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CROSSCAP NUMBER OF A KNOT

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B. E. Clark defined the crosscap number of a knot to be the minimum number of the first Betti numbers of non-orientable surfaces bounding it. In this paper, we investigate the crosscap numbers of knots. We show that the crosscap number of 7_4 is equal to 3. This gives an affirmative answer to a question given by Clark. In general, the crosscap number is not additive under the connected sum. We give a necessary and sufficient condition for the crosscap number to be additive under the connected sum.

0. Introduction.

We study knots in the 3-sphere S^3. The genus $g(K)$ of a knot K is the minimum number of the genera of Seifert surfaces for it [**11**]. Here a Seifert surface means a connected, orientable surface with boundary K. In 1978, B. E. Clark [**3**] defined the *crosscap number* $C(K)$ of K to be the minimum number of the first Betti numbers of connected, non-orientable surfaces bounding it. (For the trivial knot, it is defined to be 0 instead of 1.) He proved the following inequality and asked whether there exist knots for which the equality holds.

$$C(K) \leq 2g(K) + 1.$$

Note that since C (trivial knot) $= 0$, the equality does not hold for the trivial knot. In this paper, we give an example which satisfies the equality showing that $C(7_4) = 3$. (We use the notation of J. W. Alexander and B. G. Briggs for knots [**1**]. See also [**9**] and [**2**].)

Clark also studied how the crosscap number behaves under the connected sum. If we denote by $\Gamma(K)$ the minimum number of the first Betti numbers of connected, unoriented surfaces bounding it (an unoriented surface means a surface which is orientable or not), $\Gamma(K)$ is additive under the connected sum, i.e., $\Gamma(K_1 \sharp K_2) = \Gamma(K_1) + \Gamma(K_2)$ [**3**, Lemma 2.7] as H. Schubert proved for the genus [**10**]. Note that $\Gamma(K) = \min(2g(K), C(K))$. A proof is given by an ordinary "cut-and-paste" argument. See for example [**9**, Theorem 5A14]. But such an argument does not apply to the crosscap number because one of the two surfaces obtained from a non-orientable surface by cutting along an

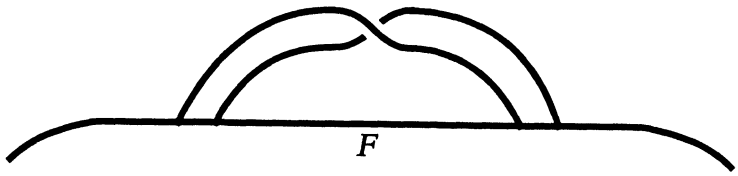

Figure 1.1.

arc may be orientable. But Clark proved that $C(K_1 \sharp K_2)$ is either $C(K_1) + C(K_2) - 1$ or $C(K_1) + C(K_2)$. More precisely, he proved that

$$C(K_1) + C(K_2) - 1 \leq C(K_1 \sharp K_2) \leq C(K_1) + C(K_2).$$

If we put $K_1 = K_2 = 7_4$, we have the first equality and if we put $K_1 = K_2 = 3_1$, we have the second equality (this is not so interesting). More generally, we can show that the second equality holds if and only if $C(K_i) = \Gamma(K_i)$ for $i = 1$ and 2.

1. Inequalities.

In this section, we will give some inequalities concerning $\Gamma(K)$ and $C(K)$.

As an upper bound for $C(K)$, Clark proved the following by adding a half-twisted band to a Seifert surface as in Figure 1.1.

Proposition 1.1. (B.E. Clark [**3**, Proposition 2.6]).

$$C(K) \leq 2g(K) + 1.$$

Now we give another upper bound in terms of the crossing number. Let $n(K)$ be the minimum crossing number of a knot K. Then we have

Proposition 1.2.

$$\Gamma(K) \leq \left[\frac{n(K)}{2}\right],$$

where $[x]$ denotes the greatest integer which does not exceed x.

Proof. Let D be a diagram in S^2 of K with minimum crossings. We colour the regions of D with black and white like a chess-board and denote by b and w the numbers of black regions and white regions respectively. Then we can construct two surfaces bounding K form the black regions and the white regions. So $\Gamma(K)$ is less than or equal to the first Betti number of these surfaces. Since the number of all the regions is $n(K)+2$, we have

$$\max(b,w) \geq \begin{cases} \dfrac{n(K)+2}{2} & \text{if } n(K) \text{ is even,} \\ \\ \dfrac{n(K)+3}{2} & \text{if } n(K) \text{ is odd.} \end{cases}$$

Now from an argument using the Euler characteristics, we have

$$\Gamma(K) \leq 1 + n(K) - \max(b,w) \leq \left[\frac{n(K)}{2}\right].$$

Thus the proof is complete. □

For the crosscap number, we have

Proposition 1.3.

$$C(K) \leq \left[\frac{n(K)}{2}\right].$$

Proof. If the surface constructed above is non-orientable, the inequality clearly holds. If it is orientable and its first Betti number is strictly less than $[n(K)/2]$, then we can add a half-twisted band as in Figure 1.1 and make it non-orientable. (This was observed by Clark in the proof of [**3**, Proposition 2.6].) Since the first Betti number increases by one, the inequality still holds. So we assume without loss of generality that the surface constructed from the black regions (black surface) is orientable and its first Betti number is equal to $[n(K)/2]$.

If $n(K) \equiv 2$ or $3 \bmod 4$, then $[n(K)/2]$ is odd. But this cannot occur since a surface with odd first Betti number must be non-orientable.

Next we consider the case that $n(K) \equiv 0 \bmod 4$. Since the first Betti number of the white surface is also $[n(K)/2] = n(K)/2$ in this case, the result follows since the white surface must be non-orientable (if $n(K) \neq 0$) as indicated in Figure 1.2 and so we can choose the white one. (It was observed by Clark in the proof of Theorem 2.1 in [**3**] that either black or white surface is non-orientable.) If $n(K) = 0$, then K is the trivial knot and the equality holds from the definition.

Finally we consider the case that $n(K) \equiv 1 \bmod 4$. Let b and w be the numbers of black regions and white regions respectively, and B and W the black and the white surfaces respectively. From the assumption, $\beta_1(B) = [n(K)/2] = (n(K)-1)/2$ and so we have $b = (n(K)+3)/2$ since $\beta_1(B) = 1 + n(K) - b$, where β_1 denotes the first Betti number. Since $b + w = n(K) + 2$, we have

$$n(K) = 2w - 1.$$

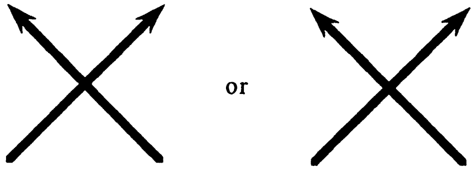

Figure 1.2.

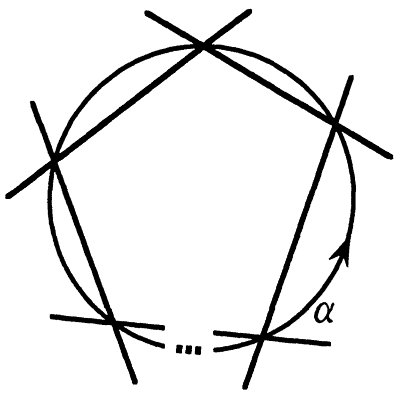

Figure 1.3.

Now we see that the number of edges of every white region is even, since otherwise an arc α indicated in Figure 1.3 is non-orientable and so is B.

Let $2x_1, 2x_2, \dots$, and $2x_w$ be the numbers of edges of white regions. Since the number of all the edges is $2n(K)$, we have

$$n(K) = \sum_{i=1}^{w} x_i.$$

Since $n(K) = 2w - 1$, we see there exists a white region which has only two edges. If we replace a neighbourhood of this region as in Figure 1.4, we have a non-orientable surface B' with the same first Betti number. So we have $C(K) \leq \beta_1(B') = \beta_1(B) = [n(K)/2]$.

So the proof is complete. □

Observation 1.4. The inequalities in Propositions 1.2 and 1.3 are best possible if $n(K) \equiv 0,\ 1,$ or $3 \bmod 4$, i.e., we have examples satisfying the equalities in these cases.

Proof. If $n(K) = 4m$ for some integer $m \geq 0$, we take $\sharp_m 4_1$, where $\sharp_m 4_1$ is the connected sum of m copies of 4_1. Since $\sharp_m 4_1$ is alternating, we have

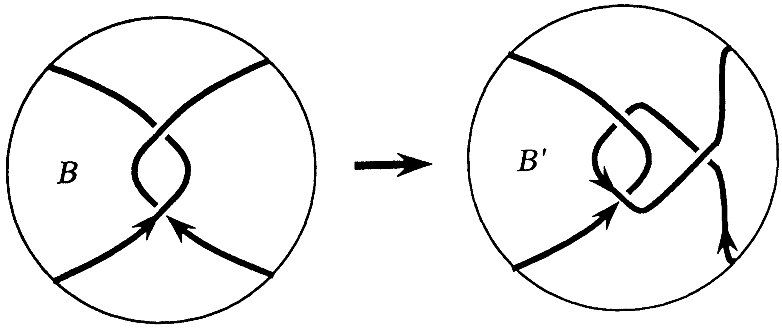

Figure 1.4.

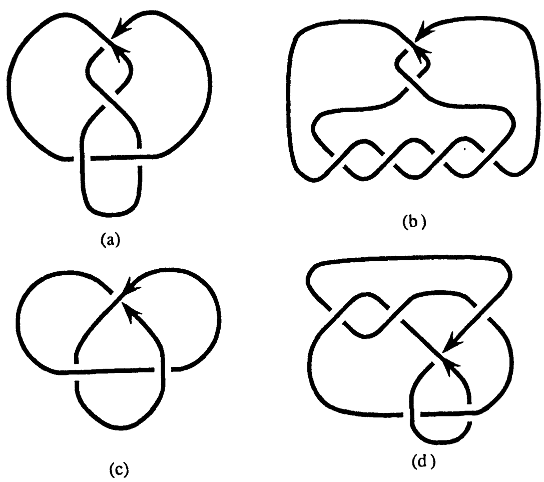

Figure 1.5.

$n(\sharp_m 4_1) = 4m$ from Tait's conjecture [**5**], [**7**], [**12**]. As indicated in Figure 1.5(a), $C(4_1) \leq 2$. From Proposition 2.1, we see that $C(4_1) = 2$ since 4_1 is not a cable knot. So we have $\Gamma(4_1) = \min(2g(4_1), C(4_1)) = 2$. Thus $C(\sharp_m 4_1) = \Gamma(\sharp_m 4_1) = 2m = [4m/2]$ from the additivity of $\Gamma(K)$. So the equalities in Propositions 1.2 and 1.3 follow in this case.

If $n(K) = 4m + 1$ $(m \geq 1)$, then we take $5_2\sharp(\sharp_{m-1}4_1)$. $n(5_2\sharp(\sharp_{m-1}$ $4_1))$ is equal to $4m + 1$ as above. Since $\Gamma(5_2) = C(5_2) = 2$ from Figure 1.5(b) and Proposition 2.1, we see that $C(5_2\sharp(\sharp_{m-1}4_1)) = \Gamma(5_2\sharp(\sharp_{m-1}4_1)) = 2m = [(4m + 1)/2]$ by the same reason as above. This gives the equalities for $n(K) = 4m + 1$.

If $n(K) = 4m + 3$ $(m \geq 0)$, we take $3_1\sharp(\sharp_m 4_1)$. Since $\Gamma(3_1) = C(3_1) = 1$ (see Figure 1.5(c)), we have $C(3_1\sharp(\sharp_m 4_1)) = \Gamma(3_1\sharp(\sharp_m 4_1)) = 2m+1 = [(4m+$

3)/2], and the proof is complete. □

Remark 1.5. As shown in Figure 1.5(d), $\Gamma(6_3) = C(6_3) \leq 3$. (Note that $g(6_3) = 2$.) But the authors do not know whether the equality holds or not. If it holds, then the equalities in Propositions 1.1 and 1.2 also hold for $n(K) \equiv 2 \bmod 4$.

Let $D(K)$ be the double branched cover of S^3 branched along a knot K and $e_2(K)$ be the minimum number of generators of $H_1(D(K); \mathbb{Z})$. Then for a lower bound, we have

Proposition 1.6. $\Gamma(K) \geq e_2(K)$. *Thus we have* $C(K) \geq e_2(K)$ *and* $2g(K) \geq e_2(K)$.

Proof. Since we can construct the double branched cover of K by cutting along an unoriented surface bounding it, we have $\Gamma(K) \geq e_2(K)$. The second and the third inequalities follow from the definition of $\Gamma(K)$. □

2. Knots with crosscap number two.

If a knot bounds a Möbius band, it is a cable knot of the centre line of the band. So we have the following proposition due to Clark [**3**, Proposition 2.2].

Proposition 2.1. (B.E. Clark). *A non-trivial knot has crosscap number one if and only if it is a* $(2,p)$*-cable of some knot for an odd integer* p.

Now consider knots with crosscap number two.

For an unoriented surface F with boundary a knot K, C. McA. Gordon and R. A. Litherland define a *Goeritz matrix* $G_F(K)$ as follows [**4**]. Choose a generator system $\{\alpha_1, \alpha_2, \dots, \alpha_d\}$ for $H_1(F; \mathbb{Z})$. Then the (i,j)-entry of $G_F(K)$ is defined to be $\mathrm{lk}(a_i, \tau a_j)$, where lk is the linking number, a_i is a 1-cycle representing α_i, and τa_j is a 1-cycle in $S^3 - F$ obtained by pushing off $2a_j$.

The *normal Euler number* $e(F)$ of F is defined to be $-lk(K, K')$, where K' is the intersection of F and the boundary of the regular neighbourhood of K in S^3 with orientation parallel to that of K.

Now a result of Gordon and Litherland is as follows [**4**, Corollary 5].

$$(*) \quad \sigma(K) = \mathrm{sign}(G_F(K)) + \frac{1}{2}e(F),$$

where $\sigma(K)$ is the signature of K and $\mathrm{sign}(G_F(K))$ is the signature of the symmetric matrix $G_F(K)$. Note that $\sigma(K)$ is by definition equal to $\mathrm{sign}(G_F(K))$ for *orientable* surface F. We also note that a Goeritz matrix is uniquely determined if one fixes a generator system for $H_1(F; \mathbb{Z})$.

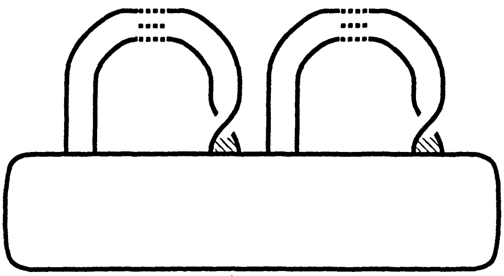

Figure 2.1.

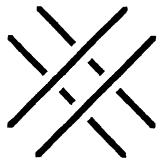

Figure 2.2.

Now we will prove

Theorem 2.2. *Suppose that a knot K bounds a non-orientable surface F with the first Betti number two. Then one can choose a generator system for $H_1(F;\mathbb{Z})$ so that the Goeritz matrix $G_F(K)$ corresponding to it is of the following form.*

$$G_F(K) = \begin{pmatrix} l & m \\ m & n \end{pmatrix}$$

with $\sigma(K) = \operatorname{sign}(G_F(K)) - (l + 2m + n)$ for some odd integers l and n and some even integer m. Note that $l + 2m + n$ is the sum of all the entries in $G_F(K)$.

Proof. We may assume that F is a disc with two non-orientable bands as indicated in Figure 2.1. We also assume that crossings of bands are as in Figure 2.2, except near their heads. Figure 2.3 shows an example.

Choosing a generator system $\{\alpha, \beta\}$ as in figure 2.4, we see that the Goeritz matrix corresponding to it is

$$\begin{pmatrix} \operatorname{lk}(a, \tau a) & \operatorname{lk}(a, \tau b) \\ \operatorname{lk}(b, \tau a) & \operatorname{lk}(b, \tau b) \end{pmatrix} = \begin{pmatrix} 2w(A, A) + 1 & w(A, B) \\ w(B, A) & 2w(B, B) + 1 \end{pmatrix}.$$

Here a and b are 1-cycles representing α and β respectively, A and B are bands α and β pass through respectively, and $w(X, Y)$ is the sum of the signs of crossings of bands X and Y with signs determined as in Figure 2.5.

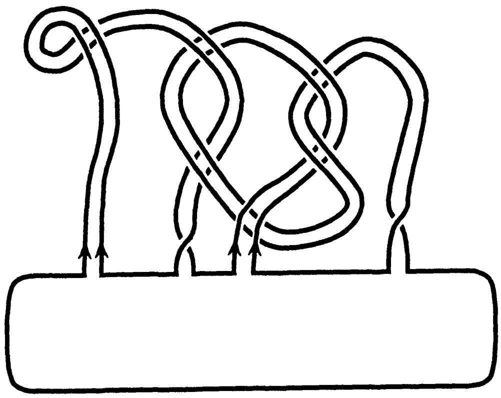

Figure 2.3.

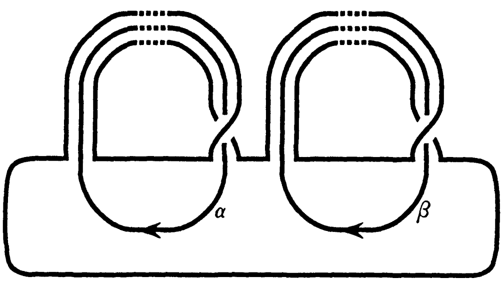

Figure 2.4.

Note that $\mathrm{lk}(a, \tau b) = \mathrm{lk}(b, \tau a) = 2\,\mathrm{lk}(a, b)$, so $w(A, B) = w(B, A)$ and both $w(A, B)$ and $w(B, A)$ are even.

We also see that the normal Euler number of F is $-2\{2w(A, A)+w(A, B)+w(B, A) + 2w(B, B)\} - 4$ (the last 4 comes from single crossings near the heads of the bands).

From $(*)$, we have

$$\begin{aligned}\sigma(K) =& \operatorname{sign}(G_F(K)) \\ &- \{2w(A, A) + w(A, B) + w(B, A) + 2w(B, B)\} - 2.\end{aligned}$$

Thus the proof is complete putting $l = 2w(A, A) + 1$, $n = 2w(B, B) + 1$, and $m = w(A, B) = w(B, A)$. □

Similarly we can prove

Theorem 2.3. *If $C(K) \leq d$, then we can choose a $d \times d$ Goeritz matrix of the following form.*

$$G_F(K) = (a_{ij})$$

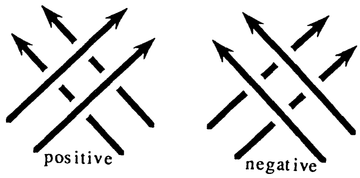

Figure 2.5.

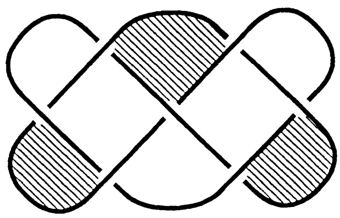

Figure 3.1.

with a_{ii} *odd,* a_{ij} *even* $(i \neq j)$*, and* $\sigma(K) = \mathrm{sign}(G_F(K)) - \sum_{i,j=1}^{d} a_{ij}$.

3. Crosscap number of 7_4.

In this section we will prove

Theorem 3.1. $C(7_4) = 3$.

As is well known and shown in Figure 3.1, we have $g(7_4) = 1$. Thus $C(7_4) \leq 3$ from Proposition 1.1. So we need to prove $C(7_4) > 2$.

Note that Proposition 1.6 is not useful at all, since it does not take orientability into account.

Proof of Theorem 3.1. Suppose that $C(7_4) \leq 2$. Then there exists a non-orientable surface F with $\partial F = 7_4$ and $H_1(F;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. The Goeritz matrix $G_F(7_4)$ determined by a generator system of $H_1(F;\mathbb{Z})$ is a 2×2-matrix. If one changes a generator system by using a unimodular matrix P, then a Goeritz matrix becomes $P^t G_F(7_4) P$, where P^t is the transposed matrix of P. Since the absolute value of the determinant of a Goeritz matrix of a knot is equal to the order of $H_1(D(7_4);\mathbb{Z})$, the determinant of $G_F(7_4)$ is ± 15. From an elementary theory of binary quadratic forms (see for example [**8**, 3.5]), there are the following seven cases to be considered after changes of generator systems.

$$\text{(i)}\,\varepsilon\begin{pmatrix}1 & 0\\ 0 & 15\end{pmatrix},\quad \text{(ii)}\,\varepsilon\begin{pmatrix}2 & 1\\ 1 & 8\end{pmatrix},\quad \text{(iii)}\,\varepsilon\begin{pmatrix}3 & 0\\ 0 & 5\end{pmatrix},\quad \text{(iv)}\,\varepsilon\begin{pmatrix}4 & 1\\ 1 & 4\end{pmatrix},$$

$$\text{(v)}\,\varepsilon\begin{pmatrix}3 & 0\\ 0 & -5\end{pmatrix},\quad \text{(vi)}\,\varepsilon\begin{pmatrix}2 & 1\\ 1 & -7\end{pmatrix},\quad \text{(vii)}\,\varepsilon\begin{pmatrix}1 & 0\\ 0 & -15\end{pmatrix},$$

with $\varepsilon = \pm 1$.

First of all, we consider the linking form $\lambda : H_1(D(7_4);\mathbb{Z}) \times H_1(D(7_4);\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$. (For the definition and the way to calculate it from the Goeritz matrix, see [**4**].) Since the Goeritz matrix corresponding to the *oriented* Seifert surface described in Figure 3.1 is $\left(\begin{smallmatrix}-4 & 1\\ 1 & -4\end{smallmatrix}\right)$ with appropriate generator system, there is a generator g of $H_1(D(7_4);\mathbb{Z})$ ($\cong \mathbb{Z}/15\mathbb{Z}$) such that $\lambda(g,g) = -4/15$ in $\mathbb{Q}/\mathbb{Z}$. From an easy calculation, we see that the cases (i) with $\varepsilon = -1$, (iv) with $\varepsilon = -1$, and (vii) with $\varepsilon = 1$ are possible. Thus we only need to consider

$$X = \begin{pmatrix}-1 & 0\\ 0 & -15\end{pmatrix},\quad Y = \begin{pmatrix}-4 & -1\\ -1 & -4\end{pmatrix},\quad \text{and } Z = \begin{pmatrix}1 & 0\\ 0 & -15\end{pmatrix}.$$

Now we use Theorem 2.2. Since $\sigma(7_4) = -2$, there exists an integral matrix $Q = \left(\begin{smallmatrix}p & q\\ r & s\end{smallmatrix}\right)$ with $ps - qr = \pm 1$ such that the diagonal entries of Q^tMQ is odd and that

$$(**)\quad \operatorname{sign}(M) - (\text{the sum of all the entries in } Q^tMQ) = -2,$$

where M is X, Y, or Z.

We note that the diagonal entries of Q^tYQ are even. This contradicts the fact that the diagonal entries of Q^tYQ must be odd. (Here the equation $(**)$ is not used.)

Since the sum of all the entries in Q^tXQ is equal to $-(p+q)^2 - 15(r+s)^2$, we have

$$-2 + (p+q)^2 + 15(r+s)^2 = -2$$

from $(**)$. But the solution to this equation is $p+q = r+s = 0$, which is impossible since Q must be unimodular.

The sum of all entries in Q^tZQ is equal to $(p+q)^2 - 15(r+s)^2$. So we have

$$(p+q)^2 - 15(r+s)^2 = 2.$$

But this is also impossible since the equation becomes

$$(p+q)^2 \equiv 2 \bmod 3.$$

Thus the proof is complete. □

Remark 3.2. As W. B. R. Lickorish mentioned in [**6**], 7_4 bounds a Möbius band in a 4-ball. So we cannot prove Theorem 3.1 by using a 4-dimensional technique.

Remark 3.3. We cannot prove that $C(6_3) > 2$ using Theorem 2.2, since the matrix $\begin{pmatrix} 7 & -6 \\ -6 & 7 \end{pmatrix}$ satisfies the condition of the theorem and the linking form determined by it coincides with that of 6_3.

4. Behaviour under the connected sum.

Clark studied the behaviour of the crosscap number under the connected sum and show the following inequality.

Proposition 4.1. (B.E. Clark [**3**, Theorem 2.8]).

$$C(K_1) + C(K_2) - 1 \leq C(K_1 \sharp K_2) \leq C(K_1) + C(K_2).$$

We will prove this proposition carefully to obtain a necessary and sufficient condition for the equalities (Proposition 4.3). Our proof is essentially the same as Clark's. Before proving this, we prepare a lemma.

Lemma 4.2.

$$C(K_1 \sharp K_2) = \min(C(K_1) + \Gamma(K_2), \Gamma(K_1) + C(K_2)).$$

Proof. It is easy to see

$$C(K_1 \sharp K_2) \leq \min(C(K_1) + \Gamma(K_2), \Gamma(K_1) + C(K_2)).$$

So we prove

$$C(K_1 \sharp K_2) \geq \min(C(K_1) + \Gamma(K_2), \Gamma(K_1) + C(K_2)).$$

Let F be a non-orientable surface bounding $K_1 \sharp K_2$ with $\beta_1(F) = C(K_1 \sharp K_2)$. Let S be a 2-sphere separating the two connected summands. We may assume that the intersection of S and F is the disjoint union of simple loops and an arc. Let D be a disc bounding an innermost loop in S with no loop or arc in it.

We see that ∂D cuts F into two connected components. For otherwise the (unoriented) surface F' obtained from "cut-and-paste" along D has the first Betti number $\beta_1(F) - 2$. On the other hand, we can construct from F'' a non-orientable surface with the first Betti number $\beta_1(F) - 1$ adding a twisted band as in Figure 1.1 if necessary, which contradicts to the minimality of F.

So we have a surface F' with boundary $K_1 \sharp K_2$ and a closed surface F'' after "cut-and-paste" along D. From the minimality, F'' is a sphere and so F' is non-orientable.

Continuing these processes, we can construct a non-orientable surface which intersects S only in an arc.

Thus we have two unoriented surfaces F_1 and F_2 bounding K_1 and K_2 respectively with $\beta_1(F_1)+\beta_1(F_2) = \beta_1(F)$. Since F is a boundary-connected-sum of F_1 and F_2, either F_1 or F_2 is non-orientable.

If F_1 is non-orientable, then we have $C(K_1) \leq \beta_1(F_1)$ and $\Gamma(K_2) \leq \beta_1(F_2)$. Thus $C(K_1 \sharp K_2) = \beta_1(F) = \beta_1(F_1) + \beta_2(F_2) \geq C(K_1) + \Gamma(K_2)$. If F_2 is non-orientable, we have $C(K_1 \sharp K_2) \geq \Gamma(K_1) + C(K_2)$. So we have the required inequality and the proof is complete. □

Proof of Proposition 4.1. Since $\Gamma(K) = \min(2g(K), C(K))$ and $C(K) \leq 2g(K) + 1$ from Proposition 1.1, we have $C(K) - \Gamma(K) \leq 1$. So $C(K_1) + \Gamma(K_2) \geq C(K_1) + C(K_2) - 1$ and $\Gamma(K_1) + C(K_2) \geq C(K_1) + C(K_2) - 1$. Therefore we have the required formula from the previous lemma. □

Now we see that $C(K_1 \sharp K_2)$ is equal to $C(K_1) + C(K_2) - 1$ or $C(K_1) + C(K_2)$. Since $\Gamma(K) \leq C(K)$, we have from Lemma 4.2 a necessary and sufficient condition to decide which value it takes.

Proposition 4.3. *$C(K_1 \sharp K_2) = C(K_1) + C(K_2)$ if and only if $C(K_1) = \Gamma(K_1)$ and $C(K_2) = \Gamma(K_2)$.*

Since $C(7_4) = 3 > \Gamma(7_4) = 2$, we have

Corollary 4.4.

$$C(K \sharp 7_4) = C(K) + C(7_4) - 1.$$

In particular, we see $C(7_4 \sharp 7_4) = 5$.

Since $g(K) \geq 1$ for a non-trivial knot K, we have

Corollary 4.5. *For any non-trivial knots $K_1, K_2, \ldots, K_n$ with $C(K_i) \leq 2 (i = 1, 2, \ldots, n)$, we have*

$$C(K_1 \sharp K_2 \sharp \ldots \sharp K_n) = C(K_1) + C(K_2) + \ldots + C(K_n).$$

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