PACIFIC JOURNAL OF MATHEMATICS
Vol. 171 No. 1 1995

ELLIPTIC FIBRATIONS ON QUARTIC K3 SURFACES WITH LARGE PICARD NUMBERS

MASATO KUWATA

Let q_1 and q_2 be two binary quartic forms. We consider the diophantine equation $q_1(x,y) = q_2(z,w)$ from the geometric view point. Under a mild condition we prove that the $K3$ surface defined by the above equation admits an elliptic fibration whose Mordell-Weil group over $\mathbf{C}(t)$ has rank at least 12. Next, we choose suitable q_1 and q_2 such that the Mordell-Weil group contains a subgroup of rank 12 defined over $\mathbb{Q}(i)$ and a subgroup of rank 8 defined over $\mathbb{Q}$.

1. Introduction.

In contrast to the arithmetic of algebraic curves, especially elliptic curves, the arithmetic theory of algebraic surfaces has not yet been well understood. In the classification theory, $K3$ surfaces occupy a position similar to that of elliptic curves in algebraic curves. Thus it is natural to expect that $K3$ surfaces will prove us a very interesting arithmetic object to study. As evidence, $K3$ surfaces arise naturally in classical diophantine problems. Since a non-singular quartic surface is a $K3$ surface, Euler's equations

$$x^4 + y^4 = z^4 + w^4$$

and

$$x^4 + y^4 + z^4 = w^4$$

define $K3$ surfaces. Also, finding two different Pythagorean triangles of the same area is in this category of diophantine problems, as it is equivalent to find the integral solutions to the equation

$$xy(x^2 - y^2) = zw(z^2 - w^2).$$

(cf. [**Br1**].) As Euler's second equation suggests, these diophantine problems are very difficult in general (cf. [**E**]). A more detailed study using L-functions has just begun for a very special type of equations, including Euler's equations (cf. [**P-Sw**]).

In this paper we study a generalization of the above equations from a geometric point of view. Let $q_1(x,y)$ and $q_2(x,y)$ be two binary quartic forms and consider the equation

$$q_1(x,y) = q_2(z,w).$$

It is easy to construct a non-trivial example of this type that has no rational solutions. For example, the equation

$$x^4 + 5y^4 = 2z^4 + 2z^2w^2 + 3w^4$$

does not have a rational solution since the equation does not have a solution modulo 5. On the other hand, once we have a rational solution, we can construct a new solution using a method classically known as the 'chord and tangent method'. This can be interpreted in terms of elliptic surfaces. In order to demonstrate that a quartic surface of the above type can have a lot of rational solutions, we will construct an elliptic fibration from the surface to the projective line that has a lot of rational sections. Our main result is:

Main Theorem. *Let λ and μ be any rational numbers that do not satisfy the equation* (2.1) (*see* §2). *Then the surface defined by*

$$(1-\mu^4)^2(x^2-y^2)(x^2-\lambda^4y^2) = (1-\lambda^4)^2(z^2-w^2)(z^2-\mu^4w^2)$$

admits an elliptic fibration whose Mordell-Weil group has rank at least 12. *It contains a subgroup of rank* 12 *that is defined over the Gaussian field* $\mathbb{Q}(i)$ *and a subgroup of rank* 8 *that is defined over the field of rational numbers* $\mathbb{Q}$.

This gives us yet another example of a family of elliptic curves of rank 8 defined over $\mathbb{Q}$ (cf. [**Sh2**]). For small values of λ and μ, the specialization gives us an elliptic curve of rank 8 with a relatively small conductor (cf. §3).

The geometry of the surfaces of the above type were first studied in detail by B. Segre [**Se**] in the 1940's, and later by Inose [**I**] in the 1970's from a modern point of view. In §1 we review their results and construct an elliptic fibration of rank 12. In §2 we take arithmetic considerations into account and search for a surface whose Mordell-Weil group has a small field of definition. Thanks to a theorem of Inose (cf. Theorem 1.2), it is not hard to show that a certain elliptic fibration has large rank. It is, however, very difficult to find many independent sections, let alone a base of the Mordell-Weil group. In order to find 12 independent sections, we use another elliptic fibration on the same surface. This is one of the special features of $K3$ surfaces, as an elliptic surface of higher geometric genus does not admit more than one ellipitc fibration.

2. Geometry of certain quartic surfaces.

Let $q_1(x,y)$ and $q_2(x,y)$ be binary quartic forms. We denote by $X(q_1,q_2)$ the surface in $\mathbb{P}^3$ defined by

$$q_1(x,y) = q_2(z,w).$$

It is easy to show that this is non-singular if and only if neither q_1 nor q_2 has a multiple factor.

In this section we study the geometry of this surface over $\mathbf{C}$. After a suitable linear change of coordinates, we may assume that the two quartic forms are of the form

$$q_1(x,y) = xy(x-y)(x-\lambda y)$$

and

$$q_2(z,w) = zw(z-w)(z-\mu w)$$

for some numbers λ and μ.

Let E_i be the elliptic curve defined by the equation

$$y^2 = q_i(x,1) \quad i = 1,2.$$

We denote by $Y(q_1,q_2)$ the Kummer surface associated to the product abelian surface $E_1 \times E_2$. Note that the surface $X(q_1,q_2)$ has the involution defined by

$$\sigma : (x:y:z:w) \mapsto (x:y:-z:-w).$$

The surface $X(q_1,q_2)$ and $Y(q_1,q_2)$ have the following relation.

Theorem 1.1 (Inose). *Let $X = X(q_1,q_2)$ and $Y = Y(q_1,q_2)$ as above. Then Y is biholomorphic to the minimal resolution of the quotient surface $X/\langle\sigma\rangle$.*

Proof. See Inose [**I**]. □

Using this relation between X and Y, we can compute the Picard number; i.e., the rank of the Néron-Severi group of X. The key ingredient is the following theorem by Inose.

Theorem 1.2 (Inose). *Let X and Y be $K3$ surfaces and $\pi : X \to Y$ be a rational map of finite degree. Then X and Y have the same Picard number.*

Proof. See Inose [**I**]. □

This theorem relates the Picard number of two $K3$ surfaces. Unfortunately, however, we cannot say much about the correspondence between the generators of the Néron-Severi groups of the two surfaces since the method of the proof of this theorem is based on a transcendental argument. Currently, we do not know a reliable method for finding a set of generators of the Néron-Severi group, or even a set of independent divisors.

Combining Theorem 1.1 and 1.2, we have

Theorem 1.3. *The Picard number of the surface $X(q_1, q_2)$ is*

(1) 18 *if E_1 and E_2 are not isogenous,*

(2) 19 *if E_1 and E_2 are isogenous but do not have complex multiplication, or*

(3) 20 *if E_1 and E_2 are isogenous and have complex multiplication.*

Proof. The Picard number of $Y(q_1, q_2)$ is computed by Shioda-Inose [**Sh-I**]. Thus Theorem 1.2 tells us the corresponding result to $X(q_1, q_2)$. □

In general, if an elliptic surface is given in the form of a Weierstrass model, it is very difficult to determine the rank of the Mordell-Weil group at its generic fiber. On the other hand, it is routine to determine the types of the bad fibers. Let $X \to C$ be an elliptic surface and E be the generic fiber. The Shioda-Tate formula (cf. [**Sh1**]) relates the rank of Mordell-Weil group and the Picard number; i.e.,

$$\rho = 2 + \operatorname{rank} E + \sum_{x \in X} (m_x - 1),$$

where ρ is the Picard number and m_x is the number of irreducible components of the fiber at x. Thus, if we find an elliptic fibration on a surface S whose Picard number is already known, determining the rank of the Mordell-Weil group is reduced to finding the types of bad fibers. Since we know the Picard number of $X(q_1, q_2)$, our task is now reduced to determining the types of the bad fibers.

The next fact will become another key ingredient to our recipe for determining the Mordell-Weil group.

Proposition 1.4. *The surface $X(q_1, q_2)$ contains exactly the following number of lines:*

(1) 16 *if E_1 and E_2 have different j-invariants,*

(2) 32 *if E_1 and E_2 have the same j-invariant different from* 0 *or* 1728*,*

(3) 48 *if both E_1 and E_2 have $j = 1728$,*

(4) 64 *if both E_1 and E_2 have $j = 0$.*

Proof. See Segre [**Se**]. Note that in his notation, the j-invariant is given by $j = \dfrac{I^3}{I^3 - J^2}$ up to a constant factor. Thus, his invariant I^3/J^2 is essentially the j-invariant. □

Now we construct an elliptic fibration on the surface $X(q_1, q_2)$. The surface $X = X(q_1, q_2)$ contains sixteen lines of the form (factor of q_1) = (factor of q_2) = 0. We denote them according to the following list.

$$\ell_1 : \begin{cases} x = 0 \\ z = 0 \end{cases} \quad \ell_2 : \begin{cases} x = 0 \\ w = 0 \end{cases} \quad \ell_3 : \begin{cases} x = 0 \\ z = w \end{cases} \quad \ell_4 : \begin{cases} x = 0 \\ z = \mu w \end{cases}$$

$$\ell_5 : \begin{cases} y = 0 \\ z = 0 \end{cases} \quad \ell_6 : \begin{cases} y = 0 \\ w = 0 \end{cases} \quad \ell_7 : \begin{cases} y = 0 \\ z = w \end{cases} \quad \ell_8 : \begin{cases} y = 0 \\ z = \mu w \end{cases}$$

$$\ell_9 : \begin{cases} x = y \\ z = 0 \end{cases} \quad \ell_{10} : \begin{cases} x = y \\ w = 0 \end{cases} \quad \ell_{11} : \begin{cases} x = y \\ z = w \end{cases} \quad \ell_{12} : \begin{cases} x = y \\ z = \mu w \end{cases}$$

$$\ell_{13} : \begin{cases} x = \lambda y \\ z = 0 \end{cases} \quad \ell_{14} : \begin{cases} x = \lambda y \\ w = 0 \end{cases} \quad \ell_{15} : \begin{cases} x = \lambda y \\ z = w \end{cases} \quad \ell_{16} : \begin{cases} x = \lambda y \\ z = \mu w \end{cases}$$

Take one line, say ℓ_1, and cut X by the planes that contain this line. The intersection of X and one of the planes is the union of a plane cubic curve and the line $x = z = 0$. If the plane is given by $z = tx$, this residual cubic curve is given by the equation

$$y(x - y)(x - \lambda y) = tw(tx - w)(tx - \mu w).$$

If we define a map $\pi : X \to \mathbb{P}^1$ by $[x, y, z, w] \mapsto [x, z]$, each fiber of this map is a cubic curve given by the above equation for $t = z/x$. Thus we have an elliptic fibration. It is easy to see that this fibration has a bad fiber of type IV at $t = 0$ and ∞. More specifically, $\pi^{-1}(0)$ consists of the lines ℓ_2, ℓ_3, and ℓ_4, whereas $\pi^{-1}(\infty)$ consists of the lines ℓ_5, ℓ_9, and ℓ_{13}.

Theorem 1.5. *Suppose E_1 and E_2 do not have the same j-invariant; i.e., E_1 and E_2 are not isomorphic over* **C**. *Then the bad fibers of the elliptic fibration described above consist of two fibers of type* IV, *and the fibers of type* I_1 *and possibly type* II. *The rank of the Mordell-Weil group of this elliptic fibration is*

(1) 12 *if E_1 and E_2 are not isogenous,*

(1) 13 *if E_1 and E_2 are isogenous but do not have complex multiplication, or*

(1) 14 *if E_1 and E_2 are isogenous and have complex multiplication.*

Proof. Note once again that each fiber of the fibration is a plane cubic curve. Thus the Kodaira type of each fiber is either I_N, $(N \leq 3)$, II, III, or IV. If a bad fiber is not of type I_1, or II, it must contain a line. Now let us examine all the sixteen lines. As we have seen, six of them, ℓ_2, ℓ_3, ℓ_4, ℓ_5, ℓ_9, and ℓ_{13}, are the irreducible components of the type IV fiber at $t = 0$ and ∞. It is easy to check that each of the remaining lines, except for ℓ_1, intersects with one and only one of the lines ℓ_2, ℓ_3, and ℓ_4. This implies that all these lines are sections of the fibration. As for ℓ_1, it intersects with all of ℓ_2, ℓ_3, ℓ_4. Thus it is not a component of a bad fiber. Therefore, we conclude that the rest of the bad fibers can not contain a line and thus are of either type I_1 or II. Now the second assertion is an easy consequence of the Shioda-Tate formula (cf. [**Sh1**]):

$$\operatorname{rank} E = \rho - 2 - (m_0 - 1) - (m_\infty - 1) = \rho - 6,$$

where ρ is the Picard number already computed in Theorem 1.3. □

Remark. If E_1 and E_2 are isomorphic, ρ increases by 1, but the rank of the Mordell-Weil group is reduced by 3 because four of the 32 lines become the components of the bad fibers of type I_2. If E_1 and E_2 have $j = 1728$, the rank is once again reduced by 3 (cf. [**K**]). If E_1 and E_2 have $j = 0$ the rank is expected to be 3.

The proof of Theorem 1.5 shows that there are 9 sections coming from the lines. We designate one of them, say ℓ_6, as the zero-section. It is not difficult to show that these sections form a group of rank 4, generated by, for example, ℓ_7, ℓ_8, ℓ_{10} and ℓ_{14}. Our next task is to find the remaining sections.

It is possible to find the Weierstrass equation of the generic fiber. However, the equation is very complicated and it is useless to write it down here. Our key idea once again comes from the fact that it is easy to find the bad fibers of a fibration. If we find another elliptic fibration on $X(q_1, q_2)$, some of the bad fibers may be sections of the first one.

As a first step, let us consider the surface defined by

$$(x^2 - y^2)(ax^2 + by^2 + cz^2 + dw^2) = (z^2 - w^2)(ex^2 + fy^2 + gz^2 + hw^2),$$

where a through h are constants. On this surface we have a fibration defined by

$$\begin{cases} (x^2 - y^2) = t(z^2 - w^2) \\ t(ax^2 + by^2 + cz^2 + dw^2) = (ex^2 + fy^2 + gz^2 + hw^2). \end{cases}$$

The generic member of this fibration is a complete intersection of two quadrics in $\mathbb{P}^3$ and thus an elliptic curve. Hence, we have an elliptic fibration. For this fibration, finding the bad fibers is particularly easy. In fact we have

Proposition 1.6. *The places of the bad fibers of this elliptic fibratin contain all the roots of the determinant of the* 2×2 *minors of the matrix*

$$\begin{pmatrix} 1 & -1 & -t & t \\ at-e & bt-f & ct-g & dt-h \end{pmatrix}.$$

Proof. This can be proved by calculating the values of t at which the fiber becomes singular. $\square$

If the values of a through h are 'generic' then the bad fibers are of type I_2, or a union of two conics. We will use these conics to obtain sections of the first elliptic fibration.

Remark. Bremner [**Br2**] studied a surface that belongs to this family. He considered the surface defined by

$$(p^2 - t^2)(p^2 + t^2 + 8r^2) = (s^2 - r^2)(s^2 + 3r^2 + 6t^2),$$

which is contained in the four-fold $x^5 + y^5 + z^5 = u^5 + v^5 + w^5$.

3. Arithmetic consideration.

In the previous section we considered the two families of $K3$ surfaces that have interesting elliptic fibrations on them. In this section we consider the surfaces that belong to both of the families. Among those surfaces we will find one which has a small field of definition for the Néron-Severi group. Specifically, we consider the surface defined by

$$S(\lambda, \mu; a) : (x^2 - y^2)(x^2 - \lambda^4 y^2) = a^2(z^2 - w^2)(z^2 - \mu^4 w^2).$$

This surface contains 16 lines. Since we are using a coordinate system different from §1, we name and list the 16 lines once again.

$$\ell_1 : \begin{cases} x = y \\ z = w \end{cases} \quad \ell_2 : \begin{cases} x = y \\ z = -w \end{cases} \quad \ell_3 : \begin{cases} x = y \\ z = \mu^2 \end{cases} \quad \ell_4 : \begin{cases} x = y \\ z = -\mu^2 w \end{cases}$$

$$\ell_5 : \begin{cases} x = -y \\ z = w \end{cases} \quad \ell_6 : \begin{cases} x = -y \\ z = -w \end{cases} \quad \ell_7 : \begin{cases} x = -y \\ z = \mu^2 \end{cases} \quad \ell_7 : \begin{cases} x = -y \\ z = -\mu^2 w \end{cases}$$

$$\ell_9 : \begin{cases} x = \lambda^2 y \\ z = w \end{cases} \quad \ell_{10} : \begin{cases} x = \lambda^2 y \\ z = -w \end{cases} \quad \ell_{11} : \begin{cases} x = \lambda^2 y \\ z = \mu^2 w \end{cases} \quad \ell_{12} : \begin{cases} x = \lambda^2 y \\ z = -\mu^2 w \end{cases}$$

$$\ell_{13} : \begin{cases} x = -\lambda^2 y \\ z = w \end{cases} \quad \ell_{14} : \begin{cases} x = -\lambda^2 y \\ z = -w \end{cases} \quad \ell_{15} : \begin{cases} x = -\lambda^2 y \\ z = \mu^2 w \end{cases} \quad \ell_{16} : \begin{cases} x = -\lambda^2 y \\ z = -\mu^2 w \end{cases}$$

We assume that the two elliptic curves defined by

$$E_1 : y^2 = (x^2 - 1)(x^2 - \lambda^4)$$

and

$$E_2 : y^2 = (x^2 - 1)(x^2 - \mu^4)$$

have different j-invariants. Since we only consider the case where λ and μ are rational numbers, the condition for this is that λ and μ do not satisfy the equation

$$\begin{aligned}(2.1)\quad & (\lambda - \mu)(\lambda + \mu)(\lambda\mu - 1)(\lambda\mu + 1) \\ & \times (\lambda\mu - \lambda - \mu - 1)(\lambda\mu - \lambda + \mu + 1)(\lambda\mu + \lambda - \mu + 1)(\lambda\mu + \lambda + \mu - 1) = 0.\end{aligned}$$

First we consider the elliptic fibration defined by

$$\begin{cases} (x^2 - y^2) = a^2 t(z^2 - w^2) \\ t(x^2 - \lambda^4 y^2) = (z^2 - \mu^4 w^2). \end{cases}$$

Looking at Proposition 1.6, we can easily find the locations of the bad fibers.

Lemma 2.1. *The above elliptic fibration has bad fibers of type* I_4 *at* $t = 0$ *and* ∞, *and those of type* I_2 *at* $t = \pm\frac{1}{a}$, $\pm\frac{\mu^2}{a}$, $\pm\frac{1}{a\lambda^2}$, *and* $\pm\frac{\mu^2}{a\lambda^2}$.

Proof. The places of the bad fibers is easily obtained by Proposition 1.6. At $t = 0$, the fiber consists of the four lines

$$\begin{cases} x &= y \\ z = \mu^2 w \end{cases}, \quad \begin{cases} x &= y \\ z = -\mu^2 w \end{cases}, \quad \begin{cases} x = -y \\ z = \mu^2 w \end{cases}, \quad \begin{cases} x &= -y \\ z = -\mu^2 w \end{cases},$$

and it is of type I_4. Similarly, the fiber at $t = \infty$ is of type I_4. Under the condition that λ and μ do not satisfy (2.1), there are 8 different values of t at which the fiber becomes singular. At each of these 8 points, the fiber consists of two plane conics:

$$\begin{aligned}
t &= \frac{1}{a} : & Q_1 + Q_2, \qquad t &= -\frac{1}{a} : & Q_2 + Q_3, \\
t &= \frac{\mu^2}{a} : & Q_4 + Q_5, \qquad t &= -\frac{\mu^2}{a} : & Q_7 + Q_8, \\
t &= \frac{1}{a\lambda^2} : & Q_9 + Q_{10}, \qquad t &= -\frac{1}{a\lambda^2} : & Q_{11} + Q_{12}, \\
t &= \frac{\mu^2}{a\lambda^2} : & Q_{13} + Q_{14}, \qquad t &= -\frac{\mu^2}{a\lambda^2} : & Q_{15} + Q_{16},
\end{aligned}$$

where Q_i's are given by

$$
Q_1 : \begin{cases} y = \nu w \\ x^2 - y^2 = a(z^2 - w^2), \end{cases} \qquad Q_2 : \begin{cases} y = -\nu w \\ x^2 - y^2 = a(z^2 - w^2), \end{cases}
$$

$$
Q_3 : \begin{cases} y = i\nu w \\ x^2 - y^2 = -a(z^2 - w^2), \end{cases} \qquad Q_4 : \begin{cases} y = -i\nu w \\ x^2 - y^2 = -a(z^2 - w^2), \end{cases}
$$

$$
Q_5 : \begin{cases} y = \frac{\nu}{\mu} z \\ x^2 - y^2 = a\mu^2(z^2 - w^2), \end{cases} \qquad Q_6 : \begin{cases} y = -\frac{\nu}{\mu} z \\ x^2 - y^2 = a\mu^2(z^2 - w^2), \end{cases}
$$

$$
Q_7 : \begin{cases} y = i\frac{\nu}{\mu} z \\ x^2 - y^2 = -a\mu^2(z^2 - w^2), \end{cases} \qquad Q_8 : \begin{cases} y = -i\frac{\nu}{\mu} z \\ x^2 - y^2 = -a\mu^2(z^2 - w^2), \end{cases}
$$

$$
Q_9 : \begin{cases} x = \nu\lambda w \\ \lambda^2(x^2 - y^2) = a(z^2 - w^2), \end{cases} \qquad Q_{10} : \begin{cases} x = -\nu\lambda w \\ \lambda^2(x^2 - y^2) = a(z^2 - w^2), \end{cases}
$$

$$
Q_{11} : \begin{cases} x = i\nu\lambda w \\ \lambda^2(x^2 - y^2) = -a(z^2 - w^2), \end{cases} \qquad Q_{12} : \begin{cases} x = -i\nu\lambda w \\ \lambda^2(x^2 - y^2) = -a(z^2 - w^2), \end{cases}
$$

$$
Q_{13} : \begin{cases} x = \frac{\nu\lambda}{\mu} z \\ \lambda^2(x^2 - y^2) = a\mu^2(z^2 - w^2), \end{cases} \qquad Q_{14} : \begin{cases} x = -\frac{\nu\lambda}{\mu} z \\ \lambda^2(x^2 - y^2) = a\mu^2(z^2 - w^2), \end{cases}
$$

$$
Q_{15} : \begin{cases} x = i\frac{\nu\lambda}{\mu} z \\ \lambda^2(x^2 - y^2) = -a\mu^2(z^2 - w^2), \end{cases} \qquad Q_{16} : \begin{cases} x = -i\frac{\nu\lambda}{\mu} z \\ \lambda^2(x^2 - y^2) = -a\mu^2(z^2 - w^2), \end{cases}.
$$

Here ν is a number such that $\nu^2 = a(1-\mu^4)/(1-\lambda^4)$. The assumption that E_1 and E_2 have different j-invariants assures that these fibers are of type I_2. For, the lines ℓ_1 through ℓ_{16} are not contained in any of these conics and the surface does not contain any other lines due to Proposition 1.4. Calculating the sum of the Euler characteristics of the bad fibers, we conclude that there are no other bad fibers. □

We now prove that we have enough divisors on the surface so that we can find 12 independent divisors among them.

Lemma 2.2. *The lines ℓ_2, ℓ_3, ℓ_4, ℓ_5, ℓ_6, ℓ_9, ℓ_{11}, ℓ_{12}, ℓ_{15} and ℓ_{16}, and the conics Q_1, Q_3, Q_5, Q_7, Q_9, Q_{11}, Q_{13} and Q_{15} are linearly independent in the Néron-Severi group.*

Proof. In order to prove that the above divisors are independent, it is enough to show that the determinant of the intersection matrix is non-zero. The calculation of the intersection number of any two divisors in the lists are straightfoward and we have the following intersection matrix with respect to the above divisors in that order:

$$\begin{pmatrix}
-2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & -2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2
\end{pmatrix}$$

The determinant of this matrix is -2^{16}. This proves that the above 18 divisors are linearly independent. $\square$

Corollary 2.3. *Suppose that $a(1-\lambda^4)(1-\mu^4)$ is a perfect square. Then the Néron-Severi group of the surface $S(\lambda, \mu; a)$ has a subgroup of rank 16 that is defined over $\mathbb{Q}(i)$, and a subgroup of rank 14 that is defined over $\mathbb{Q}$.*

Proof. If $\nu = \sqrt{a(1-\mu^4)/(1-\lambda^4)}$ is a rational number, all the conics are defined over $\mathbb{Q}(i)$, and the conics Q_i, $i \equiv 1, 2 \bmod 4$ are defined over $\mathbb{Q}$. Since all the lines are defined over $\mathbb{Q}$, all of the above 18 divisors are defined over $\mathbb{Q}(i)$ and 14 of them are defined over $\mathbb{Q}$. This completes the proof. $\square$

Now we consider the elliptic fibration obtained by cutting the surface $S(\lambda, \mu; a)$ by the plane $z - w = t(x - y)$. It follows from Proposition 1.5 that this elliptic fibration has rank 12. We now state our main result.

Theorem 2.4. *Suppose that $a(1-\lambda^4)(1-\mu^4)$ is a perfect square. Then the Mordell-Weil group of this elliptic fibration contains a subgroup of rank 12 defined over $\mathbb{Q}(i)$, and the subgroup of rank 8 defined over $\mathbb{Q}$.*

Proof. Among the divisors in Lemma 2.2 the lines ℓ_2, ℓ_3 and ℓ_4 form the type IV fiber at $t = 0$ and ℓ_5 and ℓ_9 are a part of the type IV fiber at $t = \infty$. The intersection matrix shows that each of the remaining divisors intersects once and only once with the bad fiber $\ell_2 + \ell_3 + \ell_4$. Thus they are sections of the fibration. We choose the line ℓ_6 as the zero section. Since the 18 divisors in Lemma 2.2 are linearly independent, the rest of 12 sections form a linearly independent subgroup of the Mordell-Weil group. If $a(1-\lambda^4)(1-\mu^4)$ is a perfect sqare, all of these 12 sections are defined over $\mathbb{Q}(i)$ and 8 of them are defined over $\mathbb{Q}$. This completes the proof. $\square$

4. Numerical examples.

In this section we show some numerical examples. We would like to specialize the values of λ, μ and a in $S(\lambda, \mu; a)$. Before doing so, we make a change of coordinates

$$X = x - y, \quad Y = x + y, \quad Z = z - w, \quad W = z + w.$$

The elliptic fibration in §2 is obtained by setting $Z = tX$. Dehomogenizing by setting $W = 1$, the equation becomes

$$\begin{aligned}(1-\lambda^4)\,X^2Y + 2(1+\lambda^4)\,XY^2 + (1-\lambda^4)\,Y^3 \\ - a^2(1-\mu^4)t^3\,X^2 - 2a^2(1+\mu^4)t^2\,X - a^2(1-\mu^4)t = 0.\end{aligned}$$

The line ℓ_6 intersects with this cubic curve at one of the points at infinity, $(X : Y : W) = (1 : 0 : 0)$. Other lines and conics intersects with the curve at the following points:

$$\begin{aligned}
\ell_{11} &: \left(\frac{\mu^2+1}{t(\mu^2-1)}, \frac{(\mu^2+1)(\lambda^2-1)}{t(\mu^2-1)(\lambda^2+1)}\right), \\
\ell_{12} &: \left(\frac{\mu^2-1}{t(\mu^2-1)}, \frac{(\mu^2-1)(\lambda^2-1)}{t(\mu^2+1)(\lambda^2+1)}\right), \\
\ell_{15} &: \left(\frac{\mu^2+1}{t(\mu^2-1)}, \frac{(\mu^2+1)(\lambda^2+1)}{t(\mu^2-1)(\lambda^2-1)}\right), \\
\ell_{16} &: \left(\frac{\mu^2-1}{t(\mu^2+1)}, \frac{(\mu^2-1)(\lambda^2+1)}{t(\mu^2+1)(\lambda^2-1)}\right), \\
Q_1 &: \left(\frac{at-\nu}{1-\nu t}, at\right), \quad Q_5 : \left(\frac{a\mu^3 t-\nu}{\mu+\nu t}, a\mu^2 t\right), \\
Q_9 &: \left(\frac{\lambda^3\nu - at}{\lambda^2(1+\lambda\nu t)}, \frac{at}{\lambda^2}\right), \quad Q_{13} : \left(\frac{\lambda^3\nu - a\mu^3 t}{\lambda^2(\mu-\lambda\nu t)}, \frac{a\mu^2 t}{\lambda^2}\right).
\end{aligned}$$

By a theorem of J. Silverman [**Sil2**], the rank of the Mordell-Weil group of the elliptic curve over $\mathbb{Q}$ obtained by specializing the value of t is no less than 8 except for a finite number of exceptions.

As an example, we set $\lambda = \frac{2}{3}$ and $\mu = \frac{1}{3}$. We can set $a = 13$ to make ν a rational number, and we have $\nu = 4$. The equation becomes

$$65\,X^2Y + 194\,XY^2 + 65\,Y^3 - 13520\,t^3\,X^2 - 27716\,t^2\,X - 13520\,t = 0.$$

Specializing t to 2, and converting this equation to the Weierstrass form, we have

$$y^2 = x^3 + x^2 - 4002080x + 368844285828.$$

The conductor and the discriminant of this curve are

$$\begin{aligned} \text{Conductor} &= 120922306669662024 = 2^4 \cdot 3 \cdot 5 \cdot 661 \cdot 427681 \cdot 17822711, \\ \text{Discriminant} &= -587682410414557436640000000 \\ &= -2^{10} \cdot 3^6 \cdot 5^6 \cdot 661 \cdot 427681 \cdot 17822711. \end{aligned}$$

The following eight points are obtained from the above list by substituting $t = 2$ in that order:

$$\begin{gathered} (x, y) = (-1574, 609300), \quad (-53, -607500), \quad (2026, 607500), \\ (2251, -609300), \quad \left(\tfrac{2281099}{49}, \tfrac{3448366200}{343}\right), \\ \left(\tfrac{183559}{25}, -\tfrac{107190102}{125}\right), \quad \left(\tfrac{32274886}{361}, \tfrac{183359265000}{6859}\right), \\ (18838, -2641752). \end{gathered}$$

By calculating the height matrix, we can show that these eight points are independent. Apecs, a package for the arithmetic of elliptic curves on Maple, indicates that the upperbound for the rank is 8 assuming the Taniyama-Weil conjecture, the Birch-Swinnerton-Dyer conjecture and the generalized Riemann hypothesis. Further search shows that the following integral points generate the same group generated by the above 8 rational points:

$$\begin{gathered} (-1574, 609300), \ (-53, 607500), \ (2026, 607500), \ (2251, 609300), \\ (18838, 2641752), \ (-197, 607968), \ (-1862, 608148), \ (-3794, 573960). \end{gathered}$$

Here, we can see a usual phenomenon that an elliptic curve of large rank with a relatively small conductor tends to have many integral points.

Acknowledgement. The author would like to thank Professor David Cox for useful conversations. He also would like to acknowledge the Waterloo Maple Software for its powerful Maple, and Professor Ian Connell for his excellent package for the arithmetic of elliptic curves, Apecs.

References

[Br1] Bremner, A., *Pythagorean triangles and a quartic surface,* J. Reine Angew. Math., **318** (1980), 120–125.

[Br2] ——— *A Geometric Approach to Equal Sums of Fifth Powers,* J. Number Theory, **13** (1981), 337-354.

[E] Elkies, N., *On* $A^4 + B^4 + C^4 = D^4$, Math. Comp., **51** (1988), 825–835.

[I] Inose, H., *On certain Kummer surfaces which can be realized as non-singular quartic surfaces in* $\mathbb{P}^3$, J. Fac. Sci. Univ. Tokyo, **23** (1976), 545–560.

[K] Kuwata, M., *Canonical height and elliptic K3 surfaces,* J. Number Theory, **36** (1990), 399-406.

[P-Sw] Pinch, P. G. E. and Swinnerton-Dyer, H. P. F., *Arithmetic of Diagonal Quartic Surfaces I,* to appear

[Se] Segre, B., *On Arithmetical Properties of Quartic Surfaces,* Proc. London Math. Soc., **14** (1944), 353–395.

[Sh1] Shioda, T., *On elliptic modular surfaces,* J. Math. Soc. Japan, **24** (1972), 20–59.

[Sh2] ———, *Mordell-Weil lattices and Galois representation. I,* Proc. Japan Acad., **65**, Ser. A (1989), 268–271, II, ibid, 296–299, III, ibid, 300–303.

[Sh-I] Shioda, T. and Inose, H., *On singular K3 surfaces,* Complex Analysis and Algebraic Geometry, (1977), 119–136.

[Sil1] Silverman, J. H., *The Arithmetic of Elliptic Curves*, Springer-Verlag, (1986).

[Sil2] ———, *Heights and the specialization map for families of abelian varieties,* J. Reine Angew. Math., **342** (1983), 197–251.

Received January 6, 1993 and revised October 22, 1993.

UNIVERSITÉ DE CAEN,
ESPLANADE DE LA PAIX,
F-14032 CAEN CEDEX,
FRANCE
KUWATA@MATH.UNICAEN.FR