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THE INVERSE RIEMANN MAPPING THEOREM FOR RELATIVE CIRCLE DOMAINS

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A subdomain Ω contained in a domain A in the Riemann sphere is called a *relative circle domain* in A if each component of $A - \Omega$ is either a closed disk or a point. Let Ω be a relative circle domain in the unit disk U in the complex plane $\mathbb{C}$; and let A be a simply connected proper subdomain of $\mathbb{C}$. Then Ω is conformally homeomorphic to a relative circle domain in A.

1. Introduction.

Let A be any domain in the Riemann sphere $\hat{\mathbb{C}}$. A domain Ω contained in A is called a *relative circle domain* in A if each (connected) component of $A - \Omega$ is either a closed disk or a point. When $A = \hat{\mathbb{C}}$, a relative circle domain in $\hat{\mathbb{C}}$ is called a circle domain. In [**HS2**], we have proved that if A has at most countably many boundary components, then any relative circle domain in A is conformally homeomorphic to a circle domain. In the case that A is a simply connected proper subdomain of the plane, we have the following corollary.

Riemann Mapping Theorem for Relative Circle Domains. *Let A be a simply connected domain in the complex plane $\mathbb{C}$ with $A \neq \mathbb{C}$. Let Ω be a relative circle domain in A. Then there exist a relative circle domain Ω^* in the unit disk $U = \{z : |z| < 1\}$ and a conformal homeomorphism $f : \Omega \to \Omega^*$ which extends to a locally quasiconformal homeomorphism $\widetilde{f} : A \to U$ whose complex dilatation vanishes a.e. in $\partial\Omega \cap A$.*

It also follows by [**HS2**, Lemma 5.3] that the domain Ω^* and the map f in the above theorem are unique up to Möbius transformations. In this paper, we will prove the following inverse theorem.

Theorem 1.1 (Inverse Riemann Mapping Theorem for Relative Circle Domains). *Let Ω be a relative circle domain in the unit disk U; and let $A \subseteq \mathbb{C}$ be a simply connected domain with $A \neq \mathbb{C}$. Then for any $z_0 \in \Omega$ and $z_0^* \in A$, there exist a relative circle domain Ω^* in A and a conformal homeomorphism $f : \Omega \to \Omega^*$ which extends to a locally quasiconformal homeomorphism $\widetilde{f} : U \to A$ whose complex dilatation vanishes a.e. in $\partial\Omega \cap U$, and such that*

$$f(z_0) = z_0^*, \quad f'(z_0) \in (0, +\infty). \tag{1.1}$$

We note that f is uniquely determined if Ω^* is. It follows by the fixed point index lemma in [**HS2**, §4] that Ω^* is unique if A is bounded and convex (or more generally, bounded and star-shaped around z_0^*). However, in general we do not know if Ω^* is unique. The line (1.1) can be replaced by other conditions as well. For more discussions on this, see [**Sch**].

Similar results also hold for circle packings. For example, we have the following theorem. Its proof is similar to that of Theorem 1.1 (see also [**HS1**, §10]).

Theorem 1.2. *Let P be a circle packing whose carrier is the unit disk U; and let $A \subseteq \mathbb{C}$ be a simply connected domain with $A \neq \mathbb{C}$. Then there is a circle packing P^* whose carrier is A and whose graph is combinatorially equivalent to that of P. Moreover, given a pair of tangent circles c_0 and c_1 in P and a point z_0^* in A, we can choose P^* such that c_0^* is centered at z_0^* and the point of tangency between c_0^* and c_1^* is to the right of z_0^* on the horizontal line through z_0^*, where c_0^* and c_1^* are the circles of P^* corresponding to c_0 and c_1 respectively.*

The interested reader may wish to look up the related paper by Carter and Rodin [**CR**].

The rest of the paper will be devoted to the proof of Theorem 1.1. First, the case when Ω has only a finite number of boundary components follows by the results of Brandt [**Br**] or Harrington [**Har**] (see also [**Sch**]). For completeness' sake, we will give a simple proof modelled on the approach of [**Sch**]. Next we will deal with the case when ∂U is an isolated boundary component of $\partial\Omega$. Then we proceed by taking a sequence of proper subdomains of Ω and use the earlier results to build conformal mappings from those proper subdomains to relative circle domains in A. The key point is to show that (some subsequence of) these conformal mappings converge to what we need. In our argument, we will have to use the convergence results and the Schwarz Pick lemma proved in [**HS2**].

2. The Case Where Ω is Finitely Connected.

For any conformal mapping $f : \Omega \to \Omega^*$, we will denote by f^B the bijection induced by f from the boundary components of Ω to the boundary components of $f(\Omega)$.

The following theorem is fundamental.

Brandt-Harrington Uniformization Theorem. *Let $\Omega \subset \mathbb{C}$ be a domain with finitely many boundary components, $B^0, B^1, \ldots, B^n$, and suppose that none of these components is a single point. Let $A \subsetneqq \mathbb{C}$ be a simply connected* *domain, and let* $P^1, \dots, P^n$ *be compact connected sets in* $\mathbb{C}$, *such that* P^j *contains more than a single point and* $\mathbb{C} - P^j$ *is connected for each* $j = 1, \dots, n$. *Let* $z_0 \in \Omega$ *and* $z_0^* \in A$ *be arbitrary. Then there are disjoint sets* $Q^1, \dots, Q^n \subset A$ *and a conformal homeomorphism*

$$f : \Omega \to A - \cup_{j=1}^n Q^j,$$

such that
(1) Q^j *is homothetic to* P^j *for* $j = 1, \dots, n$;
(2) $f^B(B^j) = \partial Q^j$ *for* $j = 1, \dots, n$, $f^B(B^0) = \partial A$; *and*
(3) $f(z_0) = f(z_0^*)$, $f'(z_0) \in (0, \infty)$.

Recall that a homothety is a transformation of the form $z \to az + b$ with $a > 0$ and $b \in \mathbb{C}$. Two sets are *homothetic* if one is the image of the other under a homothety.

Brandt [**Br**] and Harrington [**Har**] have independently proved this theorem, using different methods. We will now present a simple proof, somewhat in the spirit of the one given in [**Sch**].

Proof. We shall use Koebe's circle uniformization theorem. It says that there is a unique conformal map $g : \Omega \to U$ such that $g(\Omega)$ is a circle domain, $g(z_0) = 0$, $g'(z_0) > 0$, and $g^B(B^0) = \partial U$. (Alternatively, one can replace this with the similar theorem about uniformization with slit domains.)

Assume, without loss of generality, that $z_0^* = 0$. Let $A_t \subset \mathbb{C}$, $t \in [0,1]$ be a continuous one parameter family of simply connected domains, such that $A_0 = U$, $A_1 = A$, and $0 \in A_t$. (The A_t are open sets, so continuity here means that the complements $\hat{\mathbb{C}} - A_t$ vary continuously in the Hausdorff metric on $\hat{\mathbb{C}}$.) Similarly, for $j = 1, \dots, n$, let P_t^j, $t \in [0,1]$, be a continuous one parameter family of compact sets in $\mathbb{C}$, such that each P_0^j is a disk, each P_t^j, $t \in [0,1)$ is homeomorphic to a closed disk, and $P_1^j = P^j$ for $j = 1, \dots, n$. To be explicit, for $t \in (0,1)$ one may take P_t^j to be the disk in $\mathbb{C}$ with boundary $t\phi_j^{-1}(\{|z| = t\})$, where $\phi_j : \hat{\mathbb{C}} - P^j \to U$ is a conformal homeomorphism satisfying $\phi_j(\infty) = 0$; and similarly for A_t.

Set $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$, and let $X = \mathbb{R}_+^n \times \mathbb{C}^n$. For $(r, c) \in X$, $j = 1, \dots, n$ and $t \in [0,1]$, let $P_t^j(r, c)$ denote the image of P_t^j under the homothety $z \to r_j z + c_j$. For every $t \in [0,1]$ let M_t denote the set of all $(r, c) \in X$ such that $P_t^j(r, c) \subset A_t - \{0\}$ for each $j = 1, \dots, n$, and $P_t^j(r, c) \cap P_t^k(r, c) = \emptyset$ when $j, k \in \{1, \dots, n\}$ are distinct. The set $\widetilde{M} = \{(r, c, t) \in X \times [0,1] : (r, c) \in M_t\}$ is a relatively open subset of $\mathbb{R}^n \times \mathbb{C}^n \times [0,1]$.

Define a map $H : \widetilde{M} \to M_0$, as follows. Take some $p = (c, r, t) \in \widetilde{M}$. Then $D_p = A_t - \cup_{j=1}^n P_t^j(r, c)$ is a domain in $\mathbb{C}$ that contains 0 and has boundary components $\partial A_t, \partial P_t^1(r, c), \dots, \partial P_t^n(r, c)$. By Koebe's circle uniformization theorem there is a unique conformal map $h_p : D_p \to U$ such that $h_p(D_p)$ is a circle domain, $h_p(0) = 0$, $h'_p(0) > 0$ and $h_p^B(\partial A_t) = \partial U$. Since each P_0^j is a disk and $A_0 = U$, there is a unique $p' = (r', c', 0) \in M_0 \times \{0\}$ such that $h_p(D_p) = D_{p'}$ and $h_p^B(P_t^j(r, c)) = P_0^j(r', c')$. Now set $H(p) = (r', c')$. The map H is continuous and proper. ('Proper' means that the inverse image of a compact set in M_0 is compact.) It is left to the reader to check that this follows from the multiply connected version of Carathéodory's Kernel Convergence Theorem [**Gol**, V§5] (or the details may be found in [**Sch**]).

We want to prove that $H(M_1 \times \{1\}) = M_0$. Assume for the moment that H is C^∞-smooth. Let $V \subset M_0$ be any open subset of M_0 with compact closure in M_0, and let ω be a $3n$ dimensional differential form on M_0 with support contained in V, whose integral over M_0 is nonzero. Since ω is top dimensional on M_0, $d\omega = 0$. Because H is proper and ω has compact support, the pullback $\delta H(\omega)$ of ω has compact support in $\widetilde{M}$, and therefore we may apply Stokes' theorem, which gives

$$\int_{\widetilde{M}} d\delta H(\omega) = \int_{\partial \widetilde{M}} \delta H(\omega).$$

Here, $\partial \widetilde{M}$ denotes not the boundary of $\widetilde{M}$ as a subset of $X \times [0, 1]$, but the boundary of $\widetilde{M}$ as a manifold with boundary. That is, $\partial \widetilde{M} = (M_0 \times \{0\}) \cup (-M_1 \times \{1\})$, where $-M_1$ means M_1 with the orientation reversed. But $d\delta H(\omega) = \delta H(d\omega) = \delta H(0) = 0$, and therefore

$$\int_{M_0 \times \{0\}} \delta H(\omega) = \int_{M_1 \times \{1\}} \delta H(\omega). \tag{2.1}$$

For $t \in \{0, 1\}$, let $I_t : M_t \to \widetilde{M}$ denotes the map $I_t(c, r) = (c, r, t)$. Note that, by construction, $H \circ I_0 : M_0 \to M_0$ is the identity, and $\int_{M_0} \omega \neq 0$. So $\int_{M_0} \omega$ is equal to the left side of (2.1), and the right side of (2.1) is not zero. In particular, $H(M_1 \times \{1\})$ intersects the support of ω, which is contained in V. Because this is valid for any open $V \subset M_0$ with compact closure, we deduce that $H \circ I_1(M_1)$ is dense in M_0. Since $H \circ I_1$ is proper and continuous, this implies that $H \circ I_1$ is surjective, which is what we wanted.

We do not know if H is really smooth. To prove that $H \circ I_1$ is surjective without this assumption it is enough to show that H may be uniformly approximated by smooth proper maps $H^* : \widetilde{M} \to M_0$ such that $H^* \circ I_0 : M_0 \to M_0$ is the identity. Since this is straightforward, the details are left to the reader.

Recall that $g : \Omega \to U$ is the conformal map such that $g(\Omega)$ is a circle domain, $g(z_0) = 0$ and $g'(z_0) > 0$. There is a $(c', r') \in M_0$ such that $D_{p'} = g(\Omega)$, where $p' = (c', r', 0)$, and $g^B(B^j) = \partial P_0^j(c', r')$. Because $H \circ I_1$ is surjective, there is a $(c, r) \in M_1$ such that $H(c, r, 1) = (c', r')$. Now the map $f = h_p^{-1} \circ g : \Omega \to D_p$ is the required map, where $p = (c, r, 1)$. The sets Q^j are given by $Q^j = P_1^j(c, r)$. □

Corollary 2.1. *Theorem* 1.1 *holds if* Ω *has finitely many boundary components.*

Proof. There is no loss of generality in assuming that no boundary component of Ω is a single point. In the Brandt-Harrington uniformization theorem take $P^j = \overline{U}$ for each $j = 1, \dots, n$. It tells us that there is a conformal map f of Ω onto some relative circle domain in A with $f(z_0) = z_0^*$, $f'(z_0) > 0$, and $f^B(\partial U) = \partial A$. Because each circle component of $\partial\Omega$, except ∂U, is mapped to a circle, f is smooth near $\partial\Omega - \partial U$, and hence extends to a K-quasiconformal map $\widetilde{f} : U \to A$. Since $\partial\Omega$ has zero measure, the condition that the complex dilatation of $\widetilde{f}$ vanishes a.e. in $\partial\Omega \cap U$ is trivial. □

3. The Case ∂U is Isolated in $\partial\Omega$.

In this section we prove:

Lemma 3.1. *Theorem* 1.1 *holds if* ∂U *is an isolated component of* $\partial\Omega$.

Proof. Let $C_1 = \partial U, C_2, C_3, \dots$ be all the circles in $\partial\Omega$. For each $k > 1$, let Ω_k be the domain bounded by $C_1, C_2, \dots, C_k$. We have $\Omega \subseteq \Omega_{k+1} \subseteq \Omega_k$. Then Corollary 2.1 tells us that Theorem 1.1 holds for each Ω_k. Thus, there are some relative circle domains Ω_k^* and conformal homeomorphisms $f_k : \Omega_k \to \Omega_k^*$ such that $f_k^B(\partial U) = \partial A$, and $f_k(z_0) = z_0^*$, $f_k'(z_0) > 0$.

As ∂U is isolated in $\partial\Omega$, there is some $r > 0$, $r < 1$, such that $U - D(r) \subseteq \Omega$; where D(r) is the disk of radius r with center at 0. We may choose r such that z_0 is contained in $D(r)$. It follows that all f_k's are defined (and analytic) in the annulus $U - D(r)$; and if we let $s = (1+r)/2$ then the restrictions of f_k to $D(s) - D(r)$ have uniformly quasiconformal extensions $\bar{f}_k : \hat{\mathbb{C}} - D(r) \to \hat{\mathbb{C}}$. Let $\widetilde{\Omega}_k = \Omega_k \cup (\hat{\mathbb{C}} - D(r))$; and let $g_k : \widetilde{\Omega}_k \to \hat{\mathbb{C}}$ be the mapping which is equal to f_k in $\Omega_k \cap D(s)$ and identical to $\bar{f}_k$ in $\hat{\mathbb{C}} - D(r)$. Then the images of $g_k(\widetilde{\Omega}_k)$ are all circle domains; and all g_k's are uniformly quasiconformal mappings of circle domains. Using the Schottky groups generated by the inversions in the circles of the circle domains, we may extend each g_k, as in [**HS2**, §2], to some uniformly quasiconformal homeomorphism $\widetilde{g}_k : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Clearly, $g_k(z_0) = f_k(z_0) = z_0^*$; and $g_k'(z_0) = f_k'(z_0)$ are uniformly bounded from below by the Schwarz Pick Lemma of [**HS1**] (see also Lemma 4.2 below). Eventually, by replacing by subsequences, we may assume that the f_k are convergent in the interior of Ω to some conformal map $f : \Omega \to A$ and that $\widetilde{g}_k$ are uniformly convergent (in the spherical metric) in $\hat{\mathbb{C}}$ to some quasiconformal homeomorphism $\widetilde{g} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

We claim that $f(\Omega)$ is a relative circle domain in A. Let J be a component of $A - f(\Omega)$. As $\partial D(s) \subseteq \Omega$ and $f = \widetilde{g}$ near $\partial D(s)$, we have either $J \subseteq \widetilde{g}(D(s))$ or $J \subseteq \widetilde{g}(\hat{\mathbb{C}} - D(s))$. In the former case, J is a component of $\hat{\mathbb{C}} - \widetilde{g}(\widetilde{\Omega})$, and hence is a closed disk or a point. As $U - D(r) \subseteq \Omega$, the latter case would imply that $\overline{J} \cap \partial A \neq \emptyset$, and is excluded by the following lemma. $\square$

Lemma 3.2. *∂A is an isolated boundary component of $f(\Omega)$.*

Proof. Consider the quasiconformal homeomorphisms $h_k : U \to A$ defined by $h_k = \widetilde{g}_k$ in $D(s)$ and $h_k = f_k$ in $U - D(r)$. Then the h_k converge to some map $h : U \to A$ with $h = f$ in $U - D(r)$. Then the lemma follows by Carathéodory's Kernel Convergence Theorem for quasiconformal maps [**LV**, pg. 76]. $\square$

So, $f(\Omega)$ is a relative circle domain in A. Let h_k, h be the mappings constructed in the proof of Lemma 3.2, and set $\widetilde{f} = h : U \to A$. As $h_k|_\Omega = f_k|_\Omega$, by taking the limit we deduce that $h|_\Omega = f|_\Omega = f$, i.e., $\widetilde{f}|_\Omega = f$. In order to complete the proof of Lemma 3.1, it only remains to show that the complex dilatation of $\widetilde{f}$ vanishes a.e. in $\partial\Omega \cap U$. Since $\widetilde{f} = \widetilde{g}$ near $\partial\Omega \cap U$, it suffices to show that the complex dilatation of $\widetilde{g}$ vanishes a.e. in $\partial\Omega \cap U$. But this follows by [**HS2**, Lemma 3.2] restated below, since the complex dilatations of g_k vanish in Ω_k and hence a.e. in $\partial\Omega$. $\square$

Lemma 3.3. *Let A be a domain in $\hat{\mathbb{C}}$. Let $g_k : A \to \hat{\mathbb{C}}$ be a sequence of uniformly quasiconformal mappings which converge to a quasiconformal mapping $g_\infty : A \to \hat{\mathbb{C}}$. Let $\lambda_k : A \to \mathbb{C}$ denote the complex dilatation of g_k. Suppose that $\lim_{k\to\infty} \lambda_k(z)$ exists for a.e. $z \in B$; where $B \subseteq A$. Then the complex dilatation of g_∞ equals $\lim_{k\to\infty} \lambda_k(z)$ for a.e. $z \in B$.*

4. Schwarz Pick Lemma and Corollaries.

This section will prepare us for the final step in the proof of Theorem 1.1. We first recall the Schwarz Pick Lemma of [**HS2**, §5].

Lemma 4.1 (Schwarz Pick Lemma for Relative Circle Domains). *Let A and A^* be simply connected domains in $\hat{\mathbb{C}}$ such that $A \supseteq U \supseteq A^*$. Let Ω and Ω^* be relative circle domains in A and A^* respectively. Suppose that $f : \Omega \to \Omega^*$ is a conformal homeomorphism, and that $f^B(\partial A) = \partial A^*$. If f extends to a locally quasiconformal homeomorphism $g : A \to A^*$ whose complex dilatation vanishes a.e. on $\partial\Omega \cap A$, then for any p, $q \in \Omega \cap U$,*

$$d_{hyp}(f(p), f(q)) \leq d_{hyp}(p, q); \tag{4.1}$$

where d_{hyp} denotes the Poincarè metric in the unit disk. Furthermore, if equality holds for one pair $p \neq q$, then $A = A^* = U$ and f is the restriction of a hyperbolic isometry of U.

Let A be a simply connected proper subdomain in $\mathbb{C}$. Let $\delta(z, \partial A)$ denote the euclidean distance between z and ∂A; and let $|dz|$ denote the euclidean metric of $\mathbb{C}$. Define a new metric ds_A in A by

$$ds_A(z) = \frac{2|dz|}{\delta(z, \partial A)}. \tag{4.2}$$

In what follows, A will implicitly be equipped with the metric ds_A; and we will use $d_A(\cdot, \cdot)$ to denote the distance in A between any pair of points. It is easy to see that the ball $\{z \in A : d_A(z, p) \leq r\}$ is compact for any $p \in A$ and $r \in (0, \infty)$.

Now, suppose we are given a relative circle domain Ω in U, a relative circle domain Ω^* in A, and a conformal homeomorphism $f : \Omega \to \Omega^*$ which extends to a locally quasiconformal homeomorphism $\tilde{f} : U \to A$ whose complex dilatation vanishes a.e. in $\partial\Omega \cap U$. Then:

Lemma 4.2. *Considering Ω as a subset of U with the Poincarè metric, the map $f : \Omega \to A$ is locally an expansion.*

Proof. Let p be any point in Ω. We want to show that the differential $df(p)$ has norm ≥ 1. For convenience, let us assume that $p = 0$ and $f(0) = 0$. Replacing f by $g = f/\delta(0, \partial A)$, we may assume that $\delta(0, \partial A) = 1$. We need, then, to show that $|f'(0)| \geq 1$. This follows immediately from Lemma 4.1 since $U \subseteq A$. □

Lemma 4.3. *There is a universal constant $\eta > 0$ such that*

$$d_A(f(p), f(q)) \geq \min\{\eta, \eta\, d_{hyp}(p, q)\} \tag{4.3}$$

for any pair of points p, q in $\overline{\Omega} \cap U$.

Proof. It is enough to prove (4.3) for $p, q \in \Omega$. Again, we may assume $p = f(p) = 0$ and $\delta(0, \partial A) = 1$. Suppose that $d_A(f(p), f(q)) \leq \eta_1$ for some $\eta_1 > 0$, which we assume is very small, say $\leq \log(5/4)$. Then $|f(q)| \leq 1/4$, and hence $f(q) \in U$. As $U \subseteq A$, Lemma 4.1 tells us that $d_{hyp}(0, f(q)) \geq d_{hyp}(0, q)$. Hence $|q| \leq |f(q)| \leq 1/4$. As $\delta(0, \partial A) = 1$, this clearly implies that $\eta_2\, d_{hyp}(0, q) \leq d_A(0, f(q))$ for some $\eta_2 > 0$. Letting $\eta = \min\{\eta_1, \eta_2\}$, we obtain (4.3). □

5. Proof of Theorem 1.1 in the General Case.

We now complete the proof of Theorem 1.1 for arbitrary Ω. Let U_k be a sequence of Jordan domains contained in U such that $\partial U_k \subseteq \Omega$; $U_k \subseteq U_{k+1}$; and $\bigcup_k U_k = U$. Let $\Omega_k = \Omega \cap U_k$. Then Ω_k is a relative circle domain in U_k and ∂U_k is isolated in $\partial\Omega_k$. So we may apply the Riemann mapping theorem for relative circle domains and the special case of Theorem 1.1 proved in §3 to conclude that for each k, there exist a relative circle domain Ω_k^* in A and a conformal homeomorphism $f_k : \Omega_k \to \Omega_k^*$ which extends to some locally quasiconformal homeomorphism $\widetilde{f}_k : U_k \to A$ whose complex dilatation vanishes a.e. in $\partial\Omega \cap U_k$. Moreover, we may require f_k to satisfy

$$f_k(z_0) = z_0^*, \quad f_k'(z_0) > 0. \tag{5.1}$$

The extension of the sequence of maps $f_k : \Omega_k \to A$ to $\widetilde{f}_k : U_k \to A$ can be done inductively: in step 1, extend $f_k|_{U_1\cap\Omega}$ to a sequence of maps defined on U_1; then in step 2, extend $f_k|_{U_2\cap\Omega}$ to a sequence of maps defined on U_2 while agreeing in U_1 with the extensions constructed in the previous step; and so on. In this way, we can make sure that the maps $\widetilde{f}_k$ are locally uniformly quasiconformal, i.e., for any open subset W whose closure is contained in U, the restrictions $\widetilde{f}_k|_W$ are uniformly quasiconformal (see also **[HS2]** for a similar argument).

By the Schwarz Pick lemma (or Lemma 4.2), we see that the $f_k'(z_0)$ are uniformly bounded from below. So, by taking a subsequence, we may assume that f_k converges in Ω to some conformal mapping $f : \Omega \to A$, and $\widetilde{f}_k$ converge to some locally quasiconformal mapping $\widetilde{f} : U \to A$. Since the complex dilatation of $\widetilde{f}_k$ vanishes a.e. in $\partial\Omega\cap U_k$, by Lemma 3.3, the complex dilatation of $\widetilde{f}$ vanishes a.e. in $\partial\Omega \cap U$. It remains to show that $f(\Omega)$ is a relative circle domain in A.

First of all, if c is a component of $\partial\Omega$ other than ∂U, then $c \subseteq \partial\Omega_k \cap U_k$ for sufficiently big k. Then the locally uniform convergence of $\widetilde{f}_k$ to $\widetilde{f}$, or the argument of **[HS2**, §3**]**, show that $\widetilde{f}(c)$ is a circle or a point. That is, $f^B(c) = \widetilde{f}(c)$ is a circle or a point.

Finally, let us show that $f^B(\partial U) = \partial A$; or equivalently, that ∂A is a boundary component of $f(\Omega)$. Here, we will use Lemma 4.3. By dropping some U_k's, we may assume that $U_k \subseteq U_{k+1}$, and that the hyperbolic distance between ∂U_k and ∂U_{k+1} (as subsets in U) is at least one. Let k be a fixed positive integer. Then for each $j > k+1$, Lemmas 4.3 and 4.1 imply that $d_A(f_j(\partial U_k), f_j(\partial U_{k+1})) \geq \eta$. Letting $j \to \infty$, we obtain $d_A(f(\partial U_k), f(\partial U_{k+1})) \geq \eta$. As a consequence, for an arbitrary but fixed point $z \in U_1 \cap \Omega$ and for any big number $d > 0$, the d-neighborhood of $f(z)$ is contained in the Jordan domain bounded by $f(\partial U_k)$ if $k > d/\eta$. This implies that $f^B(\partial U) = \partial A$. Theorem 1.1 is thus complete. □

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