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PARTITIONS, VERTEX OPERATOR CONSTRUCTIONS AND MULTI-COMPONENT KP EQUATIONS

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For every partition of a positive integer n in k parts and every point of an infinite Grassmannian we obtain a solution of the k component differential-difference KP hierarchy and a corresponding Baker function. A partition of n also determines a vertex operator construction of the fundamental representations of the infinite matrix algebra gl_∞ and hence a τ function. We use these fundamental representations to study the Gauss decomposition in the infinite matrix group Gl_∞ and to express the Baker function in terms of τ-functions. The reduction to loop algebras is discussed.

1. Introduction.

1.1. Infinite Grassmannians and Hirota equations. Sato discovered that the Kadomtsev–Petviashvili (KP) hierarchy of soliton equations could be interpreted as the Plücker equations for the embedding of a certain infinite Grassmannian in infinite dimensional projective space, see e.g. [**Sa1, Sa2**].

Let us first recall the finite dimensional situation. The Grassmannian $Gr_j(\mathbb{C}^n)$ consists of all j-dimensional subspaces W of the n-dimensional complex linear space $\mathbb{C}^n$. Let $\{e_i \mid i = 1, 2, \dots, n\}$ be a basis for $\mathbb{C}^n$ and let $H_j \in Gr_j(\mathbb{C}^n)$ be the subspace spanned by the first j basis vectors $e_1, e_2, \dots, e_j$. The stabilizer in $Gl(n, \mathbb{C})$ of H_j is the "parabolic" subgroup P_j consisting of invertible matrices $X = \sum X_{ab}E_{ab}$, with $X_{ab} = 0$ if $a > j$ and $b \leq j$. Here E_{ab} is the elementary matrix with as only non zero entry a 1 on the $(a, b)^{\text{th}}$ place. So $Gr_j(\mathbb{C}^n)$ can be identified with the homogeneous space $Gl(n, \mathbb{C})/P_j$. Now this homogeneous space is projective, i.e., admits an embedding into a projective space. Explicitly, let $\Lambda\mathbb{C}^n$ be the exterior algebra generated by the basis elements e_a of $\mathbb{C}^n$ and $\Lambda^j\mathbb{C}^n$ the degree j part, i.e., the linear span of elementary wedges $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}$. For $W \in Gr_j(\mathbb{C}^n)$ with basis $w_1, w_2, \dots, w_j$ we have the element $w_1 \wedge w_2 \wedge \cdots \wedge w_j$ which is up to multiplication by a non zero scalar independent of the choice of basis. This then defines an embedding $\phi_j : Gr_j(\mathbb{C}^n) \to \mathbb{P}\Lambda^j\mathbb{C}^n$. (If V is a vector space $\mathbb{P}V$ denotes the associated projective space.) The image of ϕ_j is the projectivization of the $Gl(n, \mathbb{C})$ orbit of the highest weight vector $e_1 \wedge e_2 \wedge \cdots \wedge e_j$

and is described by the following quadratic equations: if $\tau \in \Lambda^j\mathbb{C}^n$, $\tau \neq 0$ then $[\tau]$, the line through τ, belongs to Im ϕ_j iff it satisfies the following equation:

$$\sum_{a=1}^{n} \psi(a)\tau \otimes \psi(a)^*\tau = 0. \tag{1.1.1}$$

Here $\psi(a)$ and $\psi(a)^*$, $a = 1, \dots, n$ are fermionic creation and annihilation operators on $\Lambda\mathbb{C}^n$ that act on elementary wedges by

(1.1.2)

$$\psi(a) \cdot (e_{a_1} \wedge e_{a_2} \wedge \cdots \wedge e_{a_j}) = e_a \wedge e_{a_1} \wedge e_{a_2} \wedge \cdots \wedge e_{a_j},$$

$$\psi^*(a) \cdot (e_{a_1} \wedge e_{a_2} \wedge \cdots \wedge e_{a_j}) = \sum_{k=1}^{j} (-1)^{k+1} \delta_{a a_k} e_{a_1} \wedge e_{a_2} \wedge \cdots \wedge \hat{e}_{a_k} \wedge \cdots \wedge e_{a_j}.$$

Here the hat $\hat{}$ denotes deletion. The equation (1.1.1) is one of the forms of the famous Plücker equations, cf., [**GH**].

The infinite dimensional situation relevant for soliton equations of KP-type is initially very much the same as in finite dimensions: one considers a group G of certain invertible infinite matrices indexed by $\mathbb{Z}$, a parabolic subgroup P and the homogeneous space $Gr = G/P$, an infinite Grassmannian. (We will be sketchy in this introduction about the precise definition of the infinite dimensional objects G, P etc; there are various choices for them, corresponding to various classes of solutions of the hierarchies.)

The group G has a central extension $0 \to \mathbb{C}^* \to \hat{G} \to G \to 0$. (Such an extension also occurs in the finite dimensional situation of $Gl(n, \mathbb{C})$, but is there necessarily trivial and is usually ignored.) There is an integrable highest weight representation L_λ for $\hat{G}$, with λ an integral dominant weight, such that the lift $\hat{P}$ of P stabilizes the highest weight vector v_λ of L_λ. Then the projectivized group orbit $\mathbb{P}(\hat{G} \cdot v_\lambda) \subset \mathbb{P}L_\lambda$ is isomorphic to G/P and so this construction gives a projective embedding of G/P. The representation L_λ can be realized explicitly as a homogeneous component (with respect to the grading by "charge") of a "semi infinite wedge space" on which fermionic creation and annihilation operators $\psi(a)$ and $\psi(a)^*$, $a \in \mathbb{Z}$ act by formulae analogous to (1.1.2). The image of G/P in $\mathbb{P}L_\lambda$ is described by (1.1.1), but with now the summation running over all integers.

To obtain the KP hierarchy one next considers the principal Heisenberg subalgebra $\hat{s}^{princ}$ of the Lie algebra $\hat{g}$ associated to the group $\hat{G}$ and one proves that L_λ, now thought of as a module for $\hat{g}$, remains irreducible under the action of the subalgebra $\hat{s}^{princ}$. By uniqueness of representations of Heisenberg algebras one concludes that L_λ is isomorphic to a polynomial algebra $\mathbb{C}[x_1, x_2, \dots]$ in an infinite number of variables. An element τ of this polynomial algebra corresponds to a point of the group orbit $\hat{G} \cdot v_\lambda$ precisely when it satisfies an infinite collection of differential equations (Hirota equations) of the form

$$(1.1.3) \qquad P(\partial/\partial x)\tau(x) \cdot \tau(x) := P(\partial/\partial y)\tau(x+y)\tau(x-y)|_{y=0} = 0,$$

for certain polynomials P. The equations (1.1.3) are a "bosonized" form of the fermionic Plücker equations (1.1.1). This then is the KP hierarchy in so called Hirota form and gives the defining equations for the projectivized group orbit.

One sees that the construction of the KP hierarchy depends essentially on the choice of the principal Heisenberg algebra to obtain a concrete, bosonic, realization of the representation L_λ. It is therefore natural to investigate what happens if one focuses one's attention to other Heisenberg subalgebras $\hat{s}$ of $\hat{g}$, that, as is well known, give rise to other so called vertex operator constructions ([**KaP2**, **Lep**]). In general the representation L_λ will not remain irreducible under other Heisenberg algebras but in our situation there is in the group $\hat{G}$ a subgroup $\hat{T}$, the *translation group*, of elements that commute with $\hat{s}$ in the representation L_λ and such that L_λ remains irreducible under the action of the pair $(\hat{s}, \hat{T})$.

Investigating examples (see for example [**tKB**], [**KaW**]) one quickly discovers that one obtains from these other constructions of L_λ in much the same way as before defining equations for the group orbit, but the equations can have a rather different character; in particular one will find hierarchies that contain also difference, as opposed to just differential, equations. For example the Toda lattice can be obtained in this way. The difference equations are "caused" by the occurrence of the translation group in the vertex operator construction sketched above. Also the differential equations that one obtains for other Heisenberg algebras look rather different: in the simplest case one obtains the Davey-Stewartson equation instead of the KP equation.

In this paper we want to discuss the hierarchies of soliton equations related to certain vertex operator constructions of the central extension $\hat{g}$ of the infinite matrix algebra. These constructions use Heisenberg algebras of $\hat{g}$ obtained from all possible Heisenberg algebras of the affine Lie algebras $\hat{gl}(n, \mathbb{C})$, where we think of $\hat{gl}(n, \mathbb{C})$ as a subalgebra of $\hat{g}$.

1.2. Lax and zero curvature form. Until now in this introduction we have described soliton equations in Hirota form, using the representation theory of a central extension of the infinite linear group.

Other approaches to these equations are the Lax and zero curvature formalisms. Let us sketch how these approaches are related to the representation theoretic one. As we discussed before the main ingredient in the recipe for the construction of Hirota equations was the choice of Heisenberg system $(\hat{s}, \hat{T})$ consisting of a Heisenberg subalgebra of $\hat{g}$ and a translation subgroup $\hat{T}$ of $\hat{G}$. Now using the sequences

$$0 \to \mathbb{C} \to \hat{g} \to g \to 0, \quad 0 \to \mathbb{C}^* \to \hat{G} \to G \to 0 \tag{1.2.1}$$

we obtain a pair (s, T) consisting of a commutative subalgebra in g and a subgroup T in G that commutes with s. The decomposition $\hat{s} = \hat{s}_+ \oplus \hat{s}_- \oplus \mathbb{C}c$ in annihilation operators, creation operators and central elements induces a decomposition $s = s_+ \oplus s_-$. Then one considers on G/P "continuous and discrete time flows" from the pair (γ_+, T), where $\gamma_+ = \exp(s_+)$. The compatibility or commutativity conditions of these flows will then be the Lax or zero curvature equations (depending on how one sets things up).

It is well known (at least for the KP hierarchy) that the Hirota type equations are equivalent to the Lax or zero curvature equations. The main point of this equivalence is the connection between the so called Baker function, or wave function, and the τ-function, well known from the Japanese literature (e.g., [**DJMK**]) and also, in the algebro-geometric situation, in Russian works (e.g., [**Kr**]). In the case of the principal Heisenberg algebra this connection was given by [**SeW**, 5.14], by a "simple but mystifying proof" in the words of [**W1**]. For the case of the homogeneous Heisenberg algebra [**Di3**] uses this connection to *define* the τ-function.

It is the aim of this paper to give a representation theoretic derivation of the connection between the τ-function and the wave function. (We will use the term wave function instead of Baker function in the main part of the paper.) In our set-up this relation is derived from the observation that the wave function consists, essentially, of one or several columns in a lower triangular matrix in the defining representation of G (for the principal Heisenberg algebra or in general, respectively). Furthermore one finds that one can calculate matrix elements of such matrices in terms of the fundamental highest weight representations of the central extension $\hat{G}$. The explicit use of matrix elements of fundamental representations of Lie algebras to solve integrable systems goes at least back to Kostant's solution of the finite non-periodic Toda lattice, [**Ko**].

Let us describe this last simple, but essential, step in a finite dimensional situation. Consider the group $Gl(n, \mathbb{C})$ acting on the vector space $H = \mathbb{C}^n$, with, as before, basis $e_1, \dots, e_n$ and H_j the subspace of $\mathbb{C}^n$ spanned by the first j basis vectors. We recall that any $g \in Gl(n, \mathbb{C})$ admits a *Gauss-decomposition* $g = g_- P g_+$, with g_- a lower triangular matrix with 1's on the diagonal, g_+ an invertible upper triangular matrix and P a permutation matrix. The permutation matrix is uniquely determined by g, but the $g_\pm$ are not, unless $P = 1_n$. We say that $g \in Gl(n, \mathbb{C})$ *belongs to the big cell for* H_j

if $PH_j = H_j$. In this case we can choose the factors in a variant $g = g_-^j P g'_+$ of the Gauss-decomposition in a unique way such that P is the permutation matrix from the regular Gauss decomposition and g_-^j is of the form

$$g_-^j = 1_{n\times n} + \sum_{\substack{r>j\\ s\leq j}} g_{rs}E_{rs}. \tag{1.2.2}$$

If we put $g_+^j = Pg'_+$ we have $g_+^j H_j = H_j$ and we see that the j-dimensional subspace $W = gH_j$ of $\mathbb{C}^n$ projects isomorphically to H_j by the natural projection. Now it is not too difficult to see that the matrix elements g_{rs} of g_-^j can all be calculated using the fundamental representation $\Lambda^j\mathbb{C}^n$ (cf. [**BaRą**, ch. 3.11]): we have

$$g_{rs} = \langle E_{rs}\cdot \mathbf{v}_j \mid g_-^j \cdot \mathbf{v}_j\rangle = \langle E_{rs}\cdot \mathbf{v}_j \mid g\cdot \mathbf{v}_j\rangle / \langle \mathbf{v}_j \mid g\cdot \mathbf{v}_j\rangle, \qquad r = 1,2,\dots,n, \quad s\leq j, \tag{1.2.3}$$

where $\mathbf{v}_j = e_1 \wedge e_2 \cdots \wedge e_j$, E_{rs} acts as if it were a group element and $\langle\,\cdot \mid \cdot\,\rangle$ is the canonical Hermitian form on $\Lambda^j\mathbb{C}^n$ such that $\langle \mathbf{v}_j \mid \mathbf{v}_j\rangle = 1$. The denominator $\tau_j = \langle \mathbf{v}_j \mid g\cdot \mathbf{v}_j\rangle$ of (1.2.3) is the finite dimensional analogue of the famous τ-function; if we write

$$g_+^j = \begin{pmatrix} A & B\\ 0 & D\end{pmatrix}, \tag{1.2.4}$$

with A of size $j\times j$, B of size $j\times n-j$ and D of size $n-j\times n-j$, then $\tau_j = \det(A)$. The most important property is that τ_j is nonzero iff g belongs to the big cell for H_j iff the projection from W to H_j is an isomorphism.

Translating (1.2.3) to our infinite dimensional situation gives Lemma 5.5.1. Recall that one can associate to every partition $\underline{n}$ of n into k parts a vertex operator construction for the infinite matrix algebra, using the technology of "bosonized k component fermion fields" (see, e.g., [**tKvdL**] and references therein). For each of these constructions we find solutions to the k-component differential-difference KP hierarchy and we obtain in theorem 5.5.2. by a straightforward calculation the relation between the wave function and the τ-function for these hierarchies. In the literature k-component KP hierarchies were introduced in [**DJKM2**] and studied in [**UT, Di1, Di3**]. However there apparently only the solutions related to the homogeneous partition $n = 1 + 1 \cdots + 1$ are considered and also the difference equations in the hierarchies seem to be included only implicitly.

It is for various reasons interesting to study more general partitions. Recall (from [**SeW**], say), that one can associate a solution of the KP hierarchy to algebro-geometric data, consisting of a Riemann surface X, a point $p \in X$

with local coordinate z^{-1} at p, a line bundle, etc. If z happens to be the n^{th} root of a global meromorphic function on X with only a pole at p we have a covering map $X \to \mathbb{P}^1$ with p as the n-tuple inverse image of the point $\infty \in \mathbb{P}^1$ and one obtains a solution of the n-KdV hierarchy. The natural generalization ([**AB1**, **AB2**]) of this construction consists in considering n-fold coverings of $\mathbb{P}^1$ such that the pull back divisor of $\infty \in \mathbb{P}^1$ is of the form $\sum_{a=1}^{k} n_a p_a$, for $p_1, \ldots, p_k$ points on X and the n_a positive integers. This gives us a partition $\underline{n}$ of n and by choosing other appropriate geometric data (line bundle, trivializations, etc) one finds a solution of the k-component KP hierarchy. In [**LiMu**] this construction is used to study the analogue of the Schottky problem for Prym varieties. In [**McI**] these type of soliton equations are studied in terms of flows on generalized Jacobians, see also [**Pr**]. In section 5 we will spend quite some time discussing the fermionic translation operators $\hat{Q}_a$, the translation group $\hat{T}$ constructed from it and the relation with the infinite Grassmannian. In the algebro-geometric language the operators $\hat{Q}_a$ correspond to tensoring the line bundle with a bundle with divisor p_a.

Another way in which the hierarchies related to arbitrary partitions might be of interest is the following. Recently there has been much renewed interest in the Hamiltonian structure of soliton equations in relation to the so called W-algebras of conformal field theory. For instance in [**FeFr**] the W_n algebras are constructed using vertex operator algebras and the (modified) n-KdV hierarchy corresponding to the principal partition of n. The Hamiltonian structure of the n-KdV hierarchy is there obtained using a remarkable duality of W algebras. It seems reasonable to expect that there exist for every partition of n (or more generally for every vertex operator construction of affine Kac-Moody algebras) a related W algebra and that using duality of W algebras one can obtain Hamiltonian structures for the corresponding soliton hierarchies. Much here remains to be worked out, but see [**BdG**, **BdGH**, **dGHM**].

There are many papers on soliton equations, so we list only a few. Our main sources have been the papers [**DS**, **DJKM**, **SeW**, **KaP1**]. For background and further references on soliton theory we refer to the books [**AbS**, **Ca**, **N**, **Di2**]. Infinite dimensional Grassmannians and infinite dimensional Lie algebras are discussed in the monographs [**PrS**, **Ka**]. Hierarchies of soliton equations in Hirota bilinear form related to Heisenberg algebras and vertex operator constructions have been discussed in [**KaW**].

2. The infinite Grassmannian.

2.1. Infinite matrix algebra and group Let $\mathbb{C}^\infty$ be the vector space over $\mathbb{C}$ with basis ϵ_i, $i \in \mathbb{Z}$. Let gl_∞ be the Lie algebra over $\mathbb{C}$ with generators the *elementary matrices* (of size $\infty \times \infty$) $\mathcal{E}_{ij}, i, j \in \mathbb{Z}$ that have as only non zero matrix entry a 1 on the i, jth place. We say, as usual, that $\mathcal{E}_{ij}$ is upper (lower) triangular if $i \leq j$ $(i \geq j)$. We have a natural action of gl_∞ on $\mathbb{C}^\infty$ given by

$$\mathcal{E}_{ij} \cdot \epsilon_k = \epsilon_i \delta_{jk}. \tag{2.1.1}$$

The group corresponding to gl_∞ is Gl_∞, consisting of infinite invertible matrices $X = \sum_{i,j\in\mathbb{Z}} X_{ij}\mathcal{E}_{ij}$ such that only a finite number of the $X_{ij} - \delta_{ij}$ is nonzero.

We will need in the sequel infinite linear combinations of the ϵ_i and the $\mathcal{E}_{ij}$. These don't occur in $\mathbb{C}^\infty, gl_\infty$ and Gl_∞ and therefore we introduce

(2.1.2)

$$\begin{aligned}
H &= \left\{ \sum_{i=-\infty}^{m} c_i\epsilon_i \mid c_i \in \mathbb{C}, m \in \mathbb{Z} \right\}, \\
gl_\infty^{lf} &= \left\{ \sum_{i,j\in\mathbb{Z}} c_{ij}\mathcal{E}_{ij} \mid c_{ij} \neq 0 \text{ for only a finite number of } m = i - j > 0 \right\}, \\
Gl_\infty^{lf} &= \Bigg\{ \sum_{i,j\in\mathbb{Z}} X_{ij}\mathcal{E}_{ij} \mid X \text{ invertible}, X_{ij} \neq 0 \text{ for only a finite number} \\
&\qquad\qquad \text{of } m = i - j > 0 \Bigg\}.
\end{aligned}$$

So the matrices we consider have only a finite number of non zero lower triangular diagonals but are for the rest arbitrary. The Lie algebra gl_∞^{lf} and the group Gl_∞^{lf} act on H by extension of the action (2.1.1). This definition ensures that the exponential map of a strictly upper triangular matrix in gl_∞^{lf} is a well defined element of Gl_∞^{lf}, which is the main use we will make of these infinite sums. To deal with matrices with an infinite number of both upper and lower triangular diagonals, for instance in applications in algebraic geometry, one could use the analytical setup of [**SeW**] or of [**ADKP**]. We warn the reader that in the literature on the Sato Grassmannian (e.g., [**Sa2**, **AdC**, **KNTY**, **Mu**]) one allows, in effect, sums that are infinite precisely in the opposite direction from our definition, e.g., infinite number of lower triangular diagonals, but a finite number of upper triangular diagonals. In this approach one cannot define the exponential of an upper triangular matrix. As these papers show, one can circumvent this technical problem, and one would, by following this path, obtain a larger infinite Grassmannian than we do and a wider class of (formal) solutions of the hierarchies we are going to construct. In this paper we prefer to avoid these technicalities, so as to be able to use later on (in Chapter 5) the results of the representation theory of [**tKvdL**], which is set up just in the present context.

2.2. Infinite Grassmannian and Gauss decompositions. Define in H for every integer j a subspace

$$H_j = \left\{ \sum_{k=-\infty}^{j} c_k \epsilon_k \right\} \subset H \tag{2.2.1}$$

We define the *infinite Grassmannian Gr* of H as the collection of subspaces W of H of the form $W = gH_j$, for $g \in Gl_\infty^{lf}$ and some $j \in \mathbb{Z}$.

The Grassmannian that we have defined here corresponds, mutatis mutandis, to what is called the *polynomial Grassmannian* in [**SeW**, **PrS**]. We will need just a few facts about our Grassmannian that can be conveniently derived from a factorization of elements of the group Gl_∞^{lf} as products of lower triangular, permutation and upper triangular matrices. This so called Gauss decomposition will also play an important rôle in the construction of the soliton hierarchies in chapter 4.

To formulate the Gauss decomposition in our infinite dimensional context let S_∞^{fw} be the group of infinite permutation matrices of finite width. So $P \in S_\infty^{fw}$ iff $P \in Gl_\infty^{lf}$ has a finite number of non zero diagonals and each row and column contains precisely one non zero entry, which is equal to 1. A permutation matrix $P \in S_\infty^{fw}$ acts on $\mathbb{C}^\infty$ by $P\epsilon_i = \epsilon_{\sigma_P(i)}$, where $\sigma_P : \mathbb{Z} \to \mathbb{Z}$ is a permutation. This gives us a bijection of S_∞^{fw} with the permutations σ of $\mathbb{Z}$ such that there exist an integer N such that $|\sigma(i) - i| < N$, for all $i \in \mathbb{Z}$.

Lemma 2.2.1 (Gauss decomposition). *Every $g \in Gl_\infty^{lf}$ can be factorized as*

$$g = g_- P g_+, \quad g_-, P, g_+ \in Gl_\infty^{lf} \tag{2.2.2}$$

where $g_- \in Gl_\infty^{lf}$, respectively $g_+ \in Gl_\infty^{lf}$, is strictly lower, respectively upper triangular, i.e., $g_- = 1 + \sum_{i>j} g_{ij}\mathcal{E}_{ij}$, and $g_+ = \sum_{i\le j} g_{ij}\mathcal{E}_{ij}$, $P \in S_\infty^{fw}$. In case g happens to belong to Gl_∞ also the factors $g_\pm$ and P do.

The proof, which is not essentially different from the finite dimensional case, is left to the reader. Note that, as in the finite dimensional situation, the permutation matrix P is uniquely determined by g, but that the factors $g_\pm$ are not, unless $P = 1_\infty$. We will need a variant of the Gauss decomposition (2.2.2) determined by the choice of an integer j. Let

(2.2.3)
$$gl^j_{\infty+} = \left\{X = \sum X_{rs}\mathcal{E}_{rs} \mid X_{rs} = 0 \text{ if } r > j \text{ and } s \le j\right\},$$
$$gl^j_{\infty-} = \left\{X = \sum X_{rs}\mathcal{E}_{rs} \mid X_{rs} = 0 \text{ if } r \le j \text{ or } s > j\right\}.$$

There is a natural projection $pr_{W,j} : W \to H_j$, given by $pr_{W,j}(f) = \sum_{i\le j} f_i\epsilon_i$ if $f = \sum_{i=-\infty}^m f_i\epsilon_i$. We will say that an element $W \in Gr$ belongs to the H_j cell when the natural projection $W \to H_j$ is an isomorphism.

Lemma 2.2.2 (Gauss decomposition adapted to H_j). *Let $g \in Gl^{lf}_\infty$ be such that $W = gH_j$ is in the H_j cell. Then there is a unique decomposition of g of the form*

$$g = g^j_- g^j_+$$

with

$$g^j_- = 1_\infty + X, \quad X \in gl^j_{\infty-}; \quad g_+ \cdot H_j = H_j.$$

Proof. By the Gauss decomposition we have $g = g_- P g_+$ and since W belongs to the H_j cell we have $Pg_+H_j = H_j$. Now write for the minus component of the Gauss decomposition $g_- = 1_\infty + \sum_{\ell>m}(g_-)_{\ell m}\mathcal{E}_{\ell m}$. Then define a matrix

$$f^j = 1_\infty + \sum_{m<\ell\le j}(g_-)_{\ell m}\mathcal{E}_{\ell m} + \sum_{\ell>m>j}(g_-)_{\ell m}\mathcal{E}_{\ell m}. \tag{2.2.4}$$

This matrix is lower triangular with ones on the diagonal, so is invertible and we can define a new decomposition

$$g = g^j_- g^j_+, \tag{2.2.5}$$

where $g^j_- = g_- \cdot (f^j)^{-1}$, $g^j_+ = f^j \cdot P \cdot g_+$. Then g^j_- is of the required form and also $g^j_+ H_j = H_j$. □

We say that the decomposition described by this lemma is *adapted to H_j*, or to j for short.

These decompositions are all related by conjugation. Indeed, if Λ is the shift matrix $\sum_{\ell\in\mathbb{Z}}\mathcal{E}_{\ell\ell+1}$ in Gl^{lf}_∞, then

$$\Lambda^{-1}\mathcal{E}_{ij}\Lambda = \mathcal{E}_{i+1,j+1}, \tag{2.2.6}$$

so we see that

(2.2.7)
$$gl^j_{\infty+} = \Lambda^{-j} gl^0_{\infty,+}\Lambda^j,$$
$$gl^j_{\infty-} = \Lambda^{-j} gl^0_{\infty,-}\Lambda^j.$$

This will be used in Section 7.

In general an element $f \in W$ is an infinite sum $f = \sum_{i=-\infty}^{m} f_i \epsilon_i$. We say that an element $f \in W$ has *finite order* (in the negative direction) if there is an integer s, called the order of f, such that

$$f = \sum_{i=s}^{m} f_i \epsilon_i, \quad f_s \neq 0. \tag{2.2.8}$$

We denote by W^{fin} the collection of finite order elements in W.

We use the Gauss decomposition to introduce a canonical basis for W^{fin}. Let $W = gH_j$ and let $g = g_- P g_+$ be the Gauss decomposition as in the lemma. Then we have $W = g_- P H_j$, since g_+ is an automorphism of H_j. Now a basis for the finite order part of H_j is given by $\{\epsilon_i \mid i \leq j\}$ and a basis for $(PH_j)^{\text{fin}}$ is provided by $\{\epsilon_{\sigma_P(i)} \mid i \leq j\}$, where σ_P is the permutation corresponding to P. Then, since g_- is strictly lower triangular, we see that by taking linear combinations of the finite order elements $g_- \cdot \epsilon_{\sigma_P(i)}$ we can obtain a canonical basis of W^{fin} given by

$$w_s = \epsilon_s + \sum_{\substack{i \notin S_P^j \\ i > s}} c_i \epsilon_i, \tag{2.2.9}$$

where s runs over the set $S_P^j = \{\sigma_P(i) \mid i \leq j\}$ of orders that occur in W, and where for each s we have (for our definition of the Grassmannian) a finite summation. S_P^j is a set of integers that is obtained from $\mathbb{Z}_{\leq j}$ by deleting a finite number of elements and adding a finite number of integers $> j$. So S_P^j contains all sufficiently small integers. Note that if s is small enough $w_s = \epsilon_s$, since there are in g_- only finitely many diagonals below the main one.

The natural projection $pr_{W,j} : W \to H_j$ has finite dimensional kernel and cokernel. This follows, for instance, easily from the remarks about the canonical basis of W^{fin} we just made. So we can define the *index* of $pr_{W,j}$ as $\text{ind}(pr_{W,j}) = \dim(\ker(pr_{W,j})) - \dim(\text{coker}(pr_{W,j}))$. We have also $\text{ind}(pr_{W,j}) = \#(S_P^j - \mathbb{Z}_{\leq j}) - \#(\mathbb{Z}_{\leq j} - S_P^j)$, so that the index depends only on the permutation matrix P occurring in the Gauss decomposition of g, where $W = gH_j$. The index of $pr_{W,0}$ is also called the *virtual dimension* of W, written $\text{Virtdim}(W)$. The Grassmannian decomposes into disjoint components of fixed virtual dimension: $Gr = \cup_{j \in \mathbb{Z}} Gr_j$ with $Gr_j = \{W \in Gr \mid \text{Virtdim}(W) = j\}$. For instance H_j belongs to Gr_j.

An element g of Gl_∞^{lf} is said to belong to the *big cell* if it has a Gauss decomposition with the permutation matrix P the identity. More generally we say that g belongs to the H_j *cell* if it has a Gauss decomposition with a permutation matrix P as middle factor, such that the corresponding permutation σ_P is a product of two (commuting) permutations σ_j, $\sigma_j^\perp$ with σ_j ($\sigma_j^\perp$) leaving the sets of integers $\{i \mid i > j\}$ ($\{i \mid i \le j\}$) pointwise fixed. An element g belongs to the big cell iff it belongs to the H_j cell for all $j \in \mathbb{Z}$.

If g belongs to the H_j cell the corresponding element $W = gH_j$ can be written as $W = g_- H_j$, and projects therefore isomorphically to H_j. In other words g belongs to the H_j cell iff the element $W = gH_j$ does. This argument also shows that the virtual dimension of W is in this case j. An element $W \in Gr$ that is in the H_j cell has a particular simple canonical basis (for its finite order part W^{fin}):

$$w_s = \epsilon_s + \sum_{i>j} c_i \epsilon_i, \quad s \le j. \tag{2.2.10}$$

Again, only a finite number of w_s differ from ϵ_s.

3. Partitions and associated Heisenberg systems.

3.1. Relabeling associated to a partition. Fix an integer $n > 1$. Let $\underline{n} = (n_1 \ge n_2 \ge \cdots \ge n_k > 0)$ be a partition of n into k parts, so that we have $n = \sum_1^k n_a$. We relabel the basis for $\mathbb{C}^\infty$ such that we have

$$\mathbb{C}^\infty = \bigoplus_{a=1}^{k} \bigoplus_{i\in\mathbb{Z}} \mathbb{C}\epsilon_a(i), \tag{3.1.1}$$

with $\epsilon_a(i) = \epsilon_j$, where $j = np + n_1 + \cdots + n_{a-1} + q$ if $i = n_a p + q$ and $1 \le q \le n_a$. We call $\epsilon_a(i)$ the type $\underline{n}$ relabeling of ϵ_j. For all positive integers n the *principal partition* $\underline{n} = n$, i.e., into one part, leads to same, trivial, relabeling: $\epsilon_j = \epsilon_1(j)$.

The relabeling of the basis for $\mathbb{C}^\infty$ induces a natural relabeling of the basis for gl_∞: an infinite matrix is then thought to be build up out of $n \times n$ matrices, each of which consists of blocks of size $n_a \times n_b$, $1 \le a, b \le k$. More explicitly we put

$$\mathcal{E}_{ab}^{n_a p+q \ n_b r+s} = \mathcal{E}_{np+n_1+\cdots+n_{a-1}+q,nr+n_1+\cdots+n_{b-1}+s}, \tag{3.1.2}$$

and we have

$$\mathcal{E}_{ab}^{ij}\epsilon_c(\ell) = \epsilon_a(i)\delta_{j\ell}\delta_{bc}. \tag{3.1.3}$$

The multiplication for the generators after relabeling reads:

$$\mathcal{E}_{ab}^{ij}\mathcal{E}_{cd}^{kl} = \mathcal{E}_{ad}^{il}\delta_{bc}\delta_{jk}. \tag{3.1.4}$$

We extend this relabeling process for vectors and matrices in the obvious way to H and gl_∞^{lf}.

3.2. The numbers $r_b(j)$. Fix an integer j and a partition $\underline{n} = (n_1, n_2, \dots, n_k)$ of n into k parts. We then can write $j = np + n_1 + n_2 + \cdots + n_{a-1} + q$, with $1 \leq q \leq n_a$, so that ϵ_j corresponds to $\epsilon_a(i)$, $i = n_a p + q$, in the relabeling of Section 3.1. We associate to these data the numbers $r_b(j)$ defined by:

$$r_b(j) = \begin{cases} n_b(p+1) & b < a \\ n_a p + q & b = a \\ n_b p & b > a. \end{cases} \tag{3.2.1}$$

In the sequel we will usually have fixed $j \in \mathbb{Z}$ and we then will write simply r_b for $r_b(j)$. These numbers satisfy:

$$\begin{gathered} j = r_1(j) + r_2(j) + \cdots + r_k(j), \\ r_b(j) + n_b \ell = r_b(j + \ell n), \quad \ell \in \mathbb{Z}. \end{gathered} \tag{3.2.2}$$

If $\underline{n} = (n)$ then $r_b(j) = r_1(j) = j$. Note that, for any partition, the numbers $r_b(0)$ are all zero, so the reader might wish to keep this simpler case in mind.

The meaning of these numbers is the following: consider the natural ordering on the basis elements of $\mathbb{C}^\infty$: $\epsilon_\ell \leq \epsilon_j$ iff $\ell \leq j$. Let $\epsilon_a(i)$ be the type $\underline{n}$ labeling of ϵ_j, so that $j = np + n_1 + \cdots + n_{a-1} + q$ and $i = n_a p + q$. Then the ordering on the relabeled basis vectors is given by

$$\epsilon_b(m) \leq \epsilon_a(i) \Longleftrightarrow m \leq r_b(j). \tag{3.2.3}$$

Another way of saying this is: $\epsilon_b(r_b)$ is the largest basis vector of type b that is smaller than $\epsilon_a(i)$ (or equal, in case $b = a$). So for instance we have:

$$\begin{aligned} H_j &= \left\{ \sum_{t=-\infty}^{j} c_t \epsilon_t \right\} \\ &= \left\{ \sum_{b=1}^{k} \sum_{s=-\infty}^{r_b} c_s^b \epsilon_b(s) \right\}. \end{aligned} \tag{3.2.4}$$

Combining the second relation of (3.2.2) and (3.2.3) we find:

$$\epsilon_{j-n} < \epsilon_b(r_b(j)) \leq \epsilon_j. \tag{3.2.5}$$

The ordering of the basis $\epsilon_a(i)$ determines which relabeled elementary matrices are upper triangular:

$$\mathcal{E}_{ba}^{mi} \text{ is upper triangular} \Longleftrightarrow m \leq r_b(j), \tag{3.2.6}$$

if ϵ_j corresponds to $\epsilon_a(i)$.

3.3. Pre-Heisenberg system of a partition. Define now shift matrices Λ_a^+, Λ_a^- in gl_∞^{lf} by

$$\Lambda_a^+ = \sum_{\ell\in\mathbb{Z}} \mathcal{E}_{aa}^{\ell\ell+1}, \quad \Lambda_a^- = \sum_{\ell\in\mathbb{Z}} \mathcal{E}_{aa}^{\ell\ell-1}. \tag{3.3.1}$$

They act on the standard basis of $\mathbb{C}^\infty$ by

$$\Lambda_a^+\epsilon_b(i) = \epsilon_a(i-1)\delta_{ab}, \quad \Lambda_a^-\epsilon_b(i) = \epsilon_a(i+1)\delta_{ab}, \quad 1 \le a \le k. \tag{3.3.2}$$

Let

$$\mathcal{H}^{\underline{n}} = \bigoplus_{a=1}^{k} \left(\left[\bigoplus_{k>0} \mathbb{C}(\Lambda_a^+)^k \right] \bigoplus \left[\bigoplus_{k>0} \mathbb{C}(\Lambda_a^-)^k \right] \right) \tag{3.3.3}$$

be the commutative subalgebra of gl_∞^{lf} generated by the shift operators. We refer to $\mathcal{H}^{\underline{n}}$ as the *pre-Heisenberg algebra* of type $\underline{n}$, since in the universal central extension of gl_∞^{lf} the lift of $\mathcal{H}^{\underline{n}}$ is indeed a Heisenberg algebra, see Section 5.1.

When $\underline{n}$ is a partition in more than one part we have besides the pre-Heisenberg algebra another ingredient in the theory: the pre-translation group. We need some definitions. We write $H = \oplus_{a=1}^k H_a$ where H_a is the subspace of H that can be written using only $\epsilon_a(i)$, $i \in \mathbb{Z}$. Let $1_a := \sum_{i\in\mathbb{Z}} \mathcal{E}_{aa}^{ii}$ be the projection operator $H \to H_a$. Define operators on H by

$$Q_a = \sum_{b\neq a} 1_b + \Lambda_a^-, \quad a = 1, 2, \dots, k. \tag{3.3.4}$$

Then Q_a is invertible and we have

$$Q_a^{-1} = \sum_{b\neq a} 1_b + \Lambda_a^+, \quad a = 1, 2, \dots, k. \tag{3.3.5}$$

Let $R = \oplus_{i=1}^{k-1}\mathbb{Z}\alpha_i$ be the root lattice of the simple Lie algebra $sl(k,\mathbb{C})$, with $\alpha_i, i = 1, 2, \dots, k-1$, the simple roots. We define a homomorphism from the additive group R to a multiplicative Abelian subgroup in Gl_∞^{lf} by

$$\alpha_i \mapsto T_{\alpha_i} := Q_i Q_{i+1}^{-1}, \quad 1 \le i \le k-1. \tag{3.3.6}$$

In particular $\alpha = \sum_{i=1}^k d_i\alpha_i$ gets mapped to $T_\alpha := \prod_{i=1}^k T_{\alpha_i}^{d_i}$. The image of R is called the *pre-translation group* (of type $\underline{n}$) and is denoted by $T^{\underline{n}}$.

One sees immediately that elements of $\mathcal{H}^{\underline{n}}$ and $T^{\underline{n}}$ commute. The pair $(\mathcal{H}^{\underline{n}}, T^{\underline{n}})$ will be called the *pre-Heisenberg system of type* $\underline{n}$.

4. Multicomponent KP equations.

4.1. Time evolution. Let n be a positive integer. We are going to associate to every partition $\underline{n}$ of n a collection of "continuous and discrete time flows" on the infinite Grassmannian Gr, using the pre-Heisenberg system $(\mathcal{H}^{\underline{n}}, T^{\underline{n}})$ of the previous section. Let $\Gamma^{\underline{n}}$ be the subgroup of elements of Gl_∞^{lf} of the form

$$w_0^{\underline{n}}(t,\alpha) = \exp\left(\sum_{i>0}\sum_{a=1}^{k} t_i^a(\Lambda_a^+)^i\right)\cdot T_\alpha. \tag{4.1.1}$$

Here $t = \{t_i^a \in \mathbb{C} \mid 1 \leq a \leq k, i > 0\}$ are the "continuous time parameters" and $\alpha \in R$, where R, the root lattice of $sl(k,\mathbb{C})$, is thought of as a "discrete time lattice". We will often identify the pair (t,α) with the element $w_0^{\underline{n}}(t,\alpha) \in \Gamma^{\underline{n}}$. The elements (4.1.1) satisfy, of course,

$$w_0^{\underline{n}}(t,\alpha+\beta) = w_0^{\underline{n}}(t,\alpha)T_\beta, \tag{4.1.2}$$

for all $\alpha, \beta \in R$.

We define the action ("time flow of type $\underline{n}$") of $w_0^{\underline{n}}(t,\alpha)$ on the Grassmannian in the following way: for $W \in Gr$ we put

$$W(t,\alpha) = w_0^{\underline{n}}(t,\alpha)^{-1}\cdot W = \exp\left(-\sum t_i^a\Lambda_a^i\right)\cdot T_{-\alpha}\cdot W. \tag{4.1.3}$$

If $W = g\cdot H_j$ then we have $W(t,\alpha) = g(t,\alpha)\cdot H_j$, where

$$g(t,\alpha) = w_0^{\underline{n}}(t,\alpha)^{-1}\cdot g. \tag{4.1.4}$$

Note that different choices of n and $\underline{n}$ might give the same flow on the Grassmannian. For example we obtain the same flow if we take for any positive integer n the *principal* partition $\underline{n} = n$ of n into one part. This are the famous KP-flows.

We denote by $\Gamma_W^{j,\underline{n}}$ the collection of points (t,α) in $\Gamma^{\underline{n}}$ such that $W(t,\alpha) = g(t,\alpha)H_j$ belongs to the big cell with respect to H_j, see Subsection 2.2.

4.2. Formal Laurent series and pseudo differential operators. The multicomponent KP equation that we are going to introduce consists of equations for a $k\times k$ matrix function of a "spectral variable" z, which appear as follows.

Fix, as always, a partition $\underline{n}$ of n into k parts. Denote by e_a, $1 \leq a \leq k$ the standard basis vector of $\mathbb{C}^k$ with a 1 on the a^{th} place and 0 elsewhere. We think of the e_a as column vectors. Similarly denote by E_{ab}, $1 \leq a,b \leq k$ the elementary matrix in $gl(k,\mathbb{C})$ with a 1 on the $(a,b)^{th}$ place and 0 elsewhere.

Let, as usual, $\mathbb{C}[[z]]$ be the integral domain of formal power series in the variable z and let $\mathbb{C}((z))$ be its quotient field, the field of formal Laurent series. Denote then by $H^{(k)} = \oplus_{a=1}^{k}\mathbb{C}((z))e_a$ the space of k-component formal Laurent series. Let now $\jmath^{\underline{n}} : H \to H^{(k)}$ be the linear isomorphism given by

$$\jmath^{\underline{n}}(\epsilon_a(i)) = z^{-i}e_a. \tag{4.2.1}$$

For any linear map $A : H \to H$ we have an induced map $A^{(k,\underline{n})} : H^{(k)} \to H^{(k)}$ given by $A^{(k,\underline{n})} = \jmath^{\underline{n}} \circ A \circ (\jmath^{\underline{n}})^{-1}$. When the partition $\underline{n}$ is clear from the context we write $\jmath$ and $A^{(k)}$.

For any $W \in Gr$ we will write $W^{(k,\underline{n})}$ (or simply $W^{(k)}$, if $\underline{n}$ is fixed) for the image $\jmath^{\underline{n}}(W) \subset H^{(k)}$. The image of $H_j \subset H$ is

$$H_j^{(k,\underline{n})} = \oplus_{b=1}^{k}\mathbb{C}[[z]]z^{-r_b}e_b = \left\{\sum_{b=1}^{k}\sum_{j=-r_b}^{\infty} c_{bj}z^je_b\right\}, \tag{4.2.2}$$

with $r_b = r_b(j)$ defined in 3.2.1. In H the subspace H_j is related to the standard subspace H_0 by $H_j = \Lambda^{-j}H_0$, for Λ the shift matrix $\sum_{i\in\mathbb{Z}}\mathcal{E}_{ii+1}$ in Gl_∞^{lf}. Similarly

$$H_j^{(k)} = \mathrm{diag}(z^{-r_1}, z^{-r_2}, \ldots, z^{-r_k})H_0^{(k)} \tag{4.2.3}$$

On $H^{(k)}$ we have a natural action of the formal loop algebra $gl(k, \mathbb{C}((z)))$. Often, in practice, it happens that the image $A^{(k)}$ of an operator $A : H \to H$ ends up lying inside $gl(k, \mathbb{C}((z)))$. This is not the case for $\Lambda^{(k)}$, in general, but for example, for $a = 1, 2, \ldots, k$ we have

$$\begin{aligned}(\Lambda_a^+)^{(k)} &= zE_{aa},\\ (\Lambda_a^-)^{(k)} &= z^{-1}E_{aa},\\ Q_a^{(k)} &= \mathrm{diag}(z^{-\delta_{ab}}),\\ T_{\alpha_i}^{(k)} &= \mathrm{diag}(z^{\delta_{i+1j}-\delta_{ij}}), \quad i = 1, 2, \ldots, k-1.\end{aligned} \tag{4.2.4}$$

To check the first relation, we note that $\Lambda_a^+\epsilon_b(i) = \delta_{ab}\epsilon_a(i-1)$, so the linear transformation induced by $\jmath$ on $H^{(k)}$ maps $\jmath(\epsilon_b(i)) = z^{-i}e_b$ to $\delta_{ab}z(z^{-i})e_a$, i.e., this induced map is multiplication by the matrix zE_{aa}. The other relations are also easily checked.

In particular the group element $w_0^{\underline{n}}$ of (4.1.1), responsible for the time evolution on Gr, corresponds to multiplication on $H^{(k)}$ by

$$w_0(z; t, \alpha) := (w_0^{\underline{n}})^{(k)} = \exp(\sum_{i>0}\sum_{a=1}^{k} t_i^a z^i E_{aa}) \cdot T_\alpha^{(k)}, \tag{4.2.5}$$

where

$$T_\alpha^{(k)} = \prod_{i=1}^{k-1} (T_i^{(k)})^{d_i}, \tag{4.2.6a}$$

if $\alpha = \sum_{i=1}^{k-1} d_i \alpha_i$. When we think of the root lattice as a sub lattice of $\oplus_{a=1}^{k} \mathbb{Z}\delta_a$, with $\alpha_i = \delta_i - \delta_{i+1}$ and $(\delta_a|\delta_b) = \delta_{ab}$, then we find

$$T_\alpha^{(k)} = \operatorname{diag}(z^{-(\alpha|\delta_1)}, z^{-(\alpha|\delta_2)}, \dots, z^{-(\alpha|\delta_k)}). \tag{4.2.6b}$$

Recall the evolution group $\Gamma^{\underline{n}}$ of Subsection 4.1 and the corresponding time flows on Gr. Of all the generators of one parameter subgroups of $\Gamma^{\underline{n}}$ we distinguish a particular one and call this the generator of the x-flow: we define,

$$\partial = \frac{\partial}{\partial x} := \sum_{a=1}^{k} \partial_a^1, \quad \partial_a^i := \frac{\partial}{\partial t_i^a}. \tag{4.2.7}$$

(See **[FNR]** for discussion of this process of singling out a particular combination of the times as "x".) We will often let this operator act from the right, in which case we write $\overleftarrow{\partial} = \frac{\overleftarrow{\partial}}{\partial x}$. This operator acts on $w_0(z;t,\alpha)$ by

$$w_0(z;t,\alpha) \cdot \overleftarrow{\partial} = w_0(z;t,\alpha) \cdot z. \tag{4.2.8}$$

We will also consider formally the inverse of $\overleftarrow{\partial}$, defined by

$$w_0(z;t,\alpha) \cdot \overleftarrow{\partial}^{-1} = w_0(z;t,\alpha) \cdot z^{-1}. \tag{4.2.9}$$

We will in the sequel have to consider k-component formal Laurent series of the form

$$w_0 \cdot X(z), \quad X(z) = \sum_{a=1}^{k} \sum_{i=s}^{m} z^i X_a^i e_a, \tag{4.2.10}$$

so that the vector $X(z)$ is a Laurent polynomial in z. We can, in this situation, "trade in" every occurrence of a power of z in $X(z)$ for the corresponding power of $\overleftarrow{\partial}$: if we write X as in (4.2.10) and define

$$\overleftarrow{X} = \sum_{a=1}^{k} \sum_{i=s}^{m} \overleftarrow{\partial}^{i} X_a^i e_a, \tag{4.2.11}$$

then, clearly, $w_0 \cdot X = w_0 \cdot \overleftarrow{X}$. This procedure introduces in the theory the (non commutative) ring of matrix pseudo differential operators, which will play an important role in the sequel.

4.3. The wave function. We will now use the concepts introduced in the previous sections to define the wave function.

Definition 4.3.1. Fix an integer j, an element $W = gH_j$ in Gr and a partition $\underline{n}$ of n into k parts.

The *wave function of type* $(j, \underline{n})$ *of* $W \in Gr$ is the $k \times k$ matrix function, defined for $(t, \alpha) \in \Gamma_W^{j,\underline{n}}$, obtained by juxtaposing the k columns with index $r_1(j), r_2(j), \dots, r_k(j)$ (these numbers are defined in (3.2.1)) of a certain infinite matrix and applying to each column the isomorphism $\jmath^{\underline{n}}$ (defined in the previous section) to get a k-component formal Laurent series:

$$w_W(z; t, \alpha) = \jmath^{\underline{n}} \left(w_0^{\underline{n}}(z; t, \alpha) \cdot g_-^j(t, \alpha) \cdot (\epsilon_1(r_1)\epsilon_2(r_2) \dots \epsilon_k(r_k)) \right)$$
$$= w_0(z; t, \alpha) \cdot \jmath^{\underline{n}}(g_-^j(t, \alpha) \cdot (\epsilon_1(r_1)\epsilon_2(r_2) \dots \epsilon_k(r_k))).$$

Mostly we will write $w_W(t, \alpha)$ or even w_W for $w_W(z; t, \alpha)$. Note that the elements $\epsilon_1(r_1), \epsilon_2(r_2), \dots, \epsilon_k(r_k)$ all belong to H_j, so the columns $g_-^j(t, \alpha) \cdot \epsilon_b(r_b)$ belong to $W(t, \alpha) = w_0^{\underline{n}}(t, \alpha)^{-1} \cdot W$, and hence the columns of the wave function w_W all belong to $W^{(k)}$. Note furthermore that the columns $g_-^j(t, \alpha) \cdot \epsilon_b(r_b)$ are of the form

$$g_-^j(t, \alpha) \cdot \epsilon_b(r_b) = \epsilon_b(r_b) + \sum_{c=1}^{k} \sum_{\ell > 0} g_{r_c+\ell, r_b}^{cb}(t, \alpha) \epsilon_c(r_c + \ell). \tag{4.3.1}$$

This is so because $g_-^j \cdot \epsilon_b(r_b) = \epsilon_b(r_b) + X \cdot \epsilon_b(r_b)$ with $X \in gl_{\infty -}^j$. Now $X \cdot \epsilon_b(r_b)$ consists of a linear combination of vectors $\epsilon_c(\ell)$ that are larger than $\epsilon_j = \epsilon_a(i)$ in the ordering of Section 4.2, as one sees using the explicit form 2.2.3 of $gl_{\infty -}^j$ and the inequality (4.2.4). Applying the isomorphism $\jmath$ to $H^{(k)}$ shows then that the columns of the wave function are of the form

$$\begin{aligned} (w_W)_b(t, \alpha) &= w_0(z; t, \alpha) \cdot \jmath^{\underline{n}}(g_-^j(t, \alpha)(\epsilon_b(r_b))), \\ &= w_0(z; t, \alpha) \cdot \left(z^{-r_b} e_b + \sum_{c=1}^{k} \sum_{\ell > 0} z^{-r_c - \ell} g_{r_c+\ell, r_b}^{cb}(t, \alpha) e_c \right), \\ &= w_0(z; t, \alpha) \cdot \mathrm{diag}(z^{-r_1}, z^{-r_2}, \dots, z^{-r_k}) \cdot \\ &\quad \cdot \left(e_b + \sum_{c=1}^{k} \sum_{\ell > 0} z^{-\ell} g_{r_c+\ell, r_b}^{cb}(t, \alpha) e_c \right). \end{aligned} \tag{4.3.2}$$

The summation over ℓ in (4.3.1-2) is finite, since in our case g_-^j contains only a finite number of non zero diagonals.

The point of introducing this wave function is that its columns form a basis for an important class of elements of $W^{(k)}$ over a ring of differential operators. Indeed, fix a $(t,\alpha) \in \Gamma^{\underline{n}}$ and define

(4.3.3)
$$W^{(k)}_{\mathrm{fin}}(t,\alpha) = \left\{ f \in W^{(k)} \mid f(z) = w_0(z;t,\alpha) \cdot \left(\sum_{b=1}^{k} \sum_{\ell=s_b}^{m_b} z^\ell f_{b\ell} e_b \right) \right\}.$$

It is clear that $W^{(k)}_{\mathrm{fin}}(t,\alpha)$ does depend trivially on $\alpha \in R$ and we will delete α here. We will often also, having fixed t, suppress the t dependence. The space $W^{(k)}_{\mathrm{fin}}$ is the image of the space $W^{(k)}(t,\alpha)^{\mathrm{fin}}$, the finite order part of $W^{(k)}(t,\alpha)$, under the isomorphism given by multiplication by $w_0(z;t,\alpha)$. Note that if $(t,\alpha),(t,\alpha+\beta) \in \Gamma^{j,\underline{n}}_W$ (so that $w_W(t,\alpha)$ and $w_W(t,\alpha+\beta)$ exist) the columns of $w_W(t,\alpha)$, $w_W(t,\alpha+\beta)$ and also of $\partial(w_W)(t,\alpha)$, $\partial(w_W)(t,\alpha+\beta)$ belong to $W^{(k)}_{\mathrm{fin}}$.

We have the following generalization of results of Drinfeld–Sokolov [**DS**], Segal–Wilson [**SW**]:

Proposition 4.3.2. *Fix $(t,\alpha) \in \Gamma^{j,\underline{n}}_W$, let $W^{(k)}$ be the image of W in the space of k-component functions $H^{(k)}$ and define $W^{(k)}_{fin}$ by (4.3.3). Then $W^{(k)}_{fin}$ is a free rank k module over the ring $\mathbb{C}[\overleftarrow{\partial}]$, with basis the columns of $w_W(t,\alpha)$. More explicitly: there exists for every $f(z) \in W^{(k)}_{fin}$ a unique k-component differential operator $\overleftarrow{P}(f) = \sum_{b=1}^{k} \overleftarrow{P}_b(f) e_b$, $P_b(f) \in \mathbb{C}[\overleftarrow{\partial}]$, such that*

$$f = w_W(t,\alpha) \cdot \overleftarrow{P}(f). \tag{4.3.4}$$

Proof. Let ϵ_j correspond to $\epsilon_a(i)$ and let r_b the numbers associated with j in Section 4.2. Suppose that for $f(z) \in W^{(k)}_{\mathrm{fin}}$ as in (4.3.3) $f_{bm_b} \neq 0$. Then we call $m_b + r_b$ *the b-order* of $f(z)$ and f_{bm_b} *the leading b-coefficient.* If all $f_{b\ell}$ are zero the b-order is $-\infty$. (This ordering comes from a refinement of the ordering (2.2.3) on $W(t,\alpha)$, via application of $\jmath$ and multiplication by $w_0(z;t,\alpha)$.)

For example, the b^{th} column of w_W has b-order 0 and leading b-coefficient is 1, whereas its c-order for $c \neq b$ is strictly negative. If $f \in W^{(k)}_{\mathrm{fin}}$ then also $f \cdot \overleftarrow{\partial} \in W^{(k)}_{\mathrm{fin}}$, with the b-order of $f \cdot \overleftarrow{\partial}$ bigger by 1 than that of f and with the leading b-coefficient unchanged, for all $b = 1,2,\dots,k$.

If $(t,\alpha) \in \Gamma^{j,\underline{n}}_W$, then we have $W(t,\alpha) = g^j_-(t,\alpha)H_j$, so that $W(t,\alpha)$ projects isomorphically to H_j. Applying $\jmath$ we see that $W^{(k)}(t,\alpha)$ maps isomorphically to $H^{(k)}_j$. In particular if $f = w_0 \cdot X \in W^{(k)}_{\mathrm{fin}}$, with $X = \sum_{b=1}^{k} \sum_{j=s_b}^{m_b} z^j f_{bj} e_b$ then $X = w_0^{-1} \cdot f \in W^{(k)}(t,\alpha)$ and X maps isomorphically to $H^{(k)}_j$ using the projection $pr^{(k)}_{W,j}$. Now if the b-order of f were negative for all b then all m_b would be smaller than $-r_b$ and, taking explicit form (4.2.2) of $H_j^{(k)}$ into account, X would project to $0 \in H_j^{(k)}$. Since this projection is an isomorphism X, and hence f, has to be zero. So we see that for $f \in W_{\mathrm{fin}}^{(k)}$ at least one of the b-orders is non negative, unless $f = 0$.

Now, of course, the idea is, given $f \in W_{\mathrm{fin}}^{(k)}$, to reduce its orders by subtraction of terms $(w_W)_b \cdot \overleftarrow{P}$, for suitable differential operator $\overleftarrow{P}$. More precisely, let $m = m_b + r_b$ be the *total order* of f, i.e., the maximum of the orders that occur, and let μ be the *multiplicity* of m, i.e., the number of components c for which the c-order is equal to the total order. Then $(w_W)_b \cdot \overleftarrow{P}$, with $\overleftarrow{P} = (\overleftarrow{\partial})^m f_{bm_b}$, has the same b-order $m = m_b + r_b$ and the same leading b-coefficient as f and the c-order, $c \neq b$, is at most $m-1$. Subtracting we obtain an element $f - (w_W)_b \cdot \overleftarrow{P}$ of $W_{\mathrm{fin}}^{(k)}$ of lower order in the b-component. If the multiplicity μ was 1 the total order of $f - (w_W)_b \cdot \overleftarrow{P}$ is strictly smaller than that of f. In case the multiplicity is larger than 1 the total order of $f - (w_W)_b \cdot \overleftarrow{P}$ will still be m, but the multiplicity is one smaller than that of f. By repeating this process we reduce the multiplicity and the total order and we can find k differential operators $\overleftarrow{P}_a$ such that $\tilde{f} := f(z) - \sum_{a=1}^{k} (w_W)_a \cdot \overleftarrow{P}_a$ has its b-order, for all b, less than 0. Then, as we argued before, $\tilde{f}$ itself must be zero, so $f(z) = \sum_{a=1}^{k} (w_W)_a \cdot \overleftarrow{P}_a$. It is easy to check that the differential operators $\overleftarrow{P}_a$ are unique. $\square$

4.4. Differential difference multi-component KP. In this subsection we derive the equations satisfied by the wave function as a function of the continuous and discrete time variables.

The wave function $w_W(t, \alpha)$ of type $j, \underline{n}$, defined whenever $(t, \alpha) \in \Gamma_W^{j,\underline{n}}$, can be written as

$$w_W(t,\alpha) = w_0(z;t,\alpha) \cdot \overleftarrow{w}_W(t,\alpha), \tag{4.4.1}$$

where $\overleftarrow{w}_W$ is a $k \times k$ matrix pseudo differential operator, called the wave operator, of the form

$$\overleftarrow{w}_W = \mathrm{diag}(z^{-r_1}, \ldots, z^{-r_k}) \cdot (1_{k\times k} + \sum_{i>0} \overleftarrow{\partial}^{-i} w_i), \tag{4.4.2}$$

with the w_i $k \times k$ matrices with entries in the ring of functions B and the numbers r_b defined in (3.2.1). To see this use the explicit form (4.3.2) for the columns of the wave function and use (4.2.9) to trade in negative powers of z for powers of $\overleftarrow{\partial}^{-1}$. The wave operator will be used later on to define resolvents.

Note that in (4.4.2), according to our definitions, the summation over i is finite. Also note that $\overleftarrow{w}_W$ is invertible as a formal pseudo differential operator (PDO), since it starts out with an invertible matrix and contains for the rest only negative powers of $\overleftarrow{\partial}$. Note finally that if we had used g_-, the component of the ordinary Gauss decomposition (Proposition 2.2.1) instead of g_-^j, the component in the Gauss decomposition adapted to j (Lemma 2.2.2), to define the wave function in (4.3.1), the first term in the expansion (4.4.2) would have been not the identity matrix but rather more complicated. The choice we make here also allows us to calculate the wave function in a rather straight forward manner in terms of the τ-function, see Theorem 5.5.2. In this sense the decomposition introduced in Lemma 2.2.2 is adapted to j.

There exist unique $k \times k$ matrix pseudo differential operators $\overleftarrow{\Lambda}_a$, $\overleftarrow{T}_{\alpha_i}$, such that

(4.4.3)
$$\partial_a^1 w_0 = w_0 \overleftarrow{\Lambda}_a,$$
$$w_0(z;t,\alpha+\alpha_i) = w_0(z;t,\alpha)\overleftarrow{T}_{\alpha_i}.$$

Explicitly we have

(4.4.4)
$$\overleftarrow{\Lambda}_a := \overleftarrow{\partial} E_{aa}$$
$$\overleftarrow{T}_{\alpha_i} := \overleftarrow{\partial} E_{ii} + \overleftarrow{\partial}^{-1} E_{i+1i+1} + \sum_{j\neq i,i+1} E_{jj}.$$

Now define

(4.4.5)
$$\mathcal{R}_a := \overleftarrow{w}_W^{-1} \cdot \overleftarrow{\Lambda}_a \cdot \overleftarrow{w}_W = \overleftarrow{\partial} E_{aa} + [E_{aa}, w_1] + \mathcal{O}(\overleftarrow{\partial}^{-1}),$$
$$\mathcal{U}_{\alpha_i} := \overleftarrow{w}_W^{-1} \cdot \overleftarrow{T}_{\alpha_i}^{-1} \cdot \overleftarrow{w}_W = \overleftarrow{w}_W^{-1} \cdot \left(\overleftarrow{\partial}^{-1} E_{ii} + \overleftarrow{\partial} E_{i+1i+1} + \sum_{j\neq i,i+1} E_{jj} \right) \cdot \overleftarrow{w}_W.$$

We refer to $\mathcal{R}_a$ and $\mathcal{U}_{\alpha_i}$ as the pseudo differential resolvents and lattice resolvents associated to $(W, j, \underline{n})$ respectively. (See **[GD]** for the concept of a resolvent. The lattice resolvent was introduced in **[BtK]**.) Note that in both $\mathcal{R}_a$ and $\mathcal{U}_{\alpha_i}$ the first diagonal factor $\operatorname{diag}(z^{-r_1}, z^{-r_2}, \ldots, z^{-r_k})$ of $\overleftarrow{w}_W$ in (4.4.2) cancels, so that resolvents and lattice resolvents have the same general form whichever H_j cell or partition into k parts we use.

The lattice resolvent $\mathcal{U}_{\alpha_i}$ is an invertible matrix pseudo differential operator. We say that an invertible $k \times k$ matrix pseudo differential operator A is *in the big cell* if it admits a decomposition $A = A_- A_+$, where A_+ is an invertible $k \times k$ matrix differential operator and $A_- = 1_{k\times k} + \mathcal{O}(\overleftarrow{\partial}^{-1})$. Such a decomposition, if it exists, is unique. The resolvent $\mathcal{R}_a$ is in general not invertible. In fact $\mathcal{R}_a \mathcal{R}_b = 0$ for $a \neq b$. (When $k = 1$ $\mathcal{R}_a$ is invertible, as a monic scalar PDO.) We will denote by $({\mathcal{R}_a}^i)_+$ the differential operator part of the i^{th} power of $\mathcal{R}_a$ and by $({\mathcal{R}_a}^i)_-$ the formal integral operator

${\mathcal{R}_a}^i - ({\mathcal{R}_a}^i)_+$, and similar for other possibly non invertible matrix pseudo differential operators. So the notation subscripts $\pm$ is not entirely unambiguous, but the meaning is hopefully clear from the context.

Proposition 4.4.2. *Let $W \in Gr_j$ and suppose that (t, α), $(t, \alpha + \alpha_i)$ belong to $\Gamma_W^{j,\underline{n}}$. Then the lattice resolvent $\mathcal{U}_{\alpha_i}$ is in the big cell and we have:*

$$\begin{aligned} \partial_b^i w_W &= w_W \cdot (\mathcal{R}_b^i)_+, \\ w_W(t, \alpha + \alpha_i) &= w_W(t, \alpha) \cdot (\mathcal{U}_{\alpha_i})_+^{-1}. \end{aligned} \tag{4.4.6}$$

Proof. Until now we have defined the action of PDO's with a finite number of negative powers of ∂ on w_0 and the action of differential operators on expressions one obtains in this way. We also need to define an action of arbitrary PDO's on expressions of the form (4.2.10): we put

$$\begin{aligned} w_0 \cdot X(z) \cdot \overleftarrow{\partial} E_{ab} &= w_0 \cdot (z + \partial) X(z) E_{ab}, \\ w_0 \cdot X(z) \cdot \overleftarrow{\partial}^{-1} E_{ab} &= w_0 \cdot (z + \partial)^{-1} X(z) E_{ab}, \\ &= w_0 \cdot z^{-1} \sum_{i=0}^{\infty} (-z^{-1}\partial)^i X(z) E_{ab}. \end{aligned} \tag{4.4.7}$$

Here we run into a little trouble: w_0 is a power series in z and there is a priori no guarantee that the product in the last line of (4.4.7) makes sense. The easiest way to circumvent this problem is not to try to calculate this product and instead interpret w_0 as an abstract free generator v_0 of a module N of expressions $v_0 \cdot \sum_{-\infty}^{m} F_i z^i$ over the matrix PDO's with action given by (4.4.7) (with w_0 replaced by v_0). (cf. [**DS**]). In the obvious way we also define differentiation with respect to the times t_ℓ^b on N. We identify then $w_W = w_0 \cdot \overleftarrow{w}_W$ with the element $v_W = v_0 \cdot \overleftarrow{w}_W$ and the proof of this Proposition takes place in the module N. It happens that for some elements of N, such as $v_W \cdot (\mathcal{R}_b^i)_+$, $v_W \cdot (\mathcal{U}_{\alpha_i})_+^{-1}$, one can give an interpretation as a formal Laurent series; in particular we can interpret $v_W \cdot (\mathcal{R}_b^i)_+$, $v_W \cdot (\mathcal{U}_{\alpha_i})_+^{-1}$ as the series $w_W \cdot (\mathcal{R}_b^i)_+$, $w_W \cdot (\mathcal{U}_{\alpha_i})_+^{-1}$. This being understood we will in the sequel just write w_0 for v_0.

As we noted before, the columns of $w_W(t, \alpha)$ belong to $W^{(k)}$ and in fact to $W_{\text{fin}}^{(k)}$. The same is true of $\partial_b^i w_W$ and of $w_W(t, \alpha + \alpha_i)$. So we can use Proposition 4.3.2 to conclude that $\partial_b^i w_W$ and $w_W(t, \alpha + \alpha_i)$ are of the form $w_W(t, \alpha) \cdot O$, respectively $w_W(t, \alpha) \cdot P$, with O, P $k \times k$ matrix differential operators. On the other hand we have by (4.4.1) and (4.4.3):

(4.4.8)
$$\begin{aligned}
\partial_b^i w_W &= w_0 \cdot (\overleftarrow{\Lambda}{}_b^i \overleftarrow{w}_W + \partial_b^i \overleftarrow{w}_W) \\
&= w_W \cdot (\overleftarrow{w}_W^{-1} \overleftarrow{\Lambda}{}_b^i \overleftarrow{w}_W + \overleftarrow{w}_W^{-1} \partial_b^i \overleftarrow{w}_W) \\
w_W(t, \alpha + \alpha_i) &= w_0(z; t, \alpha) \cdot (\overleftarrow{T}_{\alpha_i} \overleftarrow{w}_W(t, \alpha + \alpha_i)) \\
&= w_W(t, \alpha) \cdot (\overleftarrow{w}_W^{-1}(t, \alpha) \cdot \overleftarrow{T}_{\alpha_i} \cdot \overleftarrow{w}_W(t, \alpha) \cdot \overleftarrow{w}_W(t, \alpha)^{-1} \cdot \\
&\quad \cdot \overleftarrow{w}_W(t, \alpha + \alpha_i)).
\end{aligned}$$

Now w_0 and hence also w_W is a free generator for the action of PDO's, i.e., if for two PDO's X, Y we have $w_W \cdot X = w_W \cdot Y$ then $X = Y$. This implies

(4.4.9)
$$\begin{aligned}
O &= \overleftarrow{w}_W^{-1} \overleftarrow{\Lambda}{}_b^i \overleftarrow{w}_W + \overleftarrow{w}_W^{-1} \partial_b^i \overleftarrow{w}_W = \mathcal{R}_b^i + \overleftarrow{w}_W^{-1} \partial_b^i \overleftarrow{w}_W, \\
P &= \overleftarrow{w}_W^{-1}(t, \alpha) \cdot \overleftarrow{T}_{\alpha_i} \cdot \overleftarrow{w}_W(t, \alpha) \cdot \overleftarrow{w}_W(t, \alpha)^{-1} \cdot \overleftarrow{w}_W(t, \alpha + \alpha_i) \\
&= (\mathcal{U}_{\alpha_i})^{-1} \cdot \overleftarrow{w}_W(t, \alpha)^{-1} \cdot \overleftarrow{w}_W(t, \alpha + \alpha_i).
\end{aligned}$$

Note that $\overleftarrow{w}_W^{-1} \partial_b^i \overleftarrow{w}_W$ is an operator containing only negative powers of $\overleftarrow{\partial}$ while O is a differential operator. This implies

(4.4.10)
$$\overleftarrow{w}_W^{-1} \partial_b^i \overleftarrow{w}_W = -(\mathcal{R}_b^i)_-.$$

Similarly $\overleftarrow{w}_W(t, \alpha)^{-1} \cdot \overleftarrow{w}_W(t, \alpha + \alpha_i)$ is of the form $1_{k \times k} + \mathcal{O}(\overleftarrow{\partial}^{-1})$ and $P = P(\alpha)$ is an invertible differential operator. By the same argument we see that the operator $P'(\alpha)$ such that $w_W(t, \alpha - \alpha_i) = w_W(t, \alpha) \cdot P'$ is an invertible differential operator. Since w_W is a free generator we have $P'(\alpha + \alpha_i) P(\alpha) = 1$, i.e., the inverse of P is also a differential operator. From this we see that $\mathcal{U}_{\alpha_i}$ belongs to the big cell and

(4.4.11)
$$\overleftarrow{w}_W(t, \alpha)^{-1} \cdot \overleftarrow{w}_W(t, \alpha + \alpha_i) = (\mathcal{U}_{\alpha_i})_-.$$

Combining (4.4.9), (4.4.10) and (4.4.11) proves the Proposition. □

Definition 4.4.3. Let L be a matrix PDO of the form

(4.4.12)
$$L = \overleftarrow{\partial} A + \mathcal{O}(\overleftarrow{\partial}^0)$$

for A a diagonal constant matrix with distinct non zero eigenvalues A_a. Let $w(z)$ be a solution of

(4.4.13)
$$w \cdot L = zAw,$$

and introduce the resolvent and lattice resolvent associated to L by (4.4.5) using w for w_W. Then the k-component differential-difference KP hierarchy is the system of deformation equations for L:

(4.4.14)
$$\partial_b^i L = [L, (\mathcal{R}_b^i)_+],$$
$$L(t, \alpha + \alpha_i) = (\mathcal{U}_{\alpha_i})_+ \cdot L(t, \alpha) \cdot (\mathcal{U}_{\alpha_i})_+^{-1}.$$

Consider the $k \times k$ matrix pseudo differential operator

(4.4.15)
$$L_W(t, \alpha) = \overleftarrow{w}_W(t, \alpha)^{-1} \cdot \overleftarrow{\partial} A \cdot \overleftarrow{w}_W(t, \alpha) = \sum_{a=1}^{k} A_a \mathcal{R}_a,$$
$$= \overleftarrow{\partial} A + \mathcal{O}(\overleftarrow{\partial}^0).$$

Then $w_W L_W = zAw_W$ and one finds, using that L_W, $\mathcal{R}_a$ and $\mathcal{U}_{\alpha_i}$ commute, that L_W is a solution of the k-component KP hierarchy, if (t, α), $(t, \alpha + \alpha_i)$ belong to $\Gamma_W^{j,\underline{n}}$.

The compatibility equations for (4.4.6) are the "zero curvature equations" that are also useful:

(4.4.16a)
$$\partial_b^\ell ({\mathcal{R}_a}^m)_+ = \partial_a^m (\mathcal{R}_b^\ell)_+ - [({\mathcal{R}_a}^m)_+, (\mathcal{R}_b^\ell)_+],$$

(4.4.16b)
$$(\mathcal{R}_a(t, \alpha + \alpha_i)^\ell)_+ = -\partial_a^\ell (\mathcal{U}_{\alpha_i})_+ (\mathcal{U}_{\alpha_i})_+^{-1} + (\mathcal{U}_{\alpha_i})_+ \cdot (\mathcal{R}_a(t, \alpha)^\ell)_+ \cdot (\mathcal{U}_{\alpha_i})_+^{-1}.$$

4.5. An example: the Davey-Stewartson-Toda system. In this subsection we discuss a few of the equations that follow from the equations (4.4.16).

Our starting point is an element W of the Grassmannian Gr and the choice of a partition $\underline{n}$, defining a time flow $W \mapsto W(t, \alpha)$, see (4.1.3). The simplest case is obtained by choosing for any positive integer n the principal partition $\underline{n} = n$ into one part. The resulting equations form, of course, the KP hierarchy, discussed extensively in the literature, (see e.g., [**SeW**]), with the discrete part (4.4.16b) missing in this case.

The next simplest case occurs when we choose for any $n > 0$ a partition $\underline{n} = (n_1, n_2)$ into two parts. Now there will be a doubly infinite set of continuous time parameters (t_1^i, t_2^i), with $i > 0$ while the discrete parameter α lives on the rank one root lattice of sl_2 with generator α_1: $\alpha = m\alpha_1, m \in \mathbb{Z}$. We will indicate the dependence on the discrete variable by a superscript: we write $W^m(t)$ for $W(t, m\alpha_1)$, etc. We introduce some new variables

(4.5.1a)
$$x = \frac{1}{2}(t_1^1 + t_1^2), \quad \bar{x} = \frac{1}{2}(t_1^1 - t_1^2), \quad \bar{t} = \frac{1}{2}(t_2^1 + t_2^2), \quad t = \frac{1}{2}(t_2^1 - t_2^2),$$

and the following differential operators:

(4.5.1b) $$\partial = \partial_1^1 + \partial_2^1, \quad \bar{\partial} = \partial_1^1 - \partial_2^1, \quad \partial_t = \partial_1^2 - \partial_2^2.$$

Let $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We will need the following resolvents:

(4.5.2)
$$R_{\bar{x}}^m(t) = R_1^m(t) - R_2^m(t) = \overleftarrow{w}^m(t)^{-1}(\overleftarrow{\partial}\, h)\overleftarrow{w}^m(t),$$
$$R_t^m(t) = R_1^m(t)^2 - R_2^m(t)^2 = \overleftarrow{w}^m(t)^{-1}(\overleftarrow{\partial}^2 h)\overleftarrow{w}^m(t).$$

The resolvents (4.5.2) are defined iff the corresponding element $W^m(t)$ belongs to the big cell.

Lemma 4.5.1. *Let $W^m(t)$ belong to the big cell. Then the resolvents* (4.5.2) *have the expansion*

(4.5.3a)
$$R_{\bar{x}}^m = \overleftarrow{\partial}\, h + \begin{pmatrix} 0 & q^m(t) \\ r^m(t) & 0 \end{pmatrix} + \mathcal{O}(\overleftarrow{\partial}^{-1}),$$
$$R_t^m = \overleftarrow{\partial}^2 h + \overleftarrow{\partial} \begin{pmatrix} 0 & q^m(t) \\ r^m(t) & 0 \end{pmatrix} + \lambda^m(t) 1_2 + \mu^m(t) h +$$

(4.5.3b)
$$+ \begin{pmatrix} 0 & \frac{1}{2}(\partial - \bar{\partial}) q^m(t) \\ \frac{1}{2}(\partial + \bar{\partial}) r^m(t) & 0 \end{pmatrix} + \mathcal{O}(\overleftarrow{\partial}^{-1}).$$

Define $Q^m = \mu^m + \frac{1}{2} q^m r^m$. Then we have the following equations for q^m, r^m, Q^m:

(4.5.4.a) $$\partial_t q^m = -\frac{1}{2}(\partial^2 + \bar{\partial}^2) q^m + (q^m)^2 r^m - 2 q^m Q^m,$$

(4.5.4.b) $$\partial_t r^m = \frac{1}{2}(\partial^2 + \bar{\partial}^2) r^m - (r^m)^2 q^m + 2 r^m Q^m,$$

(4.5.4.c)
$$(\partial^2 - \bar{\partial}^2) Q^m = \partial^2 (q^m r^m).$$

The equations (4.5.4) form the Davey–Stewartson system ([**DaS**, **SaA**]).

Proof. We write for the wave operator and its inverse

(4.5.5) $$\overleftarrow{w}^m(t) = 1_2 + \overleftarrow{\partial}^{-1} w_1^m + \overleftarrow{\partial}^{-2} w_2^m + \dots,$$
$$\overleftarrow{w}^m(t)^{-1} = 1_2 + \overleftarrow{\partial}^{-1} v_1^m + \overleftarrow{\partial}^{-2} v_2^m + \dots,$$

where w_i^m, v_i^m are 2×2 matrices and where we have ignored the irrelevant diagonal factor $\mathrm{diag}(z^{-r_1}, z^{-r_2})$ (see the remark after (4.4.5)). Then we have

$$v_1^m = -w_1^m, \quad v_2 = -w_2^m + (w_1^m)^2. \tag{4.5.6}$$

If we write

$$\begin{aligned} R_{\bar{x}}^m &= \overleftarrow{\partial} h + \gamma^m + \mathcal{O}(\overleftarrow{\partial}^{-1}), \\ R_t^m &= \overleftarrow{\partial}^2 h + \overleftarrow{\partial}\gamma^m + \delta^m + \mathcal{O}(\overleftarrow{\partial}^{-1}), \end{aligned} \tag{4.5.7}$$

then

$$\gamma^m = [h, w_1^m], \quad \delta^m = [h, w_2^m] + w_1^m[w_1^m, h] - 2\partial w_1^m h. \tag{4.5.8}$$

This shows that γ^m is off-diagonal, so that we can write

$$\gamma^m(t) = \begin{pmatrix} 0 & q^m(t) \\ r^m(t) & 0 \end{pmatrix}, \tag{4.5.9}$$

and this proves (4.5.3.a). Next we consider the zero curvature equation

$$[\bar{\partial} + (R_{\bar{x}}^m)_+, \partial_t + (R_t^m)_+] = 0. \tag{4.5.10}$$

Equating the coefficients of powers of $\overleftarrow{\partial}$ to zero gives the equations

(4.5.11.a)
$$[h, \delta^m] = -\bar{\partial}\gamma^m - \partial\gamma^m h,$$
(4.5.11.b)
$$0 = \bar{\partial}\delta^m - \partial\delta^m.h - \partial_t\gamma^m + \partial^2\gamma^m h + \partial\gamma^m \cdot \gamma^m + [\gamma^m, \delta^m].$$

The first equation shows that we can write

$$\delta^m = \lambda^m(t)1_2 + \mu^m(t)h + \begin{pmatrix} 0 & \frac{1}{2}(\partial - \bar{\partial})q^m(t) \\ \frac{1}{2}(\partial + \bar{\partial})r^m(t) \end{pmatrix} \tag{4.5.12}$$

which proves (4.5.3.b). Substituting the Ansätze (4.5.9, 4.5.12) in (4.5.11b) we obtain the system

$$\begin{aligned} 2\bar{\partial}\lambda^m - 2\partial\mu^m + \partial(q^m r^m) &= 0, \\ -2\partial\lambda^m + 2\bar{\partial}\mu^m + \bar{\partial}(q^m r^m) &= 0, \\ -(\partial^2 + \bar{\partial}^2)q^m/2 - 2\mu^m q^m - \partial_t q^m &= 0, \\ (\partial^2 + \bar{\partial}^2)r^m/2 + 2\mu^m r^m - \partial_t r^m &= 0. \end{aligned} \tag{4.5.13}$$

Now we eliminate the variable λ^m by imposing on the first two equations of (4.5.14) the integrability condition $\bar\partial\partial(\lambda^m) = \partial\bar\partial(\lambda^m)$. This gives

$$\partial^2(2\mu^m - q^m r^m) = \bar\partial^2(2\mu^m + q^m r^m). \tag{4.5.14}$$

The last two equations of (4.5.13) combined with (4.5.14) and the definition of Q^m give then the Davey–Stewartson system (4.5.4a-c). □

To derive the discrete equations we need the lattice resolvent

$$U^m(t) = \overleftarrow{w}^m(t)^{-1}\begin{pmatrix} \overleftarrow{\partial}^{-1} & 0 \\ 0 & \overleftarrow{\partial} \end{pmatrix}\overleftarrow{w}^m(t). \tag{4.5.15}$$

Lemma 4.5.2. *Assume that $W^m(t)$ and $W^{m+1}(t)$ belong to the big cell. Then the lattice resolvent* (4.5.15) *admits a factorization $U^m = U_-^m U_+^m$ where $U_-^m = 1_2 + \mathcal{O}(\overleftarrow{\partial}^{-1})$ and*

$$U_+^m = \begin{pmatrix} 0 & 2/r^m \\ -r^m/2 & \overleftarrow{\partial} + \frac{1}{2}(\partial - \bar\partial)\log(r^m) \end{pmatrix}. \tag{4.5.16}$$

Proof. The fact that the factorization exists follows from the proof of Proposition 4.4.2. To calculate the positive factor write

(4.5.17)

$$\begin{aligned} U^m(t) = {} & \overleftarrow{\partial}E_{22} + [E_{22}, w_1^m] + \\ & + \overleftarrow{\partial}^{-1}\{E_{11} - \partial w_1^m E_{22} + [E_{22}, w_2^m] + w_1^m[w_1, E_{22}]\} + \mathcal{O}(\overleftarrow{\partial}^{-2}), \end{aligned}$$

where we have used the expansion (4.3.6). Parametrize $w_1^m = \begin{pmatrix} a^m & q^m/2 \\ -r^m/2 & b^m \end{pmatrix}$. Then we find from (4.5.9)

$$\delta^m = [h, w_2^m] + \begin{pmatrix} -\frac{1}{2}q^m r^m - 2\partial a^m & \partial q^m - a^m q^m \\ \partial r^m - b^m r^m & \frac{1}{2}q^m r^m + 2\partial b^m \end{pmatrix}. \tag{4.5.18}$$

Combining this with the explicit form (4.5.13) of δ^m gives

$$[h, w_2^m] = \begin{pmatrix} 0 & -\frac{1}{2}(\bar\partial + \partial)q^m + a^m q^m \\ -\frac{1}{2}(\partial - \bar\partial)r^m + b^m r^m & 0 \end{pmatrix}. \tag{4.5.19}$$

Since $[E_{22}, w_2^m] = -\frac{1}{2}[h, w_2^m]$ we find for the lattice resolvent the expansion

$$\begin{aligned} U^m = {} & \overleftarrow{\partial}E_{22} + \begin{pmatrix} 0 & -\frac{1}{2}q^m \\ -\frac{1}{2}r^m & 0 \end{pmatrix} + \\ & + \overleftarrow{\partial}^{-1}\begin{pmatrix} 1 + \frac{1}{4}q^m r^m & \frac{1}{4}(\bar\partial - \partial)q^m \\ \frac{1}{4}(\partial - \bar\partial)r^m & -\partial b^m - \frac{1}{4}q^m r^m \end{pmatrix} + \mathcal{O}(\overleftarrow{\partial}^{-2}). \end{aligned} \tag{4.5.20}$$

A simple calculation shows then that U_+^m has to have the form (4.5.16). $\square$

A straightforward but tedious calculation now shows that

$$(4.5.21) \quad -\bar\partial U_+^m (U_+^m)^{-1} + U_+^m (R_{\bar x}^m)_+ (U_+^m)^{-1} = \\ = \begin{pmatrix} & \overleftarrow{\partial} & -4/r^m \\ \frac{r^m}{4}(\bar\partial^2 - \partial^2)\log(r^m) - \frac{1}{4} q^m (r^m)^2 & & -\overleftarrow{\partial} \end{pmatrix}.$$

Hence, using (4.4.16b), we find

$$(4.5.22a) \qquad q^{m+1} = -4/r^m,$$

$$(4.5.22b) \qquad r^{m+1} = \frac{r^m}{4}((\bar\partial^2 - \partial^2)\log(r^m) - q^m r^m).$$

With $u^m = \log(r^m)$ the equations (4.5.23) imply

$$(4.5.23) \qquad \frac{1}{4}(\bar\partial^2 - \partial^2)u^m = \exp(u^{m+1} - u^m) - \exp(u^m - u^{m-1}),$$

the 2-dimensional Toda lattice equation. From the diagonal components of the expression $-\partial_t U_+^m (U_+^m)^{-1} + U_+^m (R_t^m)_+ (U_+^m)^{-1}$ we find

$$(4.5.24) \qquad \begin{aligned} \lambda^{m+1} &= \lambda^m + \partial\bar\partial\log(r^m), \\ \mu^{m+1} &= \mu^m + \frac{1}{2}(\partial^2 + \bar\partial^2)\log(r^m), \end{aligned}$$

from which follows

$$(4.5.23c) \qquad Q^{m+1} = Q^m + \partial^2 \log(r^m).$$

Note that, for fixed j, $W^m(t)$ will be outside the H_j cell for all t except possibly for a finite set of integers m. In fact, as in [**BtK**, Lemma 6.1.b], one can prove that this set is, in the case of the polynomial Grassmannian we use, an uninterrupted sequence $m_{\min}, m_{\min}+1, m_{\min}+2, \ldots, m_{\max}$, so that we get from this construction solutions of the *finite* 2-dimensional Toda lattice equation. To obtain solutions of the infinite 2-dimensional Toda lattice one has to choose a suitable W belonging to a bigger space, e.g., the Segal-Wilson Grassmannian.

5. The calculation of the wave function from $\Lambda^{\infty/2}\mathbb{C}^\infty$.

5.1. Semi-infinite wedge space and Fermi-Bose correspondence. In this subsection we collect some results from [**tKvdL**] on the Fermi-Bose correspondence associated to a partition of n that we will need later, see also [**DJMK, Ka**].

The space $\mathbb{C}^\infty$ is a representation for the group Gl_∞, but not a highest weight representation. The fundamental highest weight representations of Gl_∞ are contained in the *semi-infinite wedge space* $\Lambda^{\infty/2}\mathbb{C}^\infty$, the collection of (finite) linear combinations of semi-infinite exterior products of elements of $\mathbb{C}^\infty$ of the form

$$v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots, \tag{5.1.1}$$

where $\{v_i\}_{i\leq 0}$ is an *admissible set*: a set $\{v_i\}_{i\leq 0}$ of elements of H is called admissible if there exist integers $k \in \mathbb{Z}, N \leq 0$ such that $v_i = \epsilon_{k+i}$ for all $i \leq N$. Note that we might as well assume in the construction of a semi-infinite wedge that the v_i belong to $\mathbb{C}^\infty$. (This is the reason for the notation $\Lambda^{\infty/2}\mathbb{C}^\infty$ instead of $\Lambda^{\frac{\infty}{2}}H$.) For later constructions, however, we prefer to allow elements of H as members of an admissible set. The integer k is called the *charge* of the wedge $v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots$ and the semi-infinite wedge space decomposes in a direct sum of subspaces of fixed charge:

$$\Lambda^{\infty/2}\mathbb{C}^\infty = \bigoplus_{k\in\mathbb{Z}} \Lambda_k^{\frac{\infty}{2}}\mathbb{C}^\infty. \tag{5.1.2}$$

If in an admissible set $\{v_i\}_{i\leq 0}$ all the v_i are of the form $v_i = \epsilon_{j_i}$ then the corresponding element

$$\epsilon_{j_0} \wedge \epsilon_{j_{-1}} \wedge \epsilon_{j_{-2}} \cdots \tag{5.1.3}$$

of $\Lambda^{\infty/2}\mathbb{C}^\infty$ is called an *elementary wedge.*

The action of $a \in gl_\infty$, $g \in Gl_\infty$ is given as usual by

(5.1.4)
$$\begin{aligned}
\rho(a)\cdot(v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots) &= (a\cdot v_0) \wedge v_{-1} \wedge v_{-2} \wedge \dots + \\
&\qquad v_0 \wedge (a \cdot v_{-1}) \wedge v_{-2} \wedge \dots + \dots \\
\rho(g)\cdot(v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots) &= (g\cdot v_0) \wedge (g\cdot v_{-1}) \wedge (g \cdot v_{-2}) \wedge \dots .
\end{aligned}$$

This does not extend to representations of the algebra gl_∞^{lf} and the group Gl_∞^{lf}, because of the infinities that occur (e.g., in the naive action of $\lambda \sum_{i\in\mathbb{Z}} \mathcal{E}_{ii} \in Gl_\infty^{lf}$ on $\Lambda^{\infty/2}\mathbb{C}^\infty$, for $|\lambda| > 1$). However these infinities are relatively innocuous and after "renormalization" one obtains a projective representation $\hat{\rho}$ of gl_∞^{lf} and Gl_∞^{lf}, or, what is the same thing, a representation of a central extension of the algebra and group. In terms of the algebra generators it reads

$$\hat{\rho}(\mathcal{E}_{ij}) = \rho(\mathcal{E}_{ij}) - \delta_{ij}\theta_{0i}, \tag{5.1.5}$$

where $\theta_{0i} = 0$ if $i > 0$ and 1 otherwise. This defines a non trivial central extension $\hat{gl}^{lf}_{\infty} = gl^{lf}_{\infty} \oplus \mathbb{C}$ of the Lie algebra gl^{lf}_{∞}. The corresponding group will be described in the next section.

Let the *fermion operators* $\psi(i), \psi^*(i)$ be the linear operators on $\Lambda^{\infty/2}\mathbb{C}^{\infty}$ that act on elementary wedges by

(5.1.6)

$$\psi(i) \cdot (\epsilon_{i_0} \wedge \epsilon_{i_1} \wedge \epsilon_{i_2} \wedge \dots) = \epsilon_i \wedge \epsilon_{i_0} \wedge \epsilon_{i_1} \wedge \epsilon_{i_2} \wedge \dots$$

$$\psi^*(i) \cdot (\epsilon_{i_0} \wedge \epsilon_{i_1} \wedge \epsilon_{i_2} \wedge \dots) = \sum_{k=0}^{\infty} (-1)^k \delta_{i i_k} \epsilon_{i_0} \wedge \epsilon_{i_1} \wedge \epsilon_{i_2} \wedge \dots \wedge \hat{\epsilon}_{i_k} \wedge \dots$$

Now fix an integer n and a partition $\underline{n}$. Then we have, as in Section 3.1, a relabeling of the basis vectors and a corresponding relabeling of the fermion operators: the k-component fermions $\psi_a(i)$, $\psi^*_a(i)$, $i \in \mathbb{Z}, 1 \le a \le k$ correspond to $\psi(j), \psi^*(j)$ whenever $\epsilon_a(i)$ corresponds to ϵ_j. The anti-commutation relations for the relabeled fermions are

$$\{\psi_b(i), \psi_c(\ell)\} = \{\psi^*_b(i), \psi^*_c(\ell)\} = 0, \qquad (5.1.7)$$
$$\{\psi_b(i), \psi^*_c(\ell)\} = \delta_{bc}\delta_{i\ell}.$$

The action of $\mathcal{E}^{\ell m}_{bc}$ on semi-infinite wedge space is given by normal ordered fermion bilinears: if $v \in \Lambda^{\infty/2}\mathbb{C}^{\infty}$ then $\hat{\rho}(\mathcal{E}^{\ell m}_{bc})v =: \psi_b(\ell)\psi^*_c(m) : v$, where the *normal ordering* of fermions is defined by

$$: \psi_b(\ell)\psi^*_c(m) : \stackrel{\text{def}}{=} \begin{cases} \psi_b(\ell)\psi^*_c(m) & \text{if } m > 0 \\ -\psi^*_c(m)\psi_b(\ell) & \text{if } m \le 0 \end{cases}. \qquad (5.1.8)$$

Recall the operators $(\Lambda^{\pm}_a)^j, j \neq 0$ of (3.2.1). They generate, via the representation $\hat{\rho}$, on $\Lambda^{\infty/2}\mathbb{C}^{\infty}$ a Heisenberg algebra, the generators of which we will denote by

(5.1.9)

$$\alpha_a(\pm j) := \hat{\rho}((\Lambda^{\pm}_a)^j) = \sum_{\ell \in \mathbb{Z}} : \psi_a(\ell)\psi^*_a(\ell \pm j) :, \quad j > 0, a = 1, \dots, k.$$

It is natural to introduce the operator

$$\alpha_a(0) := \sum_{\ell \in \mathbb{Z}} : \psi_a(\ell)\psi^*_a(\ell) :, \quad a = 1, \dots, k. \qquad (5.1.10)$$

One checks that

$$[\alpha_a(j), \alpha_b(\ell)] = j\delta_{j+\ell,0}\delta_{ab}, \quad j, \ell \in \mathbb{Z}, 1 \le a, b \le k. \qquad (5.1.11)$$

Furthermore we need linear invertible operators $\hat{Q}_a$, $1 \le a \le k$, on $\Lambda^{\infty/2}\mathbb{C}^\infty$ that satisfy the following defining relations:

$$\begin{aligned} \hat{Q}_a \cdot \mathbf{v}_0 &= \psi_a(1) \cdot \mathbf{v}_0, \\ \hat{Q}_a \psi_b(i) \hat{Q}_a^{-1} &= \psi_b(i + \delta_{ab}), \\ \hat{Q}_a \psi_b^*(i) \hat{Q}_a^{-1} &= \psi_b^*(i + \delta_{ab}). \end{aligned} \tag{5.1.12}$$

In (5.1.12) $\mathbf{v}_0$ is a distinguished element of $\Lambda^{\infty/2}\mathbb{C}^\infty$, the 0^{th} vacuum. In general one defines the j^{th} *vacuum* by:

$$\mathbf{v}_j = \epsilon_j \wedge \epsilon_{j-1} \wedge \epsilon_{j-2} \wedge \ldots . \tag{5.1.13}$$

We have, for r_a the numbers (3.2.1) associated to j,

$$\alpha_a(0) \cdot \mathbf{v}_j = r_a \mathbf{v}_j, \quad a = 1, \ldots, k. \tag{5.1.14}$$

The operators $\hat{Q}_a$ are called *fermionic translation operators.* They satisfy the following relations ([**tKvdL**]):

$$\{\hat{Q}_a, \hat{Q}_b\} = 0, \qquad \text{for } a \neq b. \tag{5.1.15}$$

Introduce next fermionic fields, formal power series with operator coefficients:

$$\begin{aligned} \psi_a(z) &:= \sum_{\ell \in \mathbb{Z}} \psi_a(\ell) z^\ell, \\ \psi_a^*(z) &:= \sum_{\ell \in \mathbb{Z}} \psi_a^*(\ell) z^{-\ell}, \quad a = 1, \ldots, k. \end{aligned} \tag{5.1.16}$$

Now we can express the fermion fields completely in terms of the Heisenberg generators $\alpha_a(k)$ and the fermionic translation operators $\hat{Q}_a$. The *parity operator* on $\Lambda^{\infty/2}\mathbb{C}^\infty$ is defined by

$$\chi = (-1)^{\sum_b \alpha_b(0)}. \tag{5.1.17}$$

The parity operator acts as $(-1)^k$ on the charge k sector $\Lambda_k^{\frac{\infty}{2}}\mathbb{C}^\infty$ of $\Lambda^{\infty/2}\mathbb{C}^\infty$. It follows from Theorem 1.3 of [**tKvdL**] that we have the following *bosonization formulae:*

(5.1.18)

$$\psi_a(z) = \chi \hat{Q}_a (-z)^{(\alpha_a(0)+1)} \exp\left(-\sum_{\ell<0} \frac{1}{\ell} z^{-\ell} \alpha_a(\ell)\right) \exp\left(-\sum_{\ell>0} \frac{1}{\ell} z^{-\ell} \alpha_a(\ell)\right),$$

$$\psi_a^*(z) = \hat{Q}_a^{-1} \chi (-z)^{-\alpha_a(0)} \exp\left(\sum_{\ell<0} \frac{1}{\ell} z^{-\ell} \alpha_a(\ell)\right) \exp\left(\sum_{\ell>0} \frac{1}{\ell} z^{-\ell} \alpha_a(\ell)\right),$$

$$a = 1, \ldots, k.$$

So the only way the various bosonizations of k-component fermions are distinguished is through the zero-modes $\alpha_a(0)$, which, by (5.1.14), are able to detect which partition we are using. One can use this "bosonized" form for the fermions to express the whole representation of the Lie algebra gl_∞ in terms of the Heisenberg operators $\alpha_a(i)$ and the fermionic translation operators, [**tKvdL**]. We won't need this in the sequel.

5.2. Central extension of Gl_∞^{lf} and group action on $\Lambda^{\infty/2}\mathbb{C}^\infty$. We have now defined an action on $\Lambda^{\infty/2}\mathbb{C}^\infty$ of a central extension of gl_∞^{lf}, described by an exact sequence

$$0 \to \mathbb{C} \to \hat{gl}_\infty^{lf} \xrightarrow{\pi} gl_\infty^{lf} \to 0. \tag{5.2.1}$$

In this subsection we sketch the construction of the corresponding group and its action on $\Lambda^{\infty/2}\mathbb{C}^\infty$, following the approach of [**SeW**, **PrS**], to which we refer for more details.

The reason that the usual action of $g \in Gl_\infty^{lf}$ on $v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots$ by $gv_0 \wedge gv_{-1} \wedge gv_{-2} \wedge \dots$ does not work is that the set $\{gv_i\}_{i\le 0}$ is, in general, not admissible, even if $\{v_i\}_{i\le 0}$ is. Now an admissible set $\{v_i\}_{i\le 0}$ in $\mathbb{C}^\infty$, if the v_i are linearly independent, is the basis for the finite order part W^{fin} for some (unique) $W \in Gr$. In the same way $\{gv_i\}_{i\le 0}$ is a basis for $(gW)^{\text{fin}}$, (not necessarily admissible). The idea is now to replace the possibly non admissible basis $\{g.v_i\}$ by an admissible one, say $\{w_i\}$, and to replace the wedge $v_0 \wedge v_{-1} \wedge \dots$ by $w_0 \wedge w_{-1} \wedge \dots$. Because of the ambiguity in the choice of this admissible basis, we obtain in this way a projective representation of Gl_∞^{lf}, or, equivalently, a representation of a central extension of this group. Below we will make this precise.

Denote the group of invertible matrices of size $\mathbb{Z}_{\le 0} \times \mathbb{Z}_{\le 0}$ with a finite number of nonzero lower triangular diagonals by:

(5.2.2)
$$Gl(H_0)^{\text{lf}} = \left\{ a = \sum_{i,j\le 0} a_{ij}\mathcal{E}_{ij} \mid a_{ij} = 0 \text{ if } i-j \gg 0, a \text{ invertible} \right\}.$$

A subgroup of $Gl(H_0)^{\text{lf}}$ is

$$\mathcal{T} = \{t \in Gl(H_0)^{\text{lf}} \mid t = 1 + \text{finr}\}. \tag{5.2.3}$$

Here and in the sequel "finr" denotes a finite rank matrix. A finite rank matrix in $Gl(H_0)^{\text{lf}}$ is one with only a finite number of nonzero columns.

Every $W \in Gr$ has an admissible basis for W^{fin}, for instance the canonical basis of (2.2.4). It will be convenient to think of an admissible set as a matrix $\underline{v} = (\dots v_{-2}v_{-1}v_0)$ of size $\mathbb{Z} \times \mathbb{Z}_{i\le 0}$, with the v_i as columns. On such matrices we have a right action of $Gl(H_0)^{\mathrm{lf}}$. It is easy to see that if $\underline{v}$ and $\underline{v}'$ are admissible bases of W^{fin} then $\underline{v}' = \underline{v} \cdot t$ for $t \in \mathcal{T}$. We will use frequently that $\underline{v}$ and $\underline{v}'$ are bases for the finite order part W^{fin} of the same W iff the corresponding wedges $\mathbf{v}$ and $\mathbf{v}'$ differ by a constant. Moreover if $\tilde{\underline{v}}$ is any basis for W^{fin} then we can find an $a \in Gl(H_0)^{\mathrm{lf}}$ such that $\tilde{\underline{v}} \cdot a^{-1}$ is the canonical basis. In particular if, for admissible $\underline{v}$, $g \cdot \underline{v} = \{gv_i\}_{i\le 0}$ is not admissible, then we can find an $a \in Gl(H_0)^{\mathrm{lf}}$ such that $g \cdot \underline{v} \cdot a^{-1}$ is admissible.

This leads to the introduction of

$$\mathcal{E} = \{(g,a) \mid g \in Gl_\infty^{0,lf}, a \in Gl(H_0)^{\mathrm{lf}}, g_{--} - a = \mathrm{finr}\}. \tag{5.2.4}$$

Here we write $g \in Gl_\infty^{lf}$ in block form with respect to the decomposition $H = H_0 \oplus H_0^\perp$:

$$g = \begin{pmatrix} g_{--} & g_{-+} \\ g_{+-} & g_{++} \end{pmatrix}. \tag{5.2.5}$$

The subgroup of $g \in Gl_\infty^{lf}$ such that $g_{--} : H_0 \to H_0$ is a Fredholm operator of index 0, is denoted by $Gl_\infty^{0,lf}$ and is the called the identity component of Gl_∞^{lf}. One checks that $(g_1,a_1)\cdot(g_2,a_2) = (g_1g_2, a_1a_2)$ gives $\mathcal{E}$ a group structure. Suppose $(g,a) \in Gl_\infty^{0,lf} \times Gl(H_0)^{\mathrm{lf}}$ and let $\underline{v}$ be an admissible basis corresponding to a wedge of charge 0. Then $g\underline{v}a^{-1}$ is admissible iff $(g,a) \in \mathcal{E}$.

The inclusion $t \in \mathcal{T} \mapsto (1,t) \in \mathcal{E}$ gives us an exact sequence:

$$1 \to \mathcal{T} \to \mathcal{E} \to Gl_\infty^{0,lf} \to 1. \tag{5.2.6}$$

Note that every $t \in \mathcal{T}$ has a determinant.

Lemma 5.2.1. *Let*

$$\begin{aligned} \mathcal{T}_1 &= \{t \in \mathcal{T} \mid \det(t) = 1\}, \\ \mathcal{E}_1 &= \{(1,t) \in \mathcal{E} \mid t \in \mathcal{T}_1\}. \end{aligned} \tag{5.2.7}$$

Then $\mathcal{T}_1$ is normal in $Gl(H_0)^{lf}$ and $\mathcal{E}_1$ is normal in $\mathcal{E}$.

Proof. We must check that for all $a \in Gl(H_0)^{\mathrm{lf}}$ ata^{-1} belongs to $\mathcal{T}_1$ if t does. Since $t = 1 + f$, $f = \mathrm{finr}$ we have $ata^{-1} = 1 + afa^{-1} \in \mathcal{T}$, so it remains to check that $\det(ata^{-1}) = 1$. Since a might not have a determinant we need a little argument for this. It runs as follows: write $t = 1 + f$ as above and choose a basis $\{v_1, v_2, \dots\}$ of H_0 such that for $k > k_0$ all basis elements v_k belong to the kernel of f and put $V = \oplus_{i=1}^k v_k$. Then V is a finite dimensional subspace of H_0 and we get by restriction and projection a map

$t|_V : V \to V$. One checks that $\det(t|V)$ is independent of the choices made here, so we can define $\det(t) = \det(t|_V)$. Define then $V' = a \cdot V$, $t' = ata^{-1}$, so that $\det(t') = \det(t'|_{V'})$. Since $a|_V : V \to V'$ is an isomorphism of finite dimensional vector spaces we see that $\det(t) = \det(t')$, as we wanted to prove.

For the second part we must check, for all (g, a) in $\mathcal{E}$ and $(1, t)$ in $\mathcal{E}_1$, that $(g, a) \cdot (1, t) \cdot (g^{-1}, a^{-1}) = (1, ata^{-1})$ belongs to $\mathcal{E}_1$, but this follows from the first part. $\square$

Taking quotients we obtain from (5.2.6) the exact sequence:

$$1 \to \mathcal{T}/\mathcal{T}_1 \simeq \mathbb{C}^* \to \mathcal{E}/\mathcal{E}_1 \to Gl_\infty^{0,lf} \to 1. \tag{5.2.8}$$

This gives us a central extension $\hat{Gl}_\infty^{0,lf} := \mathcal{E}/\mathcal{E}_1$ of the identity component of Gl_∞^{lf}.

To get a central extension of the whole group Gl_∞^{lf} we need the shift automorphism given by

$$\sigma(g) = \Lambda \cdot g \cdot \Lambda^{-1}. \tag{5.2.9}$$

The semi-direct product $Gl_\infty^{0,lf} \ltimes_\sigma \mathbb{Z}$ (with multiplication $(g, k) \cdot (h, l) = (g\sigma^k(h), k + l))$ is isomorphic to Gl_∞^{lf} by the map $(g, k) \mapsto g \cdot \Lambda^k$. The shift automorphism σ lifts to an automorphism $\hat{\sigma}$ of $\mathcal{E}/\mathcal{E}_1$ as follows: an element of $\mathcal{E}/\mathcal{E}_1$ can be written as $(g, a\mathcal{T}_1)$, with $g \in Gl_\infty^{0,lf}$, $a \in GL(H_0)^{\text{lf}}$, such that $g_{--} - a = \text{finr}$. Then let $\hat{\sigma}((g, a\mathcal{T}_1)) := (\sigma(g), \bar{\sigma}(a)\mathcal{T}_1)$ where $\bar{\sigma}(a)$ is obtained by adding to a a row and column of zeroes and a diagonal 1:

$$\bar{\sigma}(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in Gl(H_0)^{\text{lf}}. \tag{5.2.10}$$

One checks that this is independent of the representation of the coset $a\mathcal{T}_1$. To see that this indeed defines an automorphism note that the fiber of the projection $\mathcal{E}/\mathcal{E}_1 \to Gl_\infty^{0,lf}$ over g is $\mathbb{C}^*$ and that $\hat{\sigma}$ defines a homomorphism from the fiber over g to the fiber over $\sigma(g)$ with kernel 1. Therefore this homomorphism has to be an isomorphism and hence $\hat{\sigma}$ is an automorphism.

Next one defines $\hat{Gl}_\infty^{lf} := \hat{Gl}_\infty^{0,lf} \ltimes_{\hat{\sigma}} \mathbb{Z}$ and we get an exact sequence

$$1 \to \mathbb{C}^* \to \hat{Gl}_\infty^{lf} \to Gl_\infty^{lf} \to 1, \tag{5.2.11}$$

and this is the central extension of Gl_∞^{lf} we were looking for. One checks that this exact sequence corresponds to the Lie algebra extension (5.2.1).

Next we have to define an action of $\hat{Gl}_\infty^{lf}$ on $\Lambda^{\infty/2}\mathbb{C}^\infty$. It suffices to establish an action on wedges $v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots$, since as soon that is known we can extend to all of $\Lambda^{\infty/2}\mathbb{C}^\infty$ by linearity. We first discuss wedges in some more detail.

To a non zero wedge $\mathbf{v} = v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots$ we can associate (non uniquely) a linear independent admissible set $\{v_i\}_{i\leq 0}$. If $\mathbf{v}$ is a wedge of charge k then the corresponding matrix $\underline{v}$ is of the form

$$\underline{v} = \Lambda^{-k}\begin{pmatrix} v_- \\ v_+ \end{pmatrix}, \quad v_- = 1 + \text{finr}, \quad v_+ = \text{finr}. \tag{5.2.12}$$

Here the subscripts $\pm$ refer to the decomposition of a $\mathbb{Z}\times\mathbb{Z}_{\leq 0}$ matrix into blocks induced by the decomposition $H = H_0 \oplus H_0^\perp$, cf. (5.2.5). Denote by $\mathcal{A}$ the collection of all matrices of the form (5.2.12) with linearly independent columns. On $\mathcal{A}$ we have an action from the right of the group $\mathcal{T}_1$ of (5.2.7). We can identify, in a bijective manner, a non trivial wedge $\mathbf{v}$ with an orbit of $\mathcal{T}_1$ in $\mathcal{A}$ via $\mathbf{v} \leftrightarrow \underline{v}\mathcal{T}_1$. In order to define an action of $\hat{Gl}^{lf}_\infty$ on $\Lambda^{\infty/2}\mathbb{C}^\infty$ it therefore suffices to define an action on the orbit space $\mathcal{A}/\mathcal{T}_1$.

Let $\hat{Q} \in \hat{Gl}^{lf}_\infty = ((e, e\mathcal{T}_1), 1)$ be the canonical lift of the shift matrix $\Lambda^{-1} = \sum \mathcal{E}_{i+1i} \in Gl^{lf}_\infty$. Just as any element of Gl^{lf}_∞ can be written as $g\cdot\Lambda^{-k}$, with $g \in Gl^{0,lf}_\infty$ we can write an element of $\hat{Gl}^{lf}_\infty$ as $(g, a\mathcal{T}_1)\cdot\hat{Q}^k$, $(g, a\mathcal{T}_1) \in \hat{Gl}^{0,lf}_\infty$. The action of $\hat{Q}$ reads in terms of elements of $\mathcal{A}/\mathcal{T}_1$:

$$\hat{Q}^j \cdot \left\{\Lambda^{-k}\begin{pmatrix} v_- \\ v_+ \end{pmatrix}\mathcal{T}_1\right\} = \Lambda^{-(k+j)}\begin{pmatrix} v_- \\ v_+ \end{pmatrix}\mathcal{T}_1. \tag{5.2.13}$$

Next we define the action of $(g', a'\mathcal{T}_1) \in \hat{Gl}^{0,lf}_\infty$. We should have

$$(g', a'\mathcal{T}_1)\cdot\left\{\Lambda^{-k}\begin{pmatrix} v_- \\ v_+ \end{pmatrix}\mathcal{T}_1\right\} = \hat{Q}^k\cdot\left(\hat{Q}^{-k}(g', a'\mathcal{T}_1)\hat{Q}^k\right)\cdot\begin{pmatrix} v_- \\ v_+ \end{pmatrix}\mathcal{T}_1. \tag{5.2.14}$$

Now note that $\hat{Q}^{-k}(g', a'\mathcal{T}_1)\hat{Q}^k$ is an element of $\hat{Gl}^{0,lf}_\infty$, so it is of the form $(g, a\mathcal{T}_1)$. To complete the definition of the $\hat{Gl}^{lf}_\infty$ action we put

$$(g, a\mathcal{T}_1)\cdot\begin{pmatrix} v_- \\ v_+ \end{pmatrix}\mathcal{T}_1 = g\cdot\begin{pmatrix} v_- \\ v_+ \end{pmatrix}\cdot a^{-1}\mathcal{T}_1. \tag{5.2.15}$$

We leave to the reader the easy verification that these definitions make sense, i.e., are independent of the choice of representatives for the coset $a\mathcal{T}_1$ and for the orbit $\begin{pmatrix} v_- \\ v_+ \end{pmatrix}\mathcal{T}_1$, and that the right hand side of (5.2.15) indeed belongs to $\mathcal{A}/\mathcal{T}_1$.

5.3. The projection $\pi : \hat{Gl}^{lf}_{\infty} \to Gl^{lf}_{\infty}$ and the translation group. In the previous subsection we have constructed the central extension $\hat{Gl}^{lf}_{\infty}$ of Gl^{lf}_{∞}. With the help of the fermions that act on $\Lambda^{\infty/2}\mathbb{C}^{\infty}$ we will describe the projection $\pi : \hat{Gl}^{lf}_{\infty} \to Gl^{lf}_{\infty}$ very explicitly. Using this we show that the fermionic translation operator $\hat{Q}_a$ (5.1.10) that occurs in the bosonization formula (5.1.16) belongs to the group $\hat{Gl}^{lf}_{\infty}$ and projects to the shift operator Q_a of (3.2.4). The fermionic translation operators $\hat{Q}_a$ are the ingredients for the lift of the translation group $T^{\underline{n}} \subset Gl^{lf}_{\infty}$ to the central extension $\hat{Gl}^{lf}_{\infty}$.

Proposition 5.3.1. *Let $g = \sum g_{ij}\mathcal{E}_{ij} \in Gl^{lf}_{\infty}$ and let $\hat{g}$ be any lift of g in $\hat{Gl}^{lf}_{\infty}$. Then, for all $i \in \mathbb{Z}$, we have, as operators on $\Lambda^{\infty/2}\mathbb{C}^{\infty}$,*

$$\hat{g} \cdot \psi(i) \cdot \hat{g}^{-1} = \sum_j g_{ji}\psi(j).$$

Proof. We write as before $(g, a\mathcal{T}_1) \cdot \hat{Q}^k$ for an element of $\hat{Gl}^{lf}_{\infty}$, with $\hat{Q}$ the canonical lift of the shift matrix Λ^{-1}, see (5.2.13). It is clear that

$$\hat{Q} \cdot \psi(i) \cdot \hat{Q}^{-1} = \psi(i+1), \quad i \in \mathbb{Z}. \tag{5.3.1}$$

Hence the proposition holds for elements of $\hat{Gl}^{lf}_{\infty}$ of the form $\hat{g} = \hat{Q}^k$, $k \in \mathbb{Z}$. It remains to prove the theorem for $(g, a\mathcal{T}_1) \in \hat{Gl}^{0,lf}_{\infty}$.

If we represent a wedge $\mathbf{v} \in \Lambda^{\infty/2}\mathbb{C}^{\infty}$ by an orbit $\underline{v}\mathcal{T}_1$ then the action (5.1.6) of $\psi(i)$ amounts to adding the vector ϵ_i on the right to $\underline{v}$:

$$\psi(i) \cdot \underline{v}\mathcal{T}_1 = (\underline{v} \mid \epsilon_i)\mathcal{T}_1. \tag{5.3.2}$$

This is independent of the choice of $\underline{v}$: If $t \in \mathcal{T}_1$ then $(\underline{v}t \mid \epsilon_i)\mathcal{T}_1 = (\underline{v} \mid \epsilon_i)\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\mathcal{T}_1 = (\underline{v} \mid \epsilon_i)\mathcal{T}_1$. The proposition then states that the endomorphism $\hat{g} \cdot \psi(i) \cdot \hat{g}^{-1}$ acts on an orbit $\underline{v}\mathcal{T}_1$ by adding on the right the column vector $g \cdot \epsilon_i$. This is a small calculation:

(5.3.3)
$$\begin{aligned}
(g, a\mathcal{T}_1) \cdot \psi(i) \cdot (g^{-1}, a^{-1}\mathcal{T}_1) \cdot \underline{v}\mathcal{T}_1 &= (g, a\mathcal{T}_1) \cdot \psi(i) \cdot (g^{-1}\underline{v}a)\mathcal{T}_1 \\
&= (g, a\mathcal{T}_1) \cdot (g^{-1}\underline{v}a \mid \epsilon_i)\mathcal{T}_1 \\
&= \hat{Q} \cdot \left(\hat{Q}^{-1}(g, a\mathcal{T}_1)\hat{Q}\right) \cdot \hat{Q}^{-1} \cdot (g^{-1}\underline{v}a \mid \epsilon_i)\mathcal{T}_1 \\
&= \hat{Q} \cdot (\Lambda g \Lambda^{-1}, \bar{\sigma}(a)\mathcal{T}_1) \cdot (\Lambda g^{-1}\underline{v}a \mid \Lambda\epsilon_i)\mathcal{T}_1 \\
&= \hat{Q} \cdot (\Lambda\underline{v}a \mid \Lambda g \cdot \epsilon_i)\bar{\sigma}(a)^{-1}\mathcal{T}_1 \\
&= \hat{Q} \cdot (\Lambda\underline{v} \mid \Lambda g \cdot \epsilon_i)\mathcal{T}_1 \\
&= (\underline{v} \mid g \cdot \epsilon_i)\mathcal{T}_1.
\end{aligned}$$

$\square$

On $\Lambda^{\infty/2}\mathbb{C}^\infty$ there exists a unique positive definite Hermitian form $\langle\,.\mid.\,\rangle$ such that the elementary wedges (5.1.3) are orthonormal. With respect to this form the adjoint of $\psi(i)$ is $\psi^*(i)$ and the adjoint of $\hat\rho(\mathcal{E}_{ij})$ is $\hat\rho(\mathcal{E}_{ij}^\dagger) = \hat\rho(\mathcal{E}_{ji})$. To discuss the adjoint action of $\hat{Gl}_\infty^{lf}$ on $\psi^*(i)$ we need a concrete description of this Hermitian form.

Lemma 5.3.2. *Let* $\mathbf{v}$, $\mathbf{w}$ *be wedges in* $\Lambda^{\infty/2}\mathbb{C}^\infty$ *with charge* k, l *respectively. Let* $\underline{v}\mathcal{T}_1$ *and* $\underline{w}\mathcal{T}_1$ *be the corresponding orbits in* $\mathcal{A}/\mathcal{T}_1$. *Then*

$$\langle \mathbf{v} \mid \mathbf{w}\rangle = \delta_{kl}\det(\underline{v}^\dagger \underline{w}). \tag{5.3.4}$$

Proof. Let us define for the moment by $H(\mathbf{v},\mathbf{w}) = \delta_{kl}\det(\underline{v}^\dagger\underline{w})$ a Hermitian form on $\Lambda^{\infty/2}\mathbb{C}^\infty$. To show that H coincides with the standard Hermitian form we must check that the elementary wedges are orthonormal for H. To this end let, for $i \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$, $A_i(\lambda) = (\exp(\lambda\mathcal{E}_{ii}), 1\mathcal{T}_1) \in \hat{Gl}_\infty^{0,lf}$. Then we have

(5.3.5)
$$H(A_i(\lambda)\mathbf{v},\mathbf{w}) = \delta_{kl}\det((A_i(\lambda)\underline{v})^\dagger\underline{w}) = \delta_{kl}\det(\underline{v}^\dagger A_i(\lambda^*)\underline{w}) = H(\mathbf{v}, A_i(\lambda^*)\mathbf{w}).$$

For an elementary wedge $\mathbf{v} = \epsilon_{i_0}\wedge\epsilon_{i_{-1}}\wedge\epsilon_{i_{-2}}\wedge\dots$, with $i_0 > i_{-1} > i_{-2} > \dots$, we have

$$A_i(\lambda)\cdot\mathbf{v} = \begin{cases} e^\lambda\mathbf{v} & \text{if } i \in \{i_0, i_{-1}, i_{-2}, \dots\}, \\ \mathbf{v} & \text{if } i \notin \{i_0, i_{-1}, i_{-2}, \dots\}. \end{cases} \tag{5.3.6}$$

Let $\mathbf{w} = \epsilon_{j_0}\wedge\epsilon_{j_{-1}}\wedge\epsilon_{j_{-2}}\wedge\dots$, with $j_0 > j_{-1} > j_{-2} > \dots$, be another elementary wedge and let $i = \max(i_0, j_0)$. If $i_0 > j_0$ we have

$$\exp(\lambda)H(\mathbf{v},\mathbf{w}) = H(A_i(\lambda^*)\mathbf{v},\mathbf{w}) = H(\mathbf{v},A_i(\lambda)\mathbf{w}) = H(\mathbf{v},\mathbf{w}) \tag{5.3.7}$$

and hence $H(\mathbf{v},\mathbf{w}) = 0$, whereas when $i_0 < j_0$ we have

$$\exp(\lambda)H(\mathbf{v},\mathbf{w}) = H(\mathbf{v},A_i(\lambda)\mathbf{w}) = H(A_i(\lambda^*)\mathbf{v},\mathbf{w}) = H(\mathbf{v},\mathbf{w}) \tag{5.3.8}$$

and also in this case $H(\mathbf{v},\mathbf{w}) = 0$. Repeating this argument for all other pairs (i_{-1}, j_{-1}), (i_{-2}, j_{-2}), $\dots$, we find that $H(\mathbf{v},\mathbf{w}) \neq 0$ implies $\mathbf{v} = \mathbf{w}$. Finally let $\mathbf{v} = \epsilon_{i_0}\wedge\epsilon_{i_{-1}}\wedge\dots\wedge\epsilon_{i_{-N-1}}\wedge\epsilon_{-N-k}\wedge\epsilon_{-N-k-1}\wedge\dots$ be an elementary wedge of charge k, with the first $-N$ exterior factors different from those of the k^{th} vacuum. Then $H(\mathbf{v},\mathbf{v}) = \det(M^\dagger M)$, where M is the $\mathbb{Z}\times N$ matrix with columns $\epsilon_{i_{-N-1}}, \dots, \epsilon_{i_{-1}}, \epsilon_{i_0}$. It is clear that $M^\dagger M = 1$, proving that H makes the elementary wedges orthonormal. $\square$

If $g \in Gl_\infty^{lf}$ then the necessary and sufficient condition for the Hermitian conjugate matrix $g^\dagger$ also to belong to Gl_∞^{lf} is that g contains only a finite number of nonzero upper triangular diagonals. If this condition is satisfied we say that g has *finite width.* Suppose now that g has finite width and let $\hat{g} = (g, a\mathcal{T}_1)$ be a lift of g. Then $a \in Gl(H_0)^{\mathrm{lf}}$ automatically also has finite width (and $a^\dagger \in Gl(H_0)^{\mathrm{lf}}$). We calculate the adjoint of $\hat{g}$ with respect to the Hermitian form of $\Lambda^{\infty/2}\mathbb{C}^\infty$:

$$\begin{aligned}\langle \hat{g}\mathbf{v} \mid \mathbf{w}\rangle &= \det((g\underline{v}a^{-1})^\dagger \underline{w}) \\ &= \det((a^{-1})^\dagger \underline{v}^\dagger g^\dagger \underline{w}) \\ &= \det(\underline{v}^\dagger g^\dagger \underline{w}(a^{-1})^\dagger) \\ &= \langle \mathbf{v} \mid \hat{g}^\dagger \mathbf{w}\rangle,\end{aligned} \tag{5.3.9}$$

with

$$\hat{g}^\dagger := (g^\dagger, a^\dagger \mathcal{T}_1) \in \hat{Gl}_\infty^{0,lf}. \tag{5.3.10}$$

So we see that the adjoint of $\hat{g}$ with respect to the standard Hermitian form of $\Lambda^{\infty/2}\mathbb{C}^\infty$ is a lift of the Hermitian conjugate matrix $g^\dagger$.

Let now $\hat{g}$ be the lift of a finite width element. Then we have the following analogue of the Proposition 5.3.1

$$\hat{g} \cdot \psi^*(i) \cdot \hat{g}^{-1} = \sum_j (g^{-1})_{ij} \psi^*(j). \tag{5.3.11}$$

Indeed

$$\begin{aligned}\hat{g} \cdot \psi^*(i) \cdot \hat{g}^{-1} &= ((\hat{g}^{-1})^\dagger \cdot \psi(i) \cdot \hat{g}^\dagger)^\dagger \\ &= \left(\sum_j (g^{-1})^\dagger_{ji} \psi(j)\right)^\dagger \\ &= \sum_j (g^{-1})_{ij} \psi^*(j).\end{aligned} \tag{5.3.12}$$

We have derived here (5.3.10) under the assumption that g has finite width, the only situation in which we will use this formula. We leave it to the reader to prove this result for general g.

Recall now the elements Q_a of Gl_∞^{lf} defined in (3.2.4). These act on the relabeled basis introduced in Section 3.1 by

$$Q_a \cdot \epsilon_b(j) = \epsilon_b(j + \delta_{ab}), \quad a, b = 1, 2, \ldots, k, \quad j \in \mathbb{Z}. \tag{5.3.13}$$

Denote by $\hat{Q}'_a$ any lift of Q_a to $\hat{Gl}^{lf}_{\infty}$. Then by proposition 5.3.1 and equation (5.3.10) we find

$$\begin{aligned} \hat{Q}'_a\psi_b(i)(\hat{Q}'_a)^{-1} &= \psi_b(i+\delta_{ab}), \\ \hat{Q}'_a\psi^*_b(i)(\hat{Q}'_a)^{-1} &= \psi^*_b(i+\delta_{ab}). \end{aligned} \tag{5.3.14}$$

Comparing this with (5.1.10) we see that $\hat{Q}'_a$ has the same adjoint action on the fermions as the fermionic translation operators introduced in Section 5.1, that had a priori no relation to $\hat{Gl}^{lf}_{\infty}$. To compare the action of $\hat{Q}'_a$ and $\hat{Q}_a$ on $\Lambda^{\infty/2}\mathbb{C}^{\infty}$ we need a lemma.

Lemma 5.3.3. *Let $\hat{Q}'_a$ be a lift of Q_a. Then*

$$\hat{Q}'_a \cdot \mathbf{v}_0 = \nu\psi_a(1)\mathbf{v}_0, \quad \nu \in \mathbb{C}^*. \tag{5.3.15}$$

Proof. The 0^{th} vacuum $\mathbf{v}_0 = \epsilon_0 \wedge \epsilon_{-1} \wedge \epsilon_{-2} \wedge \dots$ corresponds to the orbit $\underline{v}_0\mathcal{T}_1$, where we can choose $\underline{v}_0 = (\dots\epsilon_{-2}\epsilon_{-1}\epsilon_0)$. The columns for this matrix $\underline{v}_0$ form an admissible basis for

$$H_0^{\text{fin}} = \bigoplus_{j\le 0}\mathbb{C}\epsilon_j = \bigoplus_{b=1}^{k}\bigoplus_{i\le 0}\mathbb{C}\epsilon_b(i). \tag{5.3.16}$$

Now if $\hat{Q}'_a$ is any lift of Q_a then the wedge $\hat{Q}'_a \cdot \mathbf{v}_0$ corresponds to the orbit $\underline{v}'\mathcal{T}_1$, with the columns of $\underline{v}'$ an admissible basis for $(Q_a \cdot H_0)^{\text{fin}}$. Since $Q_a = \sum_{b\neq a} 1_b + \Lambda_a^-$, and $\Lambda_a^-\epsilon_b(i) = \epsilon_b(i+\delta_{ab})$, we see that

$$(Q_a \cdot H_0)^{\text{fin}} = \mathbb{C}\epsilon_a(1) \oplus \bigoplus_{b=1}^{k}\bigoplus_{i\le 0}\mathbb{C}\epsilon_b(i). \tag{5.3.17}$$

Hence a particular simple admissible basis for $(Q_a \cdot H_0)^{\text{fin}}$ is $\{\epsilon_a(1)\} \cup \{\epsilon_b(i) \mid b = 1,2,\dots,k; i \le 0\}$, corresponding to the orbit $(\underline{v}_0 \mid \epsilon_a(1))\mathcal{T}_1$ and to the wedge $\psi_a(1)\mathbf{v}_0$. Any two admissible bases of $(Q_a \cdot H_0)^{\text{fin}}$ correspond to wedges that differ by at most a non zero factor. □

This lemma, combined with (5.3.13) and (5.1.10), shows that for any lift $\hat{Q}'_a$ of Q_a we have $\hat{Q}_a = \nu\hat{Q}'_a$ for some $\nu \in \mathbb{C}^*$. Hence

Proposition 5.3.4. *Let $\hat{Q}_a$ be the fermionic translation operators on $\Lambda^{\infty/2}\mathbb{C}^{\infty}$ defined in* (5.1.10). *Then $\hat{Q}_a \in \hat{Gl}^{lf}_{\infty}$ and $\hat{Q}_a$ projects to $Q_a \in Gl^{lf}_{\infty}$.*

Next we turn to the translation operators T_α responsible for the discrete time evolution on the Grassmannian. Since in the central extension the $\hat{Q}_a$

no longer commute (as the Q_a do) we have to make a choice here. We think of the root lattice $R = \oplus_{i=1}^{k-1}\mathbb{Z}\alpha_i$ as a sublattice of the rank k lattice $\oplus_{i=1}^{k}\mathbb{Z}\delta_i$ through the identification $\alpha_i = \delta_i - \delta_{i+1}$, $i = 1, 2, \ldots, k$. The δ_j are orthogonal for a symmetric bilinear form $(\cdot|\cdot)$. We can then write uniquely for any $\alpha \in R$

$$\alpha = \sum_{j=1}^{\ell} p_{i_j}\delta_{i_j} + \sum_{j=\ell+1}^{k} n_{i_j}\delta_{i_j}, \tag{5.3.18}$$

with the p_{i_j} non negative and the n_{i_j} strictly negative. We make then a choice of the ordering in the two sets of subscripts:

$$i_1 < i_2 < \ldots i_\ell; \quad i_{\ell+1} > \cdots > i_k. \tag{5.3.19}$$

Let $\hat{F} = \langle \hat{Q}_i, i = 1, 2, \ldots, k\rangle$ be the group generated by the fermionic translation operators. We then define, using the ordering (5.3.19), a map $R \to \hat{F}$, $\alpha \mapsto \hat{T}_\alpha$ by

$$\hat{T}_\alpha := \hat{Q}_{i_1}^{p_{i_1}}\hat{Q}_{i_2}^{p_{i_2}} \ldots \hat{Q}_{i_\ell}^{p_{i_\ell}}\hat{Q}_{i_{\ell+1}}^{n_{i_{\ell+1}}} \ldots \hat{Q}_{i_k}^{n_{i_k}}. \tag{5.3.20}$$

Clearly $\hat{T}_\alpha$ is then a lift of the translation operator T_α (defined in Subsection 3.2) and the group $\hat{T}^{\underline{n}}$ generated by the $\hat{T}_{\alpha_i}$ is a central extension of the Abelian group $T^{\underline{n}}$ by $\mathbb{Z}_2$, defined by a cocycle $\epsilon : R \times R \to \mathbb{Z}_2$ given by

$$\hat{T}_\alpha\hat{T}_\beta = \epsilon(\alpha, \beta)\hat{T}_{\alpha+\beta}. \tag{5.3.21}$$

So ϵ satisfies the cocycle properties $\epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma)$ and $\epsilon(\alpha, 0) = \epsilon(0, \alpha) = 1$. The choice (5.3.19) was made to ensure simple properties of the cocycle ϵ, as described in the following Lemma.

Lemma 5.3.5. *The cocycle ϵ satisfies for all $\alpha, \beta \in R$:*

1. $\epsilon(\alpha, -\alpha) = 1$,
2. $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha|\beta)}$,
3. $\epsilon(-\alpha, -\beta) = (-1)^{(\alpha|\beta)}\epsilon(\alpha, \beta)$.

The proof of the Lemma consists of simple computations using the fact that the $\hat{Q}_i$ satisfy the anti-commutation relations (5.1.15) and the fact that the sum of the coefficients p_{i_j}, n_{i_j} is zero. Note that part 1 of the Lemma implies the useful relation $\hat{T}_\alpha^{-1} = \hat{T}_{-\alpha}$ for all $\alpha \in R$.

5.4. Group decomposition and τ-functions. In this section we will study the Gauss decomposition in Gl_∞^{lf} by means of the action of the central extension $\hat{Gl}_\infty^{lf}$ on $\Lambda^{\infty/2}\mathbb{C}^\infty$. In fact we will need a slight refinement: if

$g \in Gl^{lf}_{\infty}$ with Gauss decomposition $g = g_- P g_+$ then we can write (uniquely) $g_+ = h \cdot g^s_+$, with h a diagonal matrix and g^s_+ a upper triangular matrix with ones on the diagonal to get a decomposition

$$g = g_- P h g^s_+. \tag{5.4.1}$$

Note that in the Gauss decomposition the factors g_-, g_+ are in general not uniquely defined. However, the element h here *is* uniquely determined; later on we will see that its entries are essentially the τ-functions of $W = gH_j$ (when W is in the H_j cell), to be defined in Definition 5.4.3 below. For any lift $\hat{g}$ of g we can find elements $\hat{g}_-$, $\hat{P}$, $\hat{h}$ and $\hat{g}^s_+$ projecting to the corresponding elements without a hat to get

$$\hat{g} = \hat{g}_- \hat{P} \hat{h} \hat{g}^s_+. \tag{5.4.2}$$

This decomposition is however not unique: each of the factors can be multiplied by a non zero complex constant as long as the product of these factors is one. The following lemma fixes a normalization.

Lemma 5.4.1. *Let $g = g_- P h g^s_+$ be a fixed factorization* (5.4.1). *Assume that $PH_j = H_j$. Then, for any lift $\hat{g}$ of g there is a unique factorization as in* (5.4.2) *such that*

1. $\hat{g}^s_+ \mathbf{v}_j = \mathbf{v}_j$.
2. $\hat{P}\mathbf{v}_j = \mathbf{v}_j$.
3. $\langle \mathbf{v}_j \mid \hat{g}_- \mathbf{v}_j \rangle = 1$.

Proof. For the first part note that for any lift $\hat{g}^s_+$ of g^s_+ the wedge $\hat{g}^s_+ \mathbf{v}_j$ corresponds to an admissible basis for $g^s_+ H_j = H_j$, since g^s_+ is upper triangular. As before we use that two wedges corresponding to two admissible bases of the same point of Gr differ by a non zero factor, so that $\hat{g}^s_+ \mathbf{v}_j = \nu \mathbf{v}_j$. By changing the lift we can make $\nu = 1$. The proof for the second part is the same. For the third part we note that g_- has finite width, and $g_-^\dagger$ is an upper triangular element of Gl^{lf}_{∞} that by (1) has a unique lift $\hat{g}_-^\dagger \in \hat{Gl}^{lf}_{\infty}$ such that $\hat{g}_-^\dagger \mathbf{v}_j = \mathbf{v}_j$. Then the adjoint of $\hat{g}_-^\dagger$ is the unique lift of g_- that satisfies (3). □

To decide whether or not $PH_j = H_j$ one can use the following Lemma.

Lemma 5.4.2. *Let $\hat{P}$ be any lift of a permutation matrix P. Then*

$$\langle \mathbf{v}_j \mid \hat{P}\mathbf{v}_j \rangle \neq 0 \iff PH_j = H_j.$$

Proof. The wedge $\hat{P}\mathbf{v}_j$ corresponds to some admissible basis for PH_j. Since P is a permutation matrix it is clear that PH_j^{fin} has a basis consisting of basis vectors $\{\epsilon_{\sigma_P(i)}\}_{i\leq j}$, where σ_P is the permutation associated to P. Let $S_P^j = \{\sigma_P(i) \mid i \leq j\}$ and order the elements of S_P^j as $i_0 > i_{-1} > i_{-2} > \dots$. Then $\{v_{i_m}\}_{m\leq 0}$ is an admissible basis of PH_j^{fin} and we have

$$\hat{P}\mathbf{v}_j = \nu\epsilon_{i_0} \wedge \epsilon_{i_{-1}} \wedge \epsilon_{i_{-2}} \wedge \dots, \quad \nu \in \mathbb{C}^*. \tag{5.4.3}$$

Since elementary wedges are orthogonal we have

(5.4.4)

$$\begin{aligned}\langle \mathbf{v}_j \mid \hat{P}\mathbf{v}_j\rangle \neq 0 &\iff \hat{P}\mathbf{v}_j = \nu\mathbf{v}_j, \quad \nu \in \mathbb{C}^* \\ &\iff \sigma_P \text{ restricts to a permutation of } \{i \mid i \leq j\} \\ &\iff PH_j = H_j.\end{aligned}$$

□

Consider next the following lift to $\hat{Gl}_\infty^{lf}$ of $w_0^{\underline{n}}(t,\alpha)$:

$$\hat{w}_0^{\underline{n}}(t,\alpha) = \exp\left(\sum_{i>0}\sum_{a=1}^{k} t_a^i \alpha_a(i)\right) \cdot \hat{T}_\alpha \in \hat{Gl}_\infty^{lf}. \tag{5.4.5}$$

Let $\hat{g} \in \hat{Gl}_\infty^{lf}$ be any lift of $g \in Gl_\infty^{lf}$. Then we define

$$\hat{g}(t,\alpha) = \hat{w}_0^{\underline{n}}(t,\alpha)^{-1}\hat{g}. \tag{5.4.6}$$

Definition 5.4.3. Fix an integer j, a positive integer n, a partition $\underline{n}$ of n and let $W = gH_j \in Gr$. Fix a lift $\hat{g}$ of g. The *τ-function of type $j, \underline{n}$ of W* is the function on $\Gamma^{\underline{n}}$ given as "vacuum expectation value":

$$\tau_W^{j,\underline{n}}(t,\alpha) := \langle v_j \mid \hat{g}(t,\alpha) \cdot v_j\rangle. \tag{5.4.7}$$

Note that if $\hat{g}'$ is another element of $\hat{Gl}_\infty^{lf}$ projecting to g the τ-function calculated with $\hat{g}'$ will differ from (5.4.7) by a constant. This will be irrelevant in the sequel. Also note that the τ-function is defined for all values of (t,α), also when $W(t,\alpha)$ does not belong to the big cell. In fact the τ-function of type $j, \underline{n}$ determines whether or not $W(t,\alpha)$ belongs to the H_j cell:

Proposition 5.4.4. *Let j, $\underline{n}$, W be as above. Let $\Gamma^{\underline{n}}$ be the group of evolutions of type $\underline{n}$ on Gr and let $\Gamma_W^{j,\underline{n}}$ be the subset of $(t,\alpha) \in \Gamma^{\underline{n}}$ such that $pr : W(t,\alpha) = g(t,\alpha)H_j \to H_j$ is an isomorphism. (See Section 2.2). Then for all $(t,\alpha) \in \Gamma^{\underline{n}}$ the following two statements are equivalent:*

1. $\tau_W^{j,\underline{n}}(t,\alpha) \neq 0$,

2. $(t,\alpha)\in\Gamma_W^{j,\underline{n}}$.

Proof. We use the Gauss decomposition of $\hat{g}(t,\alpha)\in\widehat{Gl_\infty^{lf}}$. We suppress for simplicity the reference to (t,α) and write $\hat{g}=\hat{g}_-\hat{P}\hat{h}\hat{g}_+^s$, with the normalization as in Lemma 5.4.1. Then, writing τ for $\tau_W^{j,\underline{n}}(t,\alpha)$:

$$\begin{aligned}\tau&=\langle \mathbf{v}_j\mid \hat{g}_-\cdot\hat{P}\cdot\hat{h}\cdot\hat{g}_+^s\cdot\mathbf{v}_j\rangle\\&=\langle(\hat{g}_-)^\dagger\cdot\mathbf{v}_j\mid\hat{P}\cdot\hat{h}\cdot\mathbf{v}_j\rangle\\&=\exp(\lambda_j)\langle\mathbf{v}_j\mid\hat{P}\cdot\mathbf{v}_j\rangle.\end{aligned}\tag{5.4.8}$$

Here we have used Lemma 5.4.1, and the fact that $\mathbf{v}_j$ is an eigenvector of elements of lifts of diagonal matrices: $\hat{h}\cdot\mathbf{v}_j=\exp(\lambda_j)\mathbf{v}_j$, for some λ_j (depending on (t,α)). Using Lemma 5.4.2 we see from (5.4.7) that $\tau_W^{j,\underline{n}}(t,\alpha)\neq 0$ iff $PH_j=H_j$. But $PH_j=H_j$ iff $W(t,\alpha)=g_-(t,\alpha)H_j$. Now $W(t,\alpha)=g_-(t,\alpha)H_j$, for $g_-(t,\alpha)$ strictly lower triangular, is equivalent to $W(t,\alpha)$ belonging to the H_j cell. □

Note that from the proof of this proposition it follows that for $(t,\alpha)\in\Gamma_W^{j,\underline{n}}$ we have, if $\hat{g}(t,\alpha)=\hat{g}_-\cdot\hat{P}\hat{h}\cdot\hat{g}_+^s$,

$$\hat{P}\hat{h}g_+^s\cdot\mathbf{v}_j=\tau_W^{j,\underline{n}}(t,\alpha)\mathbf{v}_j,\tag{5.4.9}$$

a fact that will be used in the calculation of the wave function in terms of τ-functions.

5.5. Relation between wave function and τ-function. In the construction of the wave function of a point W of the Grassmannian the lower triangular part g_-^j of the Gauss decomposition adapted to H_j occurs (see Definition 4.4.1). It is a rather trivial observation that one can calculate matrix elements of g_-^j by lifting it to the central extension $\widehat{Gl_\infty^{lf}}$ and using the semi-infinite wedge space (see the introduction for the finite dimensional situation). First note that there is a unique lift of g_-^j such that $\langle\mathbf{v}_j\mid\hat{g}_-^j\mathbf{v}_j\rangle=1$. The proof is as in Lemma 5.4.1. To find the coefficient of the matrix $\mathcal{E}_{pq}$ in g_-^j we observe that $\mathcal{E}_{pq}$ is represented on $\Lambda^{\infty/2}\mathbb{C}^\infty$ by the operator $\psi_p\psi_q^*$ (if $p\neq q$). The following Lemma is then, maybe, not too surprising.

Lemma 5.5.1. *Fix an integer j and a positive integer n and a partition $\underline{n}$ of n. Let $g\in Gl_\infty^{lf}$ and let $g(t,\alpha)$ be given by (4.1.4). Assume that $g(t,\alpha)$ belongs to the H_j cell, see section 2.2. Let $g_-^j(t,\alpha)\in Gl_\infty^{lf}$ be the minus component of the Gauss decomposition of $g(t,\alpha)$ of Lemma 2.2.2 adapted to j. Write $g_-^j(t,\alpha)=\sum g_{pq}(t,\alpha)\mathcal{E}_{pq}$. Let $\hat{g}(t,\alpha)$ be an arbitrary lift of*

$g(t,\alpha)$ and lift $g_-^j(t,\alpha)$ to the unique element $\hat{g}_-^j(t,\alpha) \in \hat{Gl}_\infty^{lf}$ that satisfies $\langle \mathbf{v}_j \mid \hat{g}_-^j(t,\alpha)\mathbf{v}_j\rangle = 1$. Then for all $p \in \mathbb{Z}, q \le j$ we have

(5.5.1.a)
$$g_{pq}(t,\alpha) = \langle (\psi_p\psi_q^*)\cdot \mathbf{v}_j \mid \hat{g}_-^j(t,\alpha)\cdot \mathbf{v}_j\rangle,$$

(5.5.1.b)
$$= \langle (\psi_p\psi_q^*)\cdot \mathbf{v}_j \mid \hat{g}(t,\alpha)\cdot \mathbf{v}_j\rangle / \tau_W^{j,\underline{n}}(t,\alpha),$$

where $\tau_W^{j,\underline{n}}(t,\alpha)$ is the τ-function of type $(j,\underline{n})$ of $W = gH_j$.

Proof. For simplicity we will mostly suppress the dependence on (t,α). If $p \neq q$, and $p \le j$, then, by definition of g_-^j, we have $g_{pq} = 0$, but also $(\psi_p\psi_q^*)\cdot \mathbf{v}_j = 0$, so in that case (5.5.1a) is true. In case $p = q \le j$ we have $g_{pp} = 1$, $(\psi_p\psi_p^*)\cdot \mathbf{v}_j = \mathbf{v}_j$ and (5.5.1a) holds because of the normalization of the lift $\hat{g}_-^j$.

Assume next that $p > j$. Then we use Lemma 5.3.1 to calculate the right hand side of (5.5.1a). The elementary wedge $(\psi_p\psi_q^*)\cdot \mathbf{v}_j = \epsilon_j \wedge \epsilon_{j-1} \wedge \cdots \wedge \epsilon_{q+1} \wedge \epsilon_p \wedge \epsilon_{q-1} \wedge \ldots$ corresponds to the orbit $\underline{v}_j^{p,q}\mathcal{T}_1$, where $\underline{v}_j^{p,q}$ is the $\mathbb{Z}\times\mathbb{Z}_{\le 0}$ matrix with columns $(\ldots \epsilon_{q-1}\epsilon_p\epsilon_{q+1}\ldots\epsilon_j)$. The wedge $\hat{g}_-^j\mathbf{v}_j$ corresponds to an orbit $\underline{v}\mathcal{T}_1$, where we can take $\underline{v}$ to be

$$\underline{v} = g_-^j\Lambda^{-j}\begin{pmatrix}1\\0\end{pmatrix}a^{-1}, \tag{5.5.2}$$

for a the identity matrix. With this choice $\underline{v}$ is a $\mathbb{Z}\times\mathbb{Z}_{\le 0}$ matrix of the form $(\ldots v_{-2}v_{-1}v_0)$ with for all $i \le 0$

$$v_i = g_-^j \cdot \epsilon_{j+i}. \tag{5.5.3}$$

Therefore

(5.5.4)
$$\begin{aligned}\langle (\psi_p\psi_q^*)\cdot \mathbf{v}_j \mid \hat{g}_-^j \cdot \mathbf{v}_j\rangle &= \det\left((\underline{v}_j^{p,q})^\dagger \underline{v}\right),\\ &= \det\begin{pmatrix} g_{pq} & g_{pq-1} & \cdots & g_{pj}\\ 0 & 1 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & 1\end{pmatrix} = g_{pq},\end{aligned}$$

proving (5.5.1.a) also in this case.

Now $g_-^j(t,\alpha)$ is a factor in the decomposition

$$g(t,\alpha) = g_-^j \cdot f^j \cdot P \cdot g_+ = g_-^j \cdot f^j \cdot P \cdot h \cdot g_+^s, \tag{5.5.5}$$

where f^j is defined in (2.2.4) and we have decomposed g_+ in a diagonal part h and an upper triangular part g_+^s with ones on the diagonal. Since $g_-^j \cdot f^j$ is the factor g_- in the Gauss decomposition and $(t,\alpha) \in \Gamma_W^{j,\underline{n}}$ we can apply Lemma 5.4.1 to get for any lift $\hat{g}(t,\alpha)$ a unique factorization

$$\hat{g}(t,\alpha) = \hat{g}_-^j \cdot \hat{f}^j \cdot \hat{P} \cdot \hat{h} \cdot \hat{g}_+^s, \tag{5.5.6}$$

such that

$$\hat{P}\mathbf{v}_j = \mathbf{v}_j, \quad \hat{g}_+^s \mathbf{v}_j = \mathbf{v}_j, \quad \langle \mathbf{v}_j \mid \hat{g}_-^j \cdot \hat{f}^j \mathbf{v}_j \rangle = 1. \tag{5.5.7}$$

Now $f^j H_j = H_j$ so we have $\hat{f}^j \mathbf{v}_j = \nu \mathbf{v}_j$, with $\nu \in \mathbb{C}^*$. But because of the normalization of $\hat{g}_-^j$ we must have in fact $\nu = 1$ and $\hat{f}^j \mathbf{v}_j = \mathbf{v}_j$. This implies that

$$\begin{aligned} \hat{g}_-^j \cdot \mathbf{v}_j &= \hat{g}(t,\alpha) \cdot (\hat{g}_+^s)^{-1} \cdot \hat{h}^{-1} \cdot \hat{P}^{-1} \cdot (\hat{f}^j)^{-1} \cdot \mathbf{v}_j \\ &= \hat{g}(t,\alpha) \cdot \mathbf{v}_j / \tau_W^{j,\underline{n}}(t,\alpha), \end{aligned} \tag{5.5.8}$$

using (5.4.8). $\square$

Note that only because we have assumed that g_-^j is the minus part of the Gauss decomposition adapted to j we are able to calculate its matrix elements g_{pq} in a simple way in the semi-infinite wedge representation: if we had used the ordinary Gauss decomposition of (2.2.2) the finite matrix in (5.5.4) would not have had zeroes below the diagonal.

In Lemma 5.5.1 we calculated the coefficients of g_-^j in an expansion in terms of the standard basis $\mathcal{E}_{pq}$. After the relabeling of type $\underline{n}$, so that $g_-^j = \sum g_{rs}^{bc} \mathcal{E}_{bc}^{rs}$, we find of course for the new coefficients an expression in terms of the relabeled fermions: if $\epsilon_c(s)$ corresponds to ϵ_q, with $q \le j$, then

$$g_{rs}^{bc} = \langle (\psi_b(r)\psi_c^*(s)) \cdot \mathbf{v}_j \mid \hat{g}(t,\alpha) \cdot \mathbf{v}_j \rangle / \tau_W^{j,\underline{n}}(t,\alpha), \tag{5.5.9}$$

for all $1 \le b \le k$, $r \in \mathbb{Z}$. In particular the columns of type c of g_-^j that occur in the definition of the wave function of type $j, \underline{n}$ have $s = r_c$ so that the coefficients that occur in the wave function are of the form $g_{rr_c}^{bc}$ for $c = 1, 2, \dots, k$. This will be used in the next Theorem.

Theorem 5.5.2. *Fix $g \in Gl_\infty^{lf}$, a lift of g to $\hat{g} \in \hat{Gl}_\infty^{lf}$, an integer j, a positive integer n and a partition $\underline{n}$ of n into k parts and let $W = gH_j \in Gr$. Let $w_W^{j,\underline{n}}(z;t,\alpha)$, $\tau_W^{j,\underline{n}}(t,\alpha)$ be the wave function and the τ-function associated to these data. Then, if we write $w_W^{j,\underline{n}}(z;t,\alpha) = \sum_{b,c=1}^k w_{bc} E_{bc}$, we have for all $(t,\alpha) \in \Gamma_W^{j,\underline{n}}$:*

$$w_{bc} = \langle \psi_c^*(r_c)\mathbf{v}_j \mid \hat{w}_0^{\underline{n}}(t,\alpha)^{-1} \psi_b^*(z) \hat{g} \rangle / \tau_W^{j,\underline{n}}(t,\alpha). \tag{5.5.10}$$

More explicitly we have

$$(5.5.11)\quad w_{bc}(z;t,\alpha) = (-1)^{r_b-r_c} z^{-(\alpha|\delta_b)} z^{-r_b} (-z)^{\delta_{bc}-1} \epsilon(\alpha,\beta) \exp\left(\sum_{\ell>0} z^\ell t_\ell^b\right) \frac{\tau_W^{j,\underline{n}}(t\langle b\rangle, \alpha+\beta)}{\tau_W^{j,\underline{n}}(t,\alpha)},$$

where

$$(5.5.12)\qquad \tau_W^{j,\underline{n}}(t\langle b\rangle,\alpha+\beta) = \exp\left(-\sum_{\ell>0} \frac{z^{-\ell}}{\ell}\frac{\partial}{\partial t_\ell^b}\right) \tau_W^{j,\underline{n}}(t,\alpha+\beta),$$

where $\beta = \delta_b - \delta_c \in R$ *is the root corresponding to the root vector* $E_{bc} \in sl(k,\mathbb{C})$, *and* ϵ *is the cocycle defined in* (5.3.21).

Proof. For simplicity we suppress the reference to (t,α) and mostly also to z in the proof. Now by definition of the wave function we have for the c^{th} column

$$(5.5.13)\qquad \begin{aligned} w_{bc} &= e_b^t \cdot w_0(z;t,\alpha) \cdot \jmath^{\underline{n}}(g_-^j \cdot \epsilon_c(r_c)), \\ &= \exp\left(\sum_{i>0} t_i^b z^i\right) z^{<\alpha,\delta_b>} \sum_{r\in\mathbb{Z}} g_{rr_c}^{bc} z^{-r} e_b. \end{aligned}$$

Hence by (5.5.9) we have

$$(5.5.14)\qquad \begin{aligned} \sum_{r\in Z} g_{rr_c}^{bc} z^{-r} &= \sum_{r\in\mathbb{Z}} z^{-r} \langle \psi_b(r)\psi_c^*(r_c)\cdot \mathbf{v}_j \mid \hat g(t,\alpha)\cdot \mathbf{v}_j\rangle / \tau_W^{j,\underline{n}}(t,\alpha), \\ &= \left\langle \sum_{r\in\mathbb{Z}} z^r \psi_b(r)\psi_c^*(r_c)\cdot \mathbf{v}_j \mid \hat g(t,\alpha)\cdot \mathbf{v}_j \right\rangle / \tau_W^{j,\underline{n}}(t,\alpha), \\ &= \langle \psi_b(z)\psi_c^*(r_c)\cdot \mathbf{v}_j \mid \hat g(t,\alpha)\cdot \mathbf{v}_j\rangle / \tau_W^{j,\underline{n}}(t,\alpha), \\ &= \langle \psi_c^*(r_c)\cdot \mathbf{v}_j \mid \psi_b^*(z)\hat g(t,\alpha)\cdot \mathbf{v}_j\rangle / \tau_W^{j,\underline{n}}(t,\alpha). \end{aligned}$$

Here we have implicitly extended the Hermitian form $\langle\cdot\mid\cdot\rangle$ on $\Lambda^{\infty/2}\mathbb{C}^\infty$ to a map $\langle\cdot\mid\cdot\rangle_1 : \Lambda^{\infty/2}\mathbb{C}^\infty[[z,z^{-1}]] \times \Lambda^{\infty/2}\mathbb{C}^\infty \to \mathbb{C}[[z,z^{-1}]]$ such that $\langle z\mathbf{u} \mid \mathbf{v}\rangle_1 = z^{-1}\langle \mathbf{u}\mid\mathbf{v}\rangle$. There is a similar map $\langle\cdot\mid\cdot\rangle_2 : \Lambda^{\infty/2}\mathbb{C}^\infty \times \Lambda^{\infty/2}\mathbb{C}^\infty[[z,z^{-1}]] \to \mathbb{C}[[z,z^{-1}]]$ such that $\langle \mathbf{u} \mid z\mathbf{v}\rangle_2 = z\langle \mathbf{u}\mid\mathbf{v}\rangle$ and we have

$$(5.5.15)\qquad \langle \psi_b(z)\mathbf{u} \mid \mathbf{v}\rangle_1 = \langle \mathbf{u} \mid \psi_b^*(z)\mathbf{v}\rangle_2.$$

In (5.5.14) we have dropped the subscripts 1, 2, as we will continue to do in the sequel. Now, $\hat g(t,\alpha) = \hat w_0^{\underline{n}}(t,\alpha)^{-1}\hat g$ and

$$(5.5.16)\quad \psi_b^*(z)\hat w_0^{\underline{n}}(t,\alpha)^{-1} = \exp\left(\sum_{i>0} -t_i^b z^i\right) z^{\langle\alpha,\delta_b\rangle}\hat w_0^{\underline{n}}(t,\alpha)^{-1}\psi_b^*(z),$$

as follows from (see (5.1.12, 5.3.20)),

$$\begin{aligned}\hat{T}_\alpha \psi_b^*(z)\hat{T}_\alpha^{-1} &= z^{\langle\alpha,\delta_b\rangle}\psi_b^*(z),\\ [\alpha_a(i),\psi_b^*(z)] &= -z^i\delta_{ab}\psi_b^*(z).\end{aligned} \tag{5.5.17}$$

Using this, and the explicit form (4.3.5-6) for $w_0(z)$, in (5.5.14) gives part (5.5.10) of the theorem.

We next continue with the calculation of $w'_{bc}(z) := \langle\psi_c^*(r_c)\cdot\mathbf{v}_j \mid \psi_b^*(z)\hat{g}(t,\alpha)\cdot\mathbf{v}_j\rangle$. The fermion operator $\psi_c^*(r_c)$ is the coefficient of z^{-r_c} in the field $\psi^*(z)$, so using (5.1.14), (5.1.17) and the bosonization formula (5.1.18) we get

$$\psi_c^*(r_c)\cdot\mathbf{v}_j = \hat{Q}_c^{-1}(-1)^{j-r_c}\cdot\mathbf{v}_j. \tag{5.5.18}$$

This gives, also using the fact that the fermionic translation operators $\hat{Q}_c$ are unitary,

$$w'_{bc}(z) = (-1)^{j-r_c}\langle\mathbf{v}_j \mid \hat{Q}_c\psi_b^*(z)\hat{g}(t,\alpha)\cdot\mathbf{v}_j\rangle. \tag{5.5.19}$$

In the fermion field $\psi_b^*(z)$ the operator $(-z)^{-\alpha_b(0)}$ occurs. We need to move this to the left to let it act on $\mathbf{v}_j$. We have in general

$$[\alpha_a(0),\hat{Q}_b] = \delta_{ab}\hat{Q}_b, \tag{5.5.20}$$

so

(5.5.21)
$$\hat{Q}_c\hat{Q}_b^{-1}(-z)^{-\alpha_b(0)} = \hat{Q}_c(-z)^{-(\alpha_b(0)+1)}\hat{Q}_b^{-1} = (-z)^{-(\alpha_b(0)+1)+\delta_{bc}}\hat{Q}_c\hat{Q}_b^{-1}.$$

This gives

(5.5.22)
$$\begin{aligned}w'_{bc}(z) &= (-1)^{j-r_c}\left\langle(-z)^{(\alpha_b(0)+1)-\delta_{bc}}\mathbf{v}_j\middle|\hat{Q}_c\hat{Q}_b^{-1}\chi\exp\left(\sum_{\ell<0}\frac{1}{\ell}z^{-\ell}\alpha_b(\ell)\right)\cdot,\right.\\ &\qquad\left.\cdot\exp\left(\sum_{\ell>0}\frac{1}{\ell}z^{-\ell}\alpha_b(\ell)\right)\hat{g}(t,\alpha)\cdot\mathbf{v}_j\right\rangle,\\ &= (-1)^{-r_c}(-z)^{\delta_{bc}-r_b-1}\left\langle\exp\left(\sum_{\ell>0}\frac{-1}{\ell}z^{-\ell}\alpha_b(\ell)\right)\mathbf{v}_j\right|\cdot\\ &\qquad\left.\cdot\hat{Q}_c\hat{Q}_b^{-1}\hat{g}(t\langle b\rangle,\alpha)\cdot\mathbf{v}_j\right\rangle,\\ &= (z)^{-r_b}(-1)^{-r_b-r_c}(-z)^{\delta_{bc}-1}\langle\mathbf{v}_j \mid \hat{Q}_c\hat{Q}_b^{-1}\hat{g}(t\langle b\rangle,\alpha)\cdot\mathbf{v}_j\rangle,\end{aligned}$$

where $t\langle b\rangle$ is as in (5.5.12) and we have used that $\alpha_b(\ell)\mathbf{v}_j = 0$ if $\ell > 0$.

Finally we have by (5.3.20) and Lemma 5.3.5:

$$
\begin{aligned}
\hat{Q}_c\hat{Q}_b^{-1}\hat{T}_\alpha^{-1} &= \hat{T}_\beta^{-1}\hat{T}_\alpha^{-1}, \\
&= \hat{T}_{-\beta}\hat{T}_{-\alpha}, \\
&= \epsilon(-\beta,-\alpha)\hat{T}_{-(\alpha+\beta)}, \\
&= \epsilon(\alpha,\beta)\hat{T}_{\alpha+\beta}^{-1}.
\end{aligned} \tag{5.5.23}
$$

Putting this all together gives then the rest of the theorem. □

Note that in approach of, say, [**DJKM**] the formula (5.5.10) would be the *definition* of the wave function, while in the approach of [**Di3**] the τ-function would be defined by a formula like (5.5.11).

The relation between the wave function and the τ-function given by Theorem 5.5.2 allows us also to express the coefficients of resolvents, lattice resolvents, etc. in terms of τ-functions. For instance for the example of the Davey-Stewartson-Toda system discussed in Section 4.6 we find that the first coefficient w_1^m of the expansion (4.6.6) of the wave operator is given by

$$
w_1^m = \begin{pmatrix} -\partial_1^1\log(\tau^m) & -\tau^{m-1}/\tau^m \\ -\tau^{m+1}/\tau^m & -\partial_2^1\log(\tau^m) \end{pmatrix}. \tag{5.5.24}
$$

(For simplicity we use the $j=0$ part of the Grassmannian and write $\tau^m(t)$ for $\tau_W^{0,\underline{n}}(t,m\alpha_1)$, see Section 4.6, and suppress the time dependence.) From (4.6.9–10) we then see that

$$
\begin{aligned}
q^m &= -2\tau^{m-1}/\tau^m, \\
r^m &= 2\tau^{m+1}/\tau^m.
\end{aligned} \tag{5.5.25}
$$

A small calculation using the diagonal part of δ^m, see (4.6.9), then proves that the expression of the variable Q^m in terms of τ-functions is given by

$$
Q^m = \partial^2\log(\tau^m). \tag{5.5.26}
$$

6. Multicomponent KP equations in bilinear form and Plücker equations.

6.0. Introduction. In the previous section we have found a relation between the wave function of a point of the (polynomial) Grassmannian and τ-functions, matrix elements of the fundamental representations of gl_∞. These wave functions are solutions of the linear equations (4.4.6). In this section we will reformulate the linear equations (4.4.6) in terms of bilinear equations of Hirota type: any (formal) solution of the linear equations is at the same time a solution to the bilinear equations (Proposition 6.1.1). This allows us to make the connection with the Plücker equations of the embedding of Gr in $\mathbb{P}\Lambda^{\infty/2}\mathbb{C}^\infty$, which are equations for the τ-function. So we get, just as in the one component case, three equivalent descriptions of the multi component KP hierarchy: the wave functions of the multi component KP hierarchy satisfy both the linear equations and the bilinear equations and they can be expressed in terms of τ-functions that satisfy their own, equivalent, system of equations.

6.1. Dual Grassmannian and Plücker equations. The equations (4.4.6) for the wave function $w_W(t,\alpha)$ can be formulated in terms of a bilinear equation involving also the so called dual wave function $w^*_W(t,\alpha)$. This bilinear equation amounts to an orthogonality relation between $W \in Gr$ and W^*, an element of the dual Grassmannian Gr^*, as was explained in [**HP**] in an analytic context for the KP-hierarchy. So we start out this subsection by briefly discussing the dualization of all our constructions. Then we derive, for completeness sake, the bilinear equations, show that the bilinear equations for the wave function and its dual are equivalent to the differential difference linear equations (4.4.6) and give the connection to the Plücker equations of the embedding $Gr_j \to \mathbb{P}\Lambda_j^{\frac{\infty}{2}}\mathbb{C}^\infty$.

The starting point of the theory developed until now was the space H of infinite column vectors, see (2.1.2). The dual notion is

$$H^* = \left\{ \sum_{i=m}^{\infty} c_i\epsilon_i^* \mid c_i \in \mathbb{C}, m \in \mathbb{Z} \right\}, \tag{6.1.1}$$

where ϵ_i^* is the linear function on H given by $\langle \epsilon_i^*, \epsilon_j \rangle = \delta_{ij}, i,j \in \mathbb{Z}$. We can think of H^* as consisting of infinite row vectors and the natural bilinear pairing $H^* \times H \to \mathbb{C}$ is then just matrix multiplication between a row and column vector.

On H^* we have the action of $g \in Gl_\infty^{lf}$ given by

$$g \cdot \epsilon_i^* = \epsilon_i^* g^{-1} = \sum_{k=m}^{\infty} \epsilon_k^*(g^{-1})_{ik}. \tag{6.1.2}$$

Define, cf. (2.2.1),

$$H_j^* = \left\{ \sum_{k=j+1}^{\infty} c_k\epsilon_k^* \right\} \subset H^*. \tag{6.1.3}$$

Then Gr^* is the collection of $W^* \subset H^*$ of the form $g \cdot H_j^* = H_j^* g^{-1}$ for some $g \in Gl_\infty^{lf}$, $j \in \mathbb{Z}$. There is a natural 1-1 correspondence between points of

Gr and Gr^*: $W = gH_j$ corresponds to $W^* = g \cdot H_j^*$. One checks that such W and W^* are orthogonal with respect to the pairing $\langle , \rangle$, since H_j and H_j^* are.

We define also, given a partition $\underline{n}$, a relabeling on H^*: put $\epsilon_j^* = \epsilon_a^*(i)$ if $\epsilon_j = \epsilon_a(i)$ in H. Then we have a map to k-component row vector Laurent series, cf. (4.2.1):

$$\jmath^{*\underline{n}} : H^* \to H^{*(k)}, \quad \jmath^{\underline{n}}(\epsilon_a^*(i)) = z^{i-1}e_a^*, \tag{6.1.4}$$

where $e_a^*, a = 1, 2, \ldots, k$ is a basis for the k-component row vectors and $H^{*(k)} = \oplus\mathbb{C}((z))e_a^*$. The bilinear pairing between H^* and H then translates into the residue pairing on $H^{*(k)} \times H^{(k)}$:

$$(f^*, g) = \mathrm{Res}_z(f^*(z)g(z)), \quad f^* \in H^{*(k)}, g \in H^{(k)}, \tag{6.1.5}$$

where Res_z is the coefficient of z^{-1} in a formal power series.

On the dual Grassmannian we have the time evolution of type $\underline{n}$ given by $W^* \mapsto W^*(t,\alpha) := w_0^{\underline{n}}(t,\alpha)^{-1} \cdot W^* = W^* w_0^{\underline{n}}(t,\alpha)$, with $w_0^{\underline{n}}(t,\alpha)$ given by (4.1.1). An element W of Gr belongs to the H_j cell iff the corresponding element W^* of the dual Grassmannian belongs to the H_j^* cell. In particular the set $\Gamma_W^{j,\underline{n}}$ of elements (t,α) of the evolution group such that $W(t,\alpha)$ belongs to the H_j cell is equal to $\Gamma_{W^*}^{j,\underline{n}}$.

The wave function w_W of type $j, \underline{n}$ was defined using the columns $g_-^j \cdot \epsilon_b(r_b(j))$, with $r_b(j)$ given by (3.2.1). The analogous dual numbers are $r_b^*(j) = r_b(j) + 1$, with the following interpretation: $\epsilon_b^*(r_b)$ is the basis vector of type b occurring in H_j^* with smallest argument. Note that

$$\jmath^{*\underline{n}}(\epsilon_b^*(r_b^*)) = z^{r_b^*-1}e_b^* = z^{r_b}e_b^*. \tag{6.1.6}$$

Now we define the dual wave function using the rows of $(g_-^j)^{-1}$ and the numbers r_b^*: for $(t,\alpha) \in \Gamma_W^{j,\underline{n}}$

$$w_W^*(t,\alpha) := \jmath^{*\underline{n}}\left((\epsilon_1^*(r_1^*)\epsilon_2^*(^*r_2)\ldots\epsilon_k^*(r_k^*))^t(g_-^j)^{-1}\right) \cdot w_0(t,\alpha)^{-1}. \tag{6.1.7}$$

The dual wave function can be written in terms of the dual wave operator, acting now from the left (cf. (4.5.1)):

(6.1.8)
$$\begin{aligned} w_W^*(t,\alpha) &= \overrightarrow{w}_W(t,\alpha) \cdot w_0(t,\alpha)^{-1}, \\ \overrightarrow{w}_W &= \left(1_{k\times k} + \sum_{i>0} w_i^* \overrightarrow{\partial}^{-i}\right) \mathrm{diag}(z^{r_1}, z^{r_2}, \ldots, z^{r_k}). \end{aligned}$$

Denote by $W^{*(k)}$ the image of W^* in $H^{*(k)}$. Then the rows of $w^*_W(t,\alpha)$ belong to $W^{*(k)}$ for all $(t,\alpha)\in\Gamma^{j,\underline{n}}_W$, just as the columns of $w_W(t,\alpha)$ belong to $W^{(k)}$. Because $W^{*(k)}$ and $W^{(k)}$ are orthogonal for the residue pairing we find the *bilinear identity*:

$$\mathrm{Res}_z(w^*_W(t,\alpha)w_W(t',\alpha'))=0,\quad (t,\alpha),(t',\alpha')\in\Gamma^{j,\underline{n}}_W. \tag{6.1.9}$$

Both the linear equations (4.5.6) for the wave function and the bilinear equations (6.1.9) are derived in apparently different ways from the geometry of the Grassmannian. In fact these equations are equivalent. Indeed, consider two matrix PDO's acting from the left and right:

$$\overrightarrow{P}=\sum p_i\partial^i,\quad \overleftarrow{Q}=\sum\overleftarrow{\partial}^i q_i,\quad p_i,q_i\in Mat_n(\mathbb{C}). \tag{6.1.10}$$

Then we define the dual operators to act in the opposite direction:

$$(\overrightarrow{P})^*:=\sum p_i(-\overleftarrow{\partial})^i,\quad (\overleftarrow{Q})^*:=\sum(-\partial)^i q_i. \tag{6.1.11}$$

Proposition 6.1.1. *Let*

$$w(z;t,\alpha)=w_0(z;t,\alpha)\cdot\overleftarrow{w},\overleftarrow{w}=diag(z^{-r_1},\dots,z^{-r_k})\cdot\left(1_{k\times k}+\sum_{i>0}\overleftarrow{\partial}^{-i}w_i\right),$$

$$w^*(z;t,\alpha)=\overrightarrow{w}\cdot w_0(z;t,\alpha)^{-1},\overrightarrow{w}=\left(1_{k\times k}+\sum_{i>0}w^*_i\overrightarrow{\partial}^{-i}\right)\cdot diag(z^{r_1},\dots,z^{r_k}),$$

with w_i, and w^*_i matrices of size k. If w, w^* are both defined and infinitely differentiable for (t,α) and $(t',\alpha+\gamma)$, then the bilinear equations

$$Res_z(w^*(z;t,\alpha)w(z;t',\alpha+\gamma))=0 \tag{6.1.12}$$

are equivalent to the equations

$$\overrightarrow{w}=(\overleftarrow{w}^{-1})^*, \tag{6.1.13a}$$

$$\partial^i_b w=w\cdot(\mathcal{R}^i_b)_+, \tag{6.1.13b}$$

$$w(z;t,\alpha+\gamma)=w(z;t,\alpha)\cdot(\mathcal{U}_\gamma)^{-1}_+, \tag{6.1.13c}$$

where $\mathcal{R}^i_b=\overleftarrow{w}^{-1}\cdot\overleftarrow{\Lambda}^i_b\cdot\overleftarrow{w}$, $\mathcal{U}_\gamma=\overleftarrow{w}^{-1}\cdot\overleftarrow{T}^{-1}_\gamma\cdot\overleftarrow{w}$.

We omit the proof, which is rather similar to the one for the KP hierarchy, cf. [**Di2**].

In Theorem 5.5.2 we found an expression for the wave function in terms of matrix elements of the semi-infinite wedge space. There is, as one might expect, a similar expression for the dual wave function in terms of some dual semi-infinite wedge space. However the situation is simpler than that. Note that w_W^* consists essentially of rows of g_-^{-1}, and since $g_- = 1 + X$, with $X \in gl_{\infty-}^j$, we have $g_-^{-1} = 1 - X$, because $X^2 = 0$. Hence we can calculate the matrix elements of g_-^{-1} and thus w_W^* in terms of the standard semi-infinite wedge space. Writing $w_W^* = \sum_{bc}^k w_{bc}^* E_{bc}$ we find by essentially the same calculation as in Theorem 5.5.2 that

$$(6.1.14) \qquad w_{bc}^* = z^{-1}\langle \psi_b^*(r_b^*)\mathbf{v}_j \mid \hat{w}_0^{\underline{n}}(t,\alpha)^{-1}\psi_c(z)\hat{g}\mathbf{v}_j\rangle / \tau_W^{j,\underline{n}}(t,\alpha).$$

Consider the orbit $\mathcal{O}_j$ of the group $\hat{Gl}_\infty^{0,lf}$ through the vacuum $\mathbf{v}_j$ in $\Lambda^{\infty/2}\mathbb{C}^\infty$. The projectivization of $\mathcal{O}_j$ can be identified with the component Gr_j of Gr. It is well known ([**KP**]) that the points of $\mathcal{O}_j$ are characterized as follows: $\tau_j \in \Lambda_j^{\frac{\infty}{2}}\mathbb{C}^\infty$ belongs to $\mathcal{O}_j$ iff

$$(6.1.15) \qquad \sum_{i\in\mathbb{Z}} \psi(i)\tau_j \otimes \psi^*(i)\tau_j = 0.$$

So the equation (6.1.15) might be called the Plücker equation for the embedding of Gr_j in $\mathbb{P}\Lambda_j^{\frac{\infty}{2}}\mathbb{C}^\infty$. Note that if we use the bosonization formulae (5.1.18) we obtain differential-difference equations for the τ-function. We refrain from writing down explicit equations.

Using the relabeled fermionic fields of (5.1.16) we can write this as

$$(6.1.16) \qquad Res_z\left[z^{-1}\sum_{c=1}^k \psi_c(z)\tau_j \otimes \psi_c^*(z)\tau_j\right] = 0.$$

Multiplying this by $\hat{w}_0^{\underline{n}}(t,\alpha) \otimes \hat{w}_0^{\underline{n}}(t',\alpha')/\tau_W^{j,\underline{n}}(t,\alpha)\tau_W^{j,\underline{n}}(t',\alpha')$ and taking the inner product with the element $\langle \psi_b(r_b^*)\mathbf{v}_j \mid \otimes \langle \psi_d^*(r_d)\mathbf{v}_j \mid$ gives

(6.1.17)
$$Res_z\left[\sum_{c=1}^k \frac{\langle \psi_b(r_b^*)\mathbf{v}_j \mid \hat{w}_0^{\underline{n}}(t,\alpha)\psi_c(z)\tau_j}{\tau_W^{j,\underline{n}}(t,\alpha)} \frac{\langle \psi_d^*(r_d)\mathbf{v}_j \mid \hat{w}_0^{\underline{n}}(t',\alpha')\psi_c^*(z)\tau_j}{\tau_W^{j,\underline{n}}(t',\alpha')}\right] = 0,$$

which is, using (5.5.10, 6.1.14), the bilinear identity (6.1.9). In other words the Plücker equation for the embedding of Gr_j in $\mathbb{P}\Lambda_j^{\frac{\infty}{2}}\mathbb{C}^\infty$ is nothing but the bilinear equation for a point W of the component Gr_j.

7. Reduction to loop groups and KdV type equations.

7.0. Introduction. In the previous sections we have found for every W in the Grassmannian and for every choice of partition of n into k parts a solution of the differential-difference k-component KP hierarchy.

In the classical 1-component KP case one can for every integer n impose constraints on solutions of the KP to obtain solutions of the n-KdV hierarchy, related to $\tilde{G}l^{lf}(n,\mathbb{C})$, the loop group of $Gl(n,\mathbb{C})$. This procedure is called n-reduction and amounts to considering $\tilde{G}l^{lf}(n,\mathbb{C})$ in a natural way as a subgroup of Gl^{lf}_{∞}. This gives rise to a subspace Gr^n of Gr, called the n-periodic Grassmannian. The embedding of $\tilde{G}l^{lf}(n,\mathbb{C})$ in Gl^{lf}_{∞} and Gr^n are discussed in Section 7.1.

In the general case that we are studying we can similarly impose for every partition $\underline{n}$ of n into k parts constraints on the k-component KP to obtain solutions of what one might call the $\underline{n}$-reduced KP-hierarchy. This is discussed in section 7.2.

In Section 7.3 the $\underline{n}$-reduced KP hierarchy is rewritten in terms of $n \times n$ matrices depending on a variable λ, so as to make the relation with the loop group $\tilde{G}l^{lf}(n,\mathbb{C})$ and the n-periodic Grassmannian explicit. The $\underline{n}$-reduced KP hierarchy rewritten in this way will be called the $\underline{n}$-KdV hierarchy. Every conjugacy class in the symmetric group is determined by a partition $\underline{n}$ and determines a gradation of $\tilde{g}l^{lf}(n,\mathbb{C})$. In [**Wi2**] Wilson proposes to construct for every gradation of $\tilde{g}l^{lf}(n,\mathbb{C})$ a hierarchy of differential equations. We show that our $\underline{n}$-KdV hierarchies are essentially the equations Wilson had in mind for the gradation corresponding to $\underline{n}$. (He was dealing with the modified equations, related to the infinite n-periodic flag manifold in a similar way as our $\underline{n}$-KdV hierarchies are related to the n-periodic Grassmannian. Also he didn't discuss the discrete part of the hierarchies.)

7.1. n-periodicity. Recall from Section 4.4 the map $\jmath^{\underline{n}}$ from the space H of infinite column vectors to $H^{(k)}$, the space of k-component formal Laurent series in the variable z, constructed using the partition $\underline{n}$. Here we will consider the space $H^{(n)}$ of n-component formal Laurent series in the variable λ: let $\tilde{e}_i$, $1 \le i \le n$ be a basis for $\mathbb{C}^n$, define

$$H^{(n)} := \oplus_{i=1}^{n} \mathbb{C}((\lambda))\tilde{e}_i, \tag{7.1.1}$$

and introduce the isomorphism $H \to H^{(n)}$ by

$$\epsilon_j \mapsto \lambda^{-p-1}\tilde{e}_q, \tag{7.1.2}$$

where $j = np + q, 1 \le q \le n$. This corresponds to the construction of $H^{(k)}$ of section 4.4 for the partition of n into n parts. Let $\tilde{g}l^{lf}(n,\mathbb{C})$ be the collection of formal loops of the form $\sum_m^{\infty} \lambda^i A_i$ with $A_i \in gl(n,\mathbb{C})$ and m some integer. This forms in the natural way a Lie algebra and we can define a (Lie) algebra homomorphism $\tilde{g}l^{lf}(n,\mathbb{C}) \to gl^{lf}_{\infty}$ by

$$\lambda^p \tilde{E}_{ij} \mapsto \sum_{\ell\in\mathbb{Z}} \mathcal{E}_{n(\ell-p)+i,n\ell+j}, \tag{7.1.3}$$

where $\tilde{E}_{ij}$ is the element of the natural basis of $n \times n$ matrices with its only nonzero entry a 1 on the ij^{th} place. (In general we will use a tilde $\tilde{}$ to indicate that an object is related to the loop algebra, loop group or anything "of size n".)

The image of this homomorphism consists of the *n-periodic* elements, i.e., those X in gl_∞^{lf} such that

$$X = \Lambda^n X \Lambda^{-n}, \tag{7.1.4}$$

where $\Lambda = \sum \mathcal{E}_{ii+1}$ is the shift matrix in Gl_∞^{lf}. If we expand X as $X = \sum X_{ij}\mathcal{E}_{ij}$ then the condition (7.1.4) implies for the coefficients $X_{i+n,j+n} = X_{ij}$ for all $i, j \in \mathbb{Z}$. In the same way we get a surjective homomorphism from the formal loop group $\tilde{Gl}^{lf}(n,\mathbb{C})$ (consisting of the invertible elements of $\tilde{gl}^{lf}(n,\mathbb{C})$) to the subset of n-periodic elements of Gl_∞^{lf}.

Since $H^{(k)}$ and $H^{(n)}$ are both isomorphic to H there is an isomorphism $\jmath_{n,k} : H^{(k)} \to H^{(n)}$ given explicitly by

$$e_a z^{-i} \mapsto \lambda^{-p-1}\tilde{e}_t, \tag{7.1.5}$$

where $i = n_a p + q$, $1 \le q \le n_a$ and $t = n_1 + \cdots + n_{a-1} + q$. Using this isomorphism we can translate, of course, the action of $k \times k$ matrices on $H^{(k)}$ into an action on $H^{(n)}$. In particular the diagonal matrix zE_{aa} acting on $H^{(k)}$ corresponds to the action of the $n \times n$ block diagonal matrix

$$\mathcal{P}_a = \operatorname{diag}(0_{n_1}, \ldots, 0_{n_{a-1}}, P_{n_a}, 0_{n_{a+1}}, \ldots, 0_{n_k}) \tag{7.1.6}$$

where 0_{n_c} is the zero matrix of size $n_c \times n_c$ and P_{n_a} is the $n_a \times n_a$ matrix

$$P_{n_a} = \begin{pmatrix} 0\,1\,0 \ldots 0\,0 \\ 0\,0\,1 \ldots 0\,0 \\ \vdots\,\vdots\,\vdots\,\ddots\,\vdots\,\vdots \\ 0\,0\,0 \ldots 1\,0 \\ 0\,0\,0 \ldots 0\,1 \\ \lambda\,0\,0 \ldots 0\,0 \end{pmatrix}. \tag{7.1.7}$$

Since we have seen in Subsection 4.4 that the generator Λ_a^+ of the pre-Heisenberg algebra $\mathcal{H}^{\underline{n}}$ of type $\underline{n}$ corresponds in $H^{(k)}$ to zE_{aa} we find that the action of the positive part of the pre-Heisenberg algebra in $H^{(n)}$ is generated by the elements $\mathcal{P}_a$. The action of the pre-translation group is similarly generated by the elements $\tilde{T}_{\alpha_i} = \mathcal{P}_{n_i}^{-1} + \mathcal{P}_{n_{i+1}} + \sum_{j \ne i,i+1} 1_{n_j}$, $1 \le i \le k$.

We have seen that the shift matrix Λ does not correspond under $\jmath^{\underline{n}}$ to an element of the formal loop group of size k, leading to the introduction of the numbers r_b of (4.3.1). Now, when we use the map (7.1.2), the situation is much simpler: the element P_n of $\tilde{Gl}(n,\mathbb{C})$ is the image of the shift matrix Λ. This means in particular that, if we denote by $H_j^{(n)}$ the image of H_j in $H^{(n)}$, we have, cf. (4.3.3),

$$H_j^{(n)} = P_n^{-j} H_0^{(n)} = P_n^{-j} \oplus_{i=1}^n \mathbb{C}[[\lambda]]\tilde{e}_i. \tag{7.1.8}$$

Let $g \in Gl_\infty^{lf}$ be n-periodic and let $W = gH_j$. Since $\Lambda^n H_j = H_{j-n} \subset H_j$ we have $\Lambda^n W = \Lambda^n g H_j = g\Lambda^n H_j \subset W$. Conversely, elements of Gr satisfying $\Lambda^n W \subset W$ can be obtained from H_j by an n-periodic group element (see [**PrS**]) and are called also n-periodic. The collection of n-periodic elements in Gr is denoted by Gr^n, and Gr^n is called the n-periodic Grassmannian.

7.2. The $\underline{n}$ reduced k-component KP hierarchy. The infinite shift matrix $\Lambda^n \in Gl_\infty^{lf}$ corresponds to multiplication by $\mathrm{diag}(z^{n_1}, \ldots, z^{n_k})$ in $H^{(k)}$ and to multiplication by $\lambda = \lambda 1_{n\times n}$ in $H^{(n)}$. For simplicity we also write often λ for $\mathrm{diag}(z^{n_1}, \ldots, z^{n_k})$ in $H^{(k)}$. So if $W \in Gr^n$ we have for the image $W^{(k)}$ in $H^{(k)}$ the relation $\lambda W^{(k)} \subset W^{(k)}$. In particular for the wave function, the columns of which belong to $W^{(k)}$ when $(t,\alpha) \in \Gamma_W^{j,\underline{n}}$, we have

$$\lambda w_W(t,\alpha) \subset W^{(k)}, \tag{7.2.1}$$

where we say that a matrix belongs to $W^{(k)}$ if its columns do. Now by Proposition 4.4.2 this means that there is for every positive integer ℓ a unique $k \times k$ matrix differential operator $M^{(\ell)}$, such that

$$\lambda^\ell w_W(t,\alpha) = w_W \cdot M^{(\ell)}(t,\alpha). \tag{7.2.2}$$

Just as w_W also $M^{(\ell)}$ is defined for $(t,\alpha) \in \Gamma_W^{j,\underline{n}}$.

One sees that the resolvent (4.5.5) satisfies

$$w_W \cdot \mathcal{R}_a = zE_{aa} w_W, \tag{7.2.3}$$

so we have, using that $\lambda = \mathrm{diag}(z^{n_1}, \ldots, z^{n_k})$,

$$w_W \cdot \left(M^{(\ell)} - \sum_{a=1}^k \mathcal{R}_a{}^{\ell n_a} \right) = 0. \tag{7.2.4}$$

By the proof of Proposition 4.4.2 this means that

$$M^{(\ell)} = \sum_{a=1}^k \mathcal{R}_a{}^{\ell n_a}, \quad \ell > 0. \tag{7.2.5}$$

So in the linear combination of resolvents on the right hand side of (7.2.5) the negative powers of $\overleftarrow{\partial}$ that might occur cancel to give a differential operator.

Introduce now the derivation

$$\partial^{\ell,\underline{n}} = \sum_{a=1}^{k} \partial_{t^a_{\ell n_a}}, \quad \ell > 0. \tag{7.2.6}$$

Then we have

$$\begin{aligned} \partial^{\ell,\underline{n}} w_W &= w_W \cdot \left(\sum_{a=1}^{k} \mathcal{R}_a^{\ell n_a} \right)_+, \\ &= w_W \cdot M^{(\ell)}. \end{aligned} \tag{7.2.7}$$

On the other hand, writing $w_W = w_0 \cdot \overleftarrow{w}_W$, with $\overleftarrow{w}_W$ the wave operator, we have

(7.2.8)

$$\begin{aligned} \partial^{\ell,\underline{n}} w_W &= w_0 \cdot \left(\sum_{a=1}^{k} z^{\ell n_a} E_{aa} \right) \cdot \overleftarrow{w}_W + w_0 \cdot \partial^{\ell,\underline{n}} \overleftarrow{w}_W, \\ &= w_W \cdot M^{(\ell)} + w_0 \cdot \partial^{\ell,\underline{n}} \overleftarrow{w}_W. \end{aligned}$$

Comparing (7.2.7) and (7.2.8) we find that the wave operator $\overleftarrow{w}_W$ and hence all resolvents $\mathcal{R}_a$, lattice resolvents $\mathcal{U}_{\alpha_i}$ ((4.7.5)) and the multi-component KP operator L of (4.5.11) are independent of the variables corresponding to $\partial^{\ell,\underline{n}}$, $\ell > 0$ (in the n-periodic case). Conversely if $W \in Gr$ produces a solution of the multi-component KP hierarchy that is independent of these variables it belongs to the n-periodic Grassmannian.

Definition 7.2.1. The $\underline{n}$-reduced k-component differential-difference KP hierarchy is the system of deformation equations for the $k \times k$ matrix PDO L of the form $L = A\overleftarrow{\partial} + \mathcal{O}(\overleftarrow{\partial}^0)$ given by

$$\begin{aligned} \partial_b^i L &= [L, (\mathcal{R}_b^i)_+], \\ L(t, \alpha + \alpha_i) &= (\mathcal{U}_{\alpha_i})_+ \cdot L(t, \alpha) \cdot (\mathcal{U}_{\alpha_i})_+^{-1} \end{aligned} \tag{7.2.9a}$$

together with the condition

$$\partial^{\ell,\underline{n}} L = 0, \quad \ell > 0. \tag{7.2.9b}$$

So we get for every n-periodic element $W \in Gr^n$ a solution $L_W = w_W^{-1} A \overleftarrow{\partial} w_W$ of the $\underline{n}$-reduced k-component KP hierarchy.

7.3. The $\underline{n}$-KdV hierarchy. In this section we want to give an alternative description of the $\underline{n}$-reduced k-component KP hierarchy. In the case $k = 1$ there are (at least) two other descriptions available of the $\underline{n}$-reduced hierarchies: the scalar Lax equation approach, involving n^{th} order scalar differential operators, and the approach using first order matrix differential operators (see, e.g., [**DS**]). We discuss both methods in the general case of an arbitrary partition. It will turn out that the scalar Lax operator method does not generalize in a satisfactory way.

The $k \times k$ matrix differential operator $M^{(\ell)}$ introduced in (7.2.2) satisfies the same equations as the pseudo-differential operator L (4.5.11):

$$\begin{aligned} \partial_i^a M^{(\ell)} &= [M^{(\ell)}, (\mathcal{R}_a{}^i)_+], \\ M^{(\ell)}(t, \alpha + \alpha_i) &= (\mathcal{U}_{\alpha_i})_+ \cdot M^{(\ell)}(t, \alpha) \cdot (\mathcal{U}_{\alpha_i})_+^{-1}. \end{aligned} \tag{7.3.1}$$

If we consider the partition of n into 1 part (the principal partition), then we have for $M := M^{(1)}$ the relation $M = L^n$ and L is the n^{th} root of the operator M. (We take $A = 1$ here.) As is well known in this case the equations (7.3.1) form the generalized n-KdV hierarchy (the discrete part is now, of course, absent) and this hierarchy is equivalent to the n-reduced 1-component KP hierarchy, cf., [**SeW**]. So the matrix M seems to be the natural generalization of the Lax operator in the principal case.

However, in general the equations (7.3.1) are just a consequence of the equations (7.2.9) and not equivalent to them, since the operator M contains less information than L. In fact in the extreme case where $\underline{n}$ is the partition of n into n parts (the homogeneous case) M is just $\overleftarrow{\partial}\, 1_{n\times n}$. This is so because then the *differential* operator M is the sum of the pseudo differential operators $\mathcal{R}_a$ that are of the form $\overleftarrow{\partial}\, E_{aa} + [E_{aa}, w_1] + \mathcal{O}(\overleftarrow{\partial}^{-1})$. So the scalar Lax operator formulation of n-reduced KP does not seem to generalize simply to the general case, cf., [**Di1**]. Therefore we now sketch how the n-reduced k-component KP hierarchy (7.2.9) fits in the framework of $n \times n$ matrix differential operators.

Recall the construction of the k-component KP hierarchy: we started with a W in the Grassmannian, mapped it to $W^{(k)}$ in $H^{(k)}$ and considered the natural flow $W^{(k)} \mapsto w_0(t,\alpha)^{-1} \cdot W^{(k)}$. Then we considered the subspace $W_{\text{fin}}^{(k)}$ of elements of $W^{(k)}$ of the form $w_0(t,\alpha)$ times a finite order (in z) k-component vector. The space $W_{\text{fin}}^{(k)}$ was stable under the action of $\overleftarrow{\partial}$ and we proved in Proposition 4.4.2 that $W_{\text{fin}}^{(k)}$ was in fact a free rank k module over $\mathbb{C}[\overleftarrow{\partial}]$, with basis the columns of the wave function w_W. This lead in Proposition 4.5.2 to linear equations for the wave function and this in turn produced the k-component KP hierarchy.

Now, in the n-periodic case, $W_{\text{fin}}^{(k)}$ is not only invariant under the action of $\overleftarrow{\partial}$ but also under the action of $\lambda = \operatorname{diag}(z^{n_1}, z^{n_2}, \ldots, z^{n_k})$, i.e., $W_{\text{fin}}^{(k)}$ is a $\mathbb{C}[\lambda]$ module. It turns out that $W_{\text{fin}}^{(k)}$ is free of rank n and we can give an explicit basis. Then the time evolution of this basis will as before lead to linear equations and we will obtain in very much the same way as before a collection of differential difference equations, the $\underline{n}$-KdV hierarchy. In the k-component KP case the objects one deals with are $k \times k$ matrices over $\overleftarrow{\partial}$ (since $W_{\text{fin}}^{(k)}$ is rank k over $\mathbb{C}[\overleftarrow{\partial}]$) whereas in the $\underline{n}$-KdV case we deal with $n \times n$ matrices over λ (since now we think of $W_{\text{fin}}^{(k)}$ as a rank n module over $\mathbb{C}[\lambda]$).

Now we turn to some of the details of the construction. Fix an integer j and consider an n-periodic element g of Gl_∞^{lf} and the corresponding element $W = gH_j$ of Gr^n. We embed W in $H^{(n)}$ using the map (7.1.2), so that we now deal with n-component vectors depending on λ.

We say that an element $\tilde{g} \in \tilde{G}l^{lf}(n, \mathbb{C})$ is in the $\tilde{H}_j$ cell if

$$\tilde{g} = \tilde{g}_-^j \cdot \tilde{g}_+^j, \tag{7.3.4}$$

where

(7.3.5)

$$\tilde{g}_-^j = P_n^{-j} \left(\tilde{1}_{n \times n} + \sum_{i<0} \lambda^i \tilde{g}_i \right) P_n^j,$$

$$\tilde{g}_+^j = P_n^{-j} \left(\sum_{i \geq 0} \lambda^i \tilde{g}_i \right) P_n^j, \quad \tilde{g}_i \in gl(n, \mathbb{C}), \tilde{g}_0 \in Gl(n, \mathbb{C}).$$

This is called the j-Birkhoff decomposition of $\tilde{g}$. An element $\tilde{g}$ is in the $\tilde{H}_j$cell iff the corresponding (under the map (7.1.3)) n-periodic element of Gl_∞^{lf} is in the H_j cell. The image of $\tilde{g}_-^j$ (resp. $\tilde{g}_+^j$) in Gl_∞^{lf} does not quite coincide with the minus component g_-^j (resp. g_+^j) of the Gauss decomposition of g adapted to H_j, since g_-^j (g_+^j) is not n-periodic, in general. So in case g is n-periodic we have two natural decompositions adapted to H_j. However the columns $j-n+1, j-n+2, \ldots, j$ of the n-periodic elements corresponding to $\tilde{g}_-^j$ and $\tilde{g}_+^j$ are equal to the same columns of the original factors g_-^j and g_+^j of the Gauss decomposition of g adapted to H_j (the other columns will differ, in general). Since this are the only columns that play a rôle it is irrelevant which decomposition of g we use. Besides the Birkhoff decomposition in the loop group we will need in the sequel also the following corresponding decomposition of the formal loop algebra:

$$\tilde{g}l(n, \mathbb{C}) = \tilde{g}l(n, \mathbb{C})_-^j \oplus \tilde{g}l(n, \mathbb{C})_+^j, \tag{7.3.6}$$

with (cf. (2.2.3) and (7.1.8)):

$$\tag{7.3.7} \begin{aligned} \tilde{gl}(n,\mathbb{C})^j_- &= P_n^{-j}\left(gl(n,\mathbb{C})\otimes\lambda^{-1}\mathbb{C}[\lambda^{-1}]\right)P_n^j,\\ \tilde{gl}(n,\mathbb{C})^j_+ &= P_n^{-j}\left(gl(n,\mathbb{C})\otimes\mathbb{C}[[\lambda]]\right)P_n^j. \end{aligned}$$

Fix also a partition $\underline{n}$ of n and consider the time flow from the corresponding Heisenberg algebra $\mathcal{H}^{\underline{n}}$ on $H^{(n)}$generated by

$$\tag{7.3.8} \tilde{w}_0^{\underline{n}}(t,\alpha) = \exp\left(\sum_{i>0}\sum_{a=1}^{k} t_i^a \mathcal{P}_a^i\right)\tilde{T}_\alpha.$$

The $n\times n$ *loop group wave function of type* $j,\underline{n}$, associated to W and defined for $(t,\alpha)\in\Gamma_W^{j,\underline{n}}$, reads:

$$\tag{7.3.9} \tilde{w}_W(t,\alpha) = \tilde{w}_0^{\underline{n}}(t,\alpha)\cdot\tilde{g}_-^j(t,\alpha),$$

where $\tilde{g}_-^j$ is the minus component in the Birkhoff decomposition for $\tilde{H}_j$ of $\tilde{g}(t,\alpha) = \tilde{w}_0^{\underline{n}}(t,\alpha)^{-1}\tilde{g}$. We can also describe $\tilde{g}_-^j$ by noting that the i^{th} column $\tilde{g}_-^j\cdot\tilde{e}_i$ is the unique element of $W^{(n)}(t,\alpha)$ of the form $P_n^{-j}(\tilde{e}_{n_i}+\mathcal{O}(\lambda^{-1}))$. So the loop group wave function consists essentially of n columns of $g_-(t,\alpha)$, whereas the wave function contains only k columns.

The point of the introduction of the loop group wave function is that its columns form a basis for $\tilde{W}_{\text{fin}}^{(k)}$, the image of $W_{\text{fin}}^{(k)}$ in $H^{(n)}$.

Proposition 7.3.2. *Let W be n-periodic. Fix $(t,\alpha)\in\Gamma_W^{j,\underline{n}}$. Then $W_{fin}^{(k)}$ is a free rank n module over the ring $\mathbb{C}[\lambda]$, with basis the columns of $\tilde{w}_W(t,\alpha)$.*

The proof of this Proposition is very much the same as that of Proposition 4.3.2 and is omitted.

Next we define $n\times n$ loop resolvents and lattice resolvents by

$$\tag{7.3.10} \begin{aligned} \tilde{\mathcal{R}}_a &:= \tilde{w}_W(t,\alpha)^{-1}\cdot\mathcal{P}_a\cdot\tilde{w}_W(t,\alpha)\\ \tilde{\mathcal{U}}_{\alpha_i} &:= \tilde{w}_W(t,\alpha)^{-1}\cdot\tilde{T}_{\alpha_i}^{-1}\cdot\tilde{w}_W(t,\alpha). \end{aligned}$$

So the resolvent $\tilde{\mathcal{R}}_a$ belongs to the loop algebra and the lattice resolvent $\tilde{\mathcal{U}}_{\alpha_i}$ to the loop group. We use the convention that subscripts on $\tilde{\mathcal{R}}_a$ will refer to the Lie algebra decomposition 7.3.7 for $j=0$, while subscripts on $\tilde{\mathcal{U}}_{\alpha_i}$, when it is in the H_0 cell, refer to the Birkhoff decomposition of type 0.

Then the analogue of Proposition 4.4.2 is

Lemma 7.3.3. *Let $W\in Gr_j$ and suppose that (t,α), $(t,\alpha+\alpha_i)$ belong to $\Gamma_W^{j,\underline{n}}$. Then the loop lattice resolvent $\tilde{\mathcal{U}}_{\alpha_i}$ is in the H_0 cell and we have:*

$$\tag{7.3.11} \begin{aligned} \partial_{t_i^b}\tilde{w}_W(t,\alpha) &= \tilde{w}_W(t,\alpha)\cdot(\tilde{\mathcal{R}}_b^i)_+\\ \tilde{w}_W(t,\alpha+\alpha_i) &= \tilde{w}_W(t,\alpha)\cdot(\tilde{\mathcal{U}}_{\alpha_i})_+^{-1}. \end{aligned}$$

The proof is the same as that of Proposition 4.4.2.

Note that the derivations with respect to the times (7.2.6) act trivially on the $\tilde{w}_W$, i.e., $\partial^{\ell,\underline{n}}\tilde{w}_W = \tilde{w}_W \cdot \sum \mathcal{P}_a^{\ell n_a} = \lambda^\ell \tilde{w}_W$.

The analogue of the operator L_W that solves the k-component KP hierarchy is the resolvent $\tilde{L} = \tilde{w}_w^{-1} \cdot \sum A_a \mathcal{P}_a \cdot \tilde{w}_W$. We could now define the $\underline{n}$-reduced KdV hierarchy in terms of deformation equations for $\tilde{L}$. However in our situation it turns out that all information of $\tilde{L}$ is already contained in the in the positive part $\tilde{L}_+$ and it is convenient to formulate the theory in terms of this.

To proceed we need a little digression on finite order automorphisms and twisted loop algebras associated to a partition $\underline{n}$. (For more details and proofs see [**tKvdL**].) We associate to every partition $\underline{n}$ a diagonal matrix in $gl(n,\mathbb{C})$:

$$H_{\underline{n}} = \sum_{a=1}^{k} H_a, \quad H_a = \frac{1}{2n_a}\sum_{j=1}^{n_a}(n_a - 2j + 1)E_{aa}^{jj}. \tag{7.3.12}$$

We use $H_{\underline{n}}$ to define an automorphism of $gl(n,\mathbb{C})$:

$$\sigma_{\underline{n}} = \exp(2\pi i ad(H_{\underline{n}})) : gl(n,\mathbb{C}) \to gl(n,\mathbb{C}). \tag{7.3.13}$$

If N' is the least common multiple of the parts of the partition $\underline{n}$ then the order N of $\sigma_{\underline{n}}$ is equal to $2N'$ in case $N'(\frac{1}{n_a} + \frac{1}{n_b})$ is odd for some pair of parts n_a, n_b and N is equal to N' otherwise. Then $\sigma_{\underline{n}}$ defines a $\mathbb{Z}/N\mathbb{Z}$ grading of $gl(n,\mathbb{C})$:

(7.3.14)
$$\begin{aligned} gl(n,\mathbb{C}) &= \oplus_{i=0}^{N-1}\mathfrak{g}_i, \\ \mathfrak{g}_i &= \{x \in gl(n,\mathbb{C}) \mid \sigma_{\underline{n}}(x) = \omega^i x\}, \quad \omega = \exp(2\pi i/N). \end{aligned}$$

Next we define the twisted loop algebra:

$$L(\mathfrak{g},\sigma_{\underline{n}}) := \left\{ \sum_{i=m}^{\infty} \mu^i y_{\bar{i}} \mid y_{\bar{i}} \in \mathfrak{g}_{\bar{i}} \right\}. \tag{7.3.15}$$

Write $\mu = \exp(i\theta/N)$ and put $\Phi_{H_{\underline{n}}} = \exp(-i\theta ad(H_{\underline{n}}))$. Then $\Phi_{H_{\underline{n}}} : L(\mathfrak{g},\sigma_{\underline{n}}) \to \tilde{g}l(n,\mathbb{C})$ is an isomorphism between the twisted loop algebra $L(\mathfrak{g},\sigma_{\underline{n}})$ and the untwisted loop algebra $\tilde{g}l(n,\mathbb{C})$, with $\lambda = \mu^N = \exp(i\theta)$ as loop variable.

Define a Cartan subalgebra $\mathfrak{h}_{\underline{n}}$ of $gl(n,\mathbb{C})$ associated to $\underline{n}$ with basis E_a^i, $a = 1,2,\dots,k$ and $i = 1,2,\dots,n_a$, where $E_a = E_{aa}^{n_a 1} + \sum_{j=1}^{n_a-1} E_{aa}^{jj+1}$. Under $\sigma_{\underline{n}}$ the Cartan subalgebra $\mathfrak{h}_{\underline{n}}$ is mapped to itself and so the decomposition (7.3.14) induces a decomposition of $\mathfrak{h}_{\underline{n}}$. We can then define as in (7.3.15) the twisted loop algebra $L(\mathfrak{h}_{\underline{n}}, \sigma_{\underline{n}})$. The image of $L(\mathfrak{h}_{\underline{n}}, \sigma_{\underline{n}})$ under the isomorphism $\Phi_{H_{\underline{n}}}$ is precisely the Heisenberg algebra $\tilde{\mathcal{H}}_{\underline{n}}$ generated by $\mathcal{P}_a$ and $\mathcal{Q}_a := \mathcal{P}_a^{-1}$. In particular the element $\mu^{N/n_a} E_a$ corresponds to $\mathcal{P}_a$. For different $\underline{n}$ one obtains distinct Heisenberg algebras in $\tilde{gl}(n, \mathbb{C})$ and one proves also that *all* Heisenberg algebras of $\tilde{gl}(n, \mathbb{C})$ (up to isomorphism) are obtained in this way.

The decomposition $gl(n, \mathbb{C}) = \mathfrak{h}_{\underline{n}} \oplus \mathfrak{h}_{\underline{n}}^{\perp}$, where $\mathfrak{h}_{\underline{n}}^{\perp}$ is the direct sum of the root spaces with respect to $\mathfrak{h}_{\underline{n}}$, induces a decomposition

$$L(\mathfrak{g}, \sigma_{\underline{n}}) = L(\mathfrak{h}_{\underline{n}}, \sigma_{\underline{n}}) \oplus L(\mathfrak{h}_{\underline{n}}^{\perp}, \sigma_{\underline{n}}). \tag{7.3.16}$$

We also need the twisted loop group $L(\mathfrak{G}, \sigma_{\underline{n}})$, the collection of units in $L(\mathfrak{g}, \sigma_{\underline{n}})$. The image of $L(\mathfrak{G}, \sigma_{\underline{n}})$ under the isomorphism $\Phi_{H_{\underline{n}}}$ is the untwisted loop group $\tilde{Gl}(n, \mathbb{C})$. If $\tilde{g} \in \tilde{gl}(n, \mathbb{C})$, or $\tilde{g} \in \tilde{Gl}(n, \mathbb{C})$, then we write $\tilde{g}^{\sigma_{\underline{n}}}$ for the inverse image in the twisted loop algebra or twisted loop group.

Lemma 7.3.4.

1. *If* $\tilde{g}_- = 1_n + \mathcal{O}(\lambda^{-1})$ *then* $\tilde{g}_-^{\sigma_{\underline{n}}} = 1_n + \mathcal{O}(\mu^{-1})$.
2. *Let* $\tilde{g}_-$ *as above. Then there exist two formal loops* $u^{\sigma_{\underline{n}}} = \sum_{i<0} \mu^i u_i$, $v^{\sigma_{\underline{n}}} = \sum_{i<0} \mu^i v_i$, *with* $u_i \in (\mathfrak{h}_{\underline{n}})_{\bar{\imath}}$, $v_i \in (\mathfrak{h}_{\underline{n}}^{\perp})_{\bar{\imath}}$, *such that*

$$\tilde{g}_-^{\sigma_{\underline{n}}} = \exp(u^{\sigma_{\underline{n}}}) \exp(v^{\sigma_{\underline{n}}}).$$

Proof. The element $H_{\underline{n}}$ induces a $\mathbb{Z}$ grading of $\tilde{gl}(n, \mathbb{C})$ as follows: if $y \in \mathfrak{g}_{\bar{m}}$, then

$$[H_{\underline{n}}, y] = sy, \quad s = \frac{\bar{m}}{N} + \ell\ , \tag{7.3.17}$$

for some $\ell \in \mathbb{Z}$, $\bar{m} = 0, 1, \dots, N-1$. We put then

$$\deg(\lambda^r y) = (r+s)N = (r+\ell)N + \bar{m}. \tag{7.3.18}$$

This is the grading that corresponds to the grading by powers of μ in $L(\mathfrak{g}, \sigma_{\underline{n}})$ under the standard isomorphism: we have: $\Phi_{H_{\underline{n}}}^{-1}(\lambda^r y) = \mu^{(r+\ell)N+\bar{m}} y$. One calculates that the eigenvalues of $H_{\underline{n}}$ are of the form

$$\frac{q}{n_b} - \frac{p}{n_a} + \frac{1}{2n_a} - \frac{1}{2n_b}, \tag{7.3.19}$$

where $1 \le p \le n_a$ and $1 \le q \le n_b$. From this one sees that the absolute value of the eigenvalues of $H_{\underline{n}}$ is strictly less than 1, and that the only possibilities in (7.3.18) are $m = 0, l = 0$ or $0 < m \le N-1$ and $l = 0, -1$. This means that a homogeneous element of the form $\lambda^{-i} y$, $i > 0$, has strictly negative degree and maps to an element of $L(\mathfrak{g}, \sigma_{\underline{n}})$ containing a strictly negative power of μ. This proves part 1. The proof of part 2 is similar to that of Lemma 5.1 in [**BtK**]. $\square$

We return to the formulation of the $\underline{n}$ reduced KP hierarchy in terms of $n \times n$ matrix operators. Let $\partial_y = \sum A_a \partial_{t_1^a}$ and $\mathcal{P}_y = \sum A_a \mathcal{P}_a$. Then one considers the operator

(7.3.20)
$$\begin{aligned} D_y(t,\alpha) &= \overleftarrow{\partial}_y - \tilde{w}_W(t,\alpha)^{-1}\partial_y(\tilde{w}_W(t,\alpha)) = \overleftarrow{\partial}_y - \tilde{L}(t,\alpha)_+, \\ &= \overleftarrow{\partial}_y - \mathcal{P}_y - q(t,\alpha), \end{aligned}$$

for some matrix $q(t,\alpha)$. The components of $q(t,\alpha)$ are then considered the fundamental fields of the theory. Note that $\mathcal{P}_y + q$ is the polynomial part (in λ) of an element in the adjoint orbit through $\mathcal{P}_y$ of the group of elements of the form $g = 1 + \mathcal{O}(\lambda^{-1})$, and hence q is constant in λ. In particular the degree (7.3.18) of the homogeneous components of q is strictly less than the maximum of $\frac{N}{n_a}$, $a = 1, \ldots, k$. The most general form for q is obtained if one chooses the constants A_a such that $\mathcal{P}_y = \sum A_a \mathcal{P}_a$ is regular, i.e., such that the centralizer of it is just $\tilde{H}_{\underline{n}}$. However the theory would work also if $\mathcal{P}_y$ is not regular: then some of the components of q would be zero.

Lemma 7.3.5. *Let $D_y(t,\alpha)$ as above and denote by $D_y^{\sigma_{\underline{n}}}$ the inverse image of this operator in the twisted loop algebra $L(\mathfrak{g}, \sigma_{\underline{n}})$: $D_y^{\sigma_{\underline{n}}} = \overleftarrow{\partial}_y - \sum A_a \mu^{\frac{N}{n_a}} E_a - q^{\sigma_{\underline{n}}}$. Then there exists a unique formal power series $v^{\sigma_{\underline{n}}} = \sum_{i<0} \mu^i v_i$, with $v_i \in (\mathfrak{h}_{\underline{n}}^{\perp})_{\bar{\imath}}$, such that*

(7.3.21)
$$\exp(ad(v^{\sigma_{\underline{n}}}))(D_y^{\sigma_{\underline{n}}}) = \overleftarrow{\partial}_y - \sum A_a \mu^{\frac{N}{n_a}} E_a + \sum_{i<\max(N/n_a)} k_i \mu^i, k_i \in (\mathfrak{h}_{\underline{n}})_{\bar{\imath}}.$$

The components of v_i and k_i are y-differential polynomials in the components of $q^{\sigma_{\underline{n}}}$ and hence in the components of $q(t,\alpha)$.

Proof. First note that trying to construct $v^{\sigma_{\underline{n}}}$ directly from (7.3.21) by expanding in powers of μ seems not to lead in the usual way (cf. [**DS**]) to a simple recursion scheme for the v_i because of the inhomogeneity of $\mathcal{P}_y$. Therefore we break up D_y in pieces corresponding to the homogeneous components of $\mathcal{P}_y$.

Indeed, let $\mathcal{P}_y + q(t,\alpha) = [Ad(\tilde{g}_-)(\mathcal{P}_y)]_+$ for $\tilde{g}_- = 1 + \mathcal{O}(\lambda^{-1})$. Define $\mathcal{P}_a + q_a = [Ad(\tilde{g}_-)(\mathcal{P}_a)]_+$ and $D_a = \overleftarrow{\partial}_{t_1^a} - \mathcal{P}_a - q_a$. We have $D_y = \sum_{a=1}^k A_a D_a$, and

$q = \sum_{a=1}^{k} A_a q_a$. We will write as in Lemma 7.3.4 $\tilde{g}_-^{\sigma_{\underline{n}}} = \exp(u^{\sigma_{\underline{n}}})\exp(v^{\sigma_{\underline{n}}})$. Define $\tilde{w} = \tilde{w}_0 \tilde{g}_-$, so that $\tilde{w}^{\sigma_{\underline{n}}} = \tilde{w}_0 \exp(u^{\sigma_{\underline{n}}})\exp(v^{\sigma_{\underline{n}}})$ and we have

$$\tilde{w}^{\sigma_{\underline{n}}} D_a^{\sigma_{\underline{n}}} = 0. \tag{7.3.22}$$

From this one easily sees that for all a:

$$\exp(ad(v^{\sigma_{\underline{n}}}))(D_a^{\sigma_{\underline{n}}}) = \overleftarrow{\partial}_{t_1^a} - \mu^{\frac{N}{n_a}} E_a + \sum_{i<N/n_a} k_i^{(a)} \mu^i, \quad k_i^{(a)} \in (\mathfrak{h}_{\underline{n}})_{\bar{\imath}}, \tag{7.3.23}$$

and hence also that $\exp(ad(v^{\sigma_{\underline{n}}}))(D_y^{\sigma_{\underline{n}}})$ is of the same form, proving therefore the existence of a $v^{\sigma_{\underline{n}}}$ such that (7.3.21) holds, cf. [**BtK**]. Now fix an a between 1 and k. Comparing the coefficient of $\mu^{N/n_a - j}$ on both sides of (7.3.23) we obtain an equation of the form

$$\begin{aligned}[v_{-j}, E_a] + k_{N/n_a - j}^{(a)} = {} & \text{polynomial in } q_a^{\sigma_{\underline{n}}} \text{ and in} \\ & v_{-l}, -l > -j, \text{ and } t_1^a \text{ derivatives.}\end{aligned} \tag{7.3.24}$$

Using the partition $\underline{n}$ we decompose v_{-j} in blocks: write $v_{-j} = \sum_{a,b=1}^{k} v_{-j,ab}$, where the matrix $v_{-j,ab}$ is zero outside the block with index ab of size $n_a \times n_b$. Then the commutator in the left hand side of (7.3.24) becomes $\sum_{b=1}^{k} A_a[v_{-j,ba} C_{n_a} - C_{n_a} v_{-j,ab}]$, where $C_p = E_{p1} + \sum E_{ii+1}$ is a cyclic matrix of size p. Now C_p acts from the left (right) as an invertible linear transformation on the space of $p \times q$ ($q \times p$) matrices for any q. Furthermore the adjoint action of C_p is an invertible linear transformation on the orthogonal complement of the linear span of the powers in C_p in the space of $p \times p$ matrices. Using these facts one can express the blocks $v_{-j,ab}$, $v_{-j,ba}$ for $b = 1, \dots, k$ and $k_{N/n_a - j}^{(a)}$ uniquely in terms of $q_a^{\sigma_{\underline{n}}}$ and the v_{-l}, $-l > -j$, and their t_1^a derivatives. Letting now also a run from 1 to k we express *all* blocks of v_{-j} in terms of these variables. Note that to find $v_{-j,ab}$ according to this method one can consider the diagonalization (7.3.23) of D_a or of D_b. These must give the same result, since the consistency of the procedure it ensured by the existence of at least one solution $v^{\sigma_{\underline{n}}}$. Induction leads now to the conclusion that $v^{\sigma_{\underline{n}}}$ and $\sum k_i \mu^i$ (with $k_i = \sum A_a k_i^{(a)}$) are polynomials in $q_a^{\sigma_{\underline{n}}}$ and t_1^a derivatives, $a = 1, \dots, k$. Next we use the fact (which follows from (7.3.22)) that

$$0 = [D_y, D_a] = [\overleftarrow{\partial}_y - \mathcal{P}_y - q, \overleftarrow{\partial}_{t_1^a} - \mathcal{P}_a - q_a]. \tag{7.3.25}$$

This gives us an expression for $\partial_{t_1^a}(q)$ in terms of derivatives of q_a with respect to y. Since the projection $q \mapsto q_a$ is just differentiation with respect A_a we find that we can express all t_1^a derivatives of q_a in terms of y derivatives and the Lemma follows. □

From the proof of the last Lemma it follows that all resolvents, being of the form $R(p) = \tilde{w}^{-1} \cdot p \cdot \tilde{w} = \tilde{g}_-^{-1} \cdot p \cdot \tilde{g}_- = \exp(-ad(v))(p)$, are differential polynomials of the fundamental field $q(t, \alpha)$. This makes the following definition reasonable:

Definition 7.3.6. The $\underline{n}$-KdV hierarchy is the collection of deformation equations of the operator D_y (of the form (7.3.20) with $\mathcal{P}_y + q(t,\alpha) = [Ad(\tilde{g}_-)(\mathcal{P}_y)]_+$ for some $\tilde{g}_- = 1 + \mathcal{O}(\lambda^{-1})$) given by

$$\begin{aligned} \partial_{t_i^b} D_y &= [(\tilde{\mathcal{R}}_b^i)_+, D_y], \\ D_y(t, \alpha + \alpha_i) &= (\tilde{\mathcal{U}}_{\alpha_i})_+ (D_y(t,\alpha))(\tilde{\mathcal{U}}_{\alpha_i})_+^{-1}. \end{aligned} \tag{7.3.26}$$

Here $(\tilde{\mathcal{R}}_b^i)_+$ and $(\tilde{\mathcal{U}}_{\alpha_i})_+$ are the positive parts of the resolvents (7.3.10), using $\tilde{w} = \tilde{w}_0 \tilde{g}_-$ instead of $\tilde{w}_W$.

Note that we have made here the simple choice $\partial_y = \sum A_a \partial_{t_1^a}$ to determine the "spatial variable" y in our system of equations (7.3.6). It will be clear from the construction that we can make much more general choices for this spatial variable and still get a reasonable set of equations, cf. [**FNR**].

As an example of a $\underline{n}$-KdV hierarchy consider the 2-reduction of the Davey-Stewartson-Toda system of Section 4.6, for the partition $2 = 1 + 1$. The resulting system is the AKNS-Toda system: the field Q^m, being the second x derivative of the τ-function, see (5.5.32), is in the 2-periodic case identically zero, since the τ-function is now independent of x. Furthermore, in the resolvents and lattice resolvents the operator $\overleftarrow{\partial}$ can be effectively replaced by the variable z; this is because

$$w_W \overleftarrow{\partial} = w_0 \cdot \overleftarrow{w}_W \overleftarrow{\partial} = z w_0 \cdot \overleftarrow{w}_W + w_0 \partial(\overleftarrow{w}_W) = w_W, \tag{7.3.27}$$

since also the wave operator is x independent in the 2-periodic case. This gives the theory described in [**BtK**].

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