partial z_2}$ satisfies

$$\gamma_* X_f = c_\gamma X_f \tag{6}$$

for some $c_\gamma \in \mathbb{C}$; and (6) is satisfied if and only if the 1-form $\theta(z_1, z_2) = q(z_1, z_2)dz_1 - p(z_1, z_2)dz_2$ satisfies $\gamma^*\theta = c_\gamma\theta$. By Lemma 2.3, one gets the corollary. □

For $G \subset \mathcal{M}$, any homogeneous polynomial $p(z_1, z_2)$ or 1-form θ is called G-invariant if it is γ-invariant for all $\gamma \in G$.

To get a meromorphic function $f(z)$ with finite $\mathrm{Aut}(f)$ one can use Corollary 2.2. Let $G = \mathrm{Aut}(f)$ and consider the orbifold S^2/G. Let $\pi : S^2 \to S^2/G$ and $\zeta \in S^2/G$. Then

$$q_{[\zeta]}(z_1, z_2) = \prod_{[\zeta_1, \zeta_2] \in \pi^{-1}(\zeta)} (\zeta_2 z_1 - \zeta_1 z_2)$$

defines a G-invariant homogeneous polynomial. Conversely, it is not hard to see that any G-invariant homogeneous polynomial is a product of such $q_{[\zeta]}$'s.

Examples of Homogeneous Invariant Polynomials:

C_n (*The Cyclic Group of Order n*). C_n is generated by the Möbius transformation $z \mapsto e^{2\pi i/n}z$. The orbifold S^2/C_n has two cone points $[0] = [1, 0]$ and $[\infty] = [0, 1]$. Thus

$$q_{[\infty]}(z_1\ z_2) = z_1, \quad q_{[0]}(z_1, z_2) = -z_2, \tag{7}$$

and for all other $\zeta = [\zeta_1, \zeta_2]$,

(8) $$q_{[\zeta]}(z_1, z_2) = \zeta_2^n z_1^n - \zeta_1^n z_2^n.$$

D_n (*The Dihedral Group of Order* $2n$). D_n is generated by the Möbius transformation $z \mapsto e^{2\pi i/n} z$ and $z \mapsto \dfrac{1}{z}$. The orbifold S^2/D_n has three cone points: $[0] = [0, 1]$, $[1] = [1, 1]$ and $[\omega_{2n}] = [e^{\frac{\pi i}{n}}, 1]$. Thus

(9) $$q_{[0]}(z_1, z_2) = z_1 z_2, \; q_{[1]}(z_1, z_2) = z_1^n - z_2^n, \; q_{[\omega_{2n}]}(z_1, z_2) = z_1^n + z_2^n,$$

and for all other $\zeta = [\zeta_1, \zeta_2]$,

(10) $$q_{[\zeta]}(z_1, z_2) = (\zeta_2^n z_1^n + \zeta_1^n z_2^n)(\zeta_2^n z_1^n - \zeta_1^n z_2^n).$$

A_4 (*The Tetrahedral Group*). A_4 contains 3 elements of order 2 and 8 elements of order 3. They are

$$z \mapsto -z, \quad z \mapsto \frac{1}{z}, \quad z \mapsto -\frac{1}{z}$$

and

$$z \mapsto (-1)^k \frac{(\pm 1 + i)z - (1 - i)}{(1 + i)z + (\pm 1 - i)}, \; z \mapsto (-1)^k \frac{(1 + i)z - (\pm 1 - i)}{(\pm 1 + i)z + (1 - i)}, \; k = 0, 1.$$

The orbifold S^2/A_4 has 3 cone points $[0] = [0, 1]$, $[v] = \left[\dfrac{\sqrt{3} - 1}{2}, 1 + i\right]$, and $[w] = \left[\dfrac{\sqrt{3} - 1}{2}, 1 - i\right]$. Thus

(11) $$\begin{aligned} q_{[0]}(z_1, z_2) &= z_1 z_2 (z_1^4 - z_2^4), \\ q_{[v]}(z_1, z_2) &= z_1^4 + 2\sqrt{3} i z_1^2 z_2^2 + z_2^4, \\ q_{[w]}(z_1, z_2) &= z_1^4 - 2\sqrt{3} i z_1^2 z_2^2 + z_2^4. \end{aligned}$$

For all other $\zeta = [\zeta_1, \zeta_2]$,

(12)
$$\begin{aligned} q_{[\zeta]}(z_1, z_2) = (\zeta_1^2 z_1^2 - \zeta_2^2 z_2^2)(\zeta_2^2 z_1^2 - \zeta_1^2 z_2^2) \cdot \{ (\zeta_1^2 + \zeta_2^2)^4 (z_1^8 + z_2^8) - \\ -(\zeta_1^2 - \zeta_2^2 - 2i\zeta_1\zeta_2)^4 z_1^4 z_2^4 - (\zeta_1^2 - \zeta_2^2 + 2i\zeta_1\zeta_2)^4 z_1^4 z_2^4 \}. \end{aligned}$$

S_4 (*The Octahedral Group*). S_4 contains A_4, and 6 more elements of order 4 which are obtained by adding one more generator $z \mapsto e^{\frac{\pi}{2} i} z$ to those of A_4.

The orbifold S^2/S_4 has 3 cone points $[0] = [0,\ 1]$, $[v] = [\sqrt{3}-1,\ 1+i], [u] = [\sqrt{2}-1,\ 1]$.

(13)
$$\begin{aligned} q_{[0]}(z_1,\ z_2) &= z_1 z_2(z_1^4 - z_2^4), \\ q_{[v]}(z_1,\ z_2) &= z_1^8 + 14 z_1^4 z_2^4 + z_2^8, \\ q_{[u]}(z_1,\ z_2) &= z_1^{12} - 33 z_1^8 z_2^4 - 33 z_1^4 z_2^8 + z_2^{12}. \end{aligned}$$

A_5 (*The Isocahedral Group*). A_5 has generators ρ, τ and σ with orders 5, 2, 3, respectively.

$$\begin{aligned} \rho : &\quad z \mapsto e^{\frac{\pi}{5} i} z, \\ \tau : &\quad z \mapsto -\frac{1}{z}, \\ \sigma : &\quad z \mapsto \frac{-(e^{\frac{4\pi}{5} i} - 1) z + (1 - e^{\frac{2\pi}{5} i})}{(1 - e^{\frac{2\pi}{5} i}) z + (e^{\frac{2\pi}{5} i} - e^{-\frac{2\pi}{5} i})}. \end{aligned}$$

The orbifold has 3 cone points $[0] = [0,\ 1]$, $[v] = \left[3 + \sqrt{5} - \sqrt{30 + 6\sqrt{5}},\ 4\right]$ and $[i] = [i,\ 1]$. Then

(14)
$$\begin{aligned} q_{[0]}(z_1,\ z_2) &= z_1 z_2 (z_1^{10} + 11 z_1^5 z_2^5 - z_2^{10}), \\ q_{[v]}(z_1,\ z_2) &= z_1^{20} - 228 z_1^{15} z_2^5 + 494 z_1^{10} z_2^{10} + 228 z_1^5 z_2^{15} + z_2^{20}, \\ q_{[i]}(z_1,\ z_2) &= z_1^{30} + 522 z_1^{25} z_2^5 - 10005 z_1^{20} z_2^{10} - 10005 z_1^{10} z_2^{20} - 522 z_1^5 z_2^{25} + z_2^{30}. \end{aligned}$$

3. Construction of symmetric minimal surfaces.

Let M be a minimal surface in $\mathbb{R}^3$ with finite total curvature, z be a local coordinate on M, $\widetilde{M}$ be the universal covering of M. It is well-known that there are 3 holomorphic functions on $\widetilde{M}$, $\phi_k(z)$, $k = 1,\ 2,\ 3$, so that the immersion of M in $\mathbb{R}^3$ is given as

(1)
$$x_k(z) = a_k + Re \int_{z_0}^{z} \phi_k(\zeta) d\zeta,$$

and

$$\phi_1^2(z) + \phi_2^2(z) + \phi_3^2(z) \equiv 0.$$

Moreover, $g(z) = \dfrac{\phi_3(z)}{\phi_1(z) - i\phi_2(z)}$ is the Gauss map of M.

By Corollary 2.2, $\text{Aut}(M) \subset SO(3)$. For any $\rho \in \text{Aut}(M)$, let A_ρ $h^{-1}(\rho) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SU_2(\mathbb{C})$ where $h : SL_2(\mathbb{C}) \to SO(3,\ \mathbb{C})$ is defined in Section 2, and $\mu_\rho(z) = \dfrac{az + b}{cz + d}$.

Suppose M has genus 0, and let E be a finite set of point on $S^2 = M^* = \widetilde{M}$. A global coordinate $z : M \to S^2 \setminus E$ is called an equivariant coordinate if for any $\rho \in \mathrm{Aut}(M)$, there is $x_\rho \in \mathbb{R}^3$ such that

$$\rho \cdot x(z) = x(\mu_\rho(z)) + x_\rho \tag{2}$$

where $x(z)$ is the minimal immersion defined in (15), x_ρ is a point in $\mathbb{R}^3$ depending only on ρ. We have

Lemma 3.1. *For any minimal surface conformally equivalent to $S^2 \setminus E$, there exists an equivariant coordinate.*

Proof. The coordinate $z : M \to S^2$ can be extended to $M^* \to S^2$. If $\rho \in \mathrm{Aut}(M)$, then $\mu_\rho = z \circ \rho \circ z^{-1}$ is a Möbius transformation of S^2. Thus $z \circ \mathrm{Aut}(M) \circ z^{-1}$ is a subgroup in $\mathcal{M}$. By the discussion in Section 2, it is conjugate with $\mathrm{Aut}(M)$. Thus z induces $\tilde{z} : M^*/\,\mathrm{Aut}(M) \to S^2/\,\mathrm{Aut}(M)$. So we have $\rho \cdot x(z) - x(\mu_\rho(z)) = x_\rho$. □

Thus we always assume that z is an equivariant coordinate on M. From (16) we have

$$\rho \cdot dx(z) = \mu_\rho^* dx(z).$$

Let

$$F(z) = \begin{bmatrix} \int_{z_0}^{z}(-\phi_1(\zeta) - i\phi_2(\zeta))d\zeta & \int_{z_0}^{z} \phi_3(\zeta)d\zeta \\ \int_{z_0}^{z} \phi_3(\zeta)d\zeta & \int_{z_0}^{z}(\phi_1(\zeta) - i\phi_2(\zeta))d\zeta \end{bmatrix},$$

and

$$dF(z) = \begin{bmatrix} (-\phi_1(z) - i\phi_2(z))dz & \phi_3(z)dz \\ \phi_3(z)dz & (\phi_1(z) - i\phi_2(z))dz \end{bmatrix}.$$

We have

Lemma 3.2. *For any $\rho \in \mathrm{Aut}(M)$,*

$$A_\rho dF(z) A_\rho^T = \mu_\rho^* dF(z),$$

where $A_\rho = h^{-1}(\rho) \in SL_2(\mathbb{C})$.

Proof. Note that

$$dx(z) = \frac{1}{2}\left(\frac{\partial x}{\partial z}dz + \frac{\partial x}{\partial \overline{z}}d\overline{z}\right) = \frac{1}{2}(\Phi(z)dz + \overline{\Phi(z)}d\overline{z}),$$

and note that $h^{-1}(\rho) = A_\rho$. Then from (16), the definition of h and the definition of dF, one gets the lemma. □

Now we are going to construct symmetric minimal surfaces with $\mathrm{Aut}(M)$ being a prescribed subgroup of $SO(3)$. Let $q(z_1, z_2)$ be a homogeneous polynomial invariant under the action of $G \subset SU(2)$,

$$q_1(z_1, z_2) = \frac{\partial q(z_1, z_2)}{\partial z_1}, \quad q_2(z_1, z_2) = \frac{\partial q(z_1, z_2)}{\partial z_2}.$$

Then by Corollary 2.2, $g(z) = -\dfrac{q_2(z_1, z_2)}{q_1(z_1, z_2)}$ is a G-invariant meromorphic function on S^2, where $z = \dfrac{z_1}{z_2}$. Let $\eta(z) = \dfrac{q_1^2(z, 1)}{q^2(z, 1)}dz$. Then it can be seen that

$$\mu_A^* \eta(z) = (cg(z) + d)^2 \eta(z),$$

for any $A \in G$. Hence by the straightforward computation we get

Proposition 3.3. *Let* $dF(z) = \begin{bmatrix} \eta(z)g^2(z) & \eta(z)g(z) \\ \eta(z)g(z) & \eta(z) \end{bmatrix}$, *then for any* $A \in G$,

$$A \cdot dF(z) \cdot A^T = \mu_A^* dF(z).$$

To "kill the periods" of the minimal surface, i.e., to guarantee that the holomorphic immersion $\psi : \widetilde{M} \to \mathbb{C}^3$ can be projected to $\mathbb{R}^3$ with image M being of finite total curvature, we have

Lemma 3.4. *If* $dF \in \Gamma(T^{(0,1)}S^2 \otimes S_{2\times 2}(\mathbb{C}))$,

$$dF(z) = \begin{bmatrix} \alpha_1(z)dz & \alpha_3(z)dz \\ \alpha_3(z)dz & \alpha_2(z)dz \end{bmatrix}$$

where $\alpha_k(z)$ *are meromorphic functions on* S^2, $\det(dF(z)) = 0$ *but* $dF(z) \neq 0$, *then*

$$Res_{\zeta_0} dF = \begin{bmatrix} Res_{\zeta_0}\alpha_1(z) & Res_{\zeta_0}\alpha_3(z) \\ Res_{\zeta_0}\alpha_3(z) & Res_{\zeta_0}\alpha_2(z) \end{bmatrix} \in \Phi(\mathbb{R}^3)$$

for any z_0, *i.e.,*

$$-Res_{\zeta_0}\alpha_1(z) = \overline{Res_{\zeta_0}\alpha_2(z)}, \quad Res_{\zeta_0}\alpha_3(z) = \overline{Res_{\zeta_0}\alpha_3(z)},$$

if and only if $Re(\Phi^{-1} \circ F) : S^2 \to \mathbb{R}^3$ *is a complete minimal surface with* *finite total curvature* $c(M) = -4\pi \deg \frac{\alpha_3(z)}{\alpha_2(z)}$.

Proof. Let

$$\begin{aligned} \phi_1(z) &= -\frac{1}{2}(\alpha_1(z) - \alpha_2(z)), \\ \phi_2(z) &= \frac{i}{2}(\alpha_1(z) + \alpha_2(z)), \\ \phi_3(z) &= \alpha_3(z). \end{aligned}$$

Then it is not hard to see

$$\phi_1^2(z) + \phi_2^2(z) + \phi_3^2(z) = 0$$

and

$$2\left[|\phi_1(z)|^2 + |\phi_2(z)|^2 + |\phi_3(z)|^2\right] = |\alpha_1(z)|^2 + |\alpha_2(z)|^2 + 2|\alpha_3(z)|^2 > 0.$$

Thus $z \mapsto Re(\Phi^{-1} \circ F(z)) = \begin{bmatrix} Re \int_{z_0}^{z} \phi_1(\zeta)d\zeta \\ Re \int_{z_0}^{z} \phi_2(\zeta)d\zeta \\ Re \int_{z_0}^{z} \phi_3(\zeta)d\zeta \end{bmatrix}$ is a complete minimal immersion with the induced metric

$$ds^2 = \frac{1}{2}\left[|\phi_1(z)|^2 + |\phi_2(z)|^2 + |\phi_3(z)|^2\right] dz d\bar{z}.$$

The minimal surface has a finite total curvature if and only if for any loop C on S^2,

$$Re \oint_C \phi_k(\zeta)d\zeta = 0,$$

for $k = 1,\ 2,\ 3$, i.e., for any $\zeta_0 \in S^2$, $Res_{\zeta_0}\phi_k(z) \in \mathbb{R}$. Thus $Res_{\zeta_0}dF \in \Phi(\mathbb{R}^3)$. When these conditions are satisfied, the total curvature $c(M) = -\int_M g^*(dS)$ where g is the Gauss map. Thus $c(M) = -4\pi m$, where $m = \deg \frac{\alpha_3(z)}{\alpha_2(z)}$. □

Lemma 3.5. *If* $dF \in \Gamma(T^{(0,\ 1)}S^2 \otimes \mathcal{S}_{2\times 2}(\mathbb{C}))$ *satisfies*

$$dF(z) \neq 0 \text{ and } \quad \det(dF(z)) = 0,$$

and there is $G \subset SU(2)$ *such that for any* $A \in G$,

$$A \cdot dF(z) \cdot A^T = \mu_A^* dF(z),$$

and let $\zeta_1, \ldots, \zeta_n$ *are poles for* dF, *and* μ_G *acts on them transitively, then* $Res_{\zeta_j} dF \in \Phi(\mathbb{R}^3)$, $j = 1, \ldots, n$, *if* $Res_{\zeta_1} dF \in \Phi(\mathbb{R}^3)$.

Proof. For any ζ_j, let $A_j \in G$ such that $\mu_{A_j}(\zeta_1) = \zeta_j$. Then

$$Res_{\zeta_j} dF = Res_{\mu_{A_j}(\zeta_1)} dF = Res_{\zeta_1} \mu^*_{A_j} dF = Res_{\zeta_1} A_j dF A_j^T = A_j Res_{\zeta_1} dF A_j^T.$$

Since $A_j \in h^{-1}(SO(3))$, A_j preserves $\Phi(\mathbb{R}^3)$. Hence $Res_{\zeta_j} dF \in \Phi(\mathbb{R}^3)$. □

Lemma 3.6. *Let* $q(z_1, z_2)$ *be a homogeneous polynomial invariant under the action of* G, *let* $q_1(z_1, z_2) = \dfrac{\partial q(z_1, z_2)}{\partial z_1}$, $q_2(z_1, z_2) = \dfrac{\partial q(z_1, z_2)}{\partial z_2}$, $g(z) = \dfrac{\phi_3(z)}{\phi_1(z) - i\phi_2(z)}$, $\eta(z) = \dfrac{q_1^2(z, 1)}{q^2(z, 1)} dz$. *If* $\zeta_0, \zeta_1, \ldots, \zeta_n$ *are zeros of* $q(z, 1)$ *and* G *contains a nontrivial subgroup which fixes* ζ_0, *then* $Res_{\zeta_0} dF \in \Phi(\mathbb{R}^3)$ *where*

$$dF(z) = \begin{bmatrix} \eta(z)g^2(z) & \eta(z)g(z) \\ \eta(z)g(z) & \eta(z) \end{bmatrix}.$$

Proof. Without loss of generality, we may assume $\zeta_0 = 0$. If we assume that $q(1, 0) \neq 0$, then $q(z_1, z_2) = \lambda z_1 \prod_{j=1}^{n} (z_1 - \zeta_j z_2)$, $\lambda \in \mathbb{C}$. Then

$$\frac{q_1(z_1, z_2)}{q(z_1, z_2)} = \frac{1}{z_1} + \sum_{j=1}^{n} \frac{1}{z_1 - \zeta_j z_2}, \quad and \quad \frac{q_2(z_1, z_2)}{q(z_1, z_2)} = -\sum_{j=1}^{n} \frac{\zeta_j}{z_1 - \zeta_j z_2}.$$

Hence

$$\eta(z) = \frac{1}{z^2} \left[1 + \sum_{j=1}^{n} \frac{z}{z - \zeta_j} \right]^2 dz,$$

$$\eta(z)g^2(z) = \left[\sum_{j=1}^{n} \frac{\zeta_j}{z - \zeta_j} \right]^2 dz,$$

$$\eta(z)g(z) = \frac{1}{z} \left[1 + \sum_{j=1}^{n} \frac{z}{z - \zeta_j} \right] \left[\sum_{j=1}^{n} \frac{\zeta_j}{z - \zeta_j} \right] dz.$$

From the above one sees

$$Res_0 \eta(z) = 2 \sum_{j=1}^{n} \frac{1}{\zeta_j},$$

$$Res_0 \eta(z) g^2(z) = 0,$$

$$Res_0 \eta(z) g(z) = -n.$$

Now suppose $H \subset G$ is a subgroup which fixes 0. Then H is generated by $\mu_0 : \mu_0(z) = e^{\frac{2\pi}{m}i}z$. Since $\sum_{k=0}^{m-1} \frac{1}{e^{\frac{2k\pi}{m}i}z} \equiv 0$, and $\{\zeta_1, \ldots, \zeta_n\}$ are invariant under the action of H, $\sum_{k=1}^{n} \frac{1}{\zeta_k} = 0$. Thus

$$Res_0 dF = \begin{bmatrix} 0 & -n \\ -n & 0 \end{bmatrix}.$$

Now if $q(1, 0) = 0$, then

$$q(z_1, z_2) = z_1 z_2 \prod_{j=1}^{n-1} (z_1 - \zeta_j z_2).$$

Hence

$$\frac{q_1(z_1, z_2)}{q(z_1, z_2)} = \frac{1}{z_1} + \sum_{j=1}^{n-1} \frac{1}{z_1 - \zeta_j z_2}, \ and \ \frac{q_2(z_1, z_2)}{q(z_1, z_2)} = \frac{1}{z_2} - \sum_{j=1}^{n-1} \frac{\zeta_j}{z_1 - \zeta_j z_2}.$$

Then by the similar procedure, one sees

$$Res_0 dF = \begin{bmatrix} 0 & -n \\ -n & 0 \end{bmatrix}.$$

□

4. Examples of symmetric minimal surfaces.

Now we can construct symmetric minimal surfaces with finite total curvature and all ends embedded.

Examples for $G = C_n$ (The Cyclic Group of Order $n > 3$). Let

$$q(z_1, z_2) = z_1(z_1^n - r^n z_2^n)$$

where $r > 0$ is to be determined. Then

$$q_1(z_1, z_2) = (n+1)z_1^n - r^n z_2^n, \ and \ \ q_2(z_1, z_2) = -nr^n z_1 z_2^{n-1}.$$

Let

$$f_1(z) = (z - r)\frac{q_1(z, 1)}{q(z, 1)} = \frac{(n+1)z^n - r^n}{z\left[\sum_{k=0}^{n-1} r^{n-k-1} z^k\right]},$$

and

$$f_2(z) = (z-r)\frac{q_2(z,\ 1)}{q(z,\ 1)} = -\frac{nr^n}{\left[\sum_{k=0}^{n-1} r^{n-k-1} z^k\right]}.$$

Then

$$f_1(r) = 1,\ f_2(r) = -r,\ f_1'(r) = \frac{n+1}{2r},\ f_2'(r) = \frac{n-1}{2}.$$

Thus since

$$\begin{aligned} \eta(z) &= \frac{1}{(z-r)^2} f_1^2(z)dz, \\ \eta(z)g^2(z) &= \frac{1}{(z-r)^2} f_2^2(z)dz, \\ \eta(z)g(z) &= -\frac{1}{(z-r)^2} f_1(z)f_2(z)dz, \end{aligned}$$

one has

$$\begin{aligned} &Res_r\eta(z) = \frac{n+1}{r}, \\ &Res_r\eta(z)g^2(z) = -r(n-1), \\ &Res_r\eta(z)g(z) = 1. \end{aligned}$$

By Lemma 3.4, one has $r = \sqrt{\frac{n+1}{n-1}}$. By Lemma 3.5, $Res_{re^{\frac{2k\pi}{n}}} dF \in \Phi(\mathbb{R}^3)$. Lemma 3.6 shows that $Res_0 dF \in \Phi(\mathbb{R}^3)$. Thus by Lemma 3.4, η and g define a complete minimal surface with $n+1$ ends and its total curvature $c(M) = -4n\pi$, and $\mathrm{Aut}(M) \supset G \cong C_n$. Furthermore, consider the zeros of $q(z,\ 1)$, one knows that when $n > 3$, $\mathrm{Aut}(M) = G$.

Proposition 4.1. *Let M be one of the minimal surfaces constructed as above, then* $\mathrm{Aut}(M) \cong C_n$ $(n > 3)$.

Proof. From the construction one can easily see that $C_n \subset \mathrm{Aut}(M)$. To see that the symmetry group is exactly C_n, one notes that when $n > 3$, if G (a closed subgroup of $SO(3)$) contains C_n, then G will either be the cyclic group C_m, dihedral group D_m (where m is a multiple of n), or, in case $n = 4$, the octahedral group S_4, or in case $n = 5$, the isocahedral group A_5. By counting the number of ends, one may exclude C_m $(m > n)$, S_4 and A_5. To see the D_m is not the symmetry group, one has only to observe that the axis of rotation of C_n is the x_3 axis which is also the axis of an end (corresponding to $z = \infty$). But $z = 0$ is not an end. □

Example for $G = C_3$ (The Cyclic Group of Order 3) Let

$$q(z_1,\ z_2) = z_1(z_1^3 - s^3 z_2^3)(z_1^3 - r^3 z_2^3),$$

where $0 < r < s$ are to be determined. Then

$$q_1(z_1,\ z_2) = 7z_1^6 - 4(s^3 + r^3)z_1^3 z_2^3 + s^3 r^3 z_2^6,$$
$$q_2(z_1,\ z_2) = -3(s^3 + r^3)z_1^4 z_2^2 + 6s^3 r^3 z_1 z_2^5.$$

Let

$$f_1(z) = (z - r)\frac{q_1(z,\ 1)}{q(z,\ 1)} = \frac{7z^6 - 4(s^3 + r^3)z^3 + s^3 r^3}{z(z^2 + sz + s^2)(z^3 - r^3)},$$
$$f_2(z) = (z - r)\frac{q_2(z,\ 1)}{q(z,\ 1)} = \frac{-3(s^3 + r^3)z^3 + 6s^3 r^3}{(z^2 + sz + s^2)(z^3 - r^3)}.$$

Then

$$f_1(s) = 1,\ f_2(s) = -s,\ f_1'(s) = \frac{5s^3 - 2r^3}{s(s^3 - r^3)},\ f_2'(s) = \frac{s^3 - 4r^3}{s^3 - r^3}.$$

Thus, like in the previous example, since

$$\eta(z) = \frac{1}{(z - s)^2} f_1^2(z)dz,$$
$$\eta(z)g^2(z) = \frac{1}{(z - s)^2} f_2^2(z)dz,$$
$$\eta(z)g(z) = -\frac{1}{(z - s)^2} f_1(z)f_2(z)dz,$$

one has

$$Res_s \eta(z) = 2 \cdot \frac{5s^3 - 2r^3}{s(s^3 - r^3)},$$
$$Res_s \eta(z)g^2(z) = -2s\frac{s^3 - 4r^3}{s^3 - r^3},$$
$$Res_s \eta(z)g(z) = \frac{4s^3 + 2r^3}{s^3 - r^3}.$$

Similarly, one also has

$$Res_r \eta(z) = 2 \cdot \frac{5r^3 - 2s^3}{r(r^3 - s^3)},$$
$$Res_r \eta(z)g^2(z) = -2r\frac{r^3 - 4s^3}{r^3 - s^3},$$
$$Res_r \eta(z)g(z) = \frac{4r^3 + 2s^3}{r^3 - s^3}.$$

By Lemma 3.4, r and s should satisfy

$$(1) \qquad \frac{5s^3 - 2r^3}{s(s^3 - r^3)} = s\frac{s^3 - 4r^3}{s^3 - r^3}, \textit{ and } \frac{5r^3 - 2s^3}{r(r^3 - s^3)} = r\frac{r^3 - 4s^3}{r^3 - s^3}.$$

To find r and s, we let $\sigma = \frac{r}{s}$, then $0 < \sigma < 1$, therefore

$$r^2(\sigma^3 - 4) = 5\sigma^3 - 2, \; s^2(1 - 4\sigma^3) = 5 - 2\sigma^3.$$

Let

$$h(\sigma) = \sigma^2 \frac{\sigma^3 - 4}{1 - 4\sigma^3} - \frac{5\sigma^3 - 2}{5 - 2\sigma^3}.$$

It is not hard to see that $h(\sigma) = 0$ has a solution σ in $\left[0, \sqrt[3]{\frac{1}{4}}\right)$. Let

$$r^2 = \frac{\sigma^3 - 4}{1 - 4\sigma^3}, \text{ and } s^2 = \frac{5\sigma^3 - 2}{5 - 2\sigma^3}.$$

Then r and s satisfy (17). (Numerically, one can find that $r \approx 0.68673$ and $s \approx 2.34565$.) Again by Lemma 3.5, $Res_{se^{\frac{2k\pi}{3}i}} dF$, $Res_{re^{\frac{2k\pi}{3}i}} dF \in \Phi(\mathbb{R}^3)$, $(k = 0, 1, 2)$. By Lemma 3.6, $Res_0 dF \in \Phi(\mathbb{R}^3)$. Thus by Lemma 3.4, η and g define a complete minimal surface with 7 ends and its total curvature is $c(M) = -24\pi$. Furthermore

Proposition 4.2. *Let M be the minimal surface constructed as above. Then* $\mathrm{Aut}(M) \cong C_3$.

Proof. One sees that $C_3 \subset \mathrm{Aut}(M)$. Since an orbit of A_4 must contain either 4, 6 or 12 elements, but M has 7 ends, A_4 is not the symmetry group. Similarly one may exclude S_4 and A_5. Also as in the proof of Proposition 4.1, one may exclude D_m. It is easy to see that C_n is not the symmetry group when $n > 3$. □

Remark. Rob Kusner [**Ku**] suggested that a simpler example, with 4 ends and C_3 symmetry can be constructed with a different method. (He also suggested a simpler 4 ended example with D_2 symmetry than the one given below.)

Examples for $G = D_n$. (The Dihedral Group of Order $2n$, $n > 2$.) Let

$$q(z_1, z_2) = z_1^n - z_2^n.$$

Then

$$q_1(z_1, z_2) = nz_1^{n-1} \textit{ and } q_2(z_1, z_2) = -nz_2^{n-1}.$$

$$\eta(z) = \frac{n^2 z^{2n-2}}{(z^n-1)^2}dz,$$
$$g(z) = \frac{1}{z^{n-1}}.$$

Let $\mu_0(w) = -\frac{w-1}{w+1}$, $\widetilde{\eta} = \mu_0^* \eta$, $\widetilde{g}(w) = \mu_0^{-1} \circ g \circ \mu_0(w)$, then the poles of $\widetilde{\eta}(w)$ are on the imaginary line $x = 0$, symmetrically distributed about 0. Furthermore, 0 is a pole for $\widetilde{\eta}(w)$. By Lemma 3.6, $Res_0 d\widetilde{F} \in \Phi(\mathbb{R}^3)$. By Lemma 3.5, $Res_{e^{\frac{2k\pi}{n}i}} dF \in \Phi(\mathbb{R}^3)$. So this is a minimal surface with n ends and its total curvature $c(M) = -4(n-1)\pi$. And when $n > 2$, $\mathrm{Aut}(M) = D_n$.

Remark. These minimal surfaces with $\mathrm{Aut}(M) = D_n$ were originally constructed by Jorge and Meeks in [**JM**].

Proposition 4.3. *Let M be one of the minimal surfaces constructed as above, then* $\mathrm{Aut}(M) \cong D_n$, $n > 2$.

Proof. One easily sees that $D_n \subset \mathrm{Aut}(M)$. As in the proof of Proposition 4.1 and 4.2, one can exclude A_4, S_4 and A_5. One can exclude D_m $(M > n)$ by counting the number of ends. □

Another Family of Examples for $G = D_n$. (The Dihedral Group of Order $2n$, $n \neq 2, 4$.) Let

$$q(z_1,\ z_2) = z_1 z_2 (z_1^n - z_2^n).$$

Then

$$q_1(z_1,\ z_2) = (n+1) z_1^n z_2 - z_2^{n+1}, \quad and \quad q_2(z_1,\ z_2) = z_1^{n+1} - (n+1) z_1 z_2^n.$$

$$\eta(z) = \left[\frac{(n+1)z^n - 1}{z(z^n-1)}\right]^2 dz,$$
$$g(z) = \frac{-z^{n+1} + (n+1)z}{(n+1)z^n - 1}.$$

Using a method similar to that in the previous example, we have $Res_{e^{\frac{2k\pi}{n}i}} dF \in \Phi(\mathbb{R}^3)$, $k = 0, \ldots, n-1$. By Lemma 3.6, $Res_0 dF$, $Res_\infty dF \in \Phi(\mathbb{R}^3)$. So this is a minimal surface with $n+2$ ends and its total curvature $c(M) = -4(n+1)\pi$. $\mathrm{Aut}(M) = D_n$ when $n > 2$.

Proposition 4.4. *Let M be one of the minimal surfaces constructed as above, then* $\mathrm{Aut}(M) \cong D_n$, $(n \neq 2,\ 4)$.

The proof is similar to the proof of Proposition 4.3.

Example for $G = D_2$. (*The Dihedral Group of Order* 4.) Let

$$q(z_1, z_2) = z_1 z_2 (z_1^2 - z_2^2)(z_1^2 - r^2 z_2^2)(r^2 z_1^2 - z_2^2)(z_1^2 + s^2 z_2^2)(s^2 z_1^2 + z_2^2),$$

where $r, s > 0$ are to be determined. The set of zeros of $q(z_1, z_2)$ on S^2 is

$$Z = \{0, \infty, 1, -1, r, -r, r^{-1}, -r^{-1}, si, -si, s^{-1}i, -s^{-1}i\}.$$

$$\frac{q_1(z_1, z_2)}{q(z_1, z_2)} = \frac{1}{z} + \frac{1}{z-1} + \frac{1}{z+1} + \frac{1}{z-r} + \frac{1}{z+r} + \frac{r}{rz-1} + \frac{r}{rz+1} + \frac{1}{z-is} + \frac{1}{z+is} + \frac{s}{sz-i} + \frac{s}{sz+i},$$

$$\frac{q_2(z_1, z_2)}{q(z_1, z_2)} = 1 + \frac{1}{z+1} - \frac{1}{z-1} + \frac{r}{z+r} - \frac{r}{z-r} + \frac{1}{rz+1} - \frac{1}{rz-1} + \frac{is}{z+is} - \frac{is}{z-is} + \frac{i}{sz+i} - \frac{i}{sz-i}.$$

By Lemma 3.5 and Lemma 3.6, $Res_0 dF$, $Res_\infty dF$, $Res_1 dF$, $Res_{-1} dF \in \Phi(\mathbb{R}^3)$. On the other hand,

$$\begin{aligned} Res_r \eta(z) &= \frac{3}{r} + \frac{4r}{r^2-1} + \frac{4r^3}{r^4-1} + \frac{4r}{r^2+s^2} + \frac{4s^2 r}{s^2 r^2 + 1}, \\ Res_r \eta(z) g^2(z) &= -r\left[3 - \frac{4}{r^2-1} - \frac{4}{r^4-1} + \frac{4s^2}{r^2+s^2} + \frac{4}{s^2r^2+1}\right], \\ Res_r \eta(z) g(z) &= 2\left[\frac{r^2+1}{r^2-1} + \frac{r^4+1}{r^4-1} + \frac{r^2-s^2}{r^2+s^2} + \frac{s^2r^2-1}{s^2r^2+1}\right]. \end{aligned}$$

And

$$\begin{aligned} Res_{is} \eta(z) &= -i\left[\frac{3}{s} + \frac{4s}{s^2+1} + \frac{4s}{r^2+s^2} + \frac{4r^2 s}{s^2r^2+1} + \frac{4s^3}{s^4-1}\right], \\ Res_{is} \eta(z) g^2(z) &= -is\left[3 + \frac{4}{s^2+1} + \frac{4r^2}{r^2+s^2} + \frac{4}{s^2r^2+1} - \frac{4}{s^4-1}\right], \\ Res_{is} \eta(z) g(z) &= 2\left[\frac{s^2-1}{1+s^2} + \frac{s^2-r^2}{r^2+s^2} + \frac{r^2s^2-1}{1+r^2s^2} + \frac{s^4+1}{s^4-1}\right]. \end{aligned}$$

Let

$$\begin{aligned} h_1(r, s) =& 2r\left[Res_r \eta(z) + \overline{Res_r \eta(z) g^2(z)}\right] \\ =& \frac{3}{2}(1-r^2) + \frac{6r^2}{r^2-1} + 2r^2(1-s^2)\left[\frac{1}{r^2+s^2} - \frac{1}{s^2r^2+1}\right]; \end{aligned}$$

$$\begin{aligned} h_2(r,\ s) =& 2is\left[Res_{is}\eta(z) + \overline{Res_{is}\eta(z)g^2(z)}\right] \\ =& \frac{3}{2}(1-s^2) + \frac{2s^2}{s^2-1} + 2(1-r^2)s^2\left[\frac{1}{r^2+s^2} - \frac{1}{s^2r^2+1}\right]. \end{aligned}$$

By Lemma 3.4, r and s must satisfy

$$h_1(r,\ s) = 0, \text{ and } \ h_2(r,\ s) = 0. \tag{2}$$

One can apply Brouwer's Fixed Point Theorem to show there is a pair of $(r,\ s)$ which solves (18). (Numerically, $r \approx 0.43300$ and $s \approx 0.63947$.) Then by Lemma 3.5 and Lemma 3.6, one has $Res_\zeta dF \in \Phi(\mathbb{R}^3)$ for all $\zeta \in Z$. By Lemma 3.4, therefore, this represents a complete minimal surface with 12 ends and total curvature $c(M) = -44\pi$.

Proposition 4.5. *Let M be the minimal surface constructed as above. Then* $\mathrm{Aut}(M) \cong D_2$.

Proof. Clearly $D_2 \subset \mathrm{Aut}(M)$. Note that the ends of the surface correspond to $z = 0,\ \infty,\ 1,\ -1,\ r,\ -r,\ si,\ -si,\ \frac{i}{s},\ -\frac{i}{s}$. Thus the surface will only allow the isometry that is a rotation of order 2 about the axis either passing 0 and ∞, or that passing 1 and -1, or that passing i and $-i$. Thus $\mathrm{Aut}(M) \cong D_2$. □

Examples for $G = A_4$. (The Tetrahedral Group.) Using $q_{[v]}(z_1,\ z_2)$ in (11) as $q(z_1,\ z_2)$, one may obtain a minimal surface with 4 embedded ends and total curvature $c(M) = -12\pi$. One may get

$$q_1(z_1,\ z_2) = 4(z_1^3 + \sqrt{3}iz_1z_2^2), \ \textit{and} \ \ q_2(z_1,\ z_2) = 4(\sqrt{3}iz_1^2z_2 + z_2^3).$$

And

$$\begin{aligned} g(z) &= -\frac{\sqrt{3}iz^2+1}{z^3+\sqrt{3}iz}, \\ \eta(z) &= \left[\frac{4(z^3+\sqrt{3}i)}{z^4+2\sqrt{3}iz^2+1}\right]^2 dz. \end{aligned}$$

Lemma 3.5 and Lemma 3.6 assure that $Res_\zeta dF \in \Phi(\mathbb{R}^3)$ for all ζ the poles of $\eta(z)$. Then by Lemma 3.4, we obtain the minimal surface.

Proposition 4.6. *Let M be the minimal surface constructed as above. Then* $\mathrm{Aut}(M) \cong A_4$.

Proof. Clearly $A_4 \subset \mathrm{Aut}(M)$. Since the ends correspond to

$$z = \sqrt{2}e^{\frac{\pi}{3}i},\ -\sqrt{2}e^{\frac{\pi}{3}i},\ \sqrt{2}e^{\frac{2\pi}{3}i},\ -\sqrt{2}e^{\frac{2\pi}{3}i},$$

forming a set which is not invariant under the action $z \mapsto e^{\frac{\pi}{2}i}$, so S_4 is not the symmetry group. □

Examples for $G = S_4$. (The Octahedral Group.) We may use the homogeneous polynomials in (13) to give 3 complete minimal surfaces with embedded ends. Like before, one needs to apply Lemma 3.5 and 3.6 to get that $Res_\zeta dF \in \Phi(\mathbb{R}^3)$ and then use Lemma 3.4 to prove they are complete minimal surfaces.

Let $q(z_1, z_2)$ be $q_{[0]}(z_1, z_2)$ in (13), we get

$$q_1(z_1, z_2) = 5z_1^4 z_2 - z_2^5, \text{ and } \quad q_2(z_1, z_2) = z_1^4 - 5z_1 z_2^4.$$

Then

$$\eta(z) = \left[\frac{5z^4 - 1}{z(z^4 - 1)}\right]^2 dz,$$
$$g(z) = \frac{-z^5 + 5z}{5z^4 - 1}.$$

This will give a minimal surface with 6 ends and the total curvature $c(M) = -20\pi$.

Let $q(z_1, z_2)$ be $q_{[v]}(z_1, z_2)$ in (13), Then

$$q_1(z_1, z_2) = 8z_1^7 + 56z_1^3 z_2^4, \text{ and } \quad q_2(z_1, z_2) = 56z_1^4 z_2^3 + 8z_2^7.$$

Therefore

$$\eta(z) = \left[\frac{8z^7 + 56}{z^8 + 14z^4 + 1}\right]^2 dz,$$
$$g(z) = -\frac{7z^4 + 1}{z^7 + 7z^3}$$

gives a minimal surface with 8 ends and the total curvature $c(M) = -28\pi$.

Let $q(z_1, z_2)$ be $q_{[u]}(z_1, z_2)$ in (13), like the above

$$\eta(z) = \left[\frac{12z^{11} - 264z^7 - 132z^3}{z^{12} - 33z^8 - 33z^4 + 1}\right]^2 dz,$$
$$g(z) = \frac{11z^8 + 22z^4 - 1}{z^{11} - 22z^7 - 11z^3}$$

will give a minimal surface with 12 ends and the total curvature $c(M) = -44\pi$.

Proposition 4.7. *Let M be one of the minimal surfaces constructed as above. Then* $\mathrm{Aut}(M) \cong S_4$.

Proof. This directly follows since S_4 is maximal in $SO(3)$. □

Examples for $G = A_5$. (*The Icosahedral Group.*) Using the homogeneous polynomials in (14), one may find 3 complete minimal surfaces with $\text{Aut}(M) = A_5$. one needs to apply Lemma 3.5 and 3.6 to get that $Res_\zeta dF \in \Phi(\mathbb{R}^3)$ and then use Lemma 3.4 to prove they are complete minimal surfaces.

Using $q_{[0]}(z_1,\ z_2)$, one has

$$\eta(z) = \left[\frac{11z^{10} + 66z^5 - 1}{z^{11} + 11z^6 - z}\right]^2 dz,$$

$$g(z) = -\frac{z^{11} + 66z^6 - 11z}{11z^{10} + 66z^5 - 1}.$$

This defines a complete minimal surface with 12 ends and the total curvature $c(M) = -44\pi$. (See Figure.) If one uses $q_{[v]}(z_1,\ z_2)$, one has

$$\eta(z) = \left[\frac{z^{19} - 171z^{14} + 247z^9 + 57z^4}{z^{20} - 228z^{15} + 494z^{10} + 228z^5 + 1}\right]^2 dz,$$
$$g(z) = \frac{57z^{15} - 247z^{10} - 171x^5 - 1}{z^{19} - 171z^{14} + 247z^9 + 57z^4}.$$

This represents a complete minimal surface with 20 ends and the total curvature $c(M) = -76\pi$. Using $q_{[i]}(z_1,\ z_2)$, one has

$$\eta(z) = \left[\frac{z^{29} + 345z^{24} - 6670z^{19} - 3335z^9 - 87z^4}{z^{30} + 522z^{25} - 10005z^{20} - 10005z^{10} - 522z^5 + 1}\right]^2 dz,$$
$$g(z) = \frac{-87z^{25} + 3335z^{20} + 6670z^{10} + 345z^5 - 1}{z^{29} + 345z^{24} - 6670z^{19} - 3335z^9 - 87z^4}.$$

This gives a complete minimal surface with 30 ends and the total curvature $c(M) = -116\pi$.

Proposition 4.8. *Let* M *be one of the minimal surfaces constructed as above. Then* $\text{Aut}(M) \cong A_5$.

Proof. This directly follows because A_5 is maximal in $SO(3)$. □

Remark. When $\text{Aut}(M) = G \subset SO(3)$ is one of the Platonic groups, i.e. one of D_n, A_4, S_4 or A_5, as pointed out by Rob Kusner [**Ku**], we can geometrically construct the Gauss map of M in the following manner: Take a (triangular) fundamental domain F of $S^2/\widetilde{G}$ on S^2, where $\widetilde{G}$ is the natural $\mathbb{Z}_2$-extension of G in $O(3)$. Then choose one of the vertices v of F to be an end. Let v_1 and v_2 be the other 2 vertices of F, and a_1 and a_2 be their antipodal points. Thus v, a_1 and a_2 form another (nonconvex) triangle $P \supset F$. By the Riemann mapping theorem, there is a holomorphic map

g which maps F onto P, such that $g(v) = v$, $g(v_1) = a_1$ and $g(v_2) = a_2$. By Schwartz reflection, g can be extended to a map $S^2 \to S^2$, which is the desired Gauss map of M^*. (Note the the degree of g depends upon which vertex is chosen to be v.)

In summary of the above discussion, we get the following theorem

Theorem 4.9. *If $G \subset SO(3)$ is a closed subgroup, $G \not\cong SO(2)$, $SO(3)$, then there is a complete genus 0 minimal surface M with finite total curvature and all ends embedded so that* $\text{Aut}(M) = G$.

Proof. We have already constructed the minimal surfaces with symmetry group being one of C_n, $(n > 2)$, D_n, A_4, S_4 and A_5. For the cases where the symmetry group is either 1 or C_2, see [**B**] or [**Lo**]. □

Remark. One knows that there is no complete minimal surface with embedded ends having $\text{Aut}(M)$ either $SO(2)$ or $SO(3)$. Indeed, if $\text{Aut}(M) \supset SO(2)$, then M must be a minimal surface of revolution. However, the only complete minimal surface of revolution is the catenoid for which $\text{Aut}(M) = SO(2) \ltimes \mathbb{Z}_2$. (Enneper's surface does have intrinsic symmetry group $SO(2)$, but its single end is not embedded, and in particular, not catenoidal, so the [**CMW**] symmetry extension theorem does not apply.)

References

[B] E. Barbanel, *Minimal Surfaces in $\mathbb{R}^3$ with Genus Zero and Total Curvature -8π*, G.A.N.G. Preprint, 1988.

[BG] J. Lucas M. Barbosa and A. Gervasio, *Minimal Surfaces in $\mathbb{R}^3$*, Springer-Verlag, Berlin, Heidelberg, 1986.

[CMW] H.I. Choi, W.H. Meeks and B. White, *A Rigidity Theorem for Properly Embedded Minimal Surfaces in $\mathbb{R}^3$*, Jour. Diff. Geom., **32** (1990) no. 1, 65-76.

[DMc] P. Doyle and C. McMullen, *Solving the Quintic by Iteration*, Acta Math., **163** (1989), 151 - 180.

[JM] L. Jorge and W. Meeks, *The Topology of Complete Minimal Surfaces of Finite Total Gaussian Curvature*, Topology, **22** (1983) no. 2, 203-221.

[K] F. Klein, Lectures on the Icosahedron and the Solutions of Equations of the Fifth Degree, (translated by G. G. Morrice), 2nd ed. Dover Publication, Inc., New York, 1913.

[Ku] R. Kusner, Personal Communication, 1993.

[KS] R. Kusner and N. Schmitt, *Spinor Representation of Immersed Surfaces, I: Regular Homotopy Classes and the Arf Invariant*, G.A.N.G. Preprint, 1993.

[L] H.B. Lawson, Lectures on Minimal Submanifolds, Volume I, Mathematical Lecture Series 9, Publish or Perish, Inc., 1980.

[Lo] F.J. Lopez, *The Classification of Complete Minimal Surfaces with Total Curvature Greater Than -12π*, Preprint, 1991.

[M] W.H. Meeks, *The Classification of Complete Minimal Surfaces with Total Curvature Greater than* -8π, Duke Math. J., **48** (1981), 523-535.

[O] R. Osserman, A Survey of Minimal Surfaces, Dover Publication, New York, 1986.

[Or] P. Orlik, Seifert Manifolds, Springer-Verlag, 1972.

[S] N. Schmitt, *Spinor Representation of Immersed Surfaces, II: Minimal Surfaces with Embedded Planar Ends*, G.A.N.G. Preprint, 1993.

[Sc] R. Schoen, *Uniqueness, Symmetry and Embeddedness of Minimal Surfaces*, J. Diff. Geom., **18** (1983), 791-809.

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